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Riemannian Geometry

Script

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Lecture 1 (April 25)

Rough goals of this course: Study *Riemannian manifolds* (M, g)

- Connect Riemannian metric g and induced path-metric d: completeness, shortest paths
- Interpretation of (analytic) curvature on (geometric) properties: Jacobi fields, divergence of geodesics
- Geometric consequences of curvature bounds: volume growth, area growth
- Topological consequences of curvature bounds: compactness, diameter bounds (positive) versus asphericity, non-Abelian behaviour (negative)
- Purely geometric interpretation of curvature: comparison geometry

Before we can start in earnest, we need two things: a quick review of differentiable manifolds, and a set of examples we will recur to regularly.

Quick review of manifolds. This does not replace a course or book, and is just intended to refresh your memory; or point you towards topics you should review. Also, this will set up notation.
We will assume familiarity with the following analytic things:

Review 0.1.

- *Manifolds M. They are assumed smooth unless specified, and are usually assumed to be connected, and without boundary.*
- *The tangent bundle $p : TM \to M$, its fibers T_pM (the tangent spaces), and differentials of smooth maps df. A section of the tangent bundle is a vector field. Vector fields differentiate functions:*

$$Xf := df(X)$$

The Lie bracket of vector fields $[X, Y]$ is defined by:

$$[X, Y]f = XYf - YXf$$

One useful thing to remember is: $df[X, Y] = [dfX, dfY]$.
- *A* connection *(or* covariant derivative*) is a way to differentiate vector fields*

$$\nabla : \Gamma(TM) \otimes \Gamma(TM) \to \Gamma(TM),$$

which is C^∞–linear in the first variable, and satisfies the Leibniz rule in the second. This is not uniquely determined, but a choice.

- *If $\gamma : [0,1] \to M$ is a path, then a vector field along γ is a section of γ^*TM, in other words: $X : [0,1] \to TM$ smooth so that $X(t) \in T_{\gamma(t)}M$. Connections induce derivatives $\frac{\nabla}{dt}$ acting on such things. They have a Leibniz rule, and*
$$\frac{\nabla}{dt}X(\gamma(t)) = \nabla_{\gamma'(t)}X.$$
- *The tensor bundles $T^{\otimes r}M \otimes (T^*M)^{\otimes s}$. A section of this bundle is an object that "eats" s vector fields and outputs a section of $T^{\otimes r}M$. Connections extend to such bundles, and satisfy "all possible product rules" (pairing a vector and a covector is a product)*
- *Most important for us: A* Riemannian metric *is a section $g \in \Gamma((T^*M)^{\otimes 2})$ which is symmetric and positive definite at each point.*
- *Such a metric uniquely determines a* Levi-Civita connection*:*
$$\nabla_X Y - \nabla_Y X = [X,Y] \quad \textit{torsion-free or symmetric}$$
$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \quad \textit{compatible with } g.$$
- *The Levi-Civita connection is determined by the* Koszul identity*:*

$$2g(\nabla_Y X, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g([X,Z],Y) - g([Y,Z],X) - g([X,Y],Z)$$

1. A zoo of examples

Here, we will collect some recurring examples that we will use to explore concepts hands-on. Also, we recall some constructions that one can use to build more examples.

Example 1.1. Euclidean space $\mathbb{R}^n$ with the Riemannian metric g given by the standard scalar product of $\mathbb{R}^n$ at each point. Here, we are identifying each tangent space $T_p\mathbb{R}^n = \mathbb{R}^n$ (this is possible since the tangent bundle of $\mathbb{R}^n$ is trivial).

Here, the Levi-Civita connection is simply the usual deriviative, where we identify $\Gamma(T\mathbb{R}^n)$ with smooth maps $\mathbb{R}^n \to \mathbb{R}^n$.

Example 1.2. The sphere:
$$S^n = \{p = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}, \|p\| = 1\}$$
Here, the metric is the restriction of the usual metric of $\mathbb{R}^n$.
Recall that for any $p \in S^n$ we have $T_p\mathbb{R}^{n+1} = \mathbb{R}p \oplus T_pS^n$, where the sum is orthogonal with respect to the flat metric.

More generally, if $f : N \to (M,g)$ is a submanifold (or image of an immersion), then N inherits a metric from M:
$$h_p(v,w) = g_{f(p)}(d_pf(v), d_pf(w))$$
(Convince yourself that this is indeed a metric on N). The Levi-Civita connection for (N,h) can be computed from the Levi-Civita connection ∇^M of (M,g). In the case of the sphere, this takes the following explicit form:

let $\Pi_p : \mathbb{R}^n \to p^\perp$ be the orthogonal projection of $\mathbb{R}^n$ to the orthogonal complement $p^\perp$ of p. Then

$$\nabla_X Y = \Pi_p(D_p Y(X)).$$

Here, we identify vector fields on S^n with maps $X : S^n \to \mathbb{R}^{n+1}$ so that $X(p) \perp p$ for all $p \in S^n$.

Example 1.3. Hyperbolic space.

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n, x_n > 0\}$$

with the metric

$$g_{(x_1,\ldots,x_n)} = \frac{1}{x_n^2} dx^1 \otimes \cdots \otimes dx^n.$$

Here, the metric makes points closer the "higher up" they are. We'll appreciate this example later.

It might be useful to figure out how the Levi-Civita connection looks in this example, at least for $n = 2$. The easiest way is to use the Koszul identity, for the vector fields b_1, b_2 given by the standard basis of $\mathbb{R}^2$ in all combinations for X, Y, Z.

Example 1.4. The torus:

$$T^n = S^1 \times \cdots \times S^1$$

with the metric coming from the embedding $T^n \subset \mathbb{C}^n$.

There is a second construction of a metric on T^n. We already have a metric on S^1 (the one-dimensional sphere). In general, if $(M, g), (N, g)$ are Riemannian manifolds, then $M \times N$ inherits a metric. Namely,

$$\langle v, w \rangle = g(d\pi_1(v), d\pi_1(w)) + h(d\pi_2(v), d\pi_2(w)).$$

Check that this is a metric. How do we compute Levi-Civita connection? *Hint: The tangent bundle of $M \times N$ is $TM \oplus TN$, so vector fields can be written as the direct sum of vector fields on M and N.*
Convince yourself that the product metric on $S^1 \times \cdots \times S^1$ is (up to scaling) the same as the metric defined above. We also get other examples like this, e.g. $S^2 \times S^2$ etc.

There is a third way to obtain the torus, using group actions. Recall that an *action of a group G on a manifold M* is a homomorphism

$$\rho : G \to \mathrm{Diff}(M).$$

An action is called *free* if $\rho(g)p = p$ for any $p \in M$ implies $g = e$. An action is called *proper* if for any compact set K the set $\{g \in G | gK \cap K \neq \emptyset\}$ is finite.
The quotient M/G as a set consists of the equivalence classes defined by the relation $x \sim y \Leftrightarrow y = gx$ for some g.
Recall

Theorem 1.5. *Suppose G acts on M properly and freely. Then there is a unique smooth structure on M/G so that the canonical projection map $M \to M/G$ is a local diffeomorphism.*

Lecture 2 (April 29)

We begin by recalling a slight strenghtening of the quotient theorem from last time. Namely, by properness of the action (and the fact that we act on a manifold), for any point $p \in M$ there is a neighbourhood U of p so that

$$gU \cap U = \emptyset,$$

for all $g \neq 1$. For the quotient map $q : M \to M/G$ this has the consequence that

$$q^{-1}(q(U)) = \coprod_{g \in G} gU,$$

and

$$q|_U : U \to q(U)$$

is a diffeomorphism. Furthermore, for any group element g one has

$$q|_{gU} = q|_U \circ g^{-1}.$$

We use this to give a third way to describe the torus. Namely, the group $G = \mathbb{Z}^n$ acts on the space $M = \mathbb{R}^n$ by translations. The action is by isometries (with respect to the usual flat metric g on $\mathbb{R}^n$) and proper.
As a consequence $N = M/G$ is a manifold. I claim that there is a unique metric h so that

$$h(dq(v), dq(w)) = g(v, w)$$

for all $v, w \in T_pM$. This property is important, and it has a name:

Definition 1.6. A smooth map $f : (M, g) \to (N, h)$ between Riemannian manifolds is called a *local isometry*, if

$$h_{f(p)}(d_pf(v), d_pf(w)) = g_p(v, w),$$

for all $p \in M$ and all $v, w \in T_pM$. It is called an *isometry*, if it is in addition a diffeomorphism.

