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Topology

Problem Set 7

1. (10 points)
   (a) Let $H \subset G$ be a subgroup. Show that the group of G-automorphisms of the G-set $H\backslash G$ (with the obvious right action) is isomorphic to $N_G(H)/H$, where $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$ is the *normalizer of H in G*.
   (b) Let $p\colon \tilde{X} \to X$ and assume both spaces are path-connected, locally path-connected, and semilocally simply-connected. For any $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0)$ set $G = \pi_1(X, x_0)$ and $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Prove that $\mathrm{Deck}(\tilde{X}) \cong N_G(H)/H$.

2. (10 points) Let X be a path-connected and locally path-connected space with finite fundamental group. Prove that every map $X \to S^1$ is homotopic to a constant map.

3. (20 points) Let m, p be coprime integers. Let $G = \mathbb{Z}_m$ and consider the G-action on $\mathbb{C}^2$ which is defined by

$$\overline{k} \cdot (v, w) = \left(e^{\frac{2\pi ik}{m}} v, e^{\frac{2\pi ipk}{m}} w\right).$$

   The quotient $L_{m,p}$ of the induced action on $S^3 \subset \mathbb{C}^2$ is called a *lens space*.
   (a) Show that two lense spaces $L_{m,p}$ and $L_{m',p'}$ are not homotopy equivalent if $m \neq m'$.
   (b) Show that the action restricts to a G-action on $S^1 \times D^2 \subset \mathbb{C}^2$ and that the quotient $(S^1 \times D^2)/G$ is homeomorphic to $S^1 \times D^2$.
   (c) Show that $L_{m,p}$ is homeomorphic to a gluing of two solid tori along their boundaries via a homeomorphism $f\colon T^2 \to T^2$, i.e. $L_{m,p} \cong (S^1 \times D^2 \coprod D^2 \times S^1)/\sim$ where the relation is generated by $x \sim f(x)$ for all $x \in T^2 \subset S^1 \times D^2$ and $f(x)$ is interpreted as an element in $D^2 \times S^1$.
   Hint: recall the decomposition from exercise 3 on sheet 4.

Please hand in your solutions on December 3 at the end of the lecture.