Functional Analysis II – Problem sheet 1

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Due: Tuesday 5.05.2009 by 10,15 a.m. in the "Funktionalanalysis II" box
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Exercise 1. Let P be a projection acting on a Hilbert space $\mathcal{H}$, that is, $P \in BL(\mathcal{H})$ and $P^2 = P$. (a) Prove that the range of P is a closed subspace of $\mathcal{H}$. (b) Prove that the following properties are equivalent:

$$(*) \qquad \mathrm{Ker}P = (\mathrm{Ran}P)^{\perp}$$
$$(**) \qquad P^*P = P = PP^*\,.$$

When $(*)$ or $(**)$ holds, P is said to be an *orthogonal* projection.

Exercise 2. With the notation of the proof of Theorem 1.4 discussed in the last lecture, prove that if the map $\Phi_0 : \mathrm{Poly}(\mathrm{spec}(A)) \to BL(\mathcal{H})$ is a $*$-algebraic homomorphism (that is, it satisfies the property (b) of the statement of Theorem 1.4) then the same holds for its extension $\Phi : C(\mathrm{spec}(A)) \to BL(\mathcal{H})$.

Exercise 3. Let A be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. Prove that $\lambda \in \mathrm{spec}(A)$ *if and only if* there exists one sequence $\{\varphi_n\}_n$ in $\mathcal{H}$ of normalised vectors ($\|\varphi_n\| = 1$) such that $\|(A-\lambda)\varphi_n\| \to 0$ as $n \to \infty$. This characterisation of the spectrum of A usually goes under the name of *Weyl's criterion*: it says that the points of $\mathrm{spec}(A)$ are "almost eigenvalues" of A (i.e., $A\varphi_n \approx \lambda\varphi_n$) up to an error that is arbitrarily small in norm. In solving this problem, the φ_n's *need not* to be taken orthogonal.