Functional Analysis

Institute of Mathematics, LMU Munich – Spring Term 2012
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HOMEWORK ASSIGNMENT no. 10, issued on Tuesday 19 June 2012
Due: Tuesday 26 June 2012 by 6 pm in the designated "FA" box on the 1st floor
Info: `www.math.lmu.de/~michel/SS12_FA.html`

Each exercise is worth a full mark of 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions either in German or in English.

Exercise 37. (A normalized system of vectors with fixed angle among them converge weakly.)

(i) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of vectors in a Hilbert space $\mathcal{H}$ such that $\|x_n\| = 1\ \forall n$ and $\langle x_n, x_m \rangle = \frac{1}{2012}$ whenever $m \neq n$. Prove that there exists $x \in \mathcal{H}$ such that

$$\langle x_n, y\rangle \xrightarrow{n\to\infty} \langle x, y\rangle \qquad \forall y \in \mathcal{H}\,.$$

(ii) Produce an example of a Hilbert space $\mathcal{H}$, a collection $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{H}$, and a vector $x \in \mathcal{H}$ with the properties described in (i).

Exercise 38. (Cesaro summability. The canonical ONB of $L^2[0, 2\pi]$.)

Consider the Hilbert space $L^2[0, 2\pi]$ and the subspace $C(\mathbb{S}^1) \equiv \{f \in C([0, 2\pi]) \,|\, f(0) = f(2\pi)\}$. Define $e_n(x) := \dfrac{e^{inx}}{\sqrt{2\pi}}$, $n \in \mathbb{Z}$. $\{e_n\}_{n\in\mathbb{Z}}$ is clearly an orthonormal set with respect to the scalar product in $L^2[0, 2\pi]$ and it is entirely contained in $C(\mathbb{S}^1)$. For every $f \in C([0, 2\pi])$ define

$$S_N(f) := \sum_{n=-N}^{N} \langle e_n, f\rangle\, e_n\,, \qquad \Sigma_N(f) := \frac{1}{N+1}\big(S_0(f) + \cdots + S_N(f)\big) \qquad (N = 0, 1, 2, \dots).$$

(i) Show that $\|\Sigma_N(f) - f\|_\infty \xrightarrow{N\to\infty} 0$ for every $f \in C(\mathbb{S}^1)$.

(ii) Show that $\|S_N(f) - f\|_2 \xrightarrow{N\to\infty} 0$ for every $f \in L^2[0, 2\pi]$ – and therefore (owing to Theorem 2.40 in class) $\{e_n\}_{n\in\mathbb{Z}}$ is an ONB of $L^2[0, 2\pi]$.

Exercise 39. (Examples of compact / non-compact subsets of Banach spaces.)

(i) Under what condition on the sequence $\{a_n\}_{n=1}^{\infty}$ in $(0,\infty)$ is the set (the "parallelepiped") $\{x=(x_1,x_2,\dots)\in\ell^2\,|\,|x_n|\leqslant a_n\,\forall n\in\mathbb{N}\}$ compact in ℓ^2? Justify your answer.

(ii) For which continuous functions $\varphi:[0,1]\to[0,\infty)$ is the set $\overline{A}$ compact in $C([0,1])$, with $A:=\{f\in C([0,1])\,|\,|f(x)|\leqslant\varphi(x)\,\forall x\in[0,1]\}$? Justify your answer.

(iii) Let $E:=\Big\{f\in C^1([0,1])\,\Big|\,|f(0)|\leqslant a\,,\displaystyle\int_0^1|f'(x)|^2\,\mathrm{d}x\leqslant b\Big\}$, for given $a,b>0$. Show that $\overline{E}$ is compact in $C([0,1])$.

Exercise 40. (Examples of L^p-distances)

For a given p consider the set $S:=\Big\{f\in L^p[0,1]\,\Big|\,\displaystyle\int_0^1 f^2=1\Big\}$. (Here $L^p[0,1]$ is a vector space on $\mathbb{C}$.) Let $f_0:[0,1]\to\mathbb{C}$ be the function $f_0(x):=x^2$. Compute the L^p-distance from f_0 to S when

(i) $p=1$,

(ii) $p=2$,

(iii) $p=\infty$.