Duality of interactive particle systems and recursive tree processes connected to mean-field limits

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Random Walks, Random Graphs and Random Media
Munich
September 9-13, 2019

Outline

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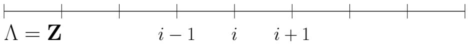
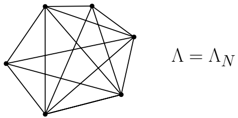
1. Interacting particle systems, graphical representations and duality
2. Pathwise duality for monotone and additive processes
3. Interacting particle system on the complete graph

Interacting particle system (IPS)

Continuous-time Markov process X on $E = \{0,1\}^\Lambda \cong \mathcal{P}(\Lambda)$
For example Λ graph with edge set $\mathcal{E}$.
The local state space is $S = \{0,1\}$.

"neighbouring sites interact"





Interacting particle system (IPS)

Classical dynamics: terminology "1" - particle "0" - empty site

One site i:	**i**	**map**
death	$1 \mapsto 0$	$\mathtt{death}_i$
birth	$0 \mapsto 1$	$\mathtt{birth}_i$
Two neighbouring sites *ij*	**ij**	**map**
branching/contact	$10 \mapsto 11$	$\mathtt{bra}_{ij}$
random walk and coalescence	$10 \mapsto 01$ $11 \mapsto 01$	$\mathtt{rw}_{ij}$
random walk and annihilation	$10 \mapsto 01$ $11 \mapsto 00$	$\mathtt{ann}_{ij}$
voter	$10 \mapsto 11$ $01 \mapsto 00$	$\mathtt{vot}_{ij}$
exclusion	$01 \mapsto 10$	$\mathtt{exc}_{ij}$
Three neighbouring sites *ijk*	**ijk**	**map**
cooperative branching	$110 \mapsto 111$	$\mathtt{cob}_{ijk}$

Markov processes and random mapping representations

With theses maps one can formulate the IPS as follows:
Let $X = (X_t)_{t \geq 0}$ be a continuous-time Markov chain with (nice) state space E and generator G. Then G can be written in the form of a **random mapping representation:**

Let $\mathcal{G} \subset \mathcal{F}(E, E) := \{m : E \to E\}$ and
let $(r_m)_{m \in \mathcal{G}}$ be nonnegative constants.

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big(f(m(x)) - f(x)\big) \quad , x \in E.$$

Note: This kind of representation is not unique.

Stochastic flow construction of Markov processes

The random mapping representation can be used for a Poissonian construction of the Markov process: **stochastic flow**.
Let Δ be a Poisson point subset of $\mathcal{G} \times \mathbb{R}$ with local intensity $r_m \mathrm{d}t$.
For $s \leq u$, set $\Delta_{s,u} := \Delta \cap (\mathcal{G} \times (s, u])$.
Define random maps $\mathbf{X}_{s,t} : E \to E$ $(s \leq t)$ by

$$\mathbf{X}_{s,t}(x) := m_n \circ \cdots \circ m_1(x) \text{ when}$$

$$\Delta_{s,t} := \{(m_1, t_1), \dots, (m_n, t_n)\}, \quad t_1 < \cdots < t_n.$$

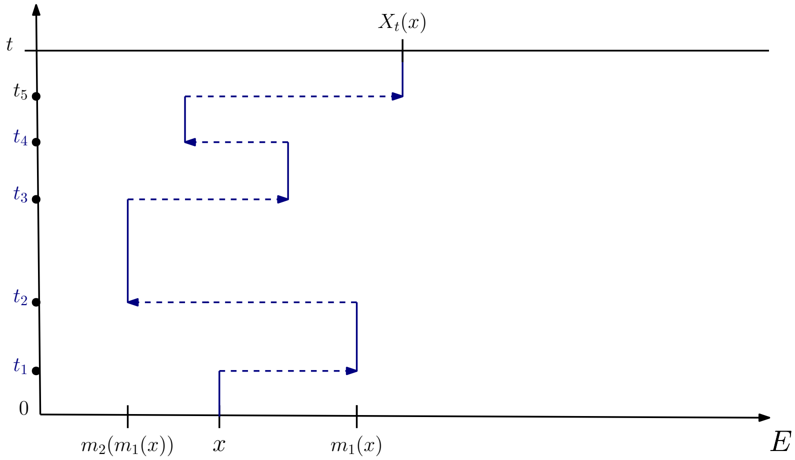
Note that $\mathbf{X}_{t,u} \circ \mathbf{X}_{s,t} = \mathbf{X}_{s,u}$ for all $s \leq t \leq u$.
Well defined for E finite, with additional conditions locally for IPS.

Poissonian construction of a Markov process
Let X_0 be an E-valued r.v., independent of Δ. Setting for $s \in \mathbb{R}$,

$$X_t := \mathbf{X}_{s,s+t}(X_0), \qquad t \geq 0$$

defines a Markov process $X = (X_t)_{t \geq 0}$ with generator G.

Stochastic flow construction of Markov processes



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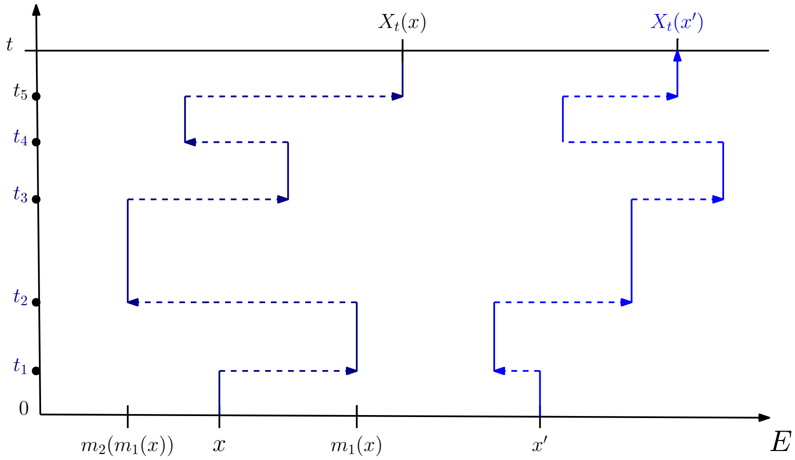
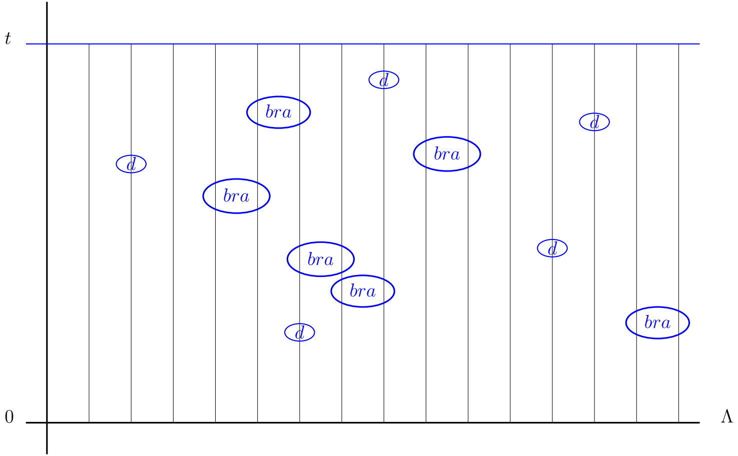


Stochastic flow construction of Markov processes

Local picture for IPS gives the graphical construction:
Contact process



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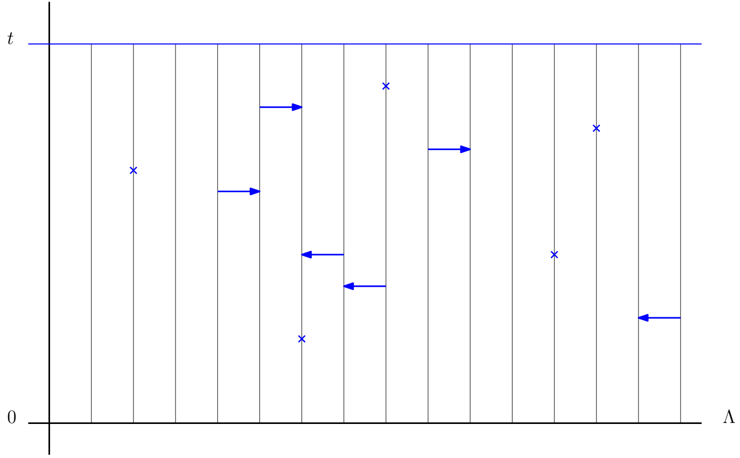

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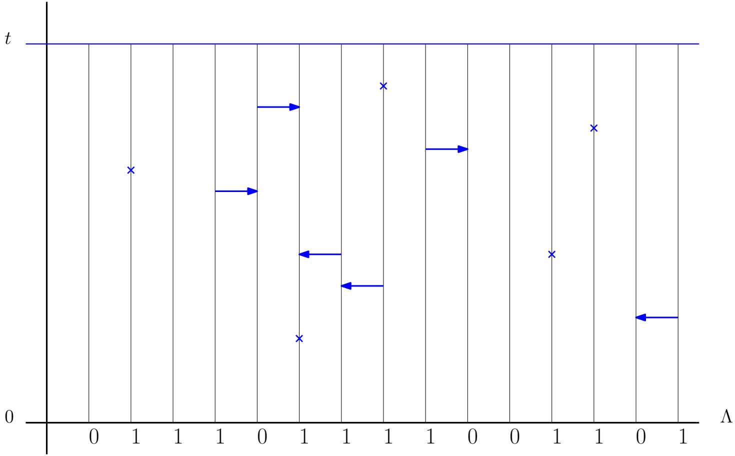

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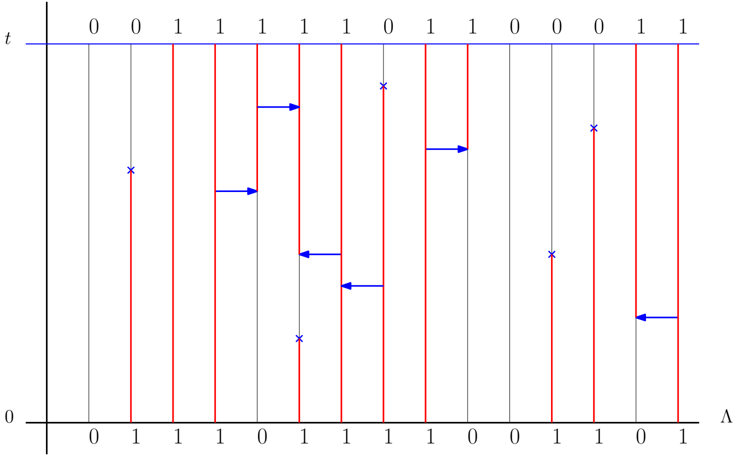
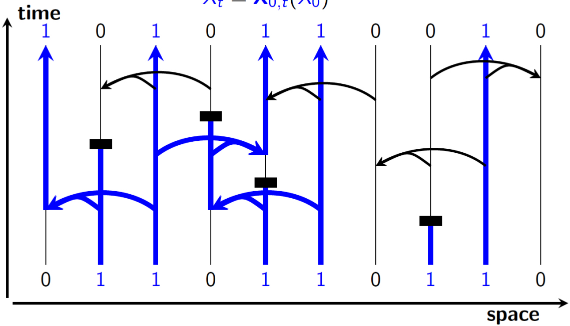


Graphical representation

Graphical representation using random maps of the cooperative branching model with deaths

$X_t = \mathbf{X}_{0,t}(X_0)$



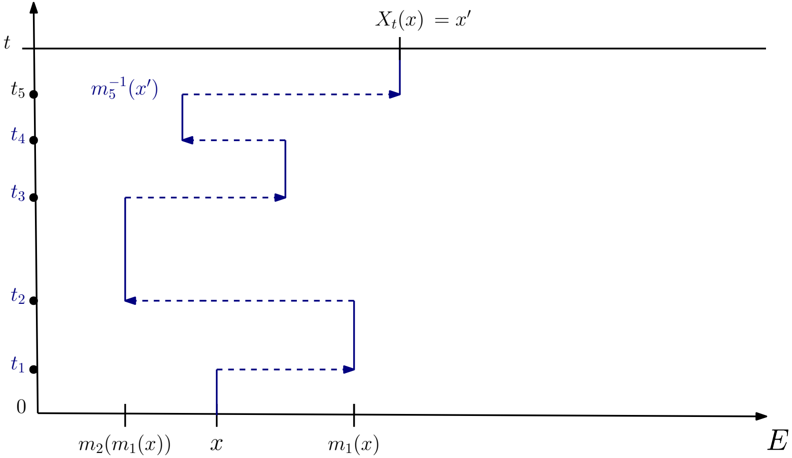
X_0

Determining X_t from the stochastic flow

Our goal is to determine

- $X_t = x'$ by looking at the relevant history of the stochastic flow in $[0, t]$ backwards in time,
- the state of one site $X_t(i)$ (or several sites) in the case of IPS by looking at the configurations at all sites that were relevant for that site.

