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Riemannian Geometry

Problem Set 6

1. *Jacobi fields and flat metrics.* Let M be a Riemannian manifold with sectional curvature identically zero. Show that for every $p \in M$, the exponential map

$$\exp_p : B_\varepsilon(0) \subseteq T_pM \longrightarrow \exp_p(B_\varepsilon(0))$$

is an isometry.

2. *Curvature of products.* We want to figure out how the curvature of a product of two Riemannian manifolds behaves.

   (a) Let M and N be Riemannian manifolds. Show that geodesics on $M \times N$ are of the form $(\alpha(t), \beta(t))$, where α and β are geodesics on M and N, respectively, or constant maps.

   (b) Consider $S^2 \times S^2$. Show that the curvature is not constant by finding two-dimensional subspaces with sectional curvature 1 and 0, respectively.

   (c) Show that on $S^2 \times S^2$, the curvature interpolates between zero and one. Generalize this idea to products $M \times N$.

3. *Curvature and parallel transport.* Let M be a Riemannian manifold with the property that for any $p, q \in M$, parallel transport from p to q is independent of the path from p to q. Show that the curvature of M is identically zero.

4. *Spread of geodesics.* Let $v, w \in T_pM$ be unit vectors, and consider the geodesic $\gamma(t) = \exp_p tv$, and the Jacobi field

$$J(t) = d_{tv} \exp_p(tw).$$

We want to study the function $f(t) = g(J(t), J(t))$.

   (a) Compute $f(0) = 0, f'(0) = 0, f''(0) = 2, f'''(0) = 0$.

   (b) Show that

$$\frac{\nabla}{dt} R(\gamma', J)\gamma'(0) = R(\gamma', \frac{\nabla}{dt} J)\gamma'(0).$$

   (*Hint: Pair the expression on the left with an arbitrary vector field* Z).

   (c) Show that $f''''(0) = -8g(R(v, w)v, w)$.

   (d) Conclude that the norm of the Jacobi field J has the Taylor expansion

$$g(J(t), J(t)) = t - \frac{1}{6} g(R(v, w)v, w)t^3 + o(t)$$

   where $o(t)/t^3 \to 0$, as $t \to 0$.

Think about what this means about the “spread of geodesics” in terms of curvature.