Functional Analysis – Exercise sheet 7

Mathematisches Institut der LMU – SS2010
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Due: Monday 21 June 2010 by 12 p.m. in the designated "Funktionalanalysis" box on the 1st floor
Students who will be attending the Mon 21 June tutorial have to hand in their solution sheets at 10:15 in class.
Info: `www.math.lmu.de/~michel/SS10_FA.html`

The full mark in each exercise is 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 25. Let $f \in H^1(S^1)$. Show that the limit

$$\lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

exists in the L^2-sense and the limit is $f'(x)$, the weak derivative of f.

Exercise 26.

(i) Show that $H^{\frac{1}{2}+\varepsilon}(S^1) \subset C(S^1)$ for any $\varepsilon > 0$.

(ii) Show that $H^{\frac{1}{2}}(S^1) \not\subset C(S^1)$.

Exercise 27. Let $f, g \in H^1(S^1)$. Show that $fg \in H^1(S^1)$ and that the Leibniz rule

$$(fg)' \;=\; f'g + fg'$$

holds as an identity in $L^2(S^1)$ (all derivatives are in the weak sense).

Exercise 28. Let A be a symmetric and positive definite $d \times d$ real matrix and $b \in \mathbb{R}^d$. Define

$$f(x) \;:=\; e^{-x\cdot Ax + b\cdot x}\,, \qquad x \in \mathbb{R}^d\,.$$

Show that the Fourier transform $\widehat{f}$ of f is given by

$$\widehat{f}(k) \;=\; \frac{1}{2^{d/2}\sqrt{\det A}}\; e^{-\frac{1}{4}(k+ib)\cdot(A^{-1}(k+ib))}\,, \qquad k \in \mathbb{R}^d\,.$$