Determining X_t from the stochastic flow



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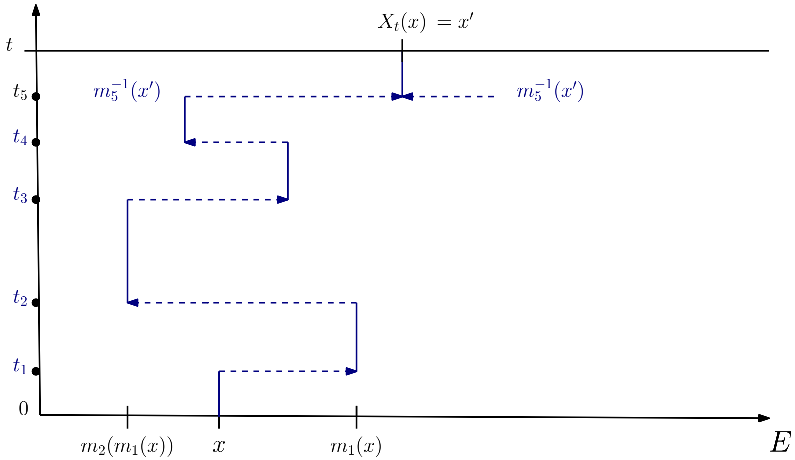
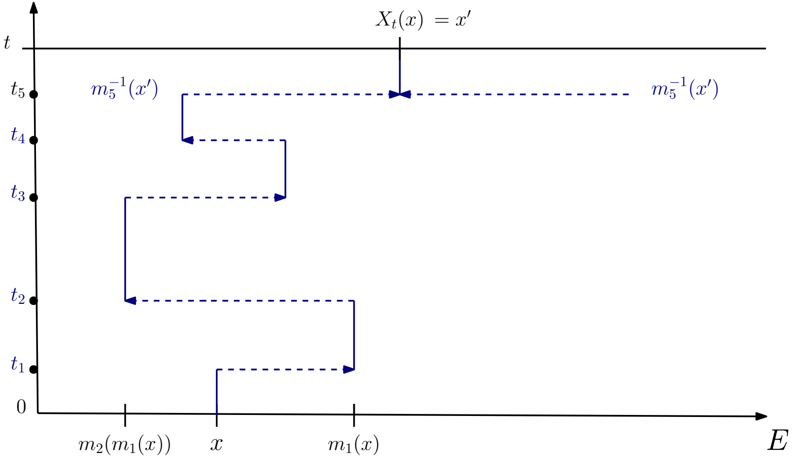


But there are many paths back into the past from x'!

Determining X_t from the stochastic flow



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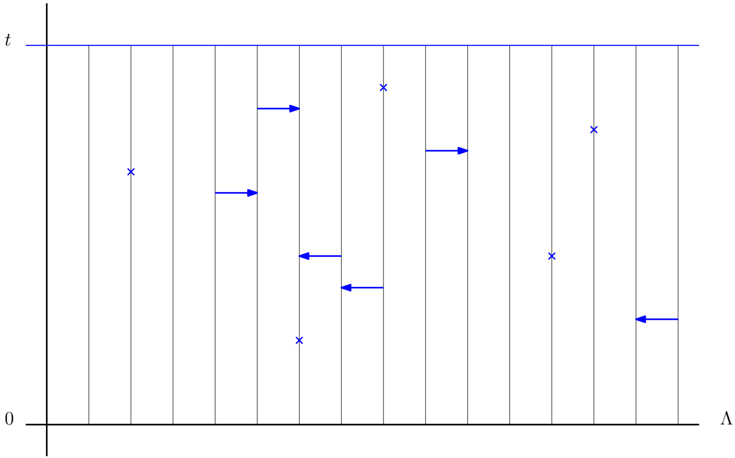
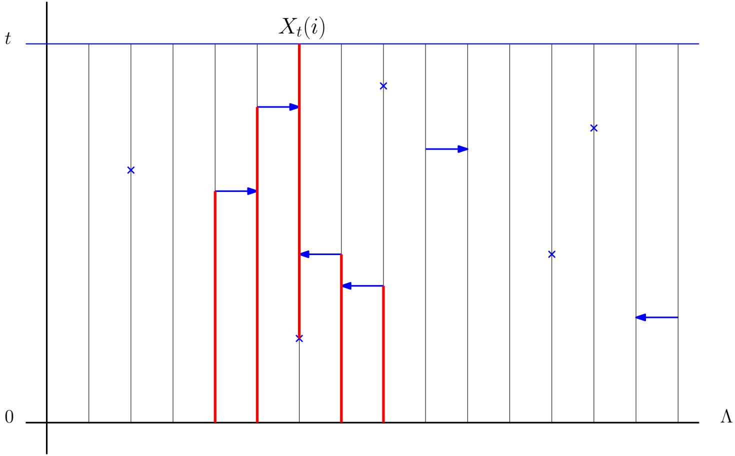


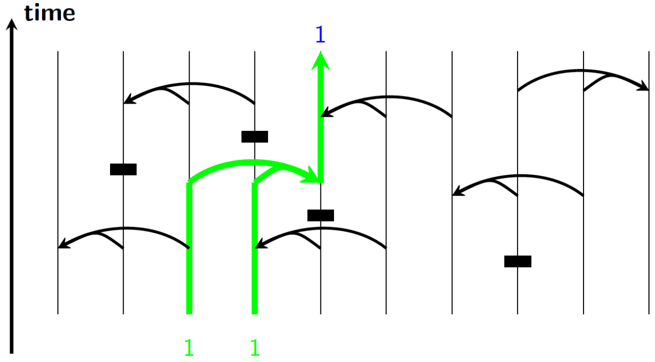
Determining X_t from the stochastic flow



To determine $X_t(i)$ trace ancestry of all (possibly) relevant sites:
Here, contact process with percolation structure.

Determining X_t from the stochastic flow

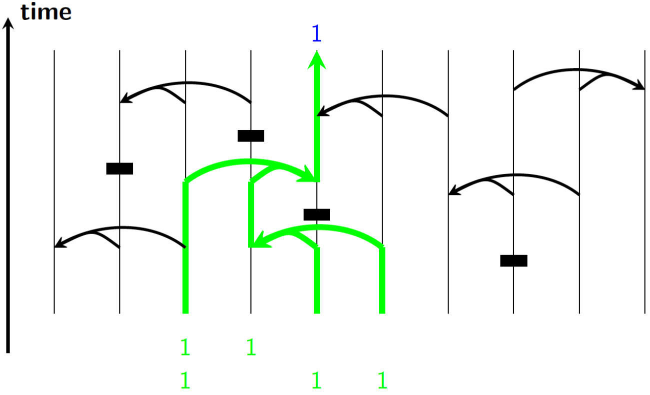
Graphical representation for cooperative branching with deaths:



Possible ancestry is more complicated than in percolation picture.

Determining X_t from the stochastic flow

Graphical representation for cooperative branching with deaths:



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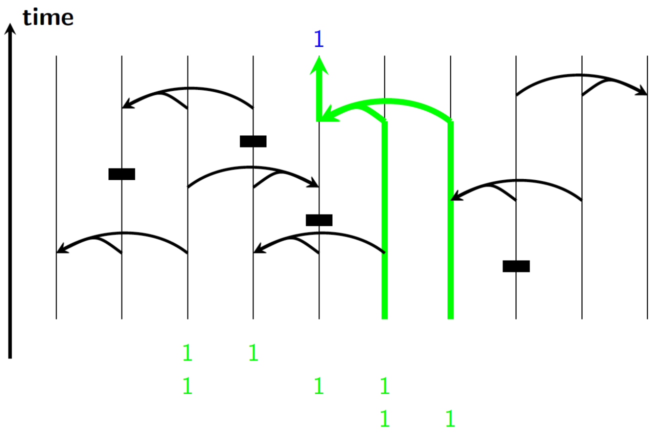

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Determining X_t from the stochastic flow

If we want to find a **process running backwards in time** that characterises a suitably large class of functions of X_t then this leads to the concept of **(pathwise) duality**.

Terminology, overview: Jansen and Kurt '14
More literature and examples later.

Pathwise duality

Let X and Y have state spaces E and E' and generators

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big(f(m(x)) - f(x)\big),$$

$$Hf(y) = \sum_{m \in \mathcal{G}} r_m \big(f(\hat{m}(y)) - f(y)\big).$$

Proposition (Pathwise duality)

Let $\psi : E \times E' \to \mathbb{R}$ be a function such that

$$(*) \qquad \psi\big(m(x), y\big) = \psi\big(x, \hat{m}(y)\big) \qquad x \in E,\ y \in E',\ m \in \mathcal{G}.$$

Then, X and Y are pathwise dual: They can be coupled such that

$$s \mapsto \psi(X_s, Y_{t-s})$$

is **almost surely constant** on $[0, t]$ with $t \geq 0$, and X_{s-} is independent of $Y_{t-s}, s \in [0, t]$.

Pathwise duality from the Poissonian construction

Let $\Delta, \hat{\Delta}$ be graphical representations for X and Y with

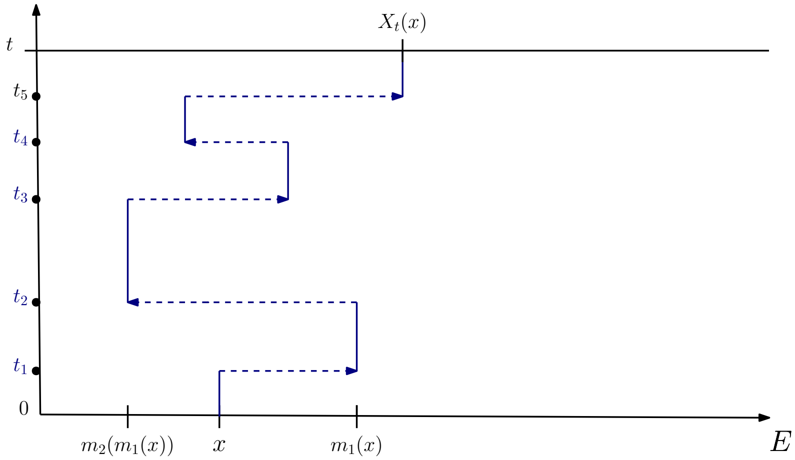
$$\hat{\Delta} := \{(\hat{m}, -t) : (m, t) \in \Delta\}.$$

Let $\mathbf{X}_{s,t-}$ and $\mathbf{Y}_{s,t}$ be the respective Poissonian constructions. Then, for all $x \in S$, $y \in S'$, the function

$$[0, t] \ni s \mapsto \psi\big(\mathbf{X}_{0,s-}(x), \mathbf{Y}_{-t,-s}(y)\big)$$

is a.s. constant.

Construction of a pathwise dual



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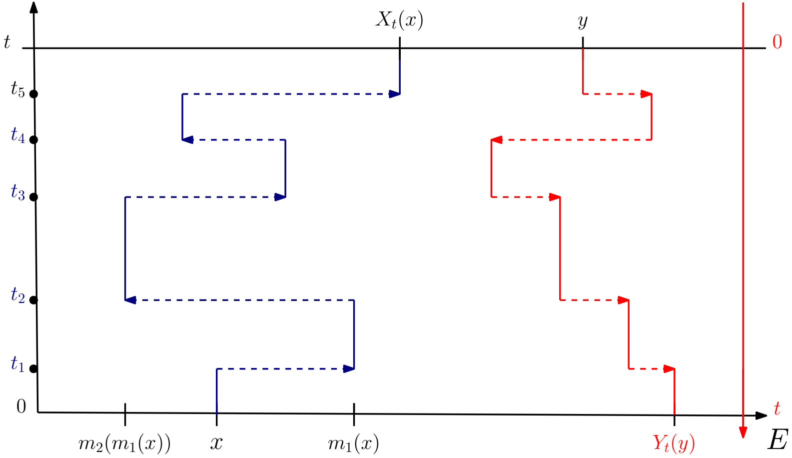

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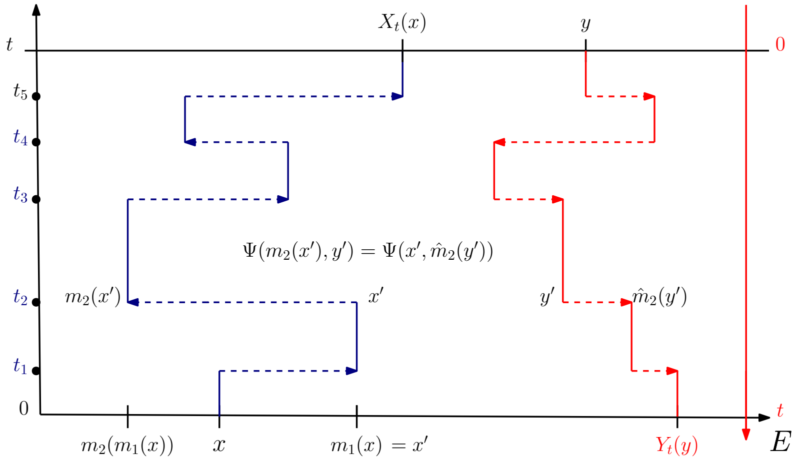


Construction of a pathwise dual

In a random mapping representation construct for any $m \in \mathcal{G}$ a dual map $\hat{m}$ and ψ such that $(*)$ holds.

$$\psi(m(x), y) = \psi(x, \hat{m}(y)).$$

General possibility

Let $E' = \mathcal{P}(E)$, the set of all subsets of E, and

$$\hat{m}(A) = m^{-1}(A) := \{x \in E : m(x) \in A\}, \quad A \in \mathcal{P}(E).$$

Then equality holds in $(*)$ with respect to the duality function

$$\psi(x, A) := 1_{\{x \in A\}}, \quad x \in E, A \in \mathcal{P}(E).$$

General duality function

$$\psi(x, A) := 1_{\{x \in A\}}, \quad x \in E, A \in \mathcal{P}(E), \quad \hat{m}(A) = m^{-1}(A)$$

Example: $E = \{0,1\}^{\Lambda}$

$$A = \{x \in E : x_i = 1\} \text{ and } 1_{\{x \in A\}} = 1_{\{x_i = 1\}}.$$

"The dual with state space $\mathcal{P}(E)$ tracks the set of configurations that a particular (set of) configuration(s) may have emerged from."

