HOMEWORK ASSIGNMENT – WEEK 11
Hand-in deadline: Thursday 26 June by 12 p.m. in the "MSP" drop box.
Info: `www.math.lmu.de/~michel/SS14_MSP.html`

Exercise 30. (The lattice BCS model.)

Consider, among the class of quantum spin systems studied in Exercise 29 in the mean field regime, the special case of a one-dimensional spin-$\frac{1}{2}$ model where the Hamiltonian H_Λ is given, with respect to the notation of Exercise 29, by

$$d=1,\quad s=\frac{1}{2},\quad A=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad B=-\lambda(C^*\otimes C+C\otimes C^*),\quad C=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \lambda\geqslant 0.$$

(i) Prove that a generic 2×2 density matrix ρ can be parametrised as follows:

$$\rho\;=\;\begin{pmatrix}\frac{1}{2}+r&\mu\\ \overline{\mu}&\frac{1}{2}-r\end{pmatrix},\qquad r\in\mathbb{R},\qquad \mu\in\mathbb{C},\qquad r^2+|\mu|^2\leqslant\frac{1}{4}.$$

(ii) Compute the reduced Hamiltonian H_ρ of the model relative to a density matrix ρ parametrised as in (i).

(iii) Compute $e^{-\beta H_\rho}$, $\beta>0$.

(iv) Write and solve the gap equation (see Exercise 29(ii)) for this model. Plot the parameter r and $|\mu|$ of the solutions as a function of β.

(v) Which of the solutions found in (iv) actually maximises f_β?

Exercise 31. (Application of RG: a quantum flute – or a bosonic string)

Consider a cigar-shaped tube of length π and assume that the pressure inside it can be modelled by a function $p(t,x)$ of the time t and of the coordinate x along the tube. Consider the classical Hamiltonian

$$H\;=\;\int_0^\pi \mathrm{d}x\,\Big(\Big(\frac{\partial p}{\partial t}\Big)^2+\Big(\frac{\partial p}{\partial x}\Big)^2\Big)\,.$$

(i) Assume Dirichlet boundary conditions $p(t,0)=p(t,\pi)=0$. Re-write H in terms of the Fourier transform of $p(t,x)$ with respect to x and prove that H takes this way the form of an infinite sum of classical harmonic oscillators of mass $\frac{1}{2}$,

$$H\;=\;\sum_{k=1}^{\infty}h_k\,,\qquad h_k:=\Big(\frac{\mathrm{d}p_k}{\mathrm{d}t}\Big)^2+\,\omega_k\,p_k^2\,,$$

of which you have to specify the frequency ω_k.

(ii) Consider the quantisation of the Hamiltonian H obtained in (i), consisting of the replacement of each mode p_k with the operator of multiplication times x_k and correspondingly of $\frac{\mathrm{d}p_k}{\mathrm{d}t}$ with the differential operator $-\mathrm{i}\frac{\partial}{\partial x_k}$. Let E_k be the ground state energy of the harmonic oscillator h_k. Prove that $\sum_{k=1}^{\infty} E_k = +\infty$.

(iii) To cure the divergence found in (ii), assume that H can be regularised to some H_{reg} in such a way that each harmonic oscillator h_k has a ground state damped by a factor e^{-a/λ_k}, where a is a reference distance (say, the typical inter-atomic distance) and λ_k is the wave length of the mode k. Correspondingly, consider $\widetilde{E} := \sum_{k=1}^{\infty} E_k e^{-a/\lambda_k}$. Prove the following asymptotics

$$\widetilde{E} \;=\; \frac{1}{a^2} - \frac{1}{12} + O(a) \qquad \text{as } a \to 0.$$

(iv) An immediate consequence of (iii) is that re-normalising H_{reg} by

$$H_{\mathrm{reg}} \;\mapsto\; H' \;:=\; H_{\mathrm{reg}} - \int_0^{\pi} \mathrm{d}x \, \frac{1}{a^2} \,,$$

namely adding a contribution which is p-independent (thus, not affecting the equation of motion) and preserves locality, the ground state energy of H' stays finite as $a \to 0$. Discuss the value of this ground state in relation to the quantity $\zeta(-1)$, where ζ is the Riemann zeta function.

Exercise 32. For each of the 10 questions below answer YES or NO and provide a brief explanation.

