Prof. Dr. Sebastian Hensel
Anna Ribelles Pérez

Riemannian Geometry

Problem Set 4

1. *From last sheet: Geodesics on the tangent bundle.* It is possible to introduce a Riemannian metric in the tangent bundle TM of a Riemannian manifold $(M, \langle\ ,\ \rangle)$ in the following manner. Let $(p_0, v_0) \in TM$ and V, W be tangent vectors in TM at (p_0, v_0). Choose curves in TM

$$\alpha : t \mapsto (p(t), v(t)), \beta : s \mapsto (q(s), w(s))$$

with $p(0) = q(0) = p_0$, $v(0) = w(0) = v_0$, and $V = \alpha'(0), W = \beta'(0)$. Define an inner product on TM by

$$\langle V, W \rangle_{(p_0,v_0)} = \langle d\pi(V), d\pi(W) \rangle_{p_0} + \langle \frac{\nabla v}{dt}(0), \frac{\nabla w}{ds}(0) \rangle_{p_0},$$

where $d\pi$ is the differential of $\pi : TM \to M$.

(a) Prove that this inner product is well-defined and introduces a Riemannian metric on TM.

(b) A vector at $(p_0, v_0) \in TM$ that is orthogonal (with respect to the metric above) to the fiber $\pi^{-1}(p) = T_pM$ is called a *horizontal vector*. A curve $\gamma : t \mapsto (p(t), v(t))$ in TM is *horizontal* if its tangent vector is horizontal for all t. Show that γ is horizontal if and only if the vector field $v(t)$ is parallel along $p(t)$ in M.

(c) Prove that the geodesic field is a horizontal vector field (i.e. it is horizontal at every point).

(d) Prove that the trajectories of the geodesic field are geodesics on TM in the metric above.
Hint: Let $\tilde{\alpha}(t) = (\alpha(t), v(t))$ be a curve in TM. Show that $l(\tilde{\alpha}) \geq l(\alpha))$ and that equality holds if v is parallel along α. Consider a trajectory of the geodesic flow passing through (p_0, v_0) which is locally of the form $\tilde{\gamma}(t) = (\gamma(t), \gamma'(t))$, where γ is a geodesic on M. Choose convex neighborhoods $U \subseteq TM$ of (p_0, v_0) and $V \subseteq M$ of p_0 such that $\pi(U) = V$. Take two points $Q_1 = (q_1, v_1), Q_2 = (q_2, v_2)$ in $\tilde{\gamma} \cap W$. If $\tilde{\gamma}$ is not a geodesic, then there exists a curve $\tilde{\alpha}$ in W passing through Q_1 and Q_2 such that $l(\tilde{\alpha}) < l(\tilde{\gamma}) = l(\gamma)$. This is a contradiction.

2. *Rectifiable curves.* We have defined a path metric d from the Riemannian metric on a manifold (M, g) as the infimum of the lengths of piecewise smooth paths between two points. We will now extend the notion of length of a curve γ to a larger family of curves.

Let $\gamma : [a, b] \to M$ be a continuous curve. If

$$L(\gamma) := \sup(\{\sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})) \mid n \in \mathbb{N},\ a = t_0 < t_1 < \ldots < t_n = b\})$$

is finite, we say γ is *rectifiable* and $L(\gamma)$ is its length. We can now define a path metric on M using rectifiable curves instead of smooth ones:

$$\tilde{d}(x, y) = \inf\{L(\gamma) \mid \gamma \text{ is a rectifiable curve from } x \text{ to } y\}.$$

Show:

(a) Piecewise smooth curves are rectifiable, and their length agrees with their length as defined in the lecture.
(b) Both definitions of length induce the same metric on M.

3. *The Poincaré disk model.* Consider the *Cayley map* on $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$

$$f(z) = \frac{z - i}{z + i},$$

which maps the upper half-plane to the unit disk.

(a) Compute the metric on the image of f making the interior of the unit disk I isometric to the hyperbolic plane $\mathbb{H}^2$ as defined in class.
(b) What do the geodesics on the disk model look like?

4. *The hyperboloid model of* $\mathbb{H}^2$. Consider

$$H := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1^2 + x_2^2 - x_3^2 = -1 \ \text{ and } \ x_3 > 0\}$$

and the symmetric bilinear form

$$(x_1, x_2, x_3) * (y_1, y_2, y_3) = x_1 y_1 + x_2 y_2 - x_3 y_3.$$

This is not positive definite on $\mathbb{R}^3$, but it *is* when restricted to the tangent space of H (why?), and so we obtain a Riemannian metric on H.

(a) Show that H is isometric to the hyperbolic plane.
(b) Describe the isometry group and geodesics on H.