Find more useful dualities with values in subspaces of $\mathcal{P}(E)$ that are invariant under the inverse image maps m^{-1} for all $m \in \mathcal{G}$.

Concentrate on E partially ordered with m monotone or additive:
Sturm, Swart JTP 2016

Outline

Little excursion: Partially ordered sets

Let $(E, \leq)$ be a (finite) partially ordered set.

- For $A \subset E$ define $A^{\downarrow} := \{x \in E : x \leq y \text{ for some } y \in A\}$.
- $\mathcal{P}_{\mathrm{dec}}(E)$ are the **decreasing** sets A with $A^{\downarrow} \subset A$.
- $\mathcal{P}_{!\mathrm{dec}}(E)$ is a **principal ideal** if it consists of A with

$$A = \{z\}^{\downarrow} \text{ for some } z \in E.$$

Define analogously $A^{\uparrow}$,
increasing sets $\mathcal{P}_{\mathrm{inc}}(E)$ and **principle filters** $\mathcal{P}_{!\mathrm{inc}}(E)$.

Little excursion: Partially ordered sets

- In a join-semilattice $\mathcal{P}_{!\mathrm{inc}}(E)$ is closed under finite intersections and the **supremum** is well defined via

$$\{x \vee y\}^{\uparrow} := \{x\}^{\uparrow} \cap \{y\}^{\uparrow}$$

- $x \vee y$ is the minimal element such that

$$x \leq x \vee y \quad \text{and} \quad y \leq x \vee y.$$

Example:
For IPS we have $E = \{0,1\}^{\Lambda}$ and $\vee$ corresponds to the coordinate wise maximum. If we consider $E \cong \mathcal{P}(\Lambda)$ then $\leq$ corresponds to $\subset$ and $\vee$ corresponds to $\cup$.

- For E a join-semilattice we have $\emptyset \neq A \in \mathcal{P}_{!\mathrm{dec}}(E) \Leftrightarrow A \in \mathcal{P}_{\mathrm{dec}}(E)$ and $x, y \in A$ implies $x \vee y \in A$.

Little excursion: Monotone and additive functions

- A function m is **monotone** if

$$x \leq y \text{ implies } m(x) \leq m(y), \quad x, y \in E.$$

- A function m is **additive** on a join-semilattice with minimal element 0 if

$$m(x \vee y) = m(x) \vee m(y), \quad x, y \in E$$

  as well as $m(0) = 0$.

Remark:

- Additive functions are monotone.
- Monotone functions are superadditive: $m(x \vee y) \geq m(x) \vee m(y)$

Invariant subspaces for monotonefunctions

Proposition (Monotone functions)

Equivalent:

- m is monotone.
- m^{-1} maps $\mathcal{P}_{\mathrm{dec}}(E)$ into itself (invariant subspace!).
- m^{-1} maps $\mathcal{P}_{\mathrm{inc}}(E)$ into itself (invariant subspace!).

- For $A \in \mathcal{P}_{\mathrm{dec}}(E)$ consider $x \leq y$ and $y \in m^{-1}(A)$.
- Then by monotonicity $m(x) \leq m(y) \in A$ and since A is decreasing $m(x) \in A$.
- It follows $x \in m^{-1}(A)$ and $m^{-1}(A)$ decreasing.

Invariant subspaces foradditive functions

Proposition (Additive functions)

Equivalent (on a finite join-semilattice with minimal element):

- m is additive.
- m^{-1} maps $\mathcal{P}_{!\mathrm{dec}}(E)$ into itself (invariant subspace!).

- $m^{-1}(A) \in \mathcal{P}_{\mathrm{dec}}(E)$ for $A \in \mathcal{P}_{!\mathrm{dec}}(E)$ (additive functions monotone)
- $x, y \in m^{-1}(A) \Rightarrow x \vee y \in m^{-1}(A)$:
  $m(x \vee y) = m(x) \vee m(y)$ and $m(x) \vee m(y) \in A$.
- Taken together this implies $m^{-1}(A) \in \mathcal{P}_{!\mathrm{dec}}(E)$.

Monotonically and additively representable processes

If a Markov process X has random mapping representation

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big(f(m(x)) - f(x) \big) \quad , x \in E$$

where

- $\mathcal{G}$ contains only monotone functions then we call X **monotonically representable.**
- $\mathcal{G}$ contains only additive functions then we call X **additively representable.**

Pathwise duality for additively representable processes

E' is a *dual* of E if there is a bijection $E \ni x \mapsto x' \in E'$ $(x'' = x)$:

$$x \leq y \quad \Leftrightarrow \quad x' \geq y'$$

Note: $m(x) \in \{y'\}^{\downarrow} \Leftrightarrow m(x) \leq y'$. So consider for $x \in E,\ y \in E'$

$$\psi(x, y) = 1_{\{x \leq y'\}} = 1_{\{y \leq x'\}}$$

Lemma (Duals to additive maps)

For additive $m : E \to E$ there exists (a unique) $m' : E' \to E'$ with

$$(*) \qquad 1_{\{m(x) \leq y'\}} = 1_{\{x \leq (m'(y))'\}}, \qquad x \in E, y \in E'.$$

- Due to additivity there exists $z \in E$ such that $m^{-1}(\{y'\}^{\downarrow}) = \{z\}^{\downarrow}$
- Set $m'(y) = z', y \in E'$ such that

$$m(x) \leq y' \quad \text{if and only if} \quad x \leq z = z'' = (m'(y))'$$

Pathwise duality for additively representable processes

Theorem (Additive systems duality)

Let E be a finite lattice and let X be a Markov process in E whose generator has a random mapping representation of the form

$$Gf(x) = \sum_{m\in\mathcal{G}} r_m\big(f(m(x)) - f(x)\big), \qquad x \in E,$$

where all maps $m \in \mathcal{G}$ are additive **(additively representable)***. Then the Markov process Y in E' with generator*

$$Hf(y) := \sum_{m\in\mathcal{G}} r_m\big(f(m'(y)) - f(y)\big), \qquad y \in E'$$

is pathwise dual to X with respect to the duality function

$$\psi(x,y) = 1_{\{x\leq y'\}}, \quad x \in E, y \in E'.$$

Pathwise duality for additively representable processes

Examples:

- **1** $E' := E$ equipped with the reversed order and $x' = x$.
- **2** For $E = \{0,1\}^\Lambda \subset \mathcal{P}(\Lambda)$ equipped with $\subset$ take for $x' := \Lambda \backslash x = x^C$, the complement of x, and $E' := \{x' : x \in E\}$.

Recall that for $x \in E,\ y \in E'$

$$\psi(x,y) = 1_{\{x \le y'\}} = 1_{\{y \le x'\}}$$

- **1** $\psi(x,y) = 1_{\{x \le y'\}} = 1_{\{x \le y\}}$
  Siegmund's duality on a totally ordered space E
  mappings monotone with $m(0) = 0$
- **2** $\psi(x,y) = 1_{\{x \subset \Lambda \backslash y\}} = 1_{\{x \cap y = \emptyset\}}$
  Additive interacting particle systems

Examples for additive maps and their dual maps

Standard additive interacting particle system dynamics on $E = \{0,1\}^\Lambda \cong \mathcal{P}(\Lambda)$

- **Voter dynamics:**
  $\mathtt{vot}_{ij}: \quad 01 \to 11, 10 \to 00$
- **Contact dynamics:**
  $\mathtt{bra}_{ij}: \quad 10 \to 11$
- **Symmetric random walk with coalescence:**
  $\mathtt{rw}_{ij}: 10, 11 \to 01$
- **Spontaneous death of particles:**
  $\mathtt{death}_i: 1 \to 0$
- **Exclusion dynamics:**
  $\mathtt{exc}_{ij}: \quad 10 \to 01, 01 \to 10$

Let $E' = E$ and $x' = x^C$. Then the duality functions are

$\mathtt{vot}'_{ij} = \mathtt{rw}_{ij}$, $\mathtt{bra}'_{ij} = \mathtt{bra}_{ji}$, $\mathtt{rw}'_{ij} = \mathtt{vot}_{ij}$, $\mathtt{death}'_i = \mathtt{death}_i$, $\mathtt{exc}'_{ij} = \mathtt{exc}_{ij}$

Percolation structure for additively representable processes

Equip $E := \mathcal{P}(\Lambda)$ with $\subset$ and let m be an additive map $E \to E$. Define $M \subset \Lambda \times \Lambda$ via

$$m(x) = \{j \in \Lambda : (i,j) \in M \text{ for some } i \in x\} \qquad \text{for all } x \in E.$$

Vice versa, any such $M \subset \Lambda \times \Lambda$ corresponds to an additive map m.

Percolation structure for additively representable processes

Let $E' = E$ and $x' = x^C$. Then we have an additive $m' : E \to E$ dual to m with the duality function

$$\psi(x, y) = 1_{\{x \subset \Lambda \setminus y\}} = 1_{\{x \cap y = \emptyset\}}, \quad x, y \in E.$$

The $M' \subset \Lambda \times \Lambda$ corresponding to m' is given by

$$M' = \{(j, i) : (i, j) \in M\}.$$

Percolation structure for additively representable processes

Percolation representation

Plot space-time $\Lambda \times \mathbb{R}$ with time upwards.
At rate r_m we consider the M associated to m and

- draw an arrow from (i, t) to (j, t) $(i \neq j)$ whenever $(i, j) \in M$
- place a "blocking symbol" at (i, t) whenever $(i, i) \notin M$

"Open paths" $\rightsquigarrow$ travel upwards along arrows and avoid blocking symbols. Then

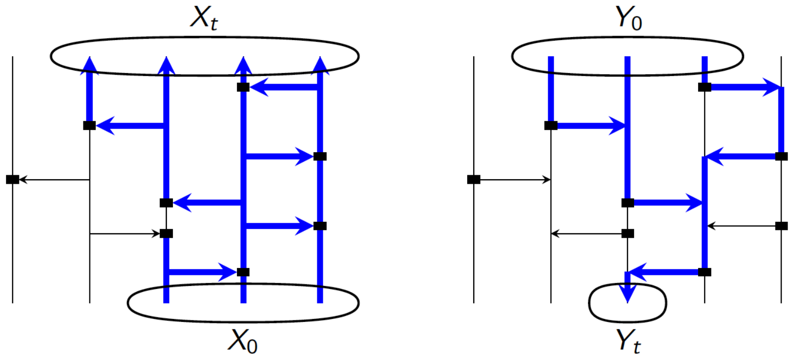
$$\mathbf{X}_{s,u}(x) = \{j \in \Lambda : (i, s) \rightsquigarrow (j, u) \text{ for some } i \in x\},$$

and the dual process is obtained via open paths using the reversed arrows (in reversed time).

Percolation structure for additively representable processes

Voter model

$E = \{0,1\}^\Lambda \cong \mathcal{P}(\Lambda)$.



Percolation structure for additively representable processes

Extensions

The above percolation structure statements also apply if

- Λ is a partially ordered set and $E = \mathcal{P}_{\mathrm{dec}}(\Lambda)$.
- E is a **distributive lattice** with

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad x, y, z \in E.$$

One can show that $E \cong \mathcal{P}_{\mathrm{dec}}(\Lambda)$ for a partially ordered set Λ by Birkhoff's representation theorem.

In this case for $i, j, i', j' \in \Lambda$

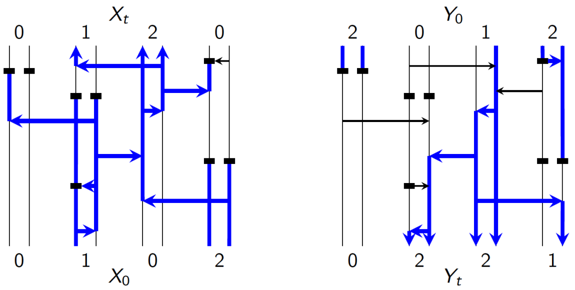
$$\text{(i)} \quad (i, j) \in M \text{ and } i \leq i' \text{ implies } (i', j) \in M,$$
$$\text{(ii)} \quad (i, j) \in M \text{ and } j \geq j' \text{ implies } (i, j') \in M.$$

Percolation structure for additively representable processes

Two stage contact process (Krone '99)

$E = \{0,1,2\}^\Lambda$ "1" younger individual "2" older individual.
Older individuals give birth to younger individuals who "grow up" and possibly die at a higher rate than older individuals.