Note that local isometries are automatically local diffeomorphisms (why?). So, I claim that in the example above, there is a unique metric h on M/G so that the quotient map q is a local isometry. We will prove this here for the general case of a quotient as in the quotient theorem from last time.

Lemma 1.7. *Suppose that G acts on (M, g) properly, freely, and by isometries. Then there is a unique metric h on $N = M/G$ which makes the quotient map $q : M \to N$ a local isometry.*

Proof. Let $p \in M$ be any point, and let U be a neighbourhood of p so that $gU \cap U = \emptyset$ for all nonidentity g. Put

$$h_x(v,w) = g_{q|_U^{-1}(x)}(d_{q|_U^{-1}(x)}q^{-1}v, d_{q|_U^{-1}(x)}q^{-1}w).$$

Observe that this defines a metric on $q(U)$, so that $q|_U : U \to q(U)$ is a local diffeomorphism, and h is uniquely determined by that latter property. Also note that h is smooth (as the right hand side depends smoothly on x).
Next, we want to show that the choice of U does not matter. Namely, if V is another neighbourhood of p, then so is $U \cap V$, and so clearly the corresponding metrics h_x are the same. Finally, we need to check that the choice of p does not matter. In light of what we proved, it suffices to see that replacing U by gU for some g does not change the metric. This follows from the fact that G acts by isometries, the chain rule, and $q|_{gU} = q|_U \circ g^{-1}$ (see above). □

We call the metric guaranteed by the lemma the *quotient metric.*

Geometric quotients are one of the most useful ways to get interesting Riemannian manifolds. For example:

Example 1.8.

$$\mathbb{R}P^n = S^n/\{\pm \mathrm{Id}\}$$

Real projective space. We'll call the quotient metric the *round metric* on $\mathbb{R}P^n$.

Observe that to do this, we couldn't consider the description $\mathbb{R}^{n+1} \setminus \{0\}/\mathbb{R}^*$ of real projective space, since the action of $\mathbb{R}^*$ on $\mathbb{R}^{n+1} \setminus \{0\}$ is neither proper, nor by isometries.

Sometimes, one can even take quotients for actions that are not as nice as properly discontinuous ones. The stereotypical example here is *complex projective space*

$$\mathbb{C}P^n = \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*.$$

This inherits a (real!) smooth structure with the same argument that we used last semester for real projective spaces. In order to define a metric, we have to be a bit more careful.
$\mathbb{C}^{n+1}$ carries a flat metric which comes from the identification $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$. In complex coordinates this takes the form

$$g(v,v) = \sum |z_i|^2, \quad v = (z_1, \dots, z_{n+1})$$

where we identified $T_p\mathbb{C}^{n+1} = \mathbb{C}^{n+1}$. From this description it is clear that the group G of unit norm complex numbers acts on $\mathbb{C}^{n+1}$ by isometries.
Now, consider the unit sphere $S \subset \mathbb{C}^{n+1}$ with respect to this metric. This inherits a (round) metric, and G acts on S by isometries. The quotient is

$$\mathbb{C}P^n = S/G.$$

Observe that G acts freely, but not properly discontinuous on S!

However, we can still induce a metric. To see this we need to understand the quotient map

$$p : S \to \mathbb{C}P^n$$

a bit better. First note that it is smooth. Next, for any $z \in S$, observe that

$$T_z\mathbb{C}^{n+1} = \mathbb{R}z \oplus T_pS$$

since $T_pS = z^\perp$. Consider the orbit $Gz \subset S$. The tangent vector to this orbit is $\mathbb{R}iz$, and we can thus further decompose

$$T_z\mathbb{C}^{n+1} = \mathbb{R}z \oplus \mathbb{R}iz \oplus H_z$$

where we define H_z to be the orthogonal complement of $\mathbb{C}z$. $H \subset TS$ is a subbundle.

Let us consider at the explicit point $x = (1, 0, \ldots, 0)$ how $p : \mathbb{C}^{n+1} \to \mathbb{C}P^n$ acts with respect to this decomposition. We can use the chart

$$\varphi([z_0 : \cdots : z_n]) = (z_1/z_0, \ldots, z_n/z_0)$$

near $p(x)$. From this description it is immediate that $dp_x : H_x \to T_{p(x)}\mathbb{C}P^n$ is an isomorphism.

But, the decomposition of $T_z\mathbb{C}^{n+1}$, as well as the projection map p are equivariant under the action of $U(n+1)$, and that unitary group acts transitively on S. Hence, we have that *for all* $x \in S$, the differential

$$dp_x : H_x \to T_{p(x)}\mathbb{C}P^n$$

is an isomorphism (p is a *submersion*).

We can use this to define a metric h on $\mathbb{C}P^n$. Namely, given $v, w \in T_{p(x)}\mathbb{C}P^n$, let $\hat{v}, \hat{w} \in H_x$ be the unique vectors so that $dp_x\hat{v} = v, dp_x\hat{w} = w$. Put

$$g_{p(x)}(v, w) = h_x(v, w).$$

This is independent of the choice of x. Namely, if $p(x) = p(y)$, then $y = gx$ for $g \in G$. But then $dg\hat{v}, dg\hat{w} \in H_y$ are the unique vectors mapping to v, w under d_yp, and we are done since G acts by isometries.

Finally, this g is smooth. To prove this, note that p has local sections: for every $q \in \mathbb{C}P^n$, there is an open neighbourhood U and a smooth map $s : U \to S$ so that $ps = \mathrm{id}$. By symmetry, it again suffices to show this for $q = [1 : 0 : \ldots : 0]$, and there the section is simply

$$s([1 : z_1 : \ldots : z_n]) = \frac{(1, z_1, \ldots, z_n)}{\|(1, z_1, \ldots, z_n)\|}.$$

Next, note that the orthogonal projection $T_pS \to H_p$ defines a smooth bundle map $\pi : TS \to H$. Then, we have

$$h_x(v, w) = g_{s(x)}(\pi d_x s(v), \pi d_x s(w))$$

which shows smoothness. This metric turns p into a *Riemannian submersion.*

Complex projective spaces are geometrically somewhat different to real projective spaces, but we will see this later.

In addition to products, immersions, quotients, there is one last tool to build manifolds, but we will get to that later.

Lecture 3 (May 2)

1.1. **Parallel Transport.** In a tiny bit more detail (since it happened at the end of last course) and to warm up how to argue with the analytic things we will recall in detail: Parallel transport.