32.1 Is there a unique KMS state for an infinite block of iron at room temperature?

❑ YES ❑ NO

32.2 Consider the C^*-algebra $\mathcal{A} = \mathcal{L}(L^2(\mathbb{R}))$ and the time evolution $\tau_t(A) = e^{\mathrm{i}tH} A e^{-\mathrm{i}tH}$, $t \in \mathbb{R}$, where H is the Hamiltonian of the 1D harmonic oscillator, namely $(Hf)(x) := -f''(x) + x^2 f(x)$ on the domain $\mathcal{D} = \{f \in L^2(\mathbb{R}) \,|\, -f'' + x^2 f \in L^2(\mathbb{R})\}$. Is τ_t asymptotically abelian?

❑ YES ❑ NO

32.3 With the usual meaning of the symbols from class and homework, consider the 1D Ising model at non-zero temperature T and with magnetic field B. Is it true that at the Renormalisation Group limit point one has $\langle S_1 S_2 \rangle = \langle S_1 \rangle \langle S_2 \rangle$?

❑ YES ❑ NO

32.4 With the usual meaning of the symbols from class and homework, consider the 3D Ising model. If you plot the logarithm of the spontaneous magnetization as a function of $\log \frac{(T-T_c)}{T_c}$ for T approaching T_c from below, do you get a straight line?

❑ YES ❑ NO

32.5 Consider the spin-$\frac{1}{2}$ 2D Heisenberg model on $\mathbb{Z}^2$ and, for every $L \in \mathbb{Z}$, consider the finite sub-lattice Λ_L consisting of the square of size L centred at the origin. Is it possible to find for every L a finite sub-lattice $\Lambda'_L \supsetneq \Lambda_L$ and a configuration on the whole $\mathbb{Z}^2$ with all spins up inside Λ_L and all spins down outside Λ'_L such that $\lim_{L\to\infty} \mathcal{E}(L)$ is finite? ($\mathcal{E}(L) :=$ energy of the configuration relative to the couple Λ_L, Λ'_L.)

❑ YES ❑ NO

32.6 Consider an “Ideal Bose Gas consisting of one-body harmonic oscillators”, i.e., with respect to Fock-space discussion of the Ideal Bose Gas done in class and homework, assume that the one-body Hamiltonian is a harmonic oscillator of frequency ω. Does BEC occur in the limit $\omega \to 0$?

❏ YES ❏ NO

32.7 Consider a process in which at each step t a particle jumps with probability $\frac{1}{3}$ from a corner of a given cube to one of the three neighbouring corners. Is this a Markov process that converges to a probability distribution in the limit $t \to \infty$?

❏ YES ❏ NO

32.8 Let $p, q \in (0,1)$ such that $p+q<1$. Consider four sites S_1, S_2, S_3, and S_4, and a process in which at each step t a particle jumps from one site to another, say $S_j \to S_k$, with this prescription:

- $S_j \to S_j$ with probability p ($j = 1, 2, 3, 4$),
- $S_1 \to S_2$ or $S_4 \to S_3$ with probability $1-p$,
- $S_2 \to S_1$ or $S_3 \to S_4$ with probability $1-p-q$,
- $S_2 \to S_3$ or $S_3 \to S_2$ with probability q.

According to what was discussed in class, this gives in fact a Markov process that converges to a limiting distribution ρ_∞ (because of aperiodicity and irreducibility) and, if $\rho(t)$ is the distribution at time t, the displacement $\rho(t) - \rho_\infty$ vanishes as $e^{-\mu t}$ for some $\mu > 0$. Is it true that $\mu = O(q)$ as $q \to 0$?

❏ YES ❏ NO

32.9 Consider a C^*-dynamical system $(\mathcal{A}, \tau^t)$, a (τ^t, β)-KMS state ω ($\beta > 0$) on $\mathcal{A}$, and a $*$-automorphism α on $\mathcal{A}$. Is it true that, if ω is α-invariant, so is τ^t?

❏ YES ❏ NO

32.10 Consider the C^*-algebra $\mathcal{A} = \mathcal{M}(3 \times 3, \mathbb{C})$, the state $\rho = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$, a dynamics τ^t on $\mathcal{A}$ such that ρ is a $(\tau^t, 1)$-KMS state, and the manifold

$$\mathcal{M}_\tau := \left\{ \widetilde{\rho} \,|\, \widetilde{\rho} \text{ is a state on } \mathcal{A} \text{ and } \widetilde{\rho} \circ \tau^t = \widetilde{\rho} \ \forall t \in \mathbb{R} \right\}$$

of all τ^t-invariant states on $\mathcal{A}$. Is it true that $\dim_{\mathbb{R}} \mathcal{M}_\tau = 4$?

❏ YES ❏ NO