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Riemannian Geometry

Problem Set 2

1. Compute the Levi-Civita connection on $\mathbb{H}^n$ in local coordinates.

2. *The flat cone.* Let $X = \{(x_1, x_2) \mid x_1 \geq 0, x_2 \geq 0\} \setminus (0,0)$ be the upper right quadrant of the Euclidean plane without the origin, and define

$$C = X/\sim,$$

where $\sim$ is the equivalence relation generated by $(x,0) \sim (0,x) \forall x > 0$. C is a topological space with the quotient topology. Show:

   (a) The flat metric on X descends to a Riemannian metric on C.
   (b) C is the same as the quotient of $\mathbb{R}^2 \setminus \{0\}$ by the group action given by rotation by $\pi/2$.
   (As an aside, think about what happens if we instead take rotation by a rational or irrational multiple of 2π, as opposed to by an angle of the form $\frac{2\pi}{n}$.)
   Moreover, C is isometric to the open (i.e. without the tip) cone embedded in $\mathbb{R}^3$ with opening angle $2\arcsin(1/4)$.
   (c) $TC = C \times \mathbb{R}^2$, i.e. show that there exist $X, Y \in \Gamma(TC)$ such that $\{X(p), Y(p)\}$ is a basis of T_pC for all $p \in C$.
   (d) Compute the parallel transport along a loop around the cone (i.e. the image of a path from $(r,0)$ to $(0,r)$ on X). Use this to compute parallel transport along a small circle of a sphere.

3. *Affine manifolds.* Let $X = \mathbb{R}^2 \setminus \{0\}/\sim$, where $\sim$ is given by the identification $\phi : x \mapsto 2x$, and let $p : \mathbb{R}^2 \setminus \{0\} \to X$ be the projection.

   (a) Show that X is diffeomorphic to the torus T^2.
   Hint: Consider an annulus around $(0,0)$ *containing one point in each equivalence class.*
   (b) Show that the flat connection D on $\mathbb{R}^2 \setminus \{0\}$ descends to a connection ∇ on X.
   (c) Compute the parallel transport along the closed curve $p \circ \gamma$ with

$$\gamma : [0,1] \to \mathbb{R}^2 \setminus \{0\}, \gamma(t) = t + 1.$$

   Conclude that ∇ cannot be the Levi-Civita connection of a Riemannian metric on X.