Suppose that $\gamma : [a, b] \to M$ is a *path*. Unless otherwise specified, all of our paths will be smooth, meaning for us that γ can be extended to a smooth map $(a - \epsilon, b + \epsilon) \to M$, and additionally $\gamma'(t) \neq 0$ for all t. One advantage of this is: locally, any vector field along γ is the restriction of a (local) vector field of the manifold.
A *piecewise smooth path* is a continuous map $\gamma : [a, b] \to M$ so that there is a partition $a = t_0 < \cdots < t_n = b$ where $\gamma|_{[t_i,t_{i+1}]}$ is smooth for all i.
A vector field $X(t)$ along a smooth path γ is called *parallel*, if $\frac{\nabla}{dt}X = 0$.

Lemma 1.9. *Given any (piecewise) smooth path γ, and any tangent vector $v_0 \in T_{\gamma(0)}M$, there is a unique parallel vector field $X(t)$ along γ with $X(0) = v_0$.*

Proof. First we prove the case where $\gamma([0, 1]) \subset U$, and $\varphi : U \to V$ is a chart. Then, let $b_i = \frac{\partial \varphi^{-1}}{\partial x^i}$ be local sections which are a basis at each point in U. Locally, we can write

$$X(t) = \sum_i f^i(t) b_i(\gamma(t)).$$

The condition $\frac{\nabla}{dt}X = 0$ becomes

$$0 = \sum_i \dot{f}^i(t) b_i(\gamma(t)) + f^i(t) \nabla_{\gamma'(t)} b_i$$

Rewriting this, this leads to a linear system of ordinary differential equations in the f^i. Hence, it has a unique solution given the initial value $X(0) = v_0$. This proves the special case.
For a general γ, cover the image with finitely many neighbourhoods, and use the uniqueness/existence from the special case. $\square$

The terminal value $X(1)$ of the output of the lemma is called the *parallel transport of v_0 along γ.*
We usually interpret parallel transport as a map

$$P^\gamma : T_{\gamma(0)}M \to T_{\gamma(1)}M,$$

and observe that it is linear (uniqueness of the solutions above and linearity of the conditions). By the same argument, we have that

$$P^\gamma = P^{\gamma|_{[c,b]}} \circ P^{\gamma|_{[a,c]}}$$

for all $c \in [a,b]$. As a consequence, we can *define* parallel transport along piecewise smooth paths as the composition of the parallel transport along smooth subpaths.
We can use parallel transport to characterise when a connection is compatible with a metric.

Lemma 1.10. *A connection ∇ is compatible with g along γ:*

$$g(X(t),Y(t))' = g\left(\frac{\nabla}{dt}X, Y\right) + g\left(X, \frac{\nabla}{dt}Y\right)$$

if and only if for any X, Y parallel along any γ we have that $g(X,Y)$ is constant.

Proof. One direction is clear. For the other, choose an orthonormal basis, extend as parallel vector fields, and then compute. □

In fact, the condition from the lemma is equivalent to compatibility of the connection. Prove this yourself (e.g. in the problem sessions).

Let's discuss parallel transport in some of our standard examples.

- In Euclidean space, it really is just a parallel shift.
- For the torus (or properly discontinuous quotients in general), one can always use the following method. Suppose that $\gamma : [a,b] \to T^n$ is a curve. Consider $p : \mathbb{R}^n \to T^n$, and choose a point $x \in \mathbb{R}^n$ so that $p(x) = \gamma(a)$. There is then a *unique curve* $\widetilde{\gamma} : [a,b] \to \mathbb{R}^n$ so that $p\widetilde{\gamma} = \gamma$, and $\widetilde{\gamma}(a) = x$ (this follows, since locally, p is a diffeomorphism).
  Similarly, if $v \in T_{\gamma(a)}T$ is given, let
  $$\widetilde{v} = d_x p^{-1}(v)$$
  We claim that
  $$P^\gamma(v) = d_{\widetilde{gamma}(b)}p(P^{\widetilde{\gamma}}(\widetilde{v}))$$
  However, this is clear since a parallel field gets sent to a parallel field by df.
  In this concrete example, this shows that parallel transport along any closed loop is trivial.
- Cones. In the problem session you'll see an example of nontrivial parallel transport along closed paths, using the same trick.
- Spheres will be handled in the problem session, using cones. Also, concretely by picture now.

The uniqueness of the Levi-Civita connection has the following consequence: suppose

$$f : (M,g) \to (N,h)$$

is an isometry. Then

$$df \nabla^M_X Y = \nabla^N_{dfX} dfY.$$

Crucial thing here is: $(df^{-1}Z)h(X,Y)(p) = Zh(X,Y)(f(p))$ (the hardest part is to interpret everything correctly).

Lecture 4 (May 6)

We've seen that parallel transport on the torus is trivial along closed loops. For future reference, here we will describe this again, in more detail, and for general quotients. Our setup is as follows.
Let (M, g) be a Riemannian metric, and let G be a group acting on M properly, freely, and by isometries. Denote by $N = M/G$, by $q : M \to N$ the quotient map, and by h the quotient metric. Suppose that $\widetilde{\gamma} : [a, b] \to M$ is a path, and denote by $\gamma = q \circ \widetilde{\gamma}$. As an aside: actually any path in N is of this form. The following is a standard result from covering space theory (which you could easily prove by hand in our setup!)

Lemma 1.11. *Suppose that $\gamma : [a, b] \to N$ is a (piecewise) smooth path, and let $\widetilde{x} \in q^{-1}(\gamma(a))$. Then there is a unique (piecewise) smooth path $\widetilde{\gamma} : [a, b] \to M$ so that $\widetilde{\gamma}(a) = x$ and $\gamma = q \circ \widetilde{\gamma}$.*

Now, we claim that

$$P^{\gamma}(d_{\widetilde{\gamma}(a)}q(v)) = d_{\widetilde{\gamma}(b)}q(P^{\widetilde{\gamma}}(v)).$$

To prove this, suppose that $\widetilde{X}$ is a parallel vector field along $\widetilde{\gamma}$ with $\widetilde{X}(a) = v$. Define a vector field along X by

$$X(t) = d_{\widetilde{\gamma}(t)}q(\widetilde{X}(t)).$$

We then have, using the fact that q is a local isometry:

$$\frac{\nabla}{dt}X = \nabla_{\gamma'}X = dq\nabla_{q^*\gamma'}q^*X = dq\nabla_{\widetilde{\gamma}'}\widetilde{X} = dq\frac{\nabla}{dt}\widetilde{X} = 0.$$

Hence, X computes parallel transport, and we have

$$P^{\gamma}(dq(v)) = X(b) = dq(\widetilde{X}(b)) = dq(P^{\widetilde{\gamma}}(v)).$$

Now, we consider the case where γ is a closed loop (i.e. $\gamma(a) = \gamma(b)$). Note that in general the path $\widetilde{\gamma}$ does not need to be closed then – this is already visible in the torus example from last time.
However, since $\gamma(a) = \gamma(b)$, there is a group element $g \in G$ so that $\widetilde{\gamma}(b) = g\widetilde{\gamma}(a)$. Let U be an open neighbourhood of $\widetilde{\gamma}(a) = x$ so that $g'U \cap U = \emptyset$ for all $g' \neq 1$. Then we have

$$q|_U = q|_{gU} \circ g.$$

and thus

$$d_xq = d_{gx}q \circ d_xg$$

Using what we proved above, we see that

$$P^{\gamma}(d_xq(v)) = d_{gx}q(P^{\widetilde{\gamma}}(v)) = d_xq \circ d_xg^{-1} \circ P^{\widetilde{\gamma}}(v).$$

From this expression we can see that the parallel transport on N has two independent ingredients: the parallel transport of M, and the group action

G. As a result, even if the parallel transport on M is very easy (e.g. trivial, as for $\mathbb{R}^n$), the parallel transport on N might be more complicated. You will explore this further in the problem sessions.