$E \cong \mathcal{P}_{\rm dec}(\Lambda \times \{1,2\})$
with $x(i) = 1 \;\cong\; (i,1) \in x$ and $x(i) = 2 \;\cong\; (i,1),(i,2) \in x$



Pathwise duality for monotonically representable processes

Now consider the duality function

$$\phi(x,B) := 1_{\{x\in\{B'\}^{\downarrow}\}} = 1_{\{x\le y' \text{ for some } y\in B\}}, \qquad x\in E,\ B\in\mathcal{P}(E').$$

$Y_0 = y \in \mathcal{P}_{\mathrm{dec}}(E)$ implies $Y_t \in \mathcal{P}_{\mathrm{dec}}(E), t\ge 0$
$\Rightarrow$ Define a process Z such that $Y_t = Z_t^{\downarrow}, t\ge 0$.

Lemma (Duals to monotone maps)

For monotone $m : E \to E$ there exist $m^* : \mathcal{P}(E') \to \mathcal{P}(E')$ with

$$(*) \qquad 1_{\{m(x)\le y' \text{ for some } y\in B\}} = 1_{\{x\le y' \text{ for some } y\in m^*(B)\}}.$$

- By monotonicity m^{-1} maps decreasing sets of the form $A = \{B'\}^{\downarrow}$ into sets of this form.
- Construct appropriate m^* :$m^*(B)' := \bigcup_{x\in B}(m^{-1}(\{x'\}^{\downarrow}))_{\max}$

Pathwise duality for monotonically representable processes

Theorem (Monotone systems duality)

Let E be a finite partially ordered set and let X be a Markov process in E whose generator has a random mapping representation of the form

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big(f(m(x)) - f(x) \big) \quad x \in E,$$

*where all maps $m \in \mathcal{G}$ are monotone (**monotonically rep.**). Then the $\mathcal{P}(E')$-valued Markov process Y^* with generator*

$$H_* f(B) = \sum_{m \in \mathcal{G}} r_m \big(f(m^*(B)) - f(B) \big), \quad B \in \mathcal{P}(E')$$

is pathwise dual to X with respect to the duality function ϕ.

Monotone/monotonically representable processes

Classical concept of monotone Markov chains:
A continuous-time Markov chain X with values in the partially ordered set E is **monotone** if

$$x \mapsto E^x(f(X_t))$$

is monotone for all monotone $f : E \to E$.

In other words:
stochastically ordered initial distributions
stay stochastically ordered for all time.

Remark:

- Monotonic representability is a stronger concept than monotonicity in the classical sense (see Fill, Machida '01).
- However, there is equivalence if E is totally ordered (see Kamae, Krengel, O'Brien '77, Fill, Machida '01).

Pathwise duality for monotonically representable processes

Example: Cooperative branching coalescent with death

IPS with state space $\{0,1\}^{\Lambda} \cong \mathcal{P}(\Lambda)$

- **Spontaneous death of particles:**
  $\texttt{death}_i : 1 \to 0$
- **Symmetric random walk with coalescence:**
  $\texttt{rw}_{ij} : 10, 11 \to 01$
- **Pairs of particles produce a new particle:**
  $\texttt{cob}_{ijk} : 110 \to 111$

Pathwise duality for cooperative branching coalescent

All maps m are monotone, all but cooperative branching are additive. Let $E' = E$ and $x' = x^C$. Then the duality function is

$$\phi(x, B) = 1_{\{x \subset y^C \text{ for some } y \in B\}} = 1_{\{x \cap y = \emptyset \text{ for some } y \in B\}}$$

for $x \in E, B \in \mathcal{P}(E)$.
For the **additive functions** m there are dual functions m' with

$$m(x) \cap y = \emptyset \Leftrightarrow x \cap m'(y) = \emptyset$$

namely

$$\mathtt{rw}'_{ij} = \mathtt{vot}_{ij} \quad \text{and} \quad \mathtt{death}'_i = \mathtt{death}_i$$

We set $m^*(B) = \{m'(x) : x \in B\}$.

Pathwise duality for cooperative branching

Notation: For $m : E \to E$ define $m : \mathcal{P}(E) \to \mathcal{P}(E)$ by $m(Y) = \{m(y) : y \in Y\}$

For the **cooperative branching map** we have

$$\mathrm{cob}^*_{ijk}(B) = b^{(1,C)}_{ijk}(B) \cup b^{(2,C)}_{ijk}(B)$$

with the definition (restricted to sites ijk)

$$b^{(1,C)} : \ 001 \to 011, \quad b^{(2,C)} : \ 001 \to 101$$

since we have $x' := x^C$ and

$$\big(\mathrm{cob}^{-1}(\{x\}^{\downarrow})\big)_{\max} = \begin{cases} \{100, 010\} & \text{if } x = 110, \\ \{x\} & \text{otherwise.} \end{cases} = b^{(1)}(x) \cup b^{(2)}(x)$$

for $b^{(1)} : 110 \to 100, \quad b^{(2)} : \ 110 \to 010.$

Pathwise duality for cooperative branching

Another natural choice is to consider $E' = E$ with $x' = x$.

Using the invariance of decreasing sets this leads to the duality function: For $x \in E, \; B \in \mathcal{P}(E)$,

$$\phi(x, B) := 1_{\{x \in \{B\}^{\downarrow}\}} = 1_{\{x \leq y \text{ for some } y \in B\}}$$

The dual maps are

$$\begin{aligned} \mathtt{rw}^*_{ij}(B) &= \mathtt{vot}_{ji}(B) \\ \mathtt{cob}^*_{ijk}(B) &= b^{(1)}_{ijk}(B) \cup b^{(2)}_{ijk}(B) \end{aligned}$$

where $b^{(1)} : 110 \to 100, \quad b^{(2)} : \; 110 \to 010.$

Pathwise duality for cooperative branching

Let now $\underline{1} = \ldots 1111111\ldots$.

With the duality we can study the particle density

$$\mathbb{P}[X_t(i) = 1] = 1 - \mathbb{P}[X_t(i) = 0] = 1 - \mathbb{P}[X_t \leq y_0^i]$$

for $y_0^i = \underline{1} - e_i = \ldots 1111101111\ldots$.

Also, for the density of particle pairs we have

$$\begin{aligned} \mathbb{P}[X_t(i) = X_t(j) = 1] &= 1 - \mathbb{P}[X_t(i) = 0 \text{ or } X_t(j) = 0] \\ &= 1 - \mathbb{P}[X_t \leq y \text{ for some } y \in Y_0] \end{aligned}$$

for $Y_0 = \{y_0^i, y_0^j\}$.

Pathwise duality for cooperative branching

But for $X_0 = \underline{1} = \ldots 1111111\ldots$ we have in either case

$$\begin{aligned} 1 - \mathbb{P}[X_t \leq y \text{ for some } y \in Y_0] &= 1 - \mathbb{P}[X_0 \leq y \text{ for some } y \in Y_t] \\ &= 1 - \mathbb{P}[\underline{1} \in Y_t] = \mathbb{P}[\underline{1} \notin Y_t] \end{aligned}$$

Thus we get a **bounds** if we consider $\underline{Y}_t$ and $\overline{Y}_t$ instead of Y_t with

$$\underline{Y}_t \subset Y_t \text{ and } Y_t \subset \overline{Y}_t$$

Pathwise duality for cooperative branching coalescent

Sturm, Swart '15

$\Lambda = \mathbb{Z}$ without spontaneous death

- $\mathtt{rw}_{ij}$: Random walk with coalescence rate 1
- $\mathtt{cob}_{ijk}$: Cooperative branching rate α

- **Results regarding phase transitions:**

$$\begin{aligned} \alpha_{\rm surv} &:= \text{inf}\{\alpha > 0 : \text{ the process survives (pairs of particles)}\}, \\ \alpha_{\rm upp} &:= \text{inf}\{\alpha > 0 : \text{ the upper invariant law is nontrivial}\}. \end{aligned}$$

  We have $1 \leq \alpha_{\rm upp}, \alpha_{\rm surv} < \infty$.

  Conjecture: $\alpha_{\rm upp} = \alpha_{\rm surv}$

- **Application of a version of this dual:**
  Decay rates of the survival probability of pairs and the density in the subcritical regime is order $t^{-1/2}$.

Pathwise duality for cooperative branching coalescent

Particle density and density of particle pairs
Without cooperative branching we get a lower bound:
Here, for Y_t there are just coalescing random walks to consider:
The interfaces of the voter dynamics for y_0^i

1 1 1 1 | 0 | 1 1 1 1 1

1 1 | 0 0 0 0 0 | 1 1 1

- **Density of particles:**
  Let $\tau^{(2)} = \tau_{i(i+1)}$ until two random walkers meet:
  $\Rightarrow \mathbb{P}[X_t(i) = 1] = \mathbb{P}[\underline{1} \notin Y_t] \geq \mathbb{P}[\tau \geq t] \sim Ct^{-1/2}$

Pathwise duality for cooperative branching coalescent

Particle density and density of particle pairs

Without cooperative branching we get a lower bound for the density of particle pairs by considering the interfaces of y_0^i and y_0^{i+1} :

$$
\begin{array}{l}
1 \quad 1 \quad 1 \quad 1 \;\mid\; 0 \;\mid\; 1 \quad 1 \quad 1 \quad 1 \quad 1 \\
1 \quad 1 \quad 1 \quad 1 \quad 1 \;\mid\; 0 \;\mid\; 1 \quad 1 \quad 1 \quad 1
\end{array}
$$

$$
\begin{array}{l}
1 \quad 1 \quad 1 \;\mid\; 0 \quad 0 \quad 0 \;\mid\; 1 \quad 1 \quad 1 \quad 1 \\
1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \;\mid\; 0 \quad 0 \quad 0 \;\mid\; 1
\end{array}
$$

- **Density of particle pairs:**
  Let $\tau^{(3)} = \tau_{i(i+1)} \wedge \tau_{(i+1)(i+2)}$ be the time for two out of three independent walkers to meet:
  $\Rightarrow \mathbb{P}[X_t(i) = X_t(i+1) = 1] \geq \mathbb{P}[\tau^{(3)} \geq t] \sim Ct^{-3/2}$

Pathwise duality for cooperative branching

With cooperative branching we add a (dependent) branching process:

```
1   1   1   1 | 0 | 1   1   1   1   1
1   1   1   1   1 | 0 | 1   1   1   1

1   1   1 | 0   0   0 | 1   1   1   1
1   1   1   1   1   0 | 0   0   0 | 1

1 | 1 | 0   0   0   0 | 1   1   1   1
1   0 | 1 | 0   0   0 | 1   1   1   1
1   1   1   1   1   0 | 0   0   0 | 1
```

Suffices to follow:

```
1   1   1 | 0   0   0 | 1   1   1   1
1   1   1   1   1   0 | 0   0   0 | 1

1 | 1 | 0   0   0   0   0   0   0   0
1   0 | 1 | 0   0   0   0   0   0   0
```

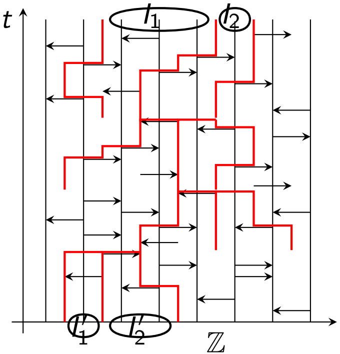
Pathwise duality for cooperative branching

With cooperative branching we have (roughly)

- triples of random walks die as soon as two out of the three meet
- with rate α a triple can give birth to a new triple of random walks started on neighbouring positions

As long as the cooperative branching rate is small enough this branching process dies out and the probability to be alive at time t decays as before without branching.

Cooperative branching coalescent: Pathwise superduality



One can also show for the survival probability of pairs

$$
\begin{aligned}
-\tfrac{\partial}{\partial t}\mathbb{P}\big[|X_t^{e_i+e_{i+1}}| \geq 2\big] &= \mathbb{P}\big[X_t^{e_i+e_{i+1}} = \{i, i+1\} \text{ for some } i \in \mathbb{Z}\big] \\
&\leq \mathbb{E}[N_t] \leq Ct^{-3/2}.
\end{aligned}
$$

where N_t is the number of three paths in the dual.

Pathwise duality for monotonically representable IPS

This kind of duality was considered by **Gray '86** for **monotone IPS with births and deaths:**

Generator:

$$\begin{aligned} Gf(x) &= \sum_{i\in\Lambda}\beta_i(x)\big(f(x+e_i)-f(x)\big) \\ &\quad+\sum_{i\in\Lambda}\delta_i(x)\big(f(x-e_i)-f(x)\big). \end{aligned}$$

Here, $\beta_i(x)$ and $-\delta_i(x)$ are assumed to be monotone.

For equivalence see **Sturm, Swart '16**.

