Institute of Mathematics, LMU Munich – Spring Term 2012
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PROBLEM IN CLASS – WEEK 3

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/SS12_FA.html`.

Problem 9. (Finite product topology: separability, countability, closed graph.)

(i) Show that the product of two separable topological spaces is separable.

(ii) Show that the product of two first (second) countable topological spaces is first (second) countable. (This proves Remark 1.24(2) stated in class.)

(iii) Show that $\mathbb{R}^2 \setminus \{(0,0)\}$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.

(iv) Let $f : X \to Y$ be a map between topological spaces X and Y and consider its GRAPH $\Gamma(f) := \{(x,y) \in X \times Y \,|\, y = f(x)\}$. Show that if Y is a Hausdorff space and f is continuous then $\Gamma(f)$ is closed in $X \times Y$ with respect to the product topology.

(v) Follow-up to (iv): is it true, conversely, that if Y is Hausdorff and $\Gamma(f)$ is closed then f is continuous? Give a proof or a counterexample.

Problem 10. (The weak topology: base, universal property. Product topology and product convergence. Pointwise convergence topology.)

Let $\mathscr{F}$ be a family of functions from a set X to a topological space $(Y, \mathcal{T}_Y)$. The $\mathscr{F}$-WEAK (or $\mathscr{F}$-INITIAL) topology $\mathcal{T}_{\mathrm{w}}$ on X is the weakest topology for which all the functions $f \in \mathscr{F}$ are continuous. ($\mathcal{T}_{\mathrm{w}}$ certainly exists, see Problem 1(iii).)

(i) Show that the family of all finite intersections of sets of the form $f^{-1}(\mathcal{O})$, where $f \in \mathscr{F}$ and $\mathcal{O} \in \mathcal{T}_Y$, is a base for the $\mathscr{F}$-weak topology $\mathcal{T}_{\mathrm{w}}$.

(ii) (Universal property.) Given another topological space $(Z, \mathcal{T}_Z)$ and a function $g : (Z, \mathcal{T}_Z) \to (X, \mathcal{T}_{\mathrm{w}})$, show that g is continuous if and only if $f \circ g : (Z, \mathcal{T}_Z) \to (Y, \mathcal{T}_Y)$ is for all $f \in \mathscr{F}$.

(iii) Given a sequence $\{x_n\}_{n=1}^{\infty}$ and a point x in X, show that $x_n \xrightarrow[n\to\infty]{(\mathcal{T}_{\mathrm{w}})} x$ if and only if $f(x_n) \xrightarrow[n\to\infty]{(\mathcal{T}_Y)} f(x)$ for all $f \in \mathscr{F}$.

Application to the product topology. Consider two topological spaces $(Z, \mathcal{T}_Z)$ and $(Y, \mathcal{T}_Y)$ and let $\pi_Z : Z \times Y \to Z$ and $\pi_Y : Z \times Y \to Y$ be the projections onto each factor.

(iv) Show that the product topology on $Z \times Y$ is precisely the $\{\pi_Z, \pi_Y\}$-weak topology.

(v) Show that a $Z \times Y \ni (z_n, y_n) \xrightarrow[n\to\infty]{} (z,y) \in Z \times Y$ in the product topology if and only if $z_n \xrightarrow[n\to\infty]{(\mathcal{T}_Z)} z$ and $y_n \xrightarrow[n\to\infty]{(\mathcal{T}_Y)} y$. (This proves Remark 1.24(3) stated in class.)

Application: on the space $C([0,1])$ of real-valued continuous functions on $[0,1]$ consider the so-called TOPOLOGY OF POINTWISE CONVERGENCE, i.e., the $\mathscr{F}$-weak topology given by the family $\mathscr{F} := \{\mathbb{E}_x \,|\, x \in [0,1]\}$, where $\mathbb{E}_x : C([0,1]) \to \mathbb{R}$ acts as $\mathbb{E}_x(f) := f(x)$.

(vi) Show that a sequence $\{f_n\}_{n=1}^{\infty}$ in $C([0,1])$ converges as $n \to \infty$ to $f \in C([0,1])$ in the topology of pointwise convergence if and only if $|f_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0$ for each $x \in [0,1]$.

Problem 11. (Examples of metrics on $\mathbb{R}^d$ and on $C([0,1])$)

(i) Which of the following $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ maps are metrics? Give a proof.

- $d_p(x,y) := \Big(\sum_{j=1}^{d} |x_j - y_j|^p\Big)^{1/p}$ for $p \in [1,\infty)$ and for $p \in (0,1)$
- $d''(p)(x,y) := \sum_{j=1}^{d} |x_j - y_j|^p$ for $p \in (0,1]$ and for $p \in (1,\infty)$
- $d_\infty(x,y) := \max_{j=1,\dots,d} |x_j - y_j|$
- $d_\phi(x,y) := \phi(d(x,y))$ where d is some metric on $\mathbb{R}^d$ and $\phi : [0,\infty) \to \mathbb{R}$ is non-decreasing, concave, and vanishing only at the origin.

(ii) Show that d_p, $p \in [1,\infty)$, and d_∞ considered above generate the same topology in $\mathbb{R}^d$.

(iii) Which of the following $C([0,1]) \times C([0,1]) \to C([0,1])$ maps are metrics? Give a proof.

- $d_1(f,g) := \int_0^1 |f(x) - g(x)|\,\mathrm{d}x$
- $d_\infty(f,g) := \max_{x\in[0,1]} |f(x) - g(x)|$
- $\widetilde{d}(f,g) := |f(x_0) - g(x_0)|$ for some $x_0 \in [0,1]$.

Problem 12. (Topological and uniform equivalence of metrics.)

Here d, d_1, d_2, etc., denote metrics on a space X. Two metrics on X are said (TOPOLOGICALLY) EQUIVALENT if they determine the same opens. Two metrics d_1 and d_2 on X are said UNIFORMLY EQUIVALENT if $c\, d_1(x,y) \leqslant d_2(x,y) \leqslant C\, d_1(x,y)$ for some constants $c, C > 0$ and for all $x \in X$. Clearly, uniform equivalence $\Rightarrow$ topological equivalence.

(i) Show that d_1 and d_2 are topologically equivalent $\Leftrightarrow$ the convergent sequences in (X,d_1) are the same as the convergent sequences in (X,d_2)

(ii) Show that the metrics d, $\min\{1,d\}$, $\frac{d}{1+d}$ are topologically (but not uniformly) equivalent. (Hence, every metric is topologically equivalent to a bounded metric.)

(iii) Show that d_1 and d_∞ considered in Problem 11(iii) are not equivalent. Is it true that one generates a finer topology than the other?