Solution to Set 8, Problem 1

We consider the variation $\widetilde{h}$ as in the problem, with associated Jacobi field $\widetilde{J} = \sin(t)\widetilde{U}(t)$ where $\widetilde{U}$ is parallel. Explicitly,

$$\widetilde{U}(t) = \widetilde{u}$$

The first step is to realise that the projection $p \circ \widetilde{h} = h$ is also a geodesic variation in $\mathbb{C}P^n$. To see this, note that every geodesic $t \mapsto \widetilde{h}(s,t)$ has horizontal velocity (by the explicit construction of $\widetilde{h}$ we can compute

$$\frac{d}{dt}\widetilde{h}(s,t) = -\sin(t)\widetilde{m} + \cos(t)(\cos(s)\widetilde{v} + \sin(s)\widetilde{u})$$

and the scalar product of $i\frac{d}{dt}\widetilde{h}(s,t)$ with $\widetilde{h}(s,t)$ is zero since $\langle \widetilde{m}, i\widetilde{m}\rangle = \langle \widetilde{u}, i\widetilde{u}\rangle = \langle \widetilde{v}, i\widetilde{v}\rangle = 0$ (where we use that $\rangle, \langle$ is the real scalar product and i acts by a rotation by $\pi/2$ in each complex coordinate) and $\langle \widetilde{m}, i\widetilde{u}\rangle = \langle \widetilde{m}, i\widetilde{v}\rangle = 0$ by assumption). Thus, by the hint, the projection stays geodesic:

$$\frac{\nabla}{dt}h(s,t) = p_*\frac{\widetilde{\nabla}}{dt}\widetilde{h}(s,t) = 0$$

Now, since h is a geodesic variation, its variational field J is a Jacobi field.

a) If $\widetilde{u}$ is orthogonal to $i\widetilde{v}$, then observe that $\widetilde{U}$ is parallel and horizontal. Thus, U is parallel again, and we can compute

$$J''(t) = -\sin(t)U(t) = -J(t)$$

By the Jacobi equation (evaluated at 0) this means

$$-u = R(v,u)v$$

which shows the claim.

b) If $\widetilde{u} = i\widetilde{v}$, then we can explicitly compute a horizontal projection $\widetilde{U}_h$ of $\widetilde{U}$ as

$$\begin{aligned}\widetilde{U}_h(t) &= \widetilde{U}(t) - \langle \widetilde{U}(t), ih(0,t)\rangle ih(0,t)\\ &= \widetilde{u} - \langle \widetilde{u}, \cos(t)i\widetilde{m} + \sin(t)i\widetilde{v}\rangle(\cos(t)i\widetilde{m} + \sin(t)i\widetilde{v})\\ &= i\widetilde{v} - \sin(t)(\cos(t)i\widetilde{m} + \sin(t)i\widetilde{v}) = \cos(t)^2 i\widetilde{v} - \cos(t)\sin(t)i\widetilde{m}\\ &= \cos(t)i\frac{d}{dt}\widetilde{h}(0,t)\end{aligned}$$

We thus have

$$\widetilde{J}(t) = \cos(t)\sin(t)i\frac{d}{dt}\widetilde{h}(0,t)$$

Consider the vector field

$$\widetilde{Y}(t) = i\frac{d}{dt}\widetilde{h}(0,t) = -\sin(t)i\widetilde{m} + \cos(t)i\widetilde{v}.$$

along $\widetilde{h}(0,t)$. It stays tangent to the sphere, so we can compute its covariant derivative as the usual derivative:

$$\widetilde{Y}'(t) = -\cos(t)i\widetilde{m} - \sin(t)i\widetilde{v}.$$

and observe that this is totally vertical (i.e. a multiple of $i\widetilde{h}$). Thus, even though $\widetilde{Y}$ is not parallel, its projection $Y = p\widetilde{Y}$ is.

So, we now have

$$J(t) = \cos(t)\sin(t)Y(t)$$

where Y is parallel, and we can compute:

$$J''(t) = -4J(t)$$

Arguing as above, this shows $R(v,u)v = 4u$ as claimed.

c) The last part is simply a small computation using symmetries and linearity of the curvature tensor.