Outline

1. Interacting particle systems, graphical representations and duality
2. Pathwise duality for monotone and additive processes
3. Interacting particle system on the complete graph

General definition

Markov process $X = (X_t)_{t \geq 0}$

- Complete graph Λ^N with vertices $[N] := \{1, \dots, N\}$
- **Polish** local state space S
- X takes values in $E = S^N : x = (x_1, \dots, x_N)$
- Dynamics are invariant under permutation of the coordinates

Random mapping representation At a certain rate choose a function g to apply to the current configuration x :

- $g : S^k \to S$ for some $k \in \mathbb{N}$
- Replace state at a randomly chosen site by g applied to the state at k distinct randomly chosen sites.

  Alternative:

  Site of replacement is part of the k randomly chosen sites.

Example: Cooperative branching with death

This is the case for **cooperative branching with death**:
We choose $S = \{0, 1\}$ and set

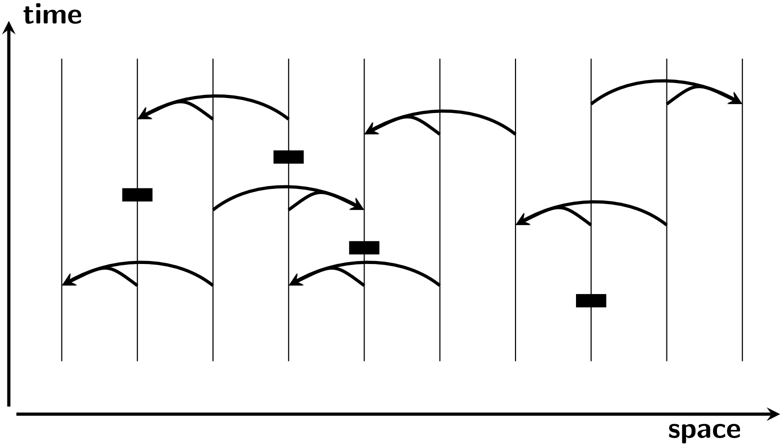
$$\begin{array}{rcll} \texttt{cob}(\mathbf{x_1}, x_2, x_3) & = & \mathbf{x_1} \vee (x_2 \wedge x_3), & S^3 \to S \\ \texttt{dth}(\varnothing) & = & 0, & S^0 \to S \end{array}$$

where the corresponding rates are

$$r_{\texttt{cob}} = \alpha \geq 0 \quad \text{and} \quad r_{\texttt{dth}} = 1.$$

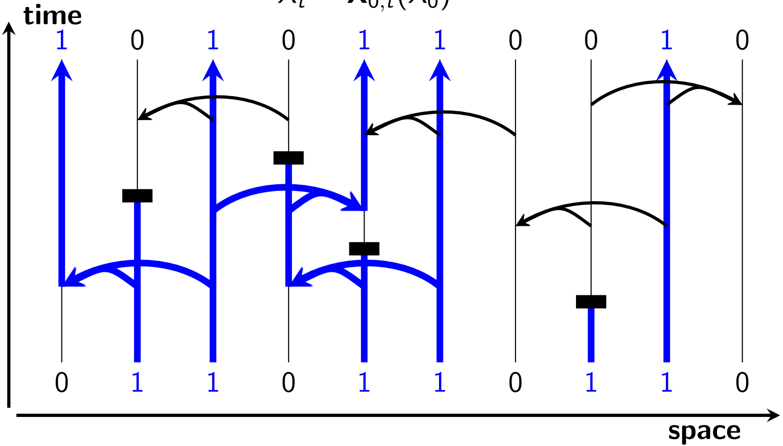
Here, the map $\texttt{cob}$ applied to $\mathbf{x_{i_1}}, x_{i_2}, x_{i_3}$ replaces $\mathbf{x_{i_1}}$ in x.

A graphical representation



A graphical representation

$$X_t = \mathbf{X}_{0,t}(X_0)$$



$$X_0$$

The Poisson events define a random map $x \mapsto \mathbf{X}_{0,t}(x)$.

Graphical/random mapping representation

Some notation

Polish space Ω models external randomness:
Consider measurable maps

- $\kappa : \Omega \to \mathbb{N}$ and $\Omega_k := \{\omega \in \Omega : \kappa(\omega) = k\}$
- $\Omega_k \times S^k \ni (\omega, x) \mapsto \gamma[\omega](x) \in S$

Let $\mathcal{G} := \{\gamma[\omega] : \omega \in \Omega\}$.

Also consider a nonzero finite measure $\mathbf{r}$ on Ω with total mass $|\mathbf{r}| := \mathbf{r}(\Omega)$ and set $r_g := \mathbf{r}\big(\{\omega \in \Omega : \gamma[\omega] = g\}\big)$ for $g \in \mathcal{G}$.

Let $[N]^{\langle k \rangle}$ denote the set of all sequences $\mathbf{i} = (i_1, \ldots, i_k)$ for which $i_1, \ldots, i_k \in [N]$ are all different.

Graphical/random mapping representation

Evolution of X :

- At the times of a Poisson process with intensity $|\mathbf{r}|$, an element $\omega \in \Omega$ is chosen according to the probability law $|\mathbf{r}|^{-1}\mathbf{r}$.
- If $\kappa(\omega) \leq N$, then $\mathbf{i} \in [N]^{\langle\kappa(\omega)\rangle}$ and $j \in [N]$ are selected independently and uniformly
- $X_{t-}(j)$ is replaced by $X_t(j) = \gamma[\omega]\big(X_{t-}(i_1), \ldots, X_{t-}(i_{\kappa(\omega)})\big)$.

Alternative: Let $j = i_1$ instead of a random choice.

(Note: In the limit $N \to \infty$ this does not make a difference.)

Stochastic flow

We can view X as a stochastic flow: For $x \in S^N$ consider

$$m_{\omega,\mathbf{i},j}(x)_{j'} := \begin{cases} \gamma[\omega](x_{i_1}, \ldots, x_{i_{\kappa(\omega)}}) & \text{if } j' = j, \\ x_{j'} & \text{otherwise,} \end{cases}$$

Let Π be a Poisson point set on

$$\big\{(\omega, \mathbf{i}, j, t) : \omega \in \Omega,\ \mathbf{i} \in [N]^{\langle \kappa(\omega) \rangle},\ j \in [N],\ t \in \mathbb{R}\big\}$$

with intensity

$$\mathbf{r}(\mathrm{d}\omega) \frac{1}{N^{\langle \kappa(\omega) \rangle}} \frac{1}{N} \, \mathrm{d}t.$$

and for $s < u$

$$\begin{aligned} \Pi_{s,u} &:= \{(\omega, \mathbf{i}, j, t) \in \Pi : s < t \leq u\} \\ &= \big\{(\omega_1, \mathbf{i}_1, j_1, t_1), \ldots, (\omega_n, \mathbf{i}_n, j_n, t_n)\big\} \end{aligned}$$

Stochastic flow

Then

$$\mathbf{X}_{s,u} = m_{\omega_n,\mathbf{i}_n,j_n} \circ \cdots \circ m_{\omega_1,\mathbf{i}_1,j_1}.$$

defines a **stochastic flow** with

$$\mathbf{X}_{s,s} = Id \quad \text{and} \quad \mathbf{X}_{t,u} \circ \mathbf{X}_{s,t} = \mathbf{X}_{s,u} \qquad (s \leq t \leq u).$$

If $X(0)$ is an S^N-valued random variable independent of Π then

$$X_t := \mathbf{X}_{0,t}\big(X(0)\big) \qquad (t \geq 0).$$

Coupling via the stochastic flow

Coupling via the stochastic flow

- Let $(X_0^1, \ldots, X_0^n)$ be a random variable with values in $(S^N)^n$, independent of $(\mathbf{X}_{s,u})_{s \le u}$:

$$(X_t^1, \ldots, X_t^n) := (\mathbf{X}_{0,t}(X_0^1), \ldots, \mathbf{X}_{0,t}(X_0^n))$$

- $(X_t^1, \ldots, X_t^n)_{t \ge 0}$ consists of n coupled Markov processes with initial states $X^1(0), \ldots, X^n(0)$.

The mean-field limit

Consider the **empirical measure**

$$\mu\{x\} := \frac{1}{N} \sum_{i \in [N]} \delta_{x_i}.$$

Since the dynamics is invariant under permutations

$$\mu_t := \mu_t^N := \mu\{X_t\} \qquad (t \geq 0)$$

defines a Markov process.

Let $\mathcal{P}(S)$ be the space of all probability measures on S, equipped with the topology of weak convergence.

Goal: Consider the limit as $N \to \infty$ with convergence in $\mathcal{P}(S)$

Note: Analogously, we can define and consider $\mu^{(n)}\{x\} \in \mathcal{P}(S^n)$ for n coupled processes with $x \in (S^N)^n$.

The mean-field equation

For any measurable map $g : S^k \to S$ we define a measurable map $T_g : \mathcal{P}(S) \to \mathcal{P}(S)$ by

$$T_g(\mu) := \text{ the law of } g(X_1, \ldots, X_k),$$

where $(X_i)_{i=1,\ldots,k}$ are i.i.d. $\mathcal{P}(S)$-valued with law μ.

Consider (weak) solutions to the **mean-field equation**

$$\frac{\partial}{\partial t}\mu_t = \int_\Omega \mathbf{r}(\mathrm{d}\omega)\big\{T_{\gamma[\omega]}(\mu_t) - \mu_t\big\} :$$

For each bounded measurable function $\phi : S \to \mathbb{R}$, the function $t \mapsto \langle \mu_t, \phi\rangle$ is continuously differentiable and

$$\frac{\partial}{\partial t}\langle \mu_t, \phi\rangle = \int_\Omega \mathbf{r}(\mathrm{d}\omega)\big\{\langle T_{\gamma[\omega]}(\mu_t), \phi\rangle - \langle \mu_t, \phi\rangle\big\}$$

Example: Cooperative branching with death

Let $S = \{0,1\}$ and $\mathcal{G} = \{\texttt{cob}, \texttt{dth}\}$ with rates α and 1.
Then the mean-field equation is

$$\frac{\partial}{\partial t}\mu_t = \alpha\big\{T_{\texttt{cob}}(\mu_t) - \mu_t\big\} + \big\{T_{\texttt{dth}}(\mu_t) - \mu_t\big\}.$$

Here, it suffices to keep track of $p_t := \mu_t(\{1\})$

$$\frac{\partial}{\partial t}p_t = \alpha p_t^2(1-p_t) - p_t \qquad (t \geq 0).$$

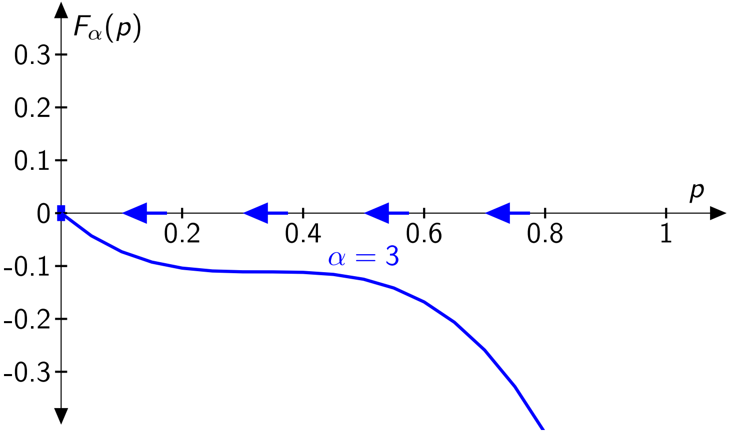
Fixed points:

- For $\alpha < 4$: $z_{\rm low} := 0$
- For $\alpha \geq 4$: $z_{\rm low}$ and

$$z_{\rm mid} := \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{\alpha}} \quad \text{and} \quad z_{\rm upp} := \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{\alpha}}.$$

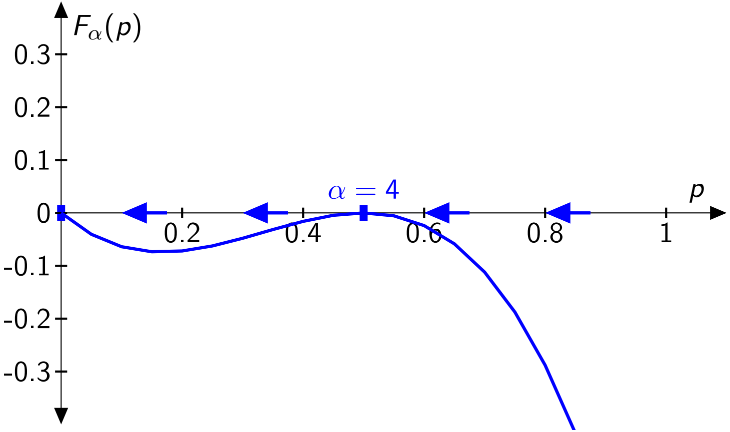
$z_{\rm low}$ and $z_{\rm upp}$ are stable, $z_{\rm mid}$ is unstable.

