Functional Analysis II – Problem sheet 9

Mathematisches Institut der LMU – SS2009
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Handout: 23.06.2009
Due: Tuesday 30.06.2009 by 1 p.m. in the "Funktionalanalysis II" box
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Grader: Ms. S. Sonner – Übungen on Wednesdays, 4,30 - 6 p.m., room C-111

Exercise 24. Let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$, the Schwartz space of smooth functions with rapid decrease in dimension $d \geqslant 1$, and let $\mathcal{F}$ be the Fourier transform[1] on $\mathcal{S}(\mathbb{R}^d)$. Prove Theorem 2.13, part (g), stated in the class. I.e., prove that $\mathcal{F}(\varphi\psi) = (2\pi)^{-d/2}\mathcal{F}(\varphi) * \mathcal{F}(\psi)$, that is, the Fourier transform of the (pointwise) product of two Schwartz functions is, up to a pre-factor, the convolution of the Fourier transform of each function. (Compare this statement with the discussion of Exercise 22.)

Exercise 25. Let $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi \not\equiv 0$. i.e., a non-zero compactly supported smooth function. Prove that $\mathcal{F}\varphi$, its Fourier transform, *cannot* have compact support. (*Hint:* assume by contradiction that $\xi \mapsto (\mathcal{F}\varphi)(\xi)$ is compactly supported, extend it from $\xi \in \mathbb{R}$ to $\xi \in \mathbb{C}$ and show that you get an holomorphic function. Then…)

Exercise 26. Let $a > 0$. Say in which sense (i.e., as elements of which space) the $\mathbb{R} \to \mathbb{R}$ functions

- $x \mapsto e^{-a|x|}$
- $x \mapsto \sqrt{\dfrac{2}{\pi}}\,\dfrac{a}{a^2 + x^2}$

admit Fourier transform and compute it. Verify the Fourier inversion formula in this cases.

[1] remember that the convention adopted in the class is $(\mathcal{F}f)(\xi) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} f(x)e^{-ix\xi}\mathrm{d}x$