Functional Analysis

Institute of Mathematics, LMU Munich – Spring Term 2012
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HOMEWORK ASSIGNMENT no. 12, issued on Tuesday 3 July 2012
Due: Tuesday 10 July 2012 by 6 pm in the designated "FA" box on the 1st floor
Info: www.math.lmu.de/~michel/SS12_FA.html

Each exercise is worth a full mark of 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions either in German or in English.

Exercise 45. (An application of Baire and Closed Graph)
Let Ω be a measure space. Let X be a closed vector subspace of $L^1(\Omega)$ such that $X \subset \bigcup_{1<q\leqslant\infty} L^q(\Omega)$.

(i) Show that there exists some $p > 1$ such that $X \subset L^p(\Omega)$.

(ii) Show that there is a constant C such that $\|f\|_p \leqslant C\|f\|_1 \;\; \forall f \in X$.

Exercise 46. (The Cantor lemma in Banach spaces)
Let X be a Banach space and let $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$ be two sequences in X such that $\forall t \in [a,b]$, with $a < b$, $\|x_n \cos nt + y_n \sin nt\| \xrightarrow{n\to\infty} 0$. Show that $x_n \xrightarrow{n\to\infty} 0$ and $y_n \xrightarrow{n\to\infty} 0$ in norm.

Exercise 47. (Extension of bounded linear functionals under further constraints)
Consider the Banach space ℓ^∞ of *real* bounded sequences, the subspace

$$c = \big\{x = (x_1, x_2, x_3, \dots) \mid x_n \in \mathbb{R}\ \forall n \text{ and } \exists \lim_{n\to\infty} x_n\big\},$$

and the points

$$\begin{aligned} a &:= (0,1,0,1,0,1,0,1,\dots) && \text{(i.e., alternating 0 and 1)} \\ b &:= (0,0,0,1,0,1,0,1,\dots) && \text{(i.e., alternating 0 and 1 from the third position)} \\ c &:= (1,0,1,0,1,0,1,0,\dots) && \text{(i.e., alternating 1 and 0)}\,. \end{aligned}$$

(i) Show that there exist bounded linear functionals λ and μ in $(\ell^\infty)'$ such that $\lambda(x) = \mu(x) = \lim_{n\to\infty} x_n \ \forall x \in c$ and $\lambda(a) = \frac{1}{2}$, $\mu(a) = -2012$.

(ii) Can it happen that the functional λ (resp. μ) considered in (i) satisfies the further condition $\lambda(b) = \frac{1}{3}$ (resp. $\mu(c) = \frac{1}{3}$)? Justify your answer.

(iii) Show that there exists a bounded linear functional $\lambda \in (\ell^\infty)'$ such that

($*$) $\liminf_n x_n \leqslant \lambda(x) \leqslant \limsup_n x_n \ \forall x \in \ell^\infty$ (and therefore $\lambda(x) = \lim_{n\to\infty} x_n \ \forall x \in c$)

($**$) $\lambda(Lx) = \lambda(x)\ \forall x \in \ell^\infty$ where $L : \ell^\infty \to \ell^\infty$ is the left-shift operator $L(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$

Exercise 48. (Explicit calculation of all the Hahn-Banach extensions)

(i) The following are given:
a normed space X, $n \in \mathbb{N}$, $\phi, \phi_1, \dots, \phi_n \in X'$, $V := \{x \in X \mid \phi_1(x) = \cdots = \phi_n(x) = 0\}$, $\eta := \phi\big|_V : V \to \mathbb{K}$, $\xi \in X'$ which extends η.
Show that there are $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ such that $\xi = \phi + \alpha_1\phi_1 + \cdots + \alpha_n\phi_n$.

(ii) Consider c_0 and ℓ^1 as *real* vector spaces. The following are given: $a = (a_1, a_2, a_3, \dots) \in \ell^1$ with $a_1 \neq 0$, $V := \left\{x = (x_1, x_2, x_3, \dots) \in c_0 \,\middle|\, \sum_{n=1}^{\infty} a_n x_n = 0\right\}$, $f : V \to \mathbb{R}$, $f(x) := x_1$.
Determine *all* Hahn-Banach extensions $\phi \in (c_0)'$ of f, i.e., bounded linear functionals $\phi : c_0 \to \mathbb{R}$ that coincide with f on V and such that $\|\phi\|_{(c_0)'} = \|f\|_{V'}$
Notice: this means that you are asked to give the explicit action $\phi(x)$ for any $x \in c_0$.