Prof. Dr. Sebastian Hensel
Dr. Leopold Zoller

Topology

Problem Set 12

1. (10 points) Consider a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \longrightarrow & B_* & \longrightarrow & C_* & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A'_* & \longrightarrow & B'_* & \longrightarrow & C'_* & \longrightarrow & 0 \end{array}$$

of complexes of Abelian groups in which the rows are exact. Show that the maps induced by the vertical arrows on the level of homology fit in between the respective long exact sequences associated to the short exact sequences of complexes such that we obtain a commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_n(A_*) & \longrightarrow & H_n(B_*) & \longrightarrow & H_n(C_*) & \longrightarrow & H_{n-1}(A_*) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H_n(A'_*) & \longrightarrow & H_n(B'_*) & \longrightarrow & H_n(C'_*) & \longrightarrow & H_{n-1}(A'_*) & \longrightarrow & \cdots \end{array}$$

One calls this *naturality of the long exact sequence.*

2. (10 points) Let $U \subset V \subset X$. Prove that there is a long exact sequence

$$\cdots \to H_n(V,U) \to H_n(X,U) \to H_n(X,V) \to H_{n-1}(V,U) \to \cdots.$$

Deduce that if $U \to V$ is a homotopy equivalence then $H_n(X,V) \to H_n(X,U)$ is an isomorphism for all n. Can you find an alternative proof of the last statement using exercise 1 and the 5-lemma?

3. (10 points) Let X_α, $\alpha \in I$ be a family of spaces with basepoints $x_\alpha \in X_\alpha$ such that x_α is a deformation retract of an open neighbourhood of x_α in X_α. Show that

$$H_n(\bigvee_{\alpha\in I} X_\alpha) \cong \bigoplus_{\alpha\in I} H_n(X_\alpha)$$

for all $n \geq 1$, where the left hand side denotes the homology of the infinite wedge sum.

4. (10 points) Compute all homology groups of $\mathbb{R}P^2$.

Please hand in your solutions on January 21 at the end of the lecture.