Functional Analysis II – Problem sheet 2

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Due: Tuesday 12.05.2009 by 10,15 a.m. in the "Funktionalanalysis II" box
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Exercise 4. Prove that any bounded measurable function on $\mathbb{R}$ can be uniformly approximated by a step function that has finitely many levels.[1] In other words, pick a bounded measurable function $f : \mathbb{R} \to \mathbb{R}$ (for a generic $f : \mathbb{R} \to \mathbb{C}$ one just repeats the argument for the real and imaginary part of f) and fix an arbitrarily small error $\varepsilon > 0$: then prove that there exists a step function $f_\varepsilon : \mathbb{R} \to \mathbb{C}$, with finitely many levels, such that $\|f - f_\varepsilon\|_\infty \leqslant \varepsilon$. (*Hint:* the natural way to proceed is to *construct* explicitly the approximating step function. To this aim, observe that the range of f can be covered by a finite and conveniently large number of balls. Consider the pre-image of these balls to define the steps of f_ε. This also shows you that the number of levels depend on ε.)

Exercise 5. (*The discrete Laplacian on $\mathbb{Z}^d$.*) Recall that the space

$$\ell^2(\mathbb{Z}^d) \;:=\; \Big\{\phi : \mathbb{Z}^d \to \mathbb{C} \;:\; \sum_{x\in\mathbb{Z}^d} |\phi(x)|^2 < \infty \Big\}$$

is a Hilbert space with the scalar product $\langle \phi, \xi\rangle = \sum_{x\in\mathbb{Z}^d} \overline{\phi(x)}\xi(x)$. You may think of an element ϕ in $\ell^2(\mathbb{Z}^d)$ just as an assignment of complex numbers, one at each point of the infinite lattice $\mathbb{Z}^d$, such that they are square summable. (You should be familiar with the one-dimensional version of it, namely $\ell^2 = \ell^2(\mathbb{Z})$.) The goal of this problem is to introduce a self-adjoint operator on $\ell^2(\mathbb{Z}^d)$ which is the discrete analogue of the usual Laplacian on $L^2(\mathbb{R}^d)$, with the key difference that the discrete version is *bounded*. As a consequence, by means of the spectral theory for bounded operators that you are supposed to know until know, you should be able to determine its spectrum.

- *Recall first* the following basic facts (if you did not know them already, you are strongly encouraged to prove them separately, although this is not part of the exercise). The operator $R : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ such that $(R\phi)(x) = \phi(x-1)$ for all $x \in \mathbb{Z}$ is called the *right shift* operator. R is unitary and its adjoint $L := R^* : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is just the *left shift* operator, that is, $(L\phi)(x) = \phi(x+1)$ for all $x \in \mathbb{Z}$. Moreover $\mathrm{Spec}(R) = \mathrm{Spec}(L) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

[1] With this nomenclature one means, for instance, that the $\mathbb{R} \to \mathbb{R}$ function

$$g(x) \;=\; \begin{cases} 1 & x \in [2k, 2k+1) \\ 0 & x \in [2k+1, 2k+2) \end{cases} \qquad (k \in \mathbb{Z})$$

is a *step function* with of course two "levels" only (0 and 1), but with an infinite number of "steps".

Here the problem starts.

5.1) By definition the *discrete Laplacian* is the operator $\Delta : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ acting on any element $\phi \in \ell^2(\mathbb{Z}^d)$ as

$$(\Delta\phi)(x) \;:=\; \sum_{\substack{y\,\in\,\mathbb{Z}^d\\ |x-y|=1}} \big(\,\phi(x) - \phi(y)\,\big)\,, \qquad x \in \mathbb{Z}^d. \tag{$*$}$$

In the notation above $|x-y|$ is the *Euclidean* distance between x and y as points of $\mathbb{R}^d$ with integer coordinates. How many terms are there in the r.h.s. of $(*)$? Rearrange the summands in the r.h.s. of $(*)$ so to express the operator Δ in terms of the identity operator and a number of shift operators.

5.2) Prove that Δ is bounded with norm $\|\Delta\| = 4d$. Prove also that $\Delta = \Delta^*$. (*Hint:* one can certainly prove both statements from the scratch and this would be fully graded as well, but it is a pain for you! (and for the grader too.) Alternatively, note that, at least for proving self-adjointness and $\|\Delta\| \leqslant 4d$, the rearrangement in point (5.1) above does the job.)

5.3) Prove the operator inequality $\mathbf{0} \leqslant \Delta \leqslant 4d$.

5.4) Prove that $\mathrm{Spec}(\Delta) \subseteq [0, 4d]$.

5.5) Prove that actually $\mathrm{Spec}(\Delta) = [0, 4d]$. (*Hint:* if you have completed the previous point, you are left with proving that $\mathrm{Spec}(\Delta) \supseteq [0, 4d]$. A possible way is to take any $\lambda \in [0, 4d]$ and to prove that λ satisfies the Weyl's criterion ($\to$ Exercise 3). To this aim, you need to identify one Weyl sequence, namely a sequence $\{\phi_n\}_{n=1}^\infty \subset \ell^2(\mathbb{Z}^d)$ such that $\|(\Delta-\lambda)\phi_n\| \to 0$ as $n \to \infty$. Here is a suggestion to construct ϕ_n explicitly. Consider the $\mathbb{Z}^d \to \mathbb{C}$ function $\widetilde{\varphi}_k(x) = e^{ik\cdot x}$ in the variable $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$, where $k = (k_1, \dots, k_d) \in \mathbb{R}^d$ is fixed and $k \cdot x = \sum_{j=1}^d k_j x_j$ is the Euclidean scalar product of k times x as points of $\mathbb{R}^d$. Observe that $\widetilde{\varphi}_k \notin \ell^2(\mathbb{Z}^d)$, nevertheless compute the *formal* action of Δ on $\widetilde{\varphi}_k$, i.e., do the computation just by means of the prescription $(*)$. This way you should see that given any $\lambda \in [0, 4d]$ you can always choose k depending on λ such that "$\Delta\widetilde{\varphi}_k = \lambda\widetilde{\varphi}_k$". Hence the "eigenvalue" λ is in the spectrum of Δ. Of course this is formal because $\widetilde{\varphi}_k$ does not belong to the domain of Δ but it should give you a hint on how to construct the Weyl sequence $\{\phi_n\}_{n=1}^\infty$ you are looking for. More concretely, what is the difference if in the formal argument above you modify $\widetilde{\varphi}_k$ setting it to give zero for all $x \in \mathbb{Z}^d$ outside a large cube centred at the origin?)