1.2. **Length metric from a Riemannian metric.**

Review 1.12.
- *Given a Riemannian metric, we have* length*:*
$$l(\gamma) = \int_0^1 \sqrt{g(\gamma'(t), \gamma'(t))}dt.$$
- *Given length, we have a* path-metric*:*
$$d(p, q) = \inf\{l(\gamma), \gamma \text{ joins } p, q\}$$

Lemma 1.13. *The path-metric d is a metric, and it determines the same topology as the manifold topology.*

Proof. For metric, the only nontrivial claim is that the distance between any two distinct point is not zero. To see this, let p, q be any distinct points, let $\varphi : U \to V$ be a chart around p and let $B \subset V$ be a closed ball so that $q \notin \varphi^{-1}(B)$. Denote by r the radius of B.
Then, there are constants c, C so that
$$cg_u(v, v) \leq \|d\varphi_u(v)\| \leq Cg_u(v, v).$$
This follows since any two metrics on $\mathbb{R}^n$ are equivalent, g is smooth, and B is compact.
Now, if γ is any path joining p to q, let t_0 denote the largest number so that $\gamma([0, t_0]) \subset \varphi^{-1}(B)$. Then
$$l(\gamma) \geq l(\gamma|_{[0,t_0]} \geq \frac{1}{C} r$$
Hence, $d(p, q) > 0$.
In the same way we can show that every metric ball contains an open set (for the manifold topology), and every open set contains a metric ball. □

We can think of d as a function
$$d : M \times M \to \mathbb{R},$$
but this function is not smooth. The square of d is, however.

In our examples, we could compute lengths now, but we defer a detailed discussion until we have more tools.

1.3. **Geodesics, Local Analytic Properties.** We now start to study one of the two central tools in Riemannian geometry: geodesics.

Definition 1.14. A curve γ is called a *geodesic* if
$$\frac{\nabla}{dt}\gamma' = 0$$
along γ.

Motivation is: velocity vector is parallel along the curve, i.e. direction does not change.

- In Euclidean space, straight lines are geodesics.
- On spheres, great circles are geodesics.
- On the tori T^n, the images of straight lines are geodesics.

Geodesics are automatically parametrised proportional to arclength:

$$\frac{d}{dt}\|\gamma'\|^2 = 2g\left(\frac{\nabla}{dt}\gamma', \gamma'\right) = 0,$$

hence there is a number c so that $l(\gamma|_{[0,t]}) = ct$ for all t.
Rescaling the speed of parametrisation does not change being a geodesic, so we will often assume that they are parametrised by arclength ($c = 1$ above).
We next want to study a) if and how many geodesics always exist, and b) what properties they have. For a), it is useful to study the geodesic equation in local coordinates on the tangent bundle. Recall that TM has local coordinates coming from the charts of M: if $\varphi : U \to V$ is a chart of M, then

$$U \times \mathbb{R}^n, \quad (u^1, \dots, u^n, v^1, \dots, v^n) \mapsto \sum v^i \left.\frac{\partial}{\partial x^i}\right|_{(u^1,\dots,u^n)}$$

gives a chart of TM.
Let now γ be a curve in M. Write

$$\gamma(t) = \varphi(c^1(t), \dots, c^n(t)),$$

and then

$$\gamma'(t) = \sum \dot{c}^i(t) \left.\frac{\partial}{\partial x^i}\right|_{c(t)}.$$

We have

$$\frac{\nabla}{dt}\gamma' = \sum \ddot{c}^i(t) \left.\frac{\partial}{\partial x^i}\right|_{c(t)} + \dot{c}^i(t)\frac{\nabla}{dt} \left.\frac{\partial}{\partial x^i}\right|_{c(t)}.$$

We have

$$\frac{\nabla}{dt} \left.\frac{\partial}{\partial x^i}\right|_{c(t)} = \nabla_{\gamma'(t)} \left.\frac{\partial}{\partial x^i}\right|_{c(t)} = \sum_k \dot{c}^k(t)\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}.$$

Recall the definition of the *Christoffel symbols*:

$$\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i} = \sum \Gamma_{ki}^j \frac{\partial}{\partial x^j}.$$

Collecting terms, we see that the geodesic equation on U is

$$\ddot{c}^j(t) + \sum \dot{c}^k(t)\dot{c}^i(t)\Gamma_{ki}^j = 0$$

Rewriting in the coordinates of TM this is equivalent to the first-order system

$$\dot{x}^j(t) = y^j(t), \quad \dot{y}^j(t) = -\sum y^k(t)y^i(t)\Gamma_{ki}^j$$

on $TU = U \times \mathbb{R}^n$. The right hand sides of these equations define a smooth vector field V on TU, called the *geodesic field*. Curves α with $\alpha'(t) =$

$V(\alpha(t))$ are called *trajectories* of a the vector field V. From the discussion above, the trajectories of the geodesic field are exactly the curves of the form $(\gamma(t), \gamma'(t))$ where γ is geodesic.
Smooth vector fields also define a flow, and the flow on TM defined by the geodesic field is called *geodesic flow*. The study of the geodesic flow is a very interesting problem in dynamics. We will not discuss this much.

Trajectories for vector fields always exist (locally), are unique, and depend smoothly on their initial conditions. We summarise the consequence for the geodesic field in the following theorem.

Lecture 5 (May 9)

The discussion of parallel transports on general quotients we did at the start of this lecture has been added to the write-up of a previous lecture.

Geodesics, Local Analytic Properties

Trajectories for vector fields always exist (locally), are unique, and depend smoothly on their initial conditions. We summarise the consequence for the geodesic field in the following theorem.

Theorem 1.15. *Let (M, g) be a Riemannian manifold, and let p be any point in M. Then there is an open neighbourhood U of $(p, 0) \in TM$, a number $\delta > 0$, and a smooth map*

$$\psi : (-\delta, \delta) \times U,$$

so that $t \mapsto \psi(t, q, v)$ is the unique trajectory of the geodesic field with initial condition $\psi(0, q, v) = (q, v)$.

Suppose now that $\gamma : (-\delta, \delta) \to M$ is a geodesic with $\gamma(0) = p, \gamma'(0) = v$. Consider the curve

$$\rho(t) = \gamma(at), \quad \rho : (-\delta/a, \delta/a) \to M.$$

This is again a geodesic, and $\rho(0) = p, \rho'(0) = av$. By the uniqueness of trajectories this also implies

$$\psi(1, q, av) = \psi(a, q, v)$$

where defined.
Hence, in Theorem 1.15 we may assume $\delta = 2$ at the cost of decreasing the size of U. Assuming this, $\psi(1, q, v)$ is defined for all $(q, v) \in U$. Denote by $p : TM \to M$ the standard projection. We define the *exponential map*

$$\exp : U \to M, \quad \exp(q, v) = \psi(1, q, v)$$

If q is any point in M, we will often also write $\exp_q : B_\delta(0) \to M, \exp_q(v) = \exp(q, v)$. From above, exp is smooth.
Geometrically, $\exp_q(v)$ is the point at time 1 of a geodesic starting in q with initial velocity v.

Lemma 1.16. *For any q, we have*

$$d_0 \exp_q = \mathrm{id}.$$

In particular, there is a small neighbourhood V of 0, so that $\exp_q : V \to \exp_q(V)$ is a diffeomorphism.

