Functional Analysis

Institute of Mathematics, LMU Munich – Spring Term 2012
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HOMEWORK ASSIGNMENT no. 1, issued on Wednesday 18 April 2012
Due: Tuesday 24 April 2012 by 6 pm in the designated "FA" box on the 1st floor
Info: `www.math.lmu.de/~michel/SS12_FA.html`

Each exercise is worth a full mark of 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions either in German or in English.

Exercise 1. (Basic facts about closed, interior, boundary.)

Let X be a topological space and $E \subset X$. Use the definitions given *in class* of closed sets in X, closure $\overline{E}$, interior $\mathring{E}$ (or $\mathrm{int}(E)$), and boundary ∂E of E.

(i) Show that any intersection of closed sets is closed, and a finite union of closed set is closed.

(ii) Show that $X \setminus \mathring{E} = \overline{X \setminus E}$.

(iii) Show that $X \setminus \overline{E} = (X \overset{\circ}{\setminus} E)$

(iv) Show that $\partial E = \overline{E} \setminus \mathring{E} = \overline{E} \cap (\overline{X \setminus E})$.

(v) Show that $\overline{E} = \mathring{E} \sqcup \partial E$ ($\sqcup$ meaning disjoint union).

Exercise 2. (The CO-FINITE TOPOLOGY.)

Let X be a set and let $\mathcal{T}$ be the family of subsets U of X such that $X \setminus U$ is finite, together with the empty set $\emptyset$.

(i) Show that $\mathcal{T}$ is a topology.

(ii) Let $E \subset X$. Find the closure $\overline{E}$ of E in the topological space $(X, \mathcal{T})$.

(iii) Take X to be the set $\mathbb{Z}$ of the integers and equip it with the topology $\mathcal{T}$ defined above. Show that the sequence $1, 2, 3, \dots$ converge in $(\mathbb{Z}, \mathcal{T})$ to *each* point of $\mathbb{Z}$.

(iv) Find *all* convergent sequences in the topological space $(\mathbb{Z}, \mathcal{T})$ considered in (iii).

Exercise 3. (Relative topology: relatively closed, relative closure, transitivity.)

Let $(X, \mathcal{T})$ be a topological space, $S \subset X$, and $(S, \mathcal{T}_S)$ be the topological space consisting of the subset S equipped with the relative topology induced by $\mathcal{T}$.

(i) Let $E \subset S$. Show that E is $\mathcal{T}_S$-closed (i.e., "relatively closed") *if and only if* $E = S \cap C$ for some $\mathcal{T}$-closed subset $C \subset X$.

(ii) Let $E \subset S$. Show that the closure of E with respect to the topology $\mathcal{T}_S$ (i.e., the "relative closure of E in S") is $\overline{E} \cap S$, where $\overline{E}$ denotes the closure of E in $(X, \mathcal{T})$.

(iii) Let $E \subset S$. Consider in E both the relative topology $\mathcal{T}_E^{(X,\mathcal{T})}$ induced by $\mathcal{T}$ and relative topology $\mathcal{T}_E^{(S,\mathcal{T}_S)}$ induced by $\mathcal{T}_S$. Show that the topological spaces $(E, \mathcal{T}_E^{(X,\mathcal{T})})$ and $(E, \mathcal{T}_E^{(S,\mathcal{T}_S)})$ coincide.

Exercise 4. (Relative topology: relative closure in general, relative convergence.)

Let $(X, \mathcal{T})$ be a topological space, $S \subset X$, and $(S, \mathcal{T}_S)$ be the topological space consisting of the subset S equipped with the relative topology induced by $\mathcal{T}$.

(i) Let $A \subset X$, not necessarily included in S. Show that the relative closure of $A \cap S$ in $(S, \mathcal{T}_S)$ is *contained* in $\overline{A} \cap S$, where $\overline{A}$ denotes the closure of A in $(X, \mathcal{T})$.

(*Hint:* Exercise 3(i).)

(ii) Follow-up to (i): Give an example where the relative closure of $A \cap S$ is a proper subset of $\overline{A} \cap S$.

(iii) Show that a sequence $\{x_n\}_{n=1}^{\infty}$ in S converges to $x \in S$ in the relative topology $\mathcal{T}_S$ of S *if and only if*, considered as a sequence in X, $\{x_n\}_{n=1}^{\infty}$ converges to x in the topology $\mathcal{T}$.