Cooperative branching



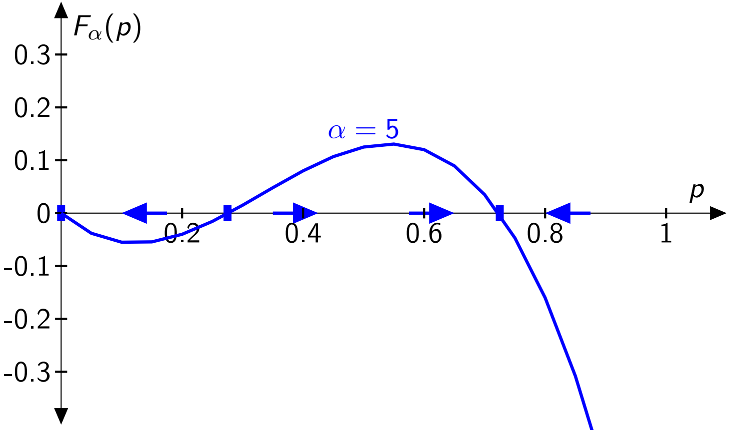
For $\alpha < 4$ the equation $\frac{\partial}{\partial t} p_t = \alpha p_t^2 (1 - p_t) - p_t =: F_\alpha(p_t)$ has a single, stable fixed point $p = 0$.

Cooperative branching



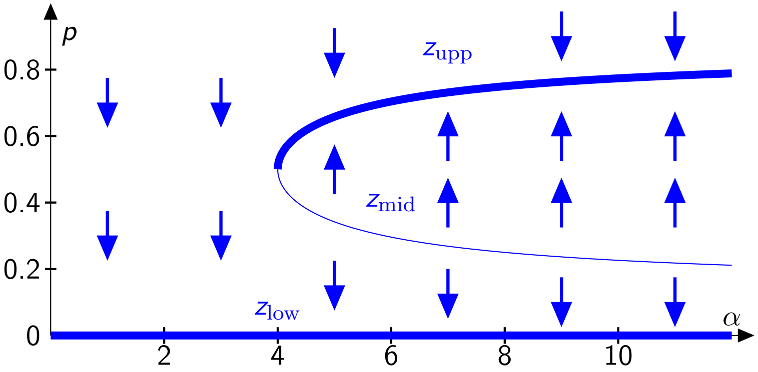
For $\alpha = 4$, a second fixed point appears at $p = 0.5$.

Cooperative branching



For $\alpha > 4$ there are two stable fixed points and one unstable fixed point, which separates the domains of attraction of the other two.

Cooperative branching



Fixed points of $\frac{\partial}{\partial t} p_t = F_\alpha(p_t)$ for different values of α.

Uniqueness of the mean-field equation

Theorem

Let $\mathbf{r}$ satisfy

$$\int_\Omega \mathbf{r}(\mathrm{d}\omega)\,\kappa(\omega) < \infty.$$

Then the mean-field equation has a unique solution $(\mu_t)_{t\geq 0}$ for each initial state $\mu_0 \in \mathcal{P}(S)$.

Convergence to the mean-field equation

Let d be a metric that corresponds to the topology of weak convergence.

Theorem

If in addition one of the following two conditions is satisfied:

- $\mathbb{P}\big[d(\mu_0^N,\mu_0)\geq\varepsilon\big]\underset{N\to\infty}{\longrightarrow}0$ for all $\varepsilon>0$, and $\mathbf{r}\big(\{\omega:\kappa(\omega)=k,\ \gamma[\omega]\text{ is discontinuous at x}\}\big)=0.$
- $\big\|\mathbb{E}[(\mu_0^N)^{\otimes n}]-\mu_0^{\otimes n}\big\|_{TV}\underset{N\to\infty}{\longrightarrow}0$ for all $n\geq 1$.

Then for $\varepsilon>0,\ T<\infty$,

$$\mathbb{P}\big[\sup_{0\leq t\leq T}d(\mu_{Nt}^N,\mu_t)\geq\varepsilon\big]\underset{N\to\infty}{\longrightarrow}0,$$

where $(\mu_t)_{t\geq 0}$ solves the mean-field equation with initial state μ_0.

Convergence to the mean-field equation

- We could more generally consider maps that change not only one but m sites simultaneously:

$$(x_1, \ldots, x_k) \mapsto (g_1(x_1, \ldots, x_k), \ldots, g_m(x_1, \ldots, x_k)) \in S^m.$$

  However, applying such a map with rate r has in the mean-field limit the same effect as independently applying $g_1(x_1, \ldots, x_k)$ to $g_m(x_1, \ldots, x_k)$ all at rate r.
- Also, the alternative of $j = i_1$ instead of a random choice leads to the same mean-field equation.

The n-variate equation

We are also interested in n **coupled mean field equations:**

For $g : S^k \to S$ we define $g^{(n)} : (S^k)^n \to S^n$ by
$g^{(n)}(x^1, \ldots, x^n) := (g(x^1), \ldots, g(x^n)) \qquad (x^1, \ldots, x^n \in S^k).$

Then the *n-variate mean field equation* is

$$\frac{\partial}{\partial t}\mu_t^{(n)} = \int_\Omega \mathbf{r}(\mathrm{d}\omega)\big\{T_{\gamma^{(n)}[\omega]}(\mu_t^{(n)}) - \mu_t^{(n)}\big\}$$

with $\mu_t^{(n)} \in \mathcal{P}(S^n)$.

For this equation invariant spaces are

- $\mathcal{P}_{\rm sym}(S^n)$: symmetric with respect to permutations
- $\mathcal{P}(S^n_{\rm diag})$: concentrated on the diagonal

Example: Cooperative branching with death

$n = 2$:

Bivariate equation for cooperative branching with deaths:

For $\alpha > 4$ there are four fixed points in $\mathcal{P}_{\mathrm{sym}}(\{0,1\}^2)$:

$$\overline{\nu}^{(2)}_{\mathrm{low}}, \quad \underline{\nu}^{(2)}_{\mathrm{mid}}, \quad \overline{\nu}^{(2)}_{\mathrm{mid}}, \quad \text{and} \quad \overline{\nu}^{(2)}_{\mathrm{upp}}.$$

which are uniquely characterized by their respective marginal means

$$z_{\mathrm{low}}, z_{\mathrm{mid}}, z_{\mathrm{mid}}, z_{\mathrm{upp}}$$

as well as the fact that $\overline{\nu}^{(2)}_{\mathrm{low}}, \overline{\nu}^{(2)}_{\mathrm{mid}}$, and $\overline{\nu}^{(2)}_{\mathrm{upp}}$ are concentrated on $\{0,1\}^2_{\mathrm{diag}} = \{(0,0),(1,1)\}$, but $\underline{\nu}^{(2)}_{\mathrm{mid}}$ is not.

A random recursive tree representation

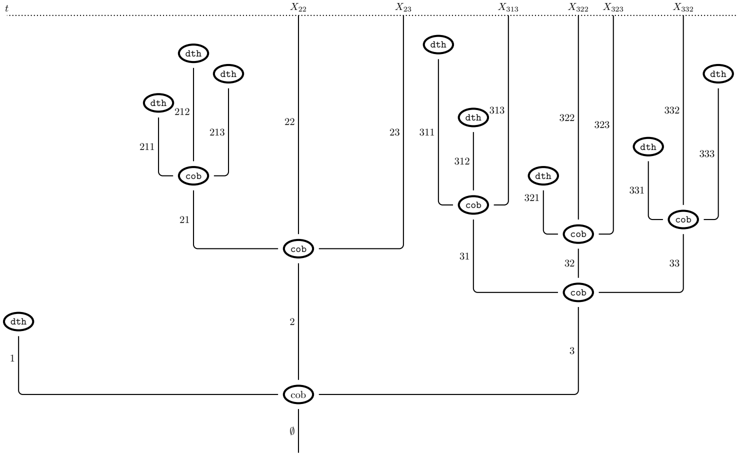
Recall: X^N is described via Poisson point process/stochastic flow.

Goal: Stochastic representation of solutions to (n-variate) mean-field equation $(\mu_t^{(n)})_{t\geq 0}$ analogous to duality:

- As $N \to \infty$ for any randomly chosen $j \in [N]$, $X_t^N(j)$ is approximately distributed as μ_t
- The state of $X_t^N(j)$ depends on the map $\gamma[\omega]$ that affected site j in the past
- It took an input from the states at site $i_1, \ldots, i_{\kappa(\omega)}$ (as $N \to \infty$ all distinct with high probability)
- Continue to determine those states...

A random recursive tree representation

Tracing back this "genealogy" leads to a representation of μ_t via a **marked branching process**.



A random recursive tree representation

Let $d \in \mathbb{N}_+ \cup \{\infty\}$ and let

$$\overline{\mathbb{T}} := \overline{\mathbb{T}}^d = \{\mathbf{i} = i_1 \cdots i_n, n \in \mathbb{N}, i_k \in [d], k \in [n]\}$$

denote the space of all finite words made up from the alphabet $[d]$.

The random subtree $\mathbb{T} \subset \overline{\mathbb{T}}$ is the family tree of a continuous -time branching process with additional structure given by the maps $\gamma[\omega_{\mathbf{i}}]$ (i.i.d. $\mathbf{r}$) attached at the branch points as well as independent lifetimes $(\sigma_{\mathbf{i}})_{\mathbf{i} \in \overline{\mathbb{T}}}$ (i.i.d. $exp(|\mathbf{r}|)$).

We also consider the random subtrees

$$\mathbb{T}_t := \left\{\mathbf{i} \in \mathbb{T} : \tau_{\mathbf{i}}^{\dagger} \leq t\right\} \quad \text{and} \quad \partial\mathbb{T}_t = \left\{\mathbf{i} \in \mathbb{T} : \tau_{\mathbf{i}}^{*} \leq t < \tau_{\mathbf{i}}^{\dagger}\right\}$$

where $\tau_{\mathbf{i}}^{*}$ and $\tau_{\mathbf{i}}^{\dagger}, \mathbf{i} \in \mathbb{T}$ are birth and death times ($\sigma_{\mathbf{i}} = \tau_{\mathbf{i}}^{\dagger} - \tau_{\mathbf{i}}^{*}$).

$(|\partial\mathbb{T}_t|)_{t \geq 0}$ is a branching process with offspring law κ and rate $|\mathbf{r}|$. The assumption $\int_\Omega \mathbf{r}(\mathrm{d}\omega)\, \kappa(\omega) < \infty$ corresponds to a finite offspring mean.

A random recursive tree representation

A stochastic flow on $\mathbb{T}$ is given by $\gamma[\omega_{\mathbf{i}}], \mathbf{i} \in \mathbb{T}$:

- For any finite subtree $\mathbb{U} \subset \mathbb{T}$ with leaves $\partial\mathbb{U}$ containing the root $\varnothing$ define inductively for each $(x_{\mathbf{i}})_{\mathbf{i}\in\partial\mathbb{U}} = x \in S^{\partial\mathbb{U}}$

$$x_{\mathbf{i}} := \gamma[\omega_{\mathbf{i}}](x_{\mathbf{i}1}, \ldots, x_{\mathbf{i}\kappa(\omega_{\mathbf{i}})}) \qquad (\mathbf{i} \in \mathbb{U}).$$

- The value $x_\varnothing$ is given by the function $G_\mathbb{U} : S^{\partial\mathbb{U}} \to S$ defined by

$$G_\mathbb{U}\big((x_{\mathbf{i}})_{\mathbf{i}\in\partial\mathbb{U}}\big) := x_\varnothing$$

- The process $x_{\mathbf{i}}$ is a kind of
  Markov process where time has a tree like structure.
  The forward time direction is towards the root. Consider

$$G_t := G_{\mathbb{T}_t} \qquad (t \geq 0)$$

A random recursive tree representation

For any random measure μ on S define $\mathbb{E}[\mu]$ via $\int \phi \, \mathrm{d}\mathbb{E}[\mu] := \mathbb{E}[\int \phi \, \mathrm{d}\mu]$ for any bounded measurable $\phi : S \to \mathbb{R}$.

Theorem

For each $\mu_0 \in \mathcal{P}(S)$, the solution $(\mu_t)_{t\geq 0}$ of the mean-field equation with initial state μ_0 is given by

$$\mu_t = \mathbb{E}\big[T_{G_t}(\mu_0)\big]$$

Interpretation as a (generalized) duality relationship between $(\mu_t)_{t\geq 0}$ and $(G_t)_{t\geq 0}$ with (generalized) duality function $H : \mathcal{G} \times \mathcal{P}(S) \to \mathcal{P}(S)$ given by

$$H(g, \mu) = T_g(\mu).$$

We have $\mu_t = H(G_0, \mu_t) = \mathbb{E}[H(G_t, \mu_0)]$ and obtain a usual real-valued duality by integrating against ϕ.