Proof. We have

$$d_0 \exp_q(v) = \frac{d}{dt}\bigg|_{t=0} \exp_q(tv) = \frac{d}{dt}\bigg|_{t=0} \psi(1, q, tv) = \frac{d}{dt}\bigg|_{t=0} \psi(t, q, v) = v.$$

This shows the first claim. The second claim follows from the first by the inverse function theorem. $\square$

Next, consider the map

$$f : TU \to M \times M, \quad (q, v) \mapsto (p(q), \exp_q(v))$$

and compute its derivative at a point $(q, 0)$. Keeping q fixed and changing v this is the computation from above, and the derivative is $(0, \mathrm{id})$. Since

$$f(q', 0) = (q', q')$$

the derivative with respect to the the first coordinate is $(\mathrm{id}, \mathrm{id})$. Hence, f also has invertible derivative at $(q, 0)$ and is a local diffeomorphism. This implies

Corollary 1.17. *For any point $p \in M$ there is a neighbourhood U so that between any two points $q, q' \in U$ there is a unique geodesic contained in U. In particular, U is in the image of the exponential map $\exp_q$ for any $q \in U$.*

Proof. By the computation above, we can find a neighbourhood V of $(p, 0)$ so that f restricts to a local diffeomorphism on it. Since geodesics are exactly the images of the exponential map, the claim follows. $\square$

2. Geodesics, First Geometric Properties

Maybe the most useful basic tool about geodesics is the following.

Lemma 2.1 (Gauss lemma)**.** *Let $p \in M$ be a point, and let $v \in T_pM$. For any $w \in T_vT_pM = T_pM$ we have*

$$g(d_v \exp_p(v), d_v \exp_p(w)) = g(v, w)$$

Before we prove this, one word of warning: it is important that the v in the argument of $d\exp$ and the v we started with are the same!

Lecture 6 (May 13)

We want to prove the Gauss lemma.

Lemma 2.2 (Gauss lemma)**.** *Let $p \in M$ be a point, and let $v \in T_pM$. For any $w \in T_vT_pM = T_pM$ we have*

$$g(d_v \exp_p(v), d_v \exp_p(w)) = g(v, w)$$

Recall from last semester the following symmetry for covariant derivatives. Let $f : (a, b) \times (c, d) \to M$ be a smooth map. Then the partial derivative $\frac{\partial f}{du}$ is a vector field along the curve $v \mapsto f(u, v)$. Similarly, $\frac{\partial f}{dv}$ is a vector field along $u \mapsto f(u, v)$. We then have

$$\frac{\nabla}{dv}\frac{\partial f}{\partial u} = \frac{\nabla}{du}\frac{\partial f}{\partial v}$$

Proof of the Gauss lemma. First consider the case where $w = v$. Then by definition of the exponential map $d_v \exp_p(v) = v$, and so the claim follows. Next consider the case where $w \perp v$ (this will prove the lemma by linearity). Choose a curve $c : (-\epsilon, \epsilon) \to T_v M$ with $c'(0) = w$, and $\|c(s)\|$ constant. Up to replacing v by δv for a small δ we may assume that $f(s, t) = \exp_p(tc(s))$ is defined for all $t \in (-2, 2), s \in (-\epsilon, \epsilon)$.
Compute

$$\frac{d}{dt} g(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}) = g(\frac{\nabla}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}) + g(\frac{\partial f}{\partial s}, \frac{\nabla}{dt}\frac{\partial f}{\partial t}).$$

By definition, $t \mapsto f(s, t)$ are geodesics, and therefore $\frac{\nabla}{dt}\frac{\partial f}{dt} = 0$. Also using the symmetry result stated before the proof began, we can continue

$$= g(\frac{\nabla}{ds}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}) = \frac{1}{2}\frac{d}{ds} g(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}).$$

Since $t \mapsto f(s, t)$ are geodesics and are therefore parametrised with constant speed, we have $g(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}) = g(c(s), c(s))$. Since c has constant norm, we thus get that $\frac{d}{dt} g(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}) = 0$.
Thus, $g(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t})$ is constant in t. But

$$\frac{\partial f}{\partial s} = d\exp(tw), \quad \frac{\partial f}{\partial t} = d\exp(c(t)).$$

Evaluating at $t = 1, s = 0$ and $t = 0, s = 0$ yields the result. $\square$

Now, as a (first) application of the Gauss lemma we prove local length minimisation for geodesics. The following terminology is (sometimes) useful.

- A *normal neighbourhood* of a point $p \in M$ is a neighbourhood $V = \exp_p U$ where $\exp_p$ is a diffeomorphism on U. Every point has one of these. Subsets containing p are again normal neighbourhoods.
- A *totally normal neighbourhood* is a set U which is a normal neighbourhood of all of its points. Every point has one of these. Subsets are again totally normal neighbourhoods.
- A *geodesic (or normal) ball* is such a neighbourhood where U is a (Euclidean) ball. Every point has one of these.
- A *geodesic (or normal) sphere* is the image of $rS^n \subset T_pM$ under $\exp_p$, if $\exp_p$ is a diffeomorphism on $U \supset rS^n$.
- A *radial geodesic* in a geodesic ball is the image under $\exp_p$ of a line segment starting in the origin. The Gauss lemma states that radial geodesics are orthogonal to geodesic spheres in any geodesic ball.

Lemma 2.3. *Suppose U is a normal neighbourhood of p, and $B \subset U$ a geodesic ball with center $p \in U$. Let (a small enough) $v \in T_pM$ be given and consider the radial geodesic $\gamma(t) = \exp_p(tv)$. If $c : [0,1] \to M$ is any piecewise smooth path with $c(0) = p, c(1) = \gamma(1)$, then*

$$l(\gamma) \leq l(c)$$

with equality if and only if $c([0,1]) = \gamma([0,1])$.

Proof. Let $B = \exp_p B_\epsilon(0)$. First assume that $\mathrm{im}(c) \subset \exp_p \overline{B}_\epsilon(0)$. We can then write

$$c(t) = \exp_p(r(t)v(t))$$

where $v : [0,1] \to T_pM$ has constant norm 1, and $r : [0,1] \to \mathbb{R}$. Define as before

$$f(s,t) = \exp_p(sv(t)).$$

and then have $c(t) = f(r(t),t)$, and therefore

$$c'(t) = \frac{\partial f}{\partial s} r'(t) + \frac{\partial f}{\partial t}$$

Since $\|v(t)\| = 1$, $v'(t) \perp v(t)$, and thus by the Gauss lemma we have

$$g(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}) = 0.$$

Thus, using that $\|\frac{\partial f}{\partial s}\|_g = 1$, we have

$$g(c'(t), c'(t)) = (r'(t))^2 + \|\frac{\partial f}{\partial t}\|_g^2$$

Hence, we get

$$\int_0^1 \sqrt{g(c'(t), c'(t))} dt \geq \int_0^1 r'(t) dt = r(1) = l(\gamma).$$

Also, if $v(t)$ is not constant, the inequality above is strict, and so γ is the only curve realising length.
Finally, if c is a curve which is not contained in $\exp_p B$, then arguing as above it follows that $l(c) > l(\gamma)$. $\square$

So: geodesics *locally minimise distance.* A converse is the following:

Lemma 2.4. *Suppose that $c : [0,1] \to M$ is a length minimiser (i.e. $l(c) = d(c(0), c(1))$). Then c is a geodesic, in particular smooth.*

Proof. First observe that c is actually length-minimising between any two of its points (otherwise there would be a global shortcut as well). Now parametrise c by arclength. Given any point $c(t_0)$, find ϵ, δ, so that at each $c(t), |t - t_0| < \epsilon$, the exponential map is a diffeomorphism on the δ–ball. Applying Lemma 2.3 we see that $c|_{[t,t+\delta]}$ is geodesic for all $|t - t_0| < \delta$. In particular, around t_0 there is an open interval on which c is geodesic. $\square$

Finally, we want to begin to discuss convexity. The technical tool is the following

Lemma 2.5. *Let $p \in M$ be given. Then there is a number $\epsilon > 0$ so that $B = \exp_p B_\epsilon(0)$ is a geodesic ball with the following property. Suppose $r < \epsilon$, and γ is a geodesic which is tangent to the geodesic sphere $S = \exp_p \partial B_r(0)$ at $q = \gamma(t)$. Then $\gamma(t-\delta, t+\delta) \cap \exp_p B_r(0) = \emptyset$ for small δ.*

A tool: the unit tangent bundle T^1M. Observe that this has compact fibers.

