Functional Analysis

Institute of Mathematics, LMU Munich – Spring Term 2012
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HOMEWORK ASSIGNMENT no. 7, issued on Tuesday 29 May 2012
Due: Tuesday 5 June 2012 by 6 pm in the designated "FA" box on the 1st floor
Info: `www.math.lmu.de/~michel/SS12_FA.html`

Each exercise is worth a full mark of 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions either in German or in English.

Exercise 25. (The Hardy's operator on ℓ^p.)

Let $p \in (1, \infty)$ and consider the linear operator H defined by

$$(Hx)_n := \frac{x_1 + \cdots + x_n}{n} \qquad (n \in \mathbb{N})$$

$\forall x = (x_1, x_2, \dots) \in c_{00} \subset \ell^p$. Show that H extends uniquely to a bounded linear operator $H : \ell^p \to \ell^p$ and compute its norm $\|H\|$.

Exercise 26. (Computation of norm of functionals.)

Compute the norm of the following linear functionals ϕ. (The normed spaces indicated here below are meant to be equipped with their usual natural norm.)

(i) $\phi : C([-1,1]) \to \mathbb{K}$, $\phi(f) := \displaystyle\int_{-1}^{1} x f(x)\,\mathrm{d}x \;\; \forall f \in C([-1,1])$

(ii) $\phi : \ell^2 \to \mathbb{K}$, $\phi(x) := \displaystyle\sum_{n=1}^{\infty} \frac{x_n}{n} \;\; \forall x = \{x_n\}_{n=1}^{\infty} \in \ell^2$

(iii) $\phi : \ell^\infty \to \mathbb{R}$ such that $\phi(x) \geqslant 0$ $\forall x = (x_1, x_2, \dots) \in \ell^\infty$ whose components are all non-negative, i.e., $x_n \geqslant 0$ $\forall n$, and such that $\phi(\mathbf{1}) = 2012$, where $\mathbf{1} = (1, 1, 1, \dots) \in \ell^\infty$. Here ℓ^∞ is meant to be the Banach space, on the field $\mathbb{R}$, of real-valued bounded sequences.

(iv) $\phi : X \to \mathbb{K}$ (where $(X, \|\ \|)$ is a generic normed space) such that $\displaystyle\inf_{\substack{x \in X \\ \phi(x)=1}} \|x\| = 2012$.

Exercise 27. (Distance from a point to a set in a normed space.)

This exercise is set in the space $C([a,b];\mathbb{R})$ of real-valued continuous functions on $[a,b]$ equipped with the usual $\| \, \|_\infty$-norm.

(i) Let $V := \left\{ f \in C([0,1];\mathbb{R}) \,\middle|\, \int_0^1 \frac{f(x)}{x+1}\,\mathrm{d}x = 1 \right\}$. Show that there is a unique element $v \in V$ that minimizes the distance from the origin to V (i.e., such that $\|v\|_\infty = \operatorname{dist}(0,V)$) and determine v explicitly.

(ii) Let $W := \left\{ f \in C([0,1];\mathbb{R}) \,\middle|\, \int_0^{1/2} x f(x)\,\mathrm{d}x = 1 \right\}$. Show that W has an infinity of elements that minimize the distance from the origin to W and determine these elements explicitly.

(iii) Consider now the space $C([-1,1];\mathbb{R})$ and, for fixed $n \in \mathbb{N}$, the subspace $P^{(n)}$ of polynomials of degree at most n. Compute $\operatorname{dist}(x^n, P^{(n-1)})$. Here x^n is the short-hand notation for the polynomial $x \mapsto x^n$.

Exercise 28. (Norm attaining and non-norm attaining bounded linear functionals.)

(i) Does $\phi(x) := \sum_{n=1}^{\infty} \frac{x_n}{2^{n-1}}$ $\forall x = (x_1, x_2, \dots) \in c_0$ define a bounded linear functional $\phi : (c_0, \| \, \|_\infty) \to \mathbb{C}$ that attains its norm? Justify your answer. (The space c_0 was defined in Exercise 20(iii)).

(ii) Does $\phi(x) := \sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) x_n$ $\forall x = (x_1, x_2, \dots) \in \ell^1$ define a bounded linear functional $\phi : (\ell^1, \| \, \|_1) \to \mathbb{C}$ that attains its norm? Justify your answer.

(iii) Does $\phi(f) := \int_0^{1/2} f(x)\,\mathrm{d}x - \int_{1/2}^1 f(x)\,\mathrm{d}x$ $\forall f \in C([0,1])$ define a bounded linear functional $\phi : (C([0,1]), \| \, \|_\infty) \to \mathbb{C}$ that attains its norm? Justify your answer.