LMU | LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN | MATHEMATISCHES INSTITUT

Spring term 2012 / *Sommersemester 2012*

# Functional Analysis – Final Test, 16.07.2012

*Funktionalanalysis – Endklausur, 16.07.2012*

**Name:**/*Name*: ______________________________

**Matriculation number:**/*Matrikelnr.:* ______________ **Semester:**/*Fachsemester:* ______________

**Degree course:**/*Studiengang:* ❏ Bachelor PO 2007 ❏ Lehramt Gymnasium (modularisiert)
❏ Bachelor PO 2010 ❏ Lehramt Gymnasium (nicht modularisiert)
❏ Diplom ❏ Master ❏ TMP ❏ ______________

**Major:**/*Hauptfach:* ❏ Mathematik ❏ Wirtschaftsm. ❏ Informatik ❏ Physik ❏ Statistik ❏ ______

**Minor:**/*Nebenfach:* ❏ Mathematik ❏ Wirtschaftsm. ❏ Informatik ❏ Physik ❏ Statistik ❏ ______

**Credits needed for:**/*Anrechnung der Credit Points für das:* ❏ Hauptfach ❏ Nebenfach (Bachelor/Master)

**Extra solution sheets submitted:**/*Zusätzlich abgegebene Lösungsblätter:* ❏ Yes ❏ No

| problem | 1 | 2 | 3 | 4 | 5 | 6 | $\sum$ |
|---|---|---|---|---|---|---|---|
| total marks | 10 | 10 | 10 | 10 | 10 | 10 | 60 |
| scored marks | | | | | | | |

| homework bonus | | final test performance | | total performance | | FINAL MARK | |
|---|---|---|---|---|---|---|---|

**INSTRUCTIONS:**

- This booklet is made of sixteen pages, including the cover, numbered from 1 to 16. The test consists of six problems. Each problem is worth the number of marks specified in the table above. 50 marks are counted as 100% performance in this test. You are free to attempt any problem and collect partial credits.
- The only material that you are allowed to use is black or blue pens/pencils and one hand-written, two-sided, A4-paper "cheat sheet" (Spickzettel). You cannot use your own paper: should you need more paper, raise your hand and you will be given extra sheets.
- Prove all your statements or refer to the standard material discussed in class.
- Work individually. Write with legible handwriting. You may hand in your solution in English or in German. Put your name on every sheet you hand in.
- You have 120 minutes.

**GOOD LUCK!**

**Fill in the form here below only if you need the certificate (Schein).**

Dieser Leistungsnachweis entspricht auch den Anforderungen
nach § Abs. Nr. Buchstabe LPO I
nach § Abs. Nr. Buchstabe LPO I

UNIVERSITÄT MÜNCHEN

# ZEUGNIS

*Der / Die Studierende der* ______

*Herr / Frau* ______ *aus* ______

*geboren am* ______ *in* ______ *hat im* **SoSe** ______ *–Halbjahr* **2012** ______

*meine Übungen* **zur Funktionalanalysis** ______

*mit* ______ *besucht.*

*Er / Sie hat* ______

*schriftliche Arbeiten geliefert, die mit ihm / ihr besprochen wurden.* ______

MÜNCHEN, *den* **16 Juli 2012**

# Name

**PROBLEM 1. (10 marks)**

Consider the linear operator $T : L^2[0, \pi] \to L^2[0, \pi]$ defined by

$$(Tf)(x) := \sin x \left( \int_0^\pi f(t) \cos t \, \mathrm{d}t \right) + \cos x \left( \int_0^\pi f(t) \sin t \, \mathrm{d}t \right) \qquad \text{for a.e. } x \in [0, \pi] \, .$$

(i) Show that $T$ is continuous.

(ii) Compute $\|T\|$, the operator norm of $T$.

**SOLUTION:**

## SOLUTION TO PROBLEM 1 (CONTINUATION):

# Name

**PROBLEM 2. (10 marks)**

Let $\mathcal{H}$ be a separable Hilbert space with norm $\| \ \|$ and scalar product $\langle \ , \ \rangle$ and let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{H}$. Show that the following two statements are equivalent.

