Functional Analysis

Institute of Mathematics, LMU Munich – Spring Term 2012
Prof. T. Ø. Sørensen Ph.D, A. Michelangeli Ph.D

PROBLEM IN CLASS – WEEK 1

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/SS12_FA.html`.

Problem 1. (Closure and continuity in the discrete and indiscrete topologies.)

Given a set X, let $\mathcal{T}_{\text{discr}}$ and $\mathcal{T}_{\text{indiscr}}$ be the DISCRETE TOPOLOGY (i.e., the topology where the opens are all the subsets of X) and the INDISCRETE TOPOLOGY (i.e., the topology where the only opens are $\emptyset$ and X) respectively.

(i) Let $E \subset X$. Find the closure $\overline{E}$ of E with respect to $\mathcal{T}_{\text{discr}}$.

(ii) Let $E \subset X$. Find the closure $\overline{E}$ of E with respect to $\mathcal{T}_{\text{indiscr}}$.

(iii) Let $(Y, \mathcal{T})$ be another topological space. Show that every map $(X, \mathcal{T}_{\text{discr}}) \to (Y, \mathcal{T})$, and every map $(Y, \mathcal{T}) \to (X, \mathcal{T}_{\text{indiscr}})$ is continuous.

(iv) Assume that the topology $\mathcal{T}$ in the topological space $(Y, \mathcal{T})$ is *not* discrete. Produce a topological space $(\widetilde{Y}, \widetilde{\mathcal{T}})$ and a function $f : (Y, \mathcal{T}) \to (\widetilde{Y}, \widetilde{\mathcal{T}})$ that is *not* continuous.

(v) Assume that the topology $\mathcal{T}$ in the topological space $(Y, \mathcal{T})$ is *not* indiscrete. Produce a topological space $(\widetilde{Y}, \widetilde{\mathcal{T}})$ and a function $f : (\widetilde{Y}, \widetilde{\mathcal{T}}) \to (Y, \mathcal{T})$ that is *not* continuous.

Problem 2. (Openness and closedness in the relative topology.)

Let $(X, \mathcal{T})$ be a topological space, $S \subset X$, and $(S, \mathcal{T}_S)$ be the topological space consisting of the subset S equipped with the relative topology induced by $\mathcal{T}$.

(i) Show that S is open in $(X, \mathcal{T})$ *if and only if* every relatively open (i.e., $\mathcal{T}_S$-open) subset of S is open in $(X, \mathcal{T})$ (i.e., is $\mathcal{T}$-open).

(ii) Is the statement in (i) true if "open" is replaced by "closed"? Justify your answer.

(iii) Let $T \subset X$ and consider also the topological space $(S \cap T, \mathcal{T}_{S\cap T})$ consisting of the subset T equipped with the relative topology induced by $\mathcal{T}$. Show that if $A \subset S$ is $\mathcal{T}_S$-open then $A \cap T$ is $\mathcal{T}_{S\cap T}$-open.

(iv) Given $R \subset X$, consider also the topological spaces $(R, \mathcal{T}_R)$ and $(S \cup R, \mathcal{T}_{S\cup R})$ with the relative topologies induced by $\mathcal{T}$. Let $E \subset S \cap R$ be both $\mathcal{T}_S$-open and $\mathcal{T}_R$-open. Is E also $\mathcal{T}_{S\cup R}$-open? Justify your answer.

Problem 3. (Kuratowski's closure axioms.) Let X be a set.

(i) Assume that X is equipped with a topology. Show that the operation $E \mapsto \overline{E}$, where $E \subset X$ and $\overline{E}$ is its closure, has the properties

   (a) $\overline{(\overline{E})} = \overline{E}$
   (b) $\overline{E_1 \cup E_2} = \overline{E_1} \cup \overline{E_2}$
   (c) $E \subset \overline{E}$
   (d) $\overline{\emptyset} = \emptyset$

   for all $E, E_1, E_2 \subset X$.

(ii) Conversely, suppose that $\mathfrak{c} : 2^X \to 2^X$ (2^X=all subsets of X) is a map obeying (a)–(d) above. Show that the family $\mathcal{T}$ of sets $U \subset X$ with $\mathfrak{c}(X \setminus U) = X \setminus U$ forms a topology in X for which the closure operation is exactly $\overline{E} = \mathfrak{c}(E)$ $\forall E \subset X$.

Problem 4. (Simple examples of homeomorphic topological spaces.)

In the following, $\mathbb{R}$, $\mathbb{R}^2$, and the other sets considered are regarded as topological spaces with the usual Euclidean metric topology.

(i) Show that all open intervals in $\mathbb{R}$ are homeomorphic.

(ii) Show that the open unit ball $\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1\}$ in $\mathbb{R}^2$ is homeomorphic to the open square $(0, 1) \times (0, 1)$ in $\mathbb{R}^2$.

(iii) Show that the punctured plane $\mathbb{R}^2 \setminus (0, 0)$ is homeomorphic to the exterior of the closed unit ball $\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leqslant 1\}$.

(iv) Show that the open right half-plane $\{(x, y) \in \mathbb{R}^2 \,|\, x > 0\}$ is homeomorphic to the open unit ball $\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1\}$.

(Comment: these examples indicate how a homeomorphism, which by definition preserves topological properties, might nonetheless drastically alter size and shape.)