# Functional Analysis II – Problem sheet 6

Mathematisches Institut der LMU – SS2009
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**Handout:** 2.06.2009
**Due:** Tuesday 9.06.2009 by 1 p.m. in the "Funktionalanalysis II" box
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**Grader:** Ms. S. Sonner – Übungen on Wednesdays, 4,30 - 6 p.m., room C-111

**Exercise 15.** [*Lebesgue* vs *Cantor measure*] The goal of this exercise is to construct a measure $\mu_{\mathcal{C}}$ on $[0,1]$ that is *singular* and *continuous* with respect to the Lebesgue measure $\mu_{\mathcal{L}}$.[1] The twofold strategy will be to construct first a set $\mathcal{C}$ with zero Lebesgue measure where the new $\mu_{\mathcal{C}}$ will be supported, and then to construct $\mu_{\mathcal{C}}$ by means of its Radon-Nikodym derivative with respect to $\mu_{\mathcal{L}}$.

15.1) Define the *Cantor set* $\mathcal{C}$ as follows. Set $\mathcal{C}_0 := [0,1]$ and remove the open interval $(\frac{1}{3},\frac{2}{3})$ from the middle to get $\mathcal{C}_1$, then remove $(\frac{1}{9},\frac{2}{9})$ and $(\frac{7}{9},\frac{8}{9})$ from the middle of each component of $\mathcal{C}_1$ to get $\mathcal{C}_2$, and so on for the following generations. With the notation

$$\begin{aligned} c + J &:= [a+c, b+c] \\ cJ &:= [ac, bc] \end{aligned}$$

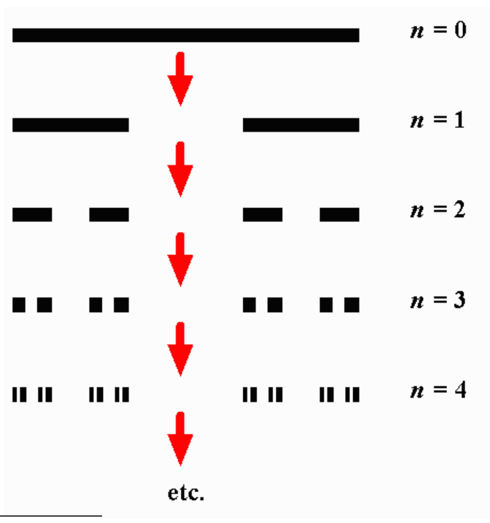
for any closed interval $J = [a,b]$ and any $c \in \mathbb{R}$, the recursive definition of $\mathcal{C}_n$ is

$$\mathcal{C}_n := \frac{\mathcal{C}_{n-1}}{3} \cup \Big(\frac{2}{3} + \frac{\mathcal{C}_{n-1}}{3}\Big), \qquad \mathcal{S}_n := [0,1] \setminus \mathcal{C}_n, \qquad n = 1,2,3,\dots$$

and

$$\mathcal{C} := \bigcap_{n=0}^{\infty} \mathcal{C}_n, \qquad \mathcal{S} := \bigcup_{n=0}^{\infty} \mathcal{S}_n = [0,1] \setminus \mathcal{C}.$$

Throughout this exercise denote the Lebesgue measure on $[0,1]$ by $\mu_{\mathcal{L}}$ or $\mathrm{d}x$ (hence $\mu_{\mathcal{L}}(B) = \int_B \mathrm{d}x$ for any Borel set in $[0,1]$). Prove that $\mathcal{C}$ is a compact Borel set and that $\mu_{\mathcal{L}}(\mathcal{C}) = 0$, i.e., the Cantor set has zero Lebesgue measure.




[1] This goes beyond the standard example of the Dirac measure, which is *singular* with respect to the Lebesgue measure, but *not continuous* (it is a *pure point* measure).

