Functional Analysis – Problems in the class, sheet 2

Mathematisches Institut der LMU – SS2010
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The problems in the class are discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for your preparation towards the final exam. Further info at `www.math.lmu.de/~michel/SS10_FA.html`.

Problem 5. Decide if the following sequences $\{f_n\}_{n=1}^{\infty}$ are compact in $C([0,1])$ with respect to the $\|\ \|_\infty$-norm topology:

(i) $f_n(x) = \sin nx$,

(ii) $f_n(x) = x^n$,

(iii) $f_n(x) = n\, e^{-n|x|}$,

(iv) $f_n(x) = \displaystyle\sum_{k=0}^{n} \frac{x^k}{k!}$.

Problem 6. Consider the family

$$\mathcal{F} \;:=\; \Big\{ f:[0,1]\to\mathbb{R} \;\Big|\; f(x) = \int_0^x g(t)\,\mathrm{d}t \quad \text{with } g \in C([0,1]) \quad \text{and } \|g\|_\infty \leqslant 1 \Big\}$$

as a subset of the space $C([0,1])$ with the $\|\ \|_\infty$-norm topology. Prove that

(i) $\mathcal{F}$ is a bounded family in $C([0,1])$,

(ii) $\mathcal{F}$ is a uniformly equicontinuous family,

(iii) $\mathcal{F}$ is not closed in $C([0,1])$,

(iv) the interior of $\mathcal{F}$ in $C([0,1])$ is empty,

(v) $\overline{\mathcal{F}}$ is compact.

Problem 7. Le $\mathcal{T}$ be a *compact* topological space.

(i) Let $C \subset \mathcal{T}$ be a closed subset of $\mathcal{T}$. Prove that C is necessarily compact in the relative topology.

(ii) Let $f:\mathcal{T}\to\mathcal{T}'$ be a continuous function ($\mathcal{T}'$ being another topological space). Prove that the image $f(\mathcal{T})$ is compact in $\mathcal{T}'$.

Problem 8.

(i) Prove that the closure in $C(\mathbb{R})$ with respect to the $\| \ \|_\infty$-norm of the space $C_0(\mathbb{R})$, the continuous functions with compact support, is the space $C_\infty(\mathbb{R})$, the continuous functions vanishing at infinity.

(ii) What is the interior of $C^1([0,1])$ in $C(\mathbb{R})$ with respect to the $\| \ \|_\infty$-norm?

(iii) What is the closure X of $\{f \in C([0,1]) \mid \|f\|_\infty = 1\}$ in $C([0,1])$ with respect to the $\| \ \|_\infty$-norm? Is X compact?