Proof. Let U be a totally normal neighbourhood of p. We may assume that the map

$$\gamma : T^1U \times (-\mu, \mu) \to M,$$

where $\gamma(q, v, t)$ is the time-t point on the geodesic starting in (q, v), is defined and smooth, and has image completely in U. Hence, we can define

$$u(q, v, t) = \exp_p^{-1}(\gamma(q, v, t)),$$

and

$$F(u, v, t) = \|u(q, v, t)\|^2 = d(p, \gamma(q, v, t))^2.$$

We have

$$\frac{\partial}{\partial t} F(u, v, t) = 2\langle \frac{\partial u}{\partial t}, u\rangle$$

$$\frac{\partial^2}{\partial t \partial t} F(u, v, t) = 2\langle \frac{\partial^2 u}{\partial t \partial t}, u\rangle + 2\|\frac{\partial u}{\partial t}\|^2$$

Choose r so that $\exp_p B_r(0)$ is a geodesic ball. Suppose that γ is as in the lemma, and $q = \gamma(q, v, 0)$. The Gauss lemma then states that

$$\langle u(q, v, 0), \frac{\partial u}{\partial t}(q, v, 0)\rangle = 0$$

Hence, $(q, v, 0)$ is a critial point of F. We want to show that (if r is small enough), it is a strict minimum.
At $q = p$, we have $\frac{\partial^2 F}{\partial t^2}(p, v, 0) = 2\|v\|^2 = 2$, and so by smoothness there is a neighbourhood V of p so that $\frac{\partial^2 F}{\partial t^2}(q, v, 0) > 0$ for all $(q, v) \in T^1V$.
Choosing ϵ small enough that $\exp_p B_\epsilon(0) \subset V$ then has the desired property. □

Lecture 7 (May 16)

More terminology:

- A set $S \subset M$ is called *strongly convex*, if for any two points $p, q \in \overline{S}$ there is a unique shortest geodesic joining p to q, and its interior is contained in S. Think of: Euclidean balls.

Lemma 2.6. *For any $p \in M$ there is an $\delta > 0$ so that the geodesic ball $B = \exp_p B_\delta(0)$ is strongly convex.*

Proof. Let ϵ be so that it satisfies as in the previous lemma, and put $\delta = \epsilon/2$. Then, suppose q_1, q_2 are two points on $\exp_p \partial B_\delta$, and let γ be a shortest geodesic between them. It is unique and is completely contained in $\exp_p B_\epsilon(0)$. Suppose that the interior γ would not be contained in $\exp_p B_\delta(0)$. Then there would be a time t_0 so that $\gamma(t_0)$ is tangent to the geodesic sphere $\exp_p \partial B_r(0)$, $r > \epsilon$, and $\gamma \subset \exp_p B_r(0)$. This violates the previous lemma. $\square$

Hyperbolic Geodesics and Isometries (I). As an application, we now compute the geodesics of $\mathbb{H}^2$ (in a tricky computation-avoiding way). Here, it is useful to think of $\mathbb{H}^2 \subset \mathbb{C}$ as the complex numbers with positive imaginary part, and use complex notation.
The first step is to find a single geodesic.

Lemma 2.7. *The geodesic joining i to $\lambda i, \lambda > 1$ in $\mathbb{H}^2$ is the vertical (Euclidean) straight line.*

Proof. Let $\alpha : [0,1] \to \mathbb{H}^2, \alpha(t) = (x(t), y(t))$ be any path joining i to λi. We then have

$$l(\alpha) = \int_0^1 \frac{1}{x(t)} \sqrt{\dot{x}^2(t) + \dot{y}^2(t)} dt \geq \int_0^1 \frac{\dot{x}(t)}{x(t)} dt = \ln(\lambda).$$

Furthermore, the inequality is strict if $\dot{y}(t) \neq 0$ for any $t \in (0,1)$. Since the straight line has hyperbolic length $\ln(\lambda)$, it is therefore the length minimiser, and thus a geodesic. $\square$

Next, we use the fact that we can guess isometries.

Lemma 2.8. *Suppose $a, b, c, d \in \mathbb{R}$ satisfy $ad - bc = 1$. Then the map*

$$f : z \mapsto \frac{az + b}{cz + d}$$

defines an isometry of $\mathbb{H}^2$.

Proof. First we check that f preserves the upper half plane. Namely,

$$\frac{az + b}{cz + d} = \frac{(az + b)(c\overline{z} + d)}{\|cz + d\|^2} = \frac{acz\overline{z} + bc\overline{z} + adz + bd}{\|cz + d\|^2}$$

Hence, this has imaginary part

$$\frac{-bc\text{Imz} + ad\text{Imz}}{\|cz + d\|^2} > 0.$$

Next, we compute the (complex) derivative.

$$f'(z) = \frac{a(cz + d) - (az + b)c}{(cz + d)^2} = \frac{acz + ad - acz - bc}{(cz + d)^2} = \frac{1}{(cz + d)^2}$$

Hence, the (real) total derivative is a rotation, followed by a scaling by $\frac{1}{(cz+d)^2}$. As this is exactly how the imaginary part of $f(z)$ is scaled compared to z, f is an isometry. $\square$

In other words, we get a map

$$\rho : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{Isom}(\mathbb{H}^2).$$

We have the following

Lemma 2.9. *The image of ρ acts transitively on $T^1\mathbb{H}^2$, the unit tangent bundle.*

Proof. Using isometries of the form $z \mapsto \lambda z, t \mapsto z + r$ for $\lambda, r \in \mathbb{R}$ it is easy to see that it acts transitively on $\mathbb{H}^2$. Hence, it suffices to show transitivity on unit tangent vectors at a single point. We do this using

$$f_\theta(z) = \frac{\cos(\theta)z + \sin(\theta)}{-\sin(\theta)z + \cos(\theta)}$$

and observe that it fixes i. Further, its derivative at i is

$$f'_\theta(i) = \frac{1}{(-\sin(\theta)i + \cos(\theta))^2}$$

so it acts as a rotation by $\pm 2\theta$. $\square$

The core to identify geodesics is the following result in "classical" (Möbius) geometry.

Lemma 2.10. *Isometries $\rho(f)$ preserve the set of vertical half-lines and half-circles orthogonal to the real line.*

Proof. First we show that circles and lines are sent to circles or lines. Recall that $\mathrm{SL}_2(\mathbb{R})$ is generated by matrices of the form

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

For the first two types the claim is clear, since the corresponding isometries are Euclidean translations or similarities. For the last type, which corresponds to the involution

$$f : z \mapsto -\frac{1}{z} = -\frac{\overline{z}}{\|z\|^2},$$

we need a different argument. Note that a circle or line is exactly the set of points satisfying

$$\epsilon\langle z, z\rangle - 2\langle x, a\rangle + t = 0,$$

for some $\epsilon, t \in \mathbb{R}, a \in \mathbb{C}$, where $\langle , \rangle$ is the usual scalar product on $\mathbb{C} = \mathbb{R}^2$. The case of lines is when $\epsilon = 0$. Now, the image of a circle or line under f is therefore the set of points satisfying

$$\epsilon\langle -\frac{\overline{z}}{\|z\|^2}, -\frac{\overline{z}}{\|z\|^2}\rangle - 2\langle -\frac{\overline{z}}{\|z\|^2}, a\rangle + t = 0,$$

The scalar product is real linear, and has $\langle \overline{z}, a\rangle = \langle z, \overline{a}\rangle$, and thus this equation is equivalent to

$$\epsilon\langle \frac{z}{\|z\|^2}, z\rangle - 2\langle z, -\overline{a}\rangle + t\|z\|^2 = 0,$$

But

$$\langle \frac{z}{\|z\|^2}, z\rangle = 1,$$

so this has the form again.

