Prof. Dr. Sebastian Hensel
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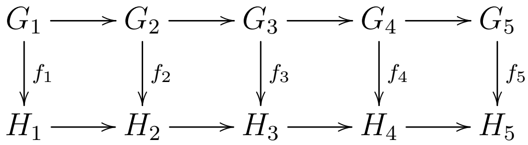
Topology

Problem Set 10

1. (10 points) Consider a commutative diagram of Abelian groups



in which the rows are exact sequences. Prove:

(a) If f_1 is surjective and f_2, f_4 are injective, then f_3 is injective.
(b) If f_2, f_4 are surjective and f_5 is injective, then f_3 is surjective.

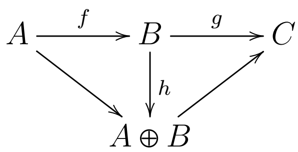
The combination of these two statements is known as the five lemma.

2. (10 points) Consider a short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

of Abelian groups. Prove that the following are equivalent:

(a) There is a homomorphism $r: B \to A$ such that $r \circ f = \mathrm{id}_A$.
(b) There is a homomorphism $s: C \to B$ such that $g \circ s = \mathrm{id}_C$.
(c) There is an isomorphism $h: B \to A \oplus B$ such that the diagram



commutes, where the non-specified maps are the canonical inclusion and projection.

3. (10 points) Let (X, x) and (Y, y) be spaces with basepoints, such that x and y are deformation retracts of respective open neighbourhoods. Show that for $n \geq 1$ we have

$$H^n(X \vee Y) \cong H^n(X) \oplus H^n(Y).$$

Hint: In this and the following exercises you may use the long exact sequence of a good pair (see Hatcher, Algebraic Topology, Theorem 2.13) even though the proof has not yet been completed in the lecture.

4. (10 points) Show that the homomorphism $\pi_1(X, x_0) \to H_1(X)$, as defined in Exercise 3 on sheet 9, is surjective if X is path-connected.

5. (10 POINTS) Let $T^2 = I^2/\sim$ be the 2-torus and $p = (\frac{1}{2}, \frac{1}{2})$. Let $A \subset T^2\backslash\{p\}$ be the intersection of $T^2\backslash\{p\}$ with the closed ball around p with radius $\frac{1}{4}$.

   (a) Show that $(T^2\backslash\{p\})/A$ is homeomorphic to T^2.
   (b) Show that the map $H^1(A) \to H^1(T^2\backslash\{p\})$ is zero.
   (c) Compute $H^k(T^2)$ for all $k \geq 0$.

6. (10 POINTS) A *christmas tree* Γ is obtained by gluing a finite number of 2-spheres (preferably shiny ones) to a finite tree by identifying a single point in each sphere with some point of the tree. Compute $H^k(\Gamma)$ for all $k \geq 0$.

Please hand in your solutions on January 7 at the end of the lecture.