PROBLEM IN CLASS – WEEK 5

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/SS12_FA.html`.

Problem 17. (A continuous function on $X \times Y$ is continuous in each entry, not vice versa. Closure and interior of the product.)

Let X, Y, and Z be topological or metric or normed spaces. Give $X \times Y$ the product topology.

(i) Is it true that if every $E \subset X$ is homeomorphic to some $F \subset Y$ and every $F \subset Y$ is homeomorphic to some $E \subset X$ then X and Y are homeomorphic? Justify your answer.

(ii) Show that if $f : X \times Y \to Z$ is continuous then it is separately continuous in each variable (i.e., for each $y \in Y$ the function $x \mapsto f(x, y)$ is continuous from X to Z, and similarly for each $x \in X$). Show with a counterexample that the converse does not hold.

(iii) Show that if $E \subset X$ and $F \subset Y$ then $\overline{E \times F} = \overline{E} \times \overline{F}$ and $(E \times F)^\circ = \mathring{E} \times \mathring{F}$.

Problem 18. (When $E + F$ is open, or closed, or compact in a normed space.)

Let $(X, \| \, \|)$ be a normed space. Let $E \subset X$ and $F \subset X$, non empty.

(i) Show that if E is open then $E + F$ is open.

(ii) Show that if E is closed and F is compact then $E + F$ is closed.

(iii) Give an example of closed sets E and F for which $E + F$ is not closed.

(iv) Show that if E and F are compact, so is $E + F$.

Problem 19. (Completion is closure, if the ambient space is complete. Compactness = complete + totally bounded.)

(i) Let (X, d) be a complete metric space. Let $E \subset X$. Identify the completion of E (up to isomorphism, of course: see Problem 15).

(ii) Let (X, d) be a metric space. Show that X is compact if and only if X is complete and totally bounded.(*) Note that this reverts Theorem 1.49 in class.

(*) To facilitate you in consulting the literature recommended for this course (see course web page), we use here TOTALLY BOUNDED as a synonymous of PRE-COMPACT. In fact, pre-compact is the term defined in class (Def. 1.48) and used by Dobrowolski and Lax, whereas for the same property Bollobas, Kaballo, Reed and Simon, Rudin, and Werner use totally bounded (while they use pre-compact for a set whose closure is compact).

Problem 20. (The closure of c_{00} in ℓ^p. When the closure of an open ball is the closed ball.)

(i) Consider the space

$$c_{00} := \Big\{ x = (x_1, x_2, x_3, \dots) \,|\, x_n \in \mathbb{C} \text{ and } x_n = 0 \text{ for all but a finite number of } n\text{'s} \Big\}.$$

Find the closure of c_{00} in ℓ^p, $1 \leqslant p \leqslant \infty$.

(ii) Show that for any metric space (X, d), the following are equivalent:

(a) $\forall x \in X$ and $\forall r > 0$, the closure of the open ball of radius r around x is the closed ball of radius r: $\overline{B_r(x)} = K_r(x)$.

(b) $\forall x, y \in X$, $x \neq y$, and $\forall \varepsilon > 0$, $\exists\, z \in X$ with $d(y, z) < \varepsilon$ and $d(x, z) < d(x, y)$.

(This supplement Remark 1.29(3) discussed in class.)

(iii) Let $(X, \| \, \|)$ be a normed space. Show that $\forall x \in X$ and $\forall r > 0$ $\overline{B_r(x)}^{\| \, \|} = K_r(x)$.

(This proves the comment stated after Theorem 2.8 in class.)