It remains to show that $\rho(f)$ preserves the angle with the real axis. Now, circles or lines orthogonal to $\mathbb{R}$ are exactly those circles or lines invariant under complex conjugation. Since $\rho(f)$ is a linear fractional map with real coefficients, it preserves this property. $\square$

As a consequence, we get

Corollary 2.11. *The geodesics of $\mathbb{H}^2$ are exactly the vertical half-lines, and the Euclidean half-circles meeting $\mathbb{R}$ orthogonally.*

Proof. A geodesic is determined by its initial point and its initial velocity. Since ρ acts transitively on $T^1\mathbb{H}^2$, the geodesics are therefore exactly the images of the imaginary half-line under elements in the image of ρ. By the lemma these are of the desired form. $\square$

Corollary 2.12. *Between any two points of $\mathbb{H}^2$ there is a unique geodesic, and it is globally length-minimising.*

Corollary 2.13. *The image of ρ is the full group of orientation-preserving isometries. In particular, isometries of $\mathbb{H}^2$ act in a well-defined way on $\partial_\infty\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$.*

Proof. Suppose F is any orientation-preserving isometry. Then, there is some A so that $\rho(A)F$ fixes i and acts trivially on $T_i\mathbb{H}^2$. Further, $\rho(A)F$ is an isometry, and so it preserves geodesics parametrised by arclength; hence it is the identity on $\mathbb{H}^2$. $\square$

Lecture 8 (May 20)

Next, we want to understand how the isometries of $\mathbb{H}^2$ act on $\mathbb{H}^2$. To do so, we recall the following from linear algebra.

Lemma 2.14. *Every nonidentity matrix in $\mathrm{SL}_2(\mathbb{R})$ is conjugate to exactly one matrix of the following types*

$$\begin{pmatrix} \lambda & 0 \\ 0\lambda^{-1} & \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

where $\lambda \neq 0, \theta \in [0, 2\pi)$.

Proof. Consider a real matrix A with determinant 1. If A has an eigenvalue not equal to 1, then it is diagonalisable, hence conjugate to the first type. If A has an eigenvalue 1, but not the identity, it is conjugate to the second type. If A has no real eigenvalue, it has a pair of complex conjugate eigenvalues of norm 1, and thus we are in the third case. $\square$

We call the isometries corresponding to the different types *hyperbolic*, *parabolic* and *elliptic*. The special form of the isometries immediately yields

Lemma 2.15. *Suppose $\phi \in \mathrm{Isom}^+(\mathbb{H}^2)$.*

i) If f is elliptic, it has a fixed point in $\mathbb{H}^2$.

ii) If f is hyperbolic, it fixes no point in $\mathbb{H}^2$, and leaves a single geodesic in $\mathbb{H}^2$ (setwise) invariant.
iii) If f is parabolic, it fixes no point in $\mathbb{H}^2$ and no geodesic in $\mathbb{H}^2$.

The geodesic left invariant by a hyperbolic element is called its *axis*. Since $\mathrm{Isom}^+(\mathbb{H}^2)$ acts transitively on $T^1\mathbb{H}^2$, it also acts transitively on the set of geodesics. Hence, every geodesic is the axis of some hyperbolic element. Namely,

$$A(\psi\phi\psi^{-1}) = \psi A(\phi)$$

Our goal is now to construct an interesting manifold as the quotient of $\mathbb{H}^2$. To do so, we need to understand how hyperbolic isometries act a bit better. We need the following simple observations

- If $A \subset \mathbb{H}^2$ is an infinite geodesic, then $\mathbb{H}^2 - A$ has two connected components, each of which we call a *halfspace*. They are convex.
- If A is a geodesic, $a \in A$ a point, then there is a unique geodesic B so that $A \cap B = \{a\}$, and A, B meet orthogonally.

Lemma 2.16 (North-South-dynamics)**.** *Let ϕ be a hyperbolic isometry of $\mathbb{H}^2$ with axis A. Let $a \in A$ be a point, and let B be the unique geodesic meeting A in a orthogonally. Let U_- be the halfspace defined by A which does not contain $\phi(A)$, and let U_+ be the halfspace defined by $\phi(A)$ which does not contain A. Then*

$$\phi(\mathbb{H}^2 - U_-) \subset U_+, \phi^{-1}(\mathbb{H}^2 - U_+) \subset U_-.$$

Proof. By conjugating ϕ, it suffices to show this for $\phi(z) = \lambda z$. Then $A, \phi(A)$ are half-circles meeting the imaginary axis orthogonally at height $a, \lambda a$. Then

$$U_- = \{z \in \mathbb{H}^2, \|z\| < a\}, \quad U_+ = \{z \in \mathbb{H}^2, \|z\| > \lambda a\}.$$

Then, the claim is clear. ☐

Now, to construct our manifold, let A, B be two geodesics meeting orthogonally, and let α, β be hyperbolic isometries with these axes. By replacing α, β by large enough powers, we may assume that the sets $U^\alpha_-, U^\alpha_+, U^\beta_-, U^\beta_+$ are all disjoint. In fact, we may assume that the 1–neighbourhoods are still disjoint.
Now consider the group $G = \langle \alpha, \beta \rangle$ generated by these two isometries.

Lemma 2.17. *For any $p \in \mathbb{H}^2$, there is some $g \in G$ so that*

$$gp \in \mathbb{H}^2 - (U^\alpha_- \cup U^\alpha_+ \cup U^\beta_- \cup U^\beta_+)$$

Proof. Denote by C the region $\mathbb{H}^2 - (U^\alpha_- \cup U^\alpha_+ \cup U^\beta_- \cup U^\beta_+)$, and let p by any point in $\mathbb{H}^2$. Suppose $p \notin C$. Choose a geodesic segment γ joining p to a point in ∂C, without intersecting C in the interior. Suppose e.g. that γ ends on ∂U^α_-. Consider $\alpha(p)$, and the path $\alpha(\gamma)$. The geodesic $\alpha(\gamma)$ now ends on ∂U^α_+, and a terminal segment of $\alpha(\gamma)$ is now contained inside C. So, either

$\alpha(p) \in C$, or we can take a subsegment of $\alpha(\gamma)$ of length $\leq l(\gamma) - 1$ joining $\alpha(p)$ to ∂C. By induction, we are done. $\square$

Lemma 2.18. *The group $G = \langle \alpha, \beta \rangle$ acts on $\mathbb{H}^2$ properly and freely.*

Proof. It suffices to show that any point $p \in \mathbb{H}^2$, there is an open neighbourhood so that

$$gU \cap U = \emptyset \forall g \neq 1.$$

By the previous lemma, it also suffices to show this for a point in C. In fact, by slightly moving the orthogonal, we may assume $p \in \operatorname{int}(C)$. We can choose a neighbourhood $V \subset C$ of p.
Now, consider any nontrivial element $g \in G$. After obvious cancellations, we may assume that it has the form

$$g = g_n \cdots g_1,$$

where each $g_i = \alpha^{\pm}, \beta^{\pm}$, and no two consecutive ones are direct inverses of each other. For each i, put $U_i = U_{\pm}^{\alpha}$ if $g_i = \alpha^{\pm}$ and $U_i = U_{\pm}^{\beta}$ if $g_i = \beta^{\pm}$. We have inductively

$$g_1 \cdots g_k(V) \subset U_i.$$

Namely, $g_1(V) \subset U_1$ as $V \subset C$. If $g_k \cdots g_1(V) \subset U_k$, assume e.g. that $U_k = U_{+}^{\alpha}$ (i.e. $g_k = \alpha$). Then, we have $g_{k+1} \neq \alpha^{-1}$, and therefore $g_{k+1}U_k \subset U_{k+1}$. Hence, $g_n \cdots g_1(V) \subset U_n$, and $U_n \cap C = \emptyset$, which proves the lemma. $\square$

What is the quotient $\mathbb{H}^2/G$? Topologically, it is a torus minus a disk. Try to prove this.