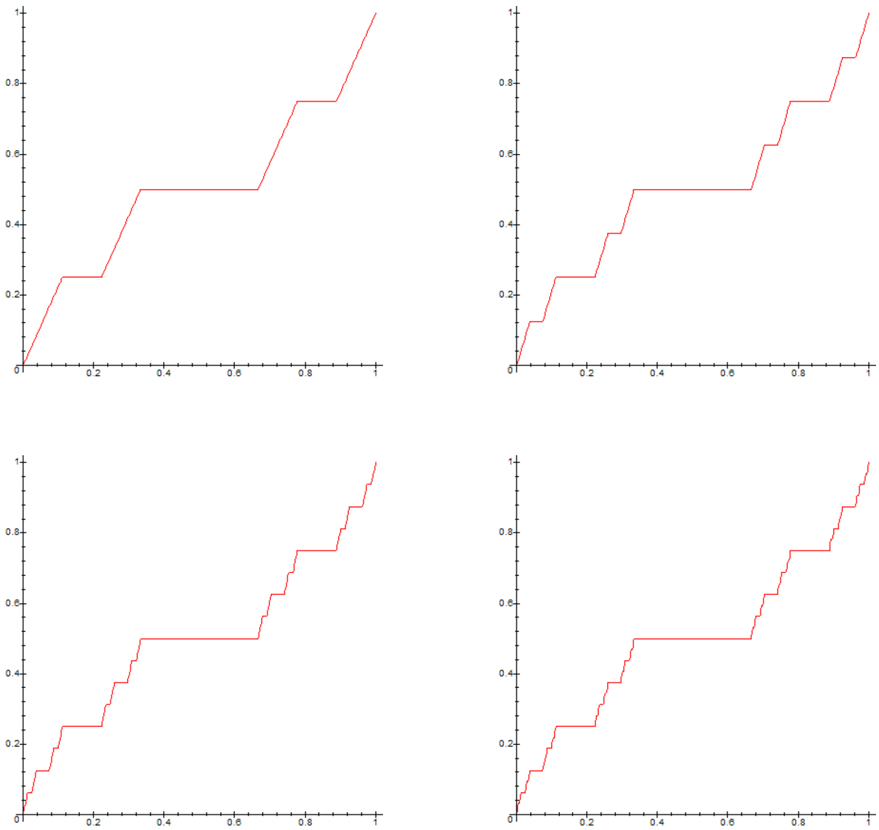
15.2) Prove that $\mathcal{C} \neq \emptyset$. Do $\frac{1}{3}, \frac{2}{3}, \frac{1}{4}$, or $\frac{3}{10}$ belong to $\mathcal{C}$? What is the difference, with respect to the recursive generation of the set $\mathcal{C}$, between $\frac{1}{3}, \frac{2}{3}$ on one side and $\frac{1}{4}, \frac{3}{10}$ on the other side?

15.3) Prove that $\mathcal{C}$ is uncountable and has the same cardinality as $\mathbb{R}$.[2] In parts 15.5 – 15.6 below you are asked to construct an explicit bijection $\mathcal{C} \leftrightarrow [0,1]$, which of course proves that $\mathcal{C}$ has the cardinality of the continuum. The idea here is that you may prove the statement with a straightforward "counting" of the points of $\mathcal{C}$.

15.4) Prove that $x \in \mathcal{C}$ if and only if $x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$ where each $a_k \in \{0, 2\}$ or, equivalently, if and only if there exists a sequence $\{n_i\}_{i=0}^{\infty}$ of integers with $0 < n_i < n_{i+1}$ such that $x = \sum_{i=0}^{\infty} \frac{2}{3^{n_i}}$. Restate this result in terms of the base-3 expansion of $x$. As a first consequence, prove that the Cantor set $\mathcal{C}$ has no interior points. As a further consequence, prove that the characteristic function $\chi_{\mathcal{C}}$ of the Cantor set is given by

$$\chi_{\mathcal{C}}(x) \;=\; \prod_{n=0}^{\infty} \chi_{_J}(3^n x \mod 1)$$

where $\chi_{_J} : [0,1] \to \mathbb{R}$ is the characteristic function of the set $J := [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.




Plot of $f_2$, $f_3$, $f_4$, and $f_5$

15.5) Now construct the *Cantor function* $f : [0,1] \to [0,1]$ as follows. For any positive integer $n$ consider the set $\mathcal{S}_n = [0,1] \setminus \mathcal{C}_n$ which consists of $2^n - 1$ open intervals (ordered from left to right) removed in the first $n$ steps of the construction of the Cantor set, and denote

[2] So one has the somewhat non-intuitive result of a set of measure zero ("very tiny") with uncountably many points ("very big"). Notice that actually the irrational numbers have the same property, but the Cantor set has the additional property of being closed, so it is not even dense in $[0,1]$, unlike the irrational numbers, which are dense everywhere.

