LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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Functional Analysis – Final test November 2010

Funktionalanalysis – Endklausur November 2010

Name: / *Name:* ____________________

Matriculation number: / *Matrikelnr.:* __________ **Semester** / *Fachsemester:* ______

Degree programme: / *Studiengang:* ❏ Bachelor PO 2007 ❏ Bachelor PO 2010 ❏ TMP ❏ Master ❏ Lehramt (modularisiert) ❏ Lehramt (nicht modularisiert) ❏ Diplom ❏ ______

Main subject: / *Hauptfach:* ❏ Mathematics ❏ Financial mathematics ❏ Physics ❏ Informatics ❏ Statistics ❏ ______

Subsidiary subject: / *Nebenfach:* ❏ Mathematics ❏ Financial mathematics ❏ Physics ❏ Informatics ❏ Statistics ❏ ______

Credits needed for: / *Anrechnung der Credit Points für das:* ❏ Hauptfach ❏ Nebenfach

Extra solution sheets submitted: / *Zusätzlich abgegebene Lösungsblätter:* ❏ ______

problem	1	2	3	4	5	6	7	8	$\sum$
total points	15	20	20	15	15	15	20	15	135
scored points									

INSTRUCTIONS:

- This booklet is made of eighteen pages, including the cover, numbered from 1 to 18. The test consists of eight problems. Each problem is worth the number of points specified in the table above. 100 points are counted as the full mark in this test. You are free to attempt any problem and collect partial credits.
- The only material that you are allowed to use is black or blue pens/pencils and one two-sided A4-paper "cheat sheet" (Spickzettel). You cannot use your own paper: should you need more paper, raise your hand and you will be given extra sheets.
- Prove all your statements or refer to the standard material discussed in class.
- Work individually. Write with legible handwriting. You may hand in your solution in English or in German. Put your name on every sheet you hand in.
- You have 150 minutes.

GOOD LUCK!

Name

PROBLEM 1. (15 points) Let $C_b(\mathbb{R})$ be the space of bounded continuous functions on $\mathbb{R}$ and $C_\infty(\mathbb{R})$ be the space of continuous functions $\mathbb{R}$ vanishing at infinity, both spaces being equipped with the supremum norm $\| \|_{\text{sup}}$. Prove that the latter is a closed subspace of the former.

SOLUTION:

Name

PROBLEM 2. (20 points) For any $f \in C([0,1])$ define

$$(Tf)(x) := \int_0^1 \frac{f(y)}{|x-y|^{1/3}}\,\mathrm{d}y\,, \qquad x \in [0,1]\,.$$

(i) Prove that $Tf \in C([0,1])$.

(ii) Equip $C([0,1])$ with the usual supremum norm and consider a bounded subset $\mathcal{B}$ of $C([0,1])$. Prove that the closure of $T(\mathcal{B})$ in $C([0,1])$ is compact.

SOLUTION:

Name

PROBLEM 3. (20 points) Find the general solution in $C^2(S^1 \times S^1)$ to the partial differential equation

$$2f_{xx} + f_{xy} + f_{yy} \;=\; \cos x \cos y\,.$$

SOLUTION:

Name

PROBLEM 4 (15 points).

(i) Prove that for all $f \in H^2(S^1)$ one has

$$\int_0^{2\pi} |f'(x)|^2 \,\mathrm{d}x \;\leqslant\; \frac{1}{2}\left(\int_0^{2\pi} |f(x)|^2 \,\mathrm{d}x + \int_0^{2\pi} |f''(x)|^2 \,\mathrm{d}x\right). \tag{$*$}$$

(ii) Find all functions $f \in H^2(S^1)$ for which $(*)$ becomes an equality.

SOLUTION:

Name

PROBLEM 5. (15 points) Let $\phi : \ell^\infty \to \mathbb{R}$ be a linear map such that $\phi(x) \geqslant 0$ for all $x = (x_1, x_2, \dots) \in \ell^\infty$ whose components are all non-negative, i.e., $x_n \geqslant 0 \ \forall n$. Prove that ϕ is bounded and compute its norm.

SOLUTION:

Name

PROBLEM 6. (15 points) Let (X,μ) be a measurable space and let $f \in L^p(X,\mu)$ $\forall p \geqslant 1$. Prove that the function

$$a \longmapsto F(a) := \ln \|f\|_{1/a} \qquad a \in (0,1)$$

is convex.

SOLUTION:

Name

PROBLEM 7. (20 points) Consider the space $P([0,1])$ of polynomials on $[0,1]$ with real coefficients, equipped with the uniform topology. Let $P_+([0,1])$ and $P_-([0,1])$ be the subset of $P([0,1])$ of polynomials whose leading coefficient is positive and negative respectively.

(i) Show that $P_+([0,1])$ and $P_-([0,1])$ are convex in $P([0,1])$.

(ii) Show that there exists no hyperplane that separates $P_+([0,1])$ and $P_-([0,1])$.

SOLUTION:

Name

PROBLEM 8. (15 points) Let X_1, X_2 be two subspaces in the Banach space X such that $X_1 \cap X_2 = \emptyset$ and $\operatorname{Span}\{X_1, X_2\} = X$. In particular, this implies that every $x \in X$ can be uniquely written as $x = x_1 + x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$. Let $P : X \to X$ be the projection onto X_1 along X_2 (i.e., if $x = x_1 + x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$ then $Px = x_1$). Prove that the operator P is bounded if and only if the subspaces X_1, X_2 are closed.

SOLUTION: