HOMEWORK ASSIGNMENT – WEEK 12
Hand-in deadline: Thursday 3 July by 12 p.m. in the "MSP" drop box.
Info: `www.math.lmu.de/~michel/SS14_MSP.html`

Exercise 33. (A toy model for the Epstein-Glaser renormalisation.)

Consider the measurable function $f : \mathbb{R}^4 \to \mathbb{R}$, $f(x) = |x|^{-4}$. Note that $f \notin L^1_{\mathrm{loc}}(\mathbb{R}^4)$ and therefore it cannot be viewed as a distribution T_f on the test functions $\mathcal{D}(\mathbb{R}^4)$ in the canonical way.

(i) Consider the subspace $\mathcal{D}_0(\mathbb{R}^4) := \{g \in \mathcal{D}(\mathbb{R}^4) \,|\, g(0) = 0\}$ and the map $T_f : \mathcal{D}_0(\mathbb{R}^4) \to \mathbb{C}$ defined by

$$T_f(\varphi) \;:=\; \int_{\mathbb{R}^4} \mathrm{d}x \, \frac{\varphi(x)}{|x|^4} \,.$$

Prove that $T_f(\varphi)$ is finite for every $\varphi \in \mathcal{D}_0(\mathbb{R}^4)$ and that the map T_f is linear and continuous in the topology of distributions.

(ii) Let $w \in \mathcal{D}(\mathbb{R}^4)$ be such that $w(0) = 1$ and define the map $\widetilde{T}_f^{(w)} : \mathcal{D}(\mathbb{R}^4) \to \mathbb{C}$ by

$$\widetilde{T}_f^{(w)}(\varphi) \;:=\; \int_{\mathbb{R}^4} \mathrm{d}x \, \frac{\varphi(x) - w(x)\varphi(0)}{|x|^4} \,.$$

Prove that $\widetilde{T}_f^{(w)} \in \mathcal{D}'(\mathbb{R}^4)$ (i.e., $\widetilde{T}_f^{(w)}$ is a distribution) and that $\widetilde{T}_f^{(w)} \equiv T_f$ on $\mathcal{D}_0(\mathbb{R}^4)$ (i.e., $\widetilde{T}_f^{(w)}$ is an extension of T_f).

(iii) Let $\lambda > 0$ and consider the scaling transformation $D_\lambda : \mathcal{D}(\mathbb{R}^4) \to \mathcal{D}(\mathbb{R}^4)$, $(D_\lambda\varphi)(x) := \varphi(\lambda x)$. Prove that $\mathcal{D}_0(\mathbb{R}^4)$ is scale-invariant and so is T_f, i.e.,

$$T_f \circ D_\lambda \;=\; T_f \qquad \text{on } \mathcal{D}_0(\mathbb{R}^4),$$

and compute the quantity $\widetilde{T}_f^{(w)}(D_\lambda\varphi) - \widetilde{T}_f^{(w)}(\varphi)$ for a generic $\varphi \in \mathcal{D}(\mathbb{R}^4)$.

Exercise 34. (Duhamel's two-point function: thermal expectation and Bogolubov inequality.)

Consider the C^*-algebra $\mathcal{A} = \mathcal{M}(n \times n, \mathbb{C})$, $n \in \mathbb{N}$, a Hamiltonian $H = H^* \in \mathcal{A}$, the corresponding Gibbs state ω_β, and the Duhamel's two-point function

$$\langle A, B\rangle_\beta \;:=\; \frac{1}{Z(\beta)} \int_0^1 \mathrm{Tr}\big(e^{-s\beta H} A \, e^{-(1-s)\beta H} B\big) \, \mathrm{d}s$$

where $\beta > 0$ and $A, B \in \mathcal{A}$.

(i) Prove that

$$\begin{aligned} \langle A,B\rangle_\beta &= \langle B,A\rangle_\beta\,, \\ |\langle A,B\rangle_\beta|^2 &\leqslant \langle A^*,A\rangle_\beta\,\langle B^*,B\rangle_\beta \end{aligned}$$

for all $A,B\in\mathcal{A}$. Is the thermal two-point function $\omega_\beta(A,B)$ also symmetric?

(ii) Express the thermal expectation value $\omega_\beta(A)$ using the Duhamel's two-point function. Conversely, set $\tau_t(A):=e^{\mathrm{i}tH}A\,e^{-\mathrm{i}tH}$, $t\in\mathbb{R}$, $A\in\mathcal{A}$ and denote by τ_z the analytic continuation of τ_t to the strip $|\mathfrak{Im}\,z|\leqslant 1$. Prove that

$$\langle A,B\rangle_\beta \;=\; \int_0^1 \omega_\beta(B\tau_{\mathrm{i}s\beta}(A))\,\mathrm{d}s\,.$$

(iii) Prove, for all $A\in\mathcal{A}$, that

$$\langle A^*,A\rangle_\beta \;\leqslant\; \frac{1}{2}\,\omega_\beta(A^*A+AA^*)\,.$$

(*Hint:* use a convexity argument for the function $h_\beta(s):=\mathrm{Tr}(e^{-s\beta H}A^*\,e^{-(1-s)\beta H}A)$.)