these intervals by $\mathcal{I}_j^{(n)}$, $j = 1, \dots, 2^n - 1$. Then define

$$f_n(x) \;:=\; \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x = 1 \\ \dfrac{j}{2^n} & \text{if } x \in \mathcal{I}_j^{(n)} \\ \text{linear} & \text{otherwise}\,. \qquad (\bullet) \end{cases}$$

In the last case $(\bullet)$, $f_n$ is meant to be defined as a linear function between two consecutive intervals $\mathcal{I}_j^{(n)}$, joining the values that $f_n$ attains in $\mathcal{I}_j^{(n)}$ and $\mathcal{I}_{j+1}^{(n)}$. Check that $f_n$ is continuous and monotone non decreasing. Prove that as $n \to \infty$ the $f_n$'s converge *uniformly* to a function $f : [0,1] \to [0,1]$, the Cantor function, that has the following somewhat "strange" behaviour:

- $f$ is continuous and monotone non decreasing,
- $f(0) = 0$, $f(1) = 1$,
- $f(x)$ is constant on every interval removed in constructing $\mathcal{C}$, i.e., on the whole set $\mathcal{S}$ that has Lebesgue measure 1!

(Due to the above properties, $f$ is usually referred to as a "Devil's ladder".)

15.6) Compute the explicit action of $f$ on the points of the Cantor set $\mathcal{C}$ in terms of their base-3 expansion (see part 15.4 above). As a consequence, prove that $f(\mathcal{C}) = [0,1]$, that is, the restriction of $f$ to the Cantor set $\mathcal{C}$ is *surjective*. (This proves once again the fact that $\mathcal{C}$ has the cardinality of the continuum.)

15.7) Prove that $f'(x)$ exists almost everywhere, with respect to the Lebesgue measure, and is zero almost everywhere.

15.8) Prove that $f'$ *well defines* a new measure $\mu_{\mathcal{C}}$ on (the Borel sets of) $[0,1]$ such that the Radon-Nikodym derivative of $\mu_{\mathcal{C}}$ with respect to the Lebesgue measure $\mu_{\mathcal{L}}$ is exactly $f'$. Prove that $\mu_{\mathcal{C}}$ is singular continuous with respect to $\mu_{\mathcal{L}}$.

**Exercise 16.** Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. Let $A$ be a bounded self-adjoint operator on $\mathcal{H}$. Prove that the essential spectrum of $A$ is not empty.

**Exercise 17.** (In this exercise the underlying measure is the Lebesgue measure on $\mathbb{R}$ and $|\Omega|$ denotes the Lebesgue measure of a Borel set $\Omega \subset \mathbb{R}$.) Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded Borel function and let $A : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the (bounded and self-adjoint) operator of multiplication by $f$, i.e., $(A\psi)(x) := f(x)\psi(x)$ for almost every $x \in \mathbb{R}$ and all $\psi \in L^2(\mathbb{R})$.

($a$) Give (necessary and sufficient) conditions on $f$ so that $\mathrm{Spec}(A) = \mathrm{Spec}_{\mathrm{ac}}(A)$. Provide also a concrete example. (*Hint:* for each $\psi \in L^2(\mathbb{R})$, with $\int |\psi(x)|^2 \mathrm{d}x = 1$, identify the spectral measure $\mu_\psi^{(A)}$ associated with the operator $A$. Then investigate when $\mu_\psi^{(A)}$ is absolutely continuous with respect to the Lebesgue measure.)

($b$) In the particular case where $f$ is the characteristic function of $[0,1]$, determine $\mathrm{Spec}(A)$ and its components ($\mathrm{Spec}_{\mathrm{dis}}$, $\mathrm{Spec}_{\mathrm{ess}}$, $\mathrm{Spec}_{\mathrm{pp}}$, $\mathrm{Spec}_{\mathrm{ac}}$, $\mathrm{Spec}_{\mathrm{sc}}$).

($c$) For a generic bounded Borel function $f : \mathbb{R} \to \mathbb{R}$, prove that $\mathrm{Spec}(A) = \text{essential-range}(f) := \{y \in \mathbb{R} \,:\, \forall \varepsilon > 0\, |f^{-1}(\,]y - \varepsilon, y + \varepsilon[\,)| > 0\}$.