(i) $x_n \rightharpoonup 0$ (i.e., $x_n$ converges weakly to 0) as $n \to \infty$,

(ii) $\langle e_m, x_n \rangle \xrightarrow{n\to\infty} 0$ for each $m \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} \|x_n\| < C$ for some constant $C > 0$.

**SOLUTION:**

**SOLUTION TO PROBLEM 2 (CONTINUATION):**

# Name

**PROBLEM 3. (10 marks)**

Let $(X, \| \|_X)$ and $(Y, \| \|_Y)$ be two Banach spaces.

(i) Let $T : X \to Y$ be a bounded linear operator such that $\|Tx\|_Y \geqslant c\|x\|_X$ $\forall x \in X$ and for some constant $c > 0$. Prove that $T$ is compact if and only if $\dim X < \infty$.

(ii) Assume that $\dim X = \infty$ and let $S : X \to Y$ be a bounded and compact linear operator. Prove that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $X$ with $\|x_n\|_X = 1$ $\forall n \in \mathbb{N}$ such that $\|Sx_n\|_Y \xrightarrow{n\to\infty} 0$.

(iii) Take $X$ and $S$ as in (ii). Let $\varepsilon > 0$. Show that there exists a linear, bounded, compact, and non-injective operator $S_\varepsilon : X \to Y$ such that $\|S - S_\varepsilon\| < \varepsilon$.

**SOLUTION:**

**SOLUTION TO PROBLEM 3 (CONTINUATION):**

# Name

**PROBLEM 4. (10 marks)**

Consider the real vector space $L^2_{\mathbb{R}}[-1,1]$ of square-integrable functions on the interval $[-1,1]$ and the non-linear functional $\phi : L^2_{\mathbb{R}}[-1,1] \to \mathbb{R}$ defined by

$$\phi(f) \ := \ \int_{-1}^{1} |f(x)|^2 \, \mathrm{d}x - 2\int_{-1}^{1} x^2 \, f(x) \, \mathrm{d}x \, .$$

Let $\mathcal{M} := \left\{ f \in L^2_{\mathbb{R}}[-1,1] \,\middle|\, \int_{-1}^{1} f(x) \, \mathrm{d}x = 0 \right\}$. Compute $\inf\limits_{f \in \mathcal{M}} \phi(f)$.

**SOLUTION:**

**SOLUTION TO PROBLEM 4 (CONTINUATION):**

# Name

**PROBLEM 5. (10 marks)**

Consider the real Banach space $C([0,1],\mathbb{R})$ of real-valued continuous functions on the interval $[0,1]$ equipped with the $\| \ \|_\infty$ norm. Let $E \subset C([0,1],\mathbb{R})$ be a closed linear subspace. Assume that every $f \in E$ is Hölder continuous, i.e., $\forall f \in E \quad \exists\, c \in \mathbb{R}$ and $\exists\, \alpha \in (0,1]$ such that $|f(x) - f(y)| \leqslant c\,|x-y|^\alpha \ \ \forall x, y \in [0,1]$.

(i) Prove that $\exists\, \gamma \in (0,1]$ and $\exists\, C \geqslant 0$ such that

$$|f(x) - f(y)| \leqslant C\,\|f\|_\infty\,|x-y|^\gamma \qquad \forall f \in E\,, \quad \forall x, y \in [0,1]\,.$$

(Thus, $\gamma$ and $C$ are independent of $f$.)

(ii) Prove that $\dim E < \infty$.

**SOLUTION:**

**SOLUTION TO PROBLEM 5 (CONTINUATION):**

## Name

### PROBLEM 6. (10 marks)

Consider the function $f : [0, 2\pi] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} x^2 & \text{if } x \in [0, \pi] \\ (x - 2\pi)^2 & \text{if } x \in (\pi, 2\pi] \, . \end{cases}$$

Use the Fourier series of $f$ to compute

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \, .$

### SOLUTION:

**SOLUTION TO PROBLEM 6 (CONTINUATION):**