A random recursive tree representation

Let $\mathcal{F}_t := \sigma\big(\partial\mathbb{T}_t, (\omega_{\mathbf{i}}, \sigma_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}_t}\big), t \geq 0$ and let $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}_t\cup\partial\mathbb{T}_t}$ be random variables defined recursively as before with

$$(X_{\mathbf{i}})_{\mathbf{i}\in\partial\mathbb{T}_t}|\mathcal{F}_t \quad \text{i.i.d with law} \quad \mu_0.$$

We then have the following consistency relationship:

Lemma

Fix $t > 0$. Then, for each $s \in [0, t]$,

(i) $(X_{\mathbf{i}})_{\mathbf{i}\in\partial\mathbb{T}_s}|\mathcal{F}_s$ are i.i.d. with common law μ_{t-s}

(ii) $X_{\mathbf{i}} = \gamma[\omega_{\mathbf{i}}]\big(X_{\mathbf{i}1}, \ldots, X_{\mathbf{i}\kappa(\omega_{\mathbf{i}})}\big) \qquad (\mathbf{i} \in \mathbb{T}_s)$,

where $(\mu_s)_{s\geq 0}$ solves the mean-field equation with initial state μ_0.

Unique ergodicity

Unique ergodicity: The mean-field equation has a unique fixed point ν and any solution μ_t started in an arbitrary initial law μ_0 satisfies that

$$||\mu_t - \nu|| \to 0, \quad t \to \infty$$

where $||\cdot||$ denotes the total variation norm.

An easy sufficient criterion:

Proposition

If we have

$$R := \int_\Omega \mathbf{r}(\mathrm{d}\omega)\,(\kappa(\omega) - 1) \leq 0$$

(and κ is not identically 1) then unique ergodicity holds.

Unique ergodicity

Proof If $R = \int_\Omega \mathbf{r}(\mathrm{d}\omega)\,(\kappa(\omega) - 1) < 0$ then $(\partial\mathbb{T}_t)_{t\geq 0}$ is a subcritical branching process, respectively for $R = 0$ a nontrivial critical branching process so that the tree $\mathbb{T}_t$ is a.s. finite.

Thus, $\partial\mathbb{T} = \emptyset$ and $G_\mathbb{T}$ is a.s. constant. Set $\nu := \mathbb{P}[G_\mathbb{T} \in \cdot]$ and observe that as $t \to 0$,

$$G_t = G_{\mathbb{T}_t} \to G_\mathbb{T} \quad a.s.$$

Unique ergodicity

For the cooperative branching model we have

$$R = \alpha \cdot (3-1) + 1 \cdot (0-1) = 2\alpha - 1$$

which gives unique ergodicity for $\alpha \leq \frac{1}{2}$.
In this case we already found that unique ergodicity holds iff $\alpha < 4$.

The previous criterion can be generalised with the same proof:

Proposition

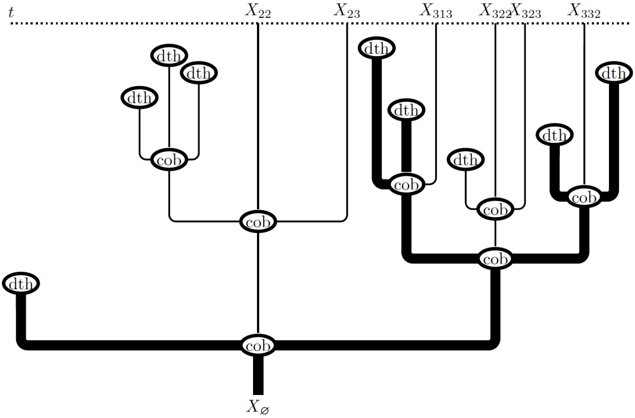
Assume that

$$\mathbb{P}[\exists t < \infty \text{ such that } G_t \text{ is constant }] = 1$$

then unique ergodicity holds.

Note: G_t is constant if there exists a finite root determining subtree of $\mathbb{T}_t$. This is a tree-valued version of **coupling from the past**.

Unique ergodicity



Example: A minimal root determining subtree. In this example, $X_{\varnothing} = 0$ regardless of the values of $X_{22}, X_{23}, X_{313}, X_{322}, X_{323}, X_{332}$. One can show that this exists a.s. iff $\alpha \leq 4$.

Unique ergodicity

This is due to the monotonicity of the maps involved.

Monotonicity is a sufficient condition for equivalence in the previous lemma:

Proposition

Assume that S is a finite partially ordered set that contains a minimal $\underline{0}$ and maximal $\underline{1}$ element, and assume that $\gamma[\omega]$ is monotone for each $\omega \in \Omega$. Then **unique ergodicity** holds if and only if

$$\mathbb{P}[\exists t < \infty \text{ such that } G_t \text{ is constant }] = 1$$

Unique ergodicity

Proof

- Due to monotonicity

$$
\begin{aligned}
X_{\varnothing}^{\mathrm{upp}} &= \lim_{t\to\infty} G_t(1,\dots,1) \\
X_{\varnothing}^{\mathrm{low}} &= \lim_{t\to\infty} G_t(0,\dots,0)
\end{aligned}
$$

exist a.s. and their laws ν_{upp} and ν_{low} are invariant such that for any other invariant law ν: $\nu_{\mathrm{low}} \leq \nu \leq \nu_{\mathrm{up}}$

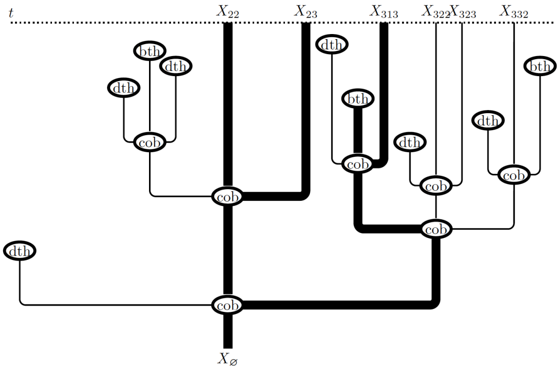
- If ν is unique then $\nu_{\mathrm{low}} = \nu_{\mathrm{up}}$ and due to monotonicity for any $x \in S^{\partial \mathbb{T}_t}$

$$G_t(0,\dots,0) \leq G_t(x) \leq G_t(1,\dots,1)$$

which implies since the left and right hand side converge to the same distribution so that for t large enough (S finite!) they need to be equal a.s.

Open subtrees

In the case of monotone maps and $S = \{0, 1\}$ we can also characterise ν_{upp} and ν_{low} via open subtrees:



An open subtree is a subtree such that for all nodes of the subtree if all inputs from branches included in the subtree is a 1 then the output of the function at the node will also be a 1.

Open subtrees

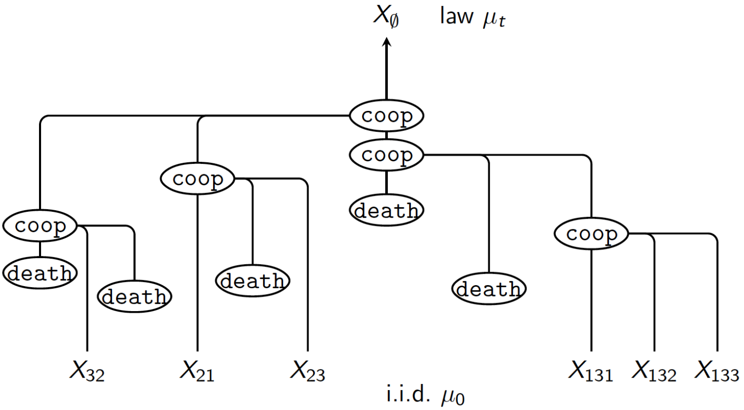
Proposition

Assume that $S = \{0, 1\}$ and $\gamma[\omega]$ is monotone for all $\omega \in \Omega$. Then

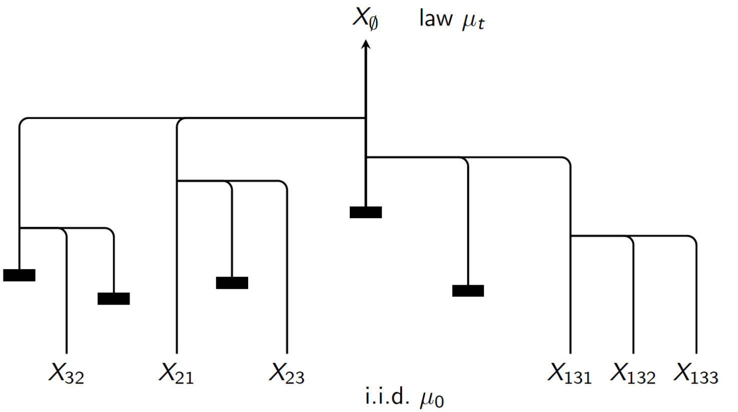
$$\begin{aligned} \nu_{\rm upp}(\{1\}) &= \mathbb{P}\big[\text{there exists an open subtree of } \mathbb{T}\big] \\ \nu_{\rm low}(\{1\}) &= \mathbb{P}\big[\text{there exists a finite open subtree of } \mathbb{T}\big]. \end{aligned}$$

- A similar statement can also be made for general finite partially ordered sets S.
- Open subtrees are closely connected to the monotone duality considered previously.

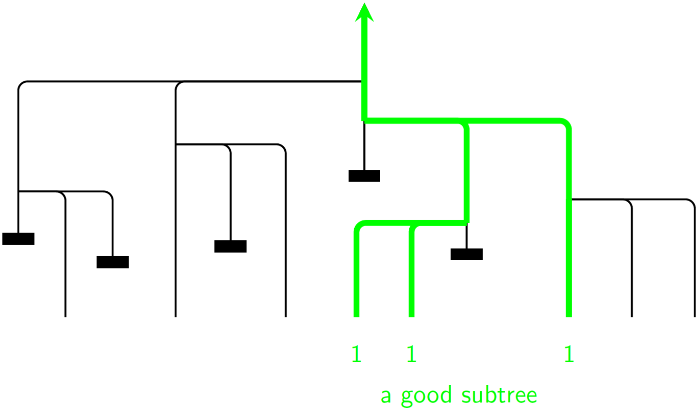
Open subtrees and monotone duality



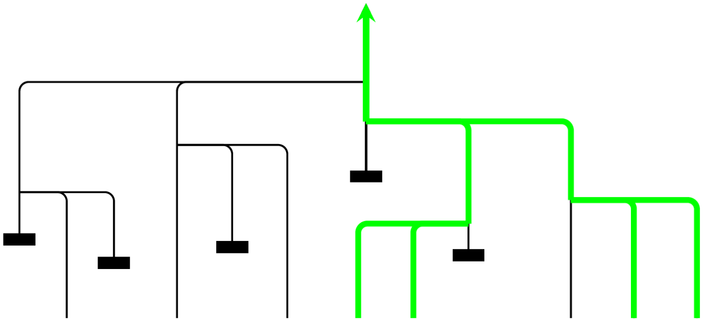
Open subtrees and monotone duality



Open subtrees and monotone duality



Open subtrees and monotone duality



dual process $\overline{Y}_t$ $\left\{ \begin{array}{llllll} 1 & 1 & & 1 & & \\ 1 & 1 & & & 1 & 1 \end{array} \right.$

Mean field fixed points and recursive tree processes

Let $\nu \in \mathcal{P}(S)$ be a **fixed point** of the mean-field equation:

$$T(\nu) := |\mathbf{r}|^{-1} \int_{\Omega} \mathbf{r}(\mathrm{d}\omega)\, T_{\gamma[\omega]}(\nu) = \nu$$

which is equivalend to $X \stackrel{\mathcal{D}}{=} \nu$ solving the
Recursive Distributional Equation (RDE)

$$X \stackrel{\mathcal{D}}{=} \gamma[\omega](X_1, \ldots, X_{\kappa(\omega)}),$$

$X_1, X_2, \ldots$ are i.i.d. copies of X and ω is an independent random variable with law $|\mathbf{r}|^{-1}\mathbf{r}$.

RDE appear in many applications, overview: Alsmeyer '12+

Mean field fixed points and recursive tree processes

We can to the fixed points to the RDE associate (continuous-time) **Recursive Tree Processes** (RTP).
Aldous and Bandyopadhyay '05 studied the discrete time case.

Theorem

Let ν be an RDE solution. Then there exist random variables $(\omega_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\overline{\mathbb{T}}}$ whose joint law is characterized by

(i) $(\omega_{\mathbf{i}})_{\mathbf{i}\in\overline{\mathbb{T}}}$ are i.i.d. with law $|\mathbf{r}|^{-1}\mathbf{r}$.

(ii) For each finite subtree $\mathbb{U} \subset \overline{\mathbb{T}}$ with $\varnothing \in \mathbb{U}$, $(X_{\mathbf{i}})_{\mathbf{i}\in\overline{\partial}\mathbb{U}}$ are i.i.d. with law ν and independent of $(\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{U}}$.

(iii) $X_{\mathbf{i}} = \gamma[\omega_{\mathbf{i}}]\big(X_{\mathbf{i}1}, \ldots, X_{\mathbf{i}\kappa(\omega_{\mathbf{i}})}\big) \qquad (\mathbf{i} \in \overline{\mathbb{T}})$.

