Functional Analysis – Problems in the class, sheet 7

Mathematisches Institut der LMU – SS2010
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The problems in the class are discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for your preparation towards the final exam. Further info at `www.math.lmu.de/~michel/SS10_FA.html`.

Problem 25. Let $\mathcal{H}$ be a complex Hilbert space. Assume that the system $\{\phi_n\}_{n=1}^{\infty}$ is such that $\|\phi_n\| = 1$ for every n and

$$\|f\|^2 \;=\; \sum_{n=1}^{\infty} |\langle \phi_n, f\rangle|^2 \qquad \forall f \in \mathcal{H}\,.$$

Prove that $\{\phi_n\}_{n=1}^{\infty}$ is an orthonormal basis.

Problem 26. By means of the Fourier series of the 2π-periodic functions

$$f_1(x) \;=\; \begin{cases} x^2 & \text{if } x \in (0,\pi] \\ (x-2\pi)^2 & \text{if } x \in (\pi, 2\pi] \end{cases} \qquad \text{and} \qquad f_2(x) \;=\; \begin{cases} x & \text{if } x \in (-\pi,\pi) \\ 0 & \text{if } x = \pi \end{cases}$$

prove that

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

(iii) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,.$

Problem 27. Recall that $C_p^k([0,2\pi])$ ($k \leqslant \infty$) is the space of C^k-functions f with periodicity conditions $f(0) = f(2\pi)$ and $f^{(j)}(0) = f^{(j)}(2\pi)$ for $j = 1, \dots, k$. Let $c_n(f)$, $n \in \mathbb{Z}$, be the n-th Fourier coefficient of f,

$$c_n(f) \;=\; \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} f(x)\,\mathrm{d}x\,.$$

(i) Show that if $f \in C_p^k([0,2\pi])$ with $k < \infty$ then $\displaystyle\sum_{n\in\mathbb{Z}} |c_n(f)|^2\,|n|^{2k} < \infty$ and $|c_n(f)| \leqslant \dfrac{C}{|n|^k}$ for every $n \in \mathbb{Z}\setminus\{0\}$ and some positive constant C.

(ii) Show that if $f \in C_p^\infty([0,2\pi])$ is such that $|f^{(k)}(x)| \leqslant C\rho^k k!$ for all $x \in [0,2\pi]$, all $k \in \mathbb{N}$, and some constants $C > 0$, $\rho \in (0,1)$, then $|c_n(f)| \leqslant \widetilde{C} e^{-\frac{|n|}{e\rho}}$ for some positive constant $\widetilde{C}$.

(iii) Show that if $f \in L^2[0,2\pi]$ is such that $\displaystyle\sum_{n\in\mathbb{Z}} |c_n(f)|\,|n|^k < \infty$ then $f \in C_p^k([0,2\pi])$.

Problem 28. Let $f, g \in H^k([0,2\pi])$ and let $c_n(f), c_n(g)$, $n \in \mathbb{Z}$ be the corresponding Fourier coefficients. Prove that the scalar products

(i) $\displaystyle\langle f, g\rangle = \sum_{n\in\mathbb{Z}} (1+n^2)^k\,\overline{c_n(f)}\,c_n(g)$

(ii) $\displaystyle\langle f, g\rangle = \sum_{n\in\mathbb{Z}} (1+n^2+\cdots+n^{2k})\,\overline{c_n(f)}\,c_n(g)$

define the same topology of $H^k([0,2\pi])$, which in class was defined assigning the scalar product $\displaystyle\langle f, g\rangle_{H^k} = \sum_{n\in\mathbb{Z}} (1+|n|^{2k})\,\overline{c_n(f)}\,c_n(g)\,.$