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Riemannian Geometry

Problem Set 3

1. *Recovering connections from parallel transport.* Let X, Y be smooth vector fields on a Riemannian manifold M. Let $p \in M$ and let $\gamma : [a,b] \to M$ be a trajectory of X through p, i.e. $\gamma(t_0) = p$ (for a $t_0 \in (a,b)$) and $\frac{d\gamma}{dt} = X(\gamma(t))$. Prove that the Levi-Civita connection of M is

$$(\nabla_X Y)(p) = \frac{d}{dt}(P^{\gamma\,-1}_{t_0,t}(Y(\gamma(t))))\Big|_{t=t_0},$$

where $P^{\gamma}_{t_0,t} : T_{\gamma(t_0)}M \to T_{\gamma(t)}$ is the parallel transport along γ from t_0 to t.

2. *Geodesics on the tangent bundle.* It is possible to introduce a Riemannian metric in the tangent bundle TM of a Riemannian manifold $(M, \langle\ ,\ \rangle)$ in the following manner. Let $(p_0, v_0) \in TM$ and V, W be tangent vectors in TM at (p_0, v_0). Choose curves in TM

$$\alpha : t \mapsto (p(t), v(t)), \beta : s \mapsto (q(s), w(s))$$

with $p(0) = q(0) = p_0$, $v(0) = w(0) = v_0$, and $V = \alpha'(0), W = \beta'(0)$. Define an inner product on TM by

$$\langle V, W\rangle_{(p_0,v_0)} = \langle d\pi(V), d\pi(W)\rangle_{p_0} + \langle \frac{\nabla v}{dt}(0), \frac{\nabla w}{ds}(0)\rangle_{p_0},$$

where $d\pi$ is the differential of $\pi : TM \to M$.

   (a) Prove that this inner product is well-defined and introduces a Riemannian metric on TM.
   (b) A vector at $(p_0, v_0) \in TM$ that is orthogonal (with respect to the metric above) to the fiber $\pi^{-1}(p) = T_pM$ is called a *horizontal vector*. A curve $\gamma : t \mapsto (p(t), v(t))$ in TM is *horizontal* if its tangent vector is horizontal for all t. Show that γ is horizontal if and only if the vector field $v(t)$ is parallel along $p(t)$ in M.
   (c) Prove that the geodesic field is a horizontal vector field (i.e. it is horizontal at every point).
   (d) Prove that the trajectories of the geodesic field are geodesics on TM in the metric above.
   Hint: Let $\tilde{\alpha}(t) = (\alpha(t), v(t))$ be a curve in TM. Show that $l(\tilde{\alpha}) \geq l(\alpha))$ and that equality holds if v is parallel along α. Consider a trajectory of the geodesic flow passing through (p_0, v_0) which is locally of the form $\tilde{\gamma}(t) = (\gamma(t), \gamma'(t))$, where γ is a geodesic on M. Choose convex neighborhoods $U \subseteq TM$ of (p_0, v_0) and $V \subseteq M$ of p_0 such that $\pi(U) = V$. Take two points $Q_1 = (q_1, v_1), Q_2 = (q_2, v_2)$ in $\tilde{\gamma} \cap W$. If $\tilde{\gamma}$ is not a geodesic, then there exists a curve $\tilde{\alpha}$ in W passing through Q_1 and Q_2 such that $l(\tilde{\alpha}) < l(\tilde{\gamma}) = l(\gamma)$. This is a contradiction.

3. *Geodesics on the hexagonal torus.* Recall the hexagonal torus we defined in Problem Sheet 1 as the quotient of $\mathbb{R}^2$ by a translation action. We showed that the resulting surface is diffeomorphic to the standard torus. In this exercise, we will study the metrics of these two manifolds.
   (a) Recall from the lecture that geodesics are completely determined by a starting point p and initial velocity v in a neighborhood of p. How can we characterize all *closed* geodesics with $\|v\| = \text{const.}$ through a point on the square and hexagonal torus, respectively?
   Hint: Consider the tilings of the plane instead of the quotient spaces.
   (b) Consider the set of shortest closed geodesics through a point. Argue that there cannot be an isometry between a square and a hexagonal torus, not even after rescaling.