Continuous time extension: If $(\sigma_{\mathbf{i}})_{\mathbf{i}\in\overline{\mathbb{T}}}$ are independent and i.i.d. exponential with mean $|\mathbf{r}|^{-1}$ then for each $t \geq 0$,

$$(X_{\mathbf{i}})_{\mathbf{i}\in\partial\mathbb{T}_t}|\mathcal{F}_t \text{ are i.i.d. with common law } \nu.$$

n-variate process

The stochastic flow $\mathbf{X}^N$ contains more information than the Markov process X^N. In particular, it allows us to describe the evolution of n coupled processes leading to the n-variate mean-field equation $(\mu_t^{(n)})_{t\geq 0}$ with associated fixed points (to $T_{(n)}$) and RTP.

Some notation and facts:

- Let $\mathcal{P}(S^n)_\mu \subset \mathcal{P}(S^n)$ have all marginals be $\mu \in \mathcal{P}(S)$.
- $\mathcal{P}_{\rm sym}(S^n)_\mu := \mathcal{P}_{\rm sym}(S^n) \cap \mathcal{P}(S^n)_\mu$.
- For $\mu \in \mathcal{P}(S)$ let $\overline{\mu}^{(n)} \in \mathcal{P}(S^n)_\mu$ be concentrated on the "diagonal" $S^n_{\rm diag} = \{x \in S^n : x_1 = \cdots = x_n\}$.
- If $T(\nu) = \nu$ then $\mathcal{P}(S^n)_\nu$ is an invariant space for $\mu_t^{(n)}$.
- $\mathcal{P}_{\rm sym}(S^n)$ and measures concentrated on $S^n_{\rm diag}$ are invariant spaces for $\mu_t^{(n)}$.

n-Variate processes

If $\nu = \mathbb{P}[X \in \cdot\,]$ solves the RDE $T(\nu) = \nu$ then

$$\overline{\nu}^{(n)} := \mathbb{P}\big[\underbrace{(X,\ldots,X)}_{n\text{ times}} \in \cdot\,\big]$$

solves the *n-variate RDE* $T^{(n)}(\nu^{(n)}) = \nu^{(n)}$.

Question:
Are all fixed points of the n-variate RDE of this form?

Example: Cooperative branching with death

Bivariate equation for cooperative branching with deaths: For $\alpha > 4$ the domains of attraction for $\mu_t^{(2)}$ are:

$$
\begin{array}{ll}
\overline{\nu}_{\text{low}}^{(2)} & \{\mu_0^{(2)} : \mu_0^{(1)}(\{1\}) < z_{\text{mid}}\}, \\
\underline{\nu}_{\text{mid}}^{(2)} & \{\mu_0^{(2)} : \mu_0^{(1)}(\{1\}) = z_{\text{mid}},\ \mu_0^{(2)} \neq \overline{\nu}_{\text{mid}}^{(2)}\}, \\
\overline{\nu}_{\text{mid}}^{(2)} & \{\overline{\nu}_{\text{mid}}^{(2)}\}, \\
\overline{\nu}_{\text{upp}}^{(2)} & \{\mu_0^{(2)} : \mu_0^{(1)}(\{1\}) > z_{\text{mid}}\}.
\end{array}
$$

This means in particular that

- $\overline{\nu}_{\text{mid}}^{(2)}$ is an unstable fixed point
- $\underline{\nu}_{\text{mid}}^{(2)}$ is a stable fixed point (as well as $\overline{\nu}_{\text{low}}^{(2)}$ and $\overline{\nu}_{\text{upp}}^{(2)}$)

Intuition for the particle system

Let $(X_t)_{t\geq 0}$ be the process in S^N with initial law $(X_0(i))_{1\leq i\leq N}$ i.i.d. with mean z_{mid}.

Let $\left(X'_t\right)_{t\geq 0}$ be a process with modified initial state: $X'_0(i) = X_0(i)$ except for an ε-fraction of sites i, which are redrawn using independent randomness.

In the mean-field limit, so intuitively when N is large:

The fraction of sites where $X'_t(i) \neq X_t(i)$ tends to a (nontrivial) limit even if ε is small.

More precisely: The joint empirical law of X_t, X'_t converges as (first $N \to \infty$ and then) $t \to \infty$ to $\underline{\nu}^{(2)}_{\mathrm{mid}}$.

Endogeny of the RTP

This kind of noise sensitivity associated to a fixed point ν is connected to **endogeny**.

An RTP $(\omega_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\overline{\mathbb{T}}}$ is called **endogenous** if

$$X_\varnothing \text{ is a.s. measurable w.r.t. } (\omega_{\mathbf{i}})_{\mathbf{i}\in\overline{\mathbb{T}}}.$$

Endogeny and bivariate uniqueness

Theorem

Let ν be a solution of the RDE.
Then the following statements are equivalent.

(i) The RTP corresponding to ν is endogenous.

(ii) $T^m_{(n)}(\mu) \underset{m\to\infty}{\Longrightarrow} \overline{\nu}^{(n)}$ for all $\mu \in \mathcal{P}(S^n)_\nu$ and $n \geq 1$.

(iii) $\overline{\nu}^{(2)}$ is the only fixed point of $T_{(2)}$ in the space $\mathcal{P}_{\rm sym}(S^2)_\nu$.

Continuous-time extension of (ii):

(iv) For any $\mu^{(n)}_0 \in \mathcal{P}(S^n)_\nu$ and $n \geq 1$, the solution $(\mu^{(n)}_t)_{t\geq 0}$ to the n-variate equation started in $\mu^{(n)}_0$ satisfies $\mu^{(n)}_t \underset{t\to\infty}{\Longrightarrow} \overline{\nu}^{(n)}$.

Example: Cooperative branching with death

Bivariate equation for cooperative branching with deaths:
Recall that for $\alpha > 4$ there are four distinct fixed points in $\mathcal{P}_{\mathrm{sym}}(\{0,1\}^2)$:

$$\overline{\nu}^{(2)}_{\mathrm{low}}, \quad \underline{\nu}^{(2)}_{\mathrm{mid}}, \quad \overline{\nu}^{(2)}_{\mathrm{mid}}, \quad \overline{\nu}^{(2)}_{\mathrm{upp}}.$$

with marginals

$$\nu_{\mathrm{low}}, \nu_{\mathrm{mid}}, \nu_{\mathrm{mid}}, \nu_{\mathrm{upp}}.$$

Thus, by our previous theorem:

- RTPs corresponding to ν_{low} and ν_{upp} are endogenous.
- RTP corresponding to ν_{mid} is not endogenous.

The higher-level equation

The n-variate map $T^{(n)}$ is defined even for $n = \infty$, and $T^{(\infty)}$ maps $\mathcal{P}_{\rm sym}(S^{\mathbb{N}_+})$ into itself.

By De Finetti's theorem, $(X_i)_{i\in\mathbb{N}_+}$ have a law in $\mathcal{P}_{\rm sym}(S^{\mathbb{N}_+})$ if and only if there exists a random probability measure ξ on S such that conditional on ξ, the $(X_i)_{i\in\mathbb{N}_+}$ are i.i.d. with law ξ.
Let $\rho := \mathbb{P}[\xi \in \cdot\,]$ the law of ξ. Then $\rho \in \mathcal{P}(\mathcal{P}(S))$.

The map $T^{(\infty)} : \mathcal{P}_{\rm sym}(S^{\mathbb{N}_+}) \to \mathcal{P}_{\rm sym}(S^{\mathbb{N}_+})$ corresponds to a **higher-level map** $\check{T} : \mathcal{P}(\mathcal{P}(S)) \to \mathcal{P}(\mathcal{P}(S))$.

The higher-level equation

For any measurable map $g : S^k \to S$ define $\check{g} : \mathcal{P}(S)^k \to \mathcal{P}(S)$ by

$$\check{g} := \text{ the law of } g(X_1, \ldots, X_k),$$

where $(X_1, \ldots, X_k)$ are independent with laws $\mu_1, \ldots, \mu_k$.

Proposition

We have

$$\check{T}(\rho) := \text{the law of } \check{\gamma}[\omega](\xi_1, \ldots, \xi_{\kappa(\omega)})$$

with ω as before and $\xi_1, \xi_2, \ldots$ i.i.d. with law ρ.

Namely, if $(\rho_t)_{t\geq 0}$ solves the **higher-level mean-field equation** corresponding to $\check{T}$, then its n-th moment measures $(\rho_t^{(n)})_{t\geq 0}$ solve the n-variate equation.

n-th moment measure of ρ: Draw a law according to ρ. Consider the law of n independent random variables drawn according to this law. One can show $\check{T}(\rho)^{(n)} = T^{(n)}(\rho^{(n)}$.

The higher-level equation

Equip $\mathcal{P}(\mathcal{P}(S))_\nu = \{\rho : \rho^{(1)} = \nu\}$ with the **convex order**

$$\rho_1 \leq_{\rm cv} \rho_2 \quad \text{iff} \quad \int \phi \, \mathrm{d}\rho_1 \leq \int \phi \, \mathrm{d}\rho_2 \quad \forall \text{ convex } \phi.$$

Define $\overline{\nu} := \mathbb{P}[\delta_X \in \cdot\,]$ with $\mathbb{P}[X \in \cdot\,] = \nu$.

Maximal and minimal elements in the convex order are $\overline{\nu}$ and δ_ν:

$$\delta_\nu \leq_{\rm cv} \rho \leq_{\rm cv} \overline{\nu} \qquad \forall \rho \in \mathcal{P}(\mathcal{P}(S))_\nu.$$

Note: The n-th moment measures of δ_ν and $\overline{\nu}$ are given by

$$\begin{aligned} \delta_\nu^{(n)} &= \mathbb{P}\big[(X_1, \ldots, X_n) \in \cdot\,\big] \\ \overline{\nu}^{(n)} &= \mathbb{P}\big[(X, \ldots, X) \in \cdot\,\big], \end{aligned}$$

where $X_1, \ldots, X_n$ are i.i.d. with common law ν and X has law ν.

The higher-level equation

Proposition

$\check{T}$ is monotone w.r.t. the convex order. There exists a solution $\underline{\nu}$ to the higher-level RDE such that

$$\check{T}^n(\delta_\nu) \underset{n\to\infty}{\Longrightarrow} \underline{\nu} \quad \text{and} \quad \check{T}_t(\delta_\nu) \underset{t\to\infty}{\Longrightarrow} \underline{\nu}$$

and any solution $\rho \in \mathcal{P}(\mathcal{P}(S))_\nu$ to the higher-level RDE satisfies

$$\underline{\nu} \leq_{\mathrm{cv}} \rho \leq_{\mathrm{cv}} \overline{\nu} \qquad \forall \rho \in \mathcal{P}(\mathcal{P}(S))_\nu.$$

The higher-level equation

Proposition

Let $(\omega_{\mathbf{i}}, \mathbf{X}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the RTP corresponding to γ and ν. Set

$$\xi_{\mathbf{i}} := \mathbb{P}\big[X_{\mathbf{i}} \in \cdot \,\big|\, (\omega_{\mathbf{ij}})_{\mathbf{j}\in\mathbb{T}}\big].$$

Then $(\omega_{\mathbf{i}}, \xi_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is an RTP corresponding to $\check{\gamma}$ and $\underline{\nu}$. Also, $(\omega_{\mathbf{i}}, \delta_{X_{\mathbf{i}}})_{\mathbf{i}\in\mathbb{T}}$ is an RTP corresponding to $\check{\gamma}$ and $\overline{\nu}$.

$$\overline{\nu} = \mathbb{P}\big[\delta_{X_\varnothing} \in \cdot\,\big]$$

corresponds to "perfect knowledge" while

$$\underline{\nu} = \mathbb{P}\big[\mathbb{P}\big[X_\varnothing \in \cdot \,\big|\, (\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}\big] \in \cdot\,\big]$$

corresponds to the knowledge about $X_\varnothing$ that is contained in the random variables $(\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$.

Corollary The RTP is endogenous iff $\underline{\nu} = \overline{\nu}$.

Example: Cooperative branching with death

Here $\mathcal{P}(\mathcal{P}(\{0,1\})) \cong \mathcal{P}[0,1]$ and for $\eta_1, \eta_2, \eta_3 \in [0,1]$,

$$\widehat{\mathtt{dth}}(\varnothing) = 0 \quad \text{and} \quad \widehat{\mathtt{cob}}(\eta_1, \eta_2, \eta_3) = \eta_1 + (1 - \eta_1)\eta_2\eta_3$$

so that the higher-level RDE is

$$\eta \stackrel{\mathcal{D}}{=} \chi \cdot \big(\eta_1 + (1 - \eta_1)\eta_2\eta_3\big),$$

where η takes values in $[0,1]$, η_1, η_2, η_3 are independent copies of η and χ is an independent Bernoulli r.v. with $\mathbb{P}[\chi = 1] = \alpha/(\alpha + 1)$.

This RDE has three "trivial" solutions

$$\overline{\nu}_{\dots} = (1 - z_{\dots})\delta_0 + z_{\dots}\delta_1 \qquad (\dots = \mathrm{low}, \mathrm{mid}, \mathrm{upp}),$$

and a nontrivial solution

$$\underline{\nu}_{\mathrm{mid}} = \lim_{n \to \infty} \check{T}^n(\delta_{z_{\mathrm{mid}}}).$$

Thank you!