(iv) Prove, for all $A,B\in\mathcal{A}$, that

$$\omega_\beta([A,B]) \;=\; \langle [A,\beta H],B\rangle_\beta$$

and deduce Bogolubov's inequality

$$|\omega_\beta([A,B])|^2 \;\leqslant\; \frac{\beta}{2}\,\omega_\beta([A^*,[H,A]])\,\omega_\beta(B^*B+BB^*)\,.$$

Exercise 35. (Reflection positivity and the Laplacian)

Consider

- the quantity

$$E(f,g) \;:=\; \iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \mathrm{d}x\,\mathrm{d}y\,\overline{f(x)}\,\frac{1}{|x-y|}\,g(y)\,, \qquad f,g\in C_0^\infty(\mathbb{R}^3,\mathbb{C})$$

  (note that $E(f,g)$, when f and g are real-valued, is nothing but the Coulomb interaction energy between two charge distributions f and g expressed in appropriate units);
- a plane Σ in the Euclidean space $\mathbb{R}^3$ and the mirror symmetry transformation $\mathcal{R}_\Sigma:\mathbb{R}^3\to\mathbb{R}^3$ with respect to the plane Σ;
- the corresponding transformation $\theta_\Sigma : C_0^\infty(\mathbb{R}^3,\mathbb{C})\to C_0^\infty(\mathbb{R}^3,\mathbb{C})$, $(\theta_\Sigma h)(x):=h(\mathcal{R}_\Sigma x)$;
- two functions $f,g\in C_0^\infty(\mathbb{R}^3,\mathbb{C})$ whose support are separated by Σ (and hence their supports lie in the two distinct open half-spaces of $\mathbb{R}^3$ determined by Σ).

Prove that

(i) $E(\theta_\Sigma g,g)\;\geqslant\;0$

(ii) $|E(f,g)|^2\;\leqslant\;E(f,\theta_\Sigma f)\,E(\theta_\Sigma g,g)\,.$

(*Remark:* albeit a non-trivial mathematical statement, *reflection-positivity* is the simple physical result that the interaction of a charge distribution and its mirror is repulsive.)

Background for Exercise 33. So far in class we have studied the *free* Bose and Fermi gases. Including interactions is more complicated and is often approached perturbatively. The argument proceeds along the following lines.

For the free gas the Hamiltonian is $\mathrm{d}\Gamma(h)$, derived from the one-particle Hamiltonian h. In terms of creation and annihilation operators that can be written as

$$H_0 \;:=\; \mathrm{d}\Gamma(h) \;=\; \sum_n E_n\, a^*(\psi_n)a(\psi_n)$$

where the ψ_n's area basis of eigenvectors of h and $h\psi_n = E_n\psi_n$. One can also define operator-valued distributions a^*_x, a_x so that

$$a(f) \;=\; \int \mathrm{d}x\, \overline{f(x)}\, a_x\,, \qquad a^*(f) \;=\; \int \mathrm{d}x\, f(x)\, a^*_x\,,$$

and re-write H_0 accordingly. In this language, a *local* interaction could be, for example,

$$H_{\mathrm{int}} \;=\; g\int \mathrm{d}x\, (a^*_x a_x)^2\,.$$

Thus, formally, a Gibbs state would look like

$$\omega_{\mathrm{int}}(A) \;\sim\; \mathrm{Tr}\big(e^{-\beta(H_0+H_{\mathrm{int}})}A\big) \;=\; \omega_0(e^{-\beta H_{\mathrm{int}}}A)$$

where ω_0 is the KMS state for the *free* bose gas.

In the next step one expands $e^{-\beta H_{\mathrm{int}}}$ in powers of the coupling g and pulls the sum out of ω_0. This leads to expressions of the form

$$\omega_0\Big(\,\Pi_k\Big(g\int \mathrm{d}x_k\, (a^*_{x_k}a_{x_k})^2\Big)A\Big)$$

which can be evaluated in terms of $\omega_0(a^*(f)a(g))$ by means of Wick's theorem (assuming that A is a polynomial in the a's and a^*'s).

For $h = -\Delta$ it turns out that

$$\omega_0(a^*_x a_x) \;\sim\; \frac{1}{|x-y|^2} \qquad \text{as } |x-y| \to 0.$$

Thus, already at the order g^2 in the expansion, expressions like $\int \mathrm{d}x_k\Big(\frac{1}{|x_k-y|^2}\Big)^2$ appear, that have to be given distributional meaning, whence the type of problems as in Exercise 33.

The result of part (iii) is that for the regularised distribution a change of scale effectively changes g, precisely in the spirit of the renormalisation group.