Functional Analysis II – Problem sheet 11

Mathematisches Institut der LMU – SS2009
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Handout: 08.07.2009
Due: Tuesday 15.07.2009 by 1 p.m. in the "Funktionalanalysis II" box
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Grader: Ms. S. Sonner – Übungen on Wednesdays, 4,30 - 6 p.m., room C-111

Exercise 29. Let T be a densely defined linear operator on a Hilbert space $\mathcal{H}$. Denote by $\mathcal{G}(T)$ the graph of T and by $\overline{\mathcal{G}(T)}$ its closure in $\mathcal{H} \times \mathcal{H}$. Assume that T is *closable*, i.e., T admits a closed extension, and denote by $\overline{T}$ its closure. Prove that

$$\mathcal{G}(\overline{T}) \;=\; \overline{\mathcal{G}(T)}\,.$$

Exercise 30. Let T be a densely defined linear operator on a Hilbert space $\mathcal{H}$. Show that

$$\rho(T) \neq \emptyset \qquad \Rightarrow \qquad T \;\text{ is closed}\,.$$

(*Hint*: apply the Closed Graph Theorem to the resolvent $(T-\lambda)^{-1}$.) Optional [freiwillig]: can you provide an example of a *non-closed* and densely defined T with empty resolvent?

Exercise 31. (This exercise proves Lemma 2.50 stated in the class.) Let $d \geqslant 1$, integer, and $s \in \mathbb{R}$, $s \geqslant 0$. Prove that

$$H^{-s}(\mathbb{R}^d) \;\cong\; H^s(\mathbb{R}^d)^*\,.$$