Lecture 9 (May 23)

Global geodesics. We have seen that geodesics always exist locally, and realise length locally. Next, we want to discuss global existence results.

Theorem 2.19 (Hopf-Rinow)**.** *Let M be a Riemannian manifold, and let $p \in M$ be a point. Then the following are equivalent:*

i) $\exp_p$ *is defined on all of* T_pM*.*
ii) Closed and bounded sets in M are compact.
iii) (M, d) is complete as a metric space.
iv) Any geodesic is defined on all of $\mathbb{R}$.

If any (hence all) of these hold, there are length realising geodesics between any two points in M.

Proof. The first step is that i) implies the existence of length realisers. Suppose $d(p,q) = r$, and let $B = \exp_p B_\delta(0)$ be a geodesic ball around p. Denote by S the boundary of B. Let x_0 be a point on S where $d(q, \cdot) : S \to \mathbb{R}$ attains a minimum. Let v be a unit vector such that $x_0 = \exp_p \delta v$, and let $\gamma(t) = \exp_p(tv)$. We want to prove that

$$d(\gamma(t), q) = r - t$$

This equation is true for $t = 0$. Let A be the set of all those times where it holds. Observe that $[0, s] \subset A$ if $s \in A$. Namely, suppose $t < s$. Then

$$d(\gamma(t), q) \leq d(\gamma(s), \gamma(t)) + d(\gamma(s), q) \leq s - t + r - s = r - t.$$

$$d(\gamma(t), q) \geq d(\gamma(s), q) - d(\gamma(s), \gamma(t)) \geq r - s - (t - s) = r - t.$$

Further, we have

$$r = d(p, q) = \delta + d(S, q) = \delta + d(x_0, q)$$

and thus $[0, \delta] \subset A$.
Also, as the condition defining it is closed, so is the set A. Hence, we only need to show that A is open in order to to show that $\gamma(r) = q$ and γ is length minimising.
To this end, let $s_0 < r$ be a time in A. Take $B_\epsilon(\gamma(s_0)$ be a geodesic ball around $\gamma(s_0)$, and let S be its boundary. Let x be a point on S minimising $d(q, \cdot)$.
This point satisfies

$$d(\gamma(s_0), q) = \epsilon + d(x, q)$$

We claim that $x = \gamma(s_0 + \epsilon)$. Namely, let ρ be a radial geodesic in $B_\epsilon(\gamma(s_0))$ ending in x. We have that

$$d(p, x) \geq d(p, q) - d(q, x) = r - (d(\gamma(s_0), q) - \epsilon) = s_0 + \epsilon$$

and $\gamma|_{[0,s_0]} * \rho$ is a path of that length. Hence, it is length minimising, therefore smooth. This implies that ρ has the same direction as γ at the point where they join, and therefore $\gamma(s_0 + \epsilon) = \rho(\epsilon)$ as claimed. Then also

$$d(\gamma(s_0 + \epsilon), q) = d(x, q) = d(\gamma(s_0), q) - \epsilon = r - s_0 - \epsilon,$$

and thus $s_0 + \epsilon \in A$.

Now assume i). Take K a closed and bounded set. By the global existence of geodesics (which we have proved), $K \subset \exp_p(B_r(0))$. Hence, K is a closed subset of the compact set $\exp_p \overline{B}_r(0)$, hence compact. Thus ii) holds.

Assume ii). Cauchy sequences are bounded, hence have bounded closure. Any point in the closure is an accumulation point of the sequence, hence a limit by Cauchy. Thus iii) holds.

Assume iii). Suppose $\gamma : [0, s) \to M$ is a geodesic which cannot be extended to s. Choose $s_i \to s$ from below. Then $\gamma(s_i)$ is a Cauchy sequence, and hence has a limit p. Choose a totally normal neighbourhood U of p. For large indices, $\gamma(s_i), \gamma(s_j)$ are contained in U, and thus $\gamma|_{[s_i,s_j]}$ is the unique geodesic in U joining these points. Since $\exp_p$ is a diffeomorphism around 0, $\gamma \cap \exp_p B$ is a radial geodesic, and therefore it extends.

Finally, clearly iii) implies i). $\square$

Curvature

First, we need to recall (correct) the analytic definition of curvature from last semester. Namely, we have

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

This is a tensor $R \in TM \otimes (T^*M)^{\otimes 3}$. Often we also use the tensor

$$R(X,Y,Z,W) = g(R(X,Y)Z, W)$$

which is in $(T^*M)^{\otimes 4}$. These have symmetries, which we showed last time.

(1) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$.
(2) $R(X,Y)Z = -R(Y,X)Z$.
(3) $R(X,Y,Z,W) = R(Z,W,X,Y)$.

Recall the definition of *sectional curvature*

$$K(x,y) = \frac{R(x,y,x,y)}{g(x,x)g(y,y) - g(x,y)^2}$$

where $x, y \in T_pM$. We proved that $K(x,y)$ depends only on the vector subspace $V = \mathrm{span}(x,y)$, and we often write it as $K(V)$.
Recall also that sectional curvature determines Riemannian curvature.
Finally, recall that curvature measures failure of symmetry of second covariant derivatives. One useful lemma is the following: if $f : U \to M$ is a smooth map, $U \subset \mathbb{R}^2$, and V is a vector field along f, then

$$\frac{\nabla}{dt}\frac{\nabla}{ds}V - \frac{\nabla}{ds}\frac{\nabla}{dt} = R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) V.$$

(obvious for a "coordinate slice", true in general: do Carmo, Chapter 4, Section 5, Lemma 4.1)
Note that local isometries preserve curvatures, in the sense that they preserve Levi-Civita connections. Sectional curvature is preserved as a number.

Lecture 10 (May 27)

We now want to join the analytic curvature (tensor) to geometric data – geodesics. In order to do this, we will begin by considering families of geodesics.
Namely, consider a family of paths $f(s,t)$, so that for any s the assigment $t \mapsto f(s,t)$ is a geodesic (e.g. in the proof of the Gauss lemma we used $f(t,s) = \exp_p tv(s)$)
The first partial derivative in the t–direction is then not very interesting (in t the curve is geodesic, with constant derivative). The other partial

$$\frac{\partial f}{\partial s} f$$

measures the "spreading" of the geodesics in the family. We will see that this is directly related to curvature.

First, we derive a differential equation which these partials satisfy.

$$J(s) = \left.\frac{\partial f}{\partial s}\right|_{(t,0)},$$

which is a vector field along the geodesic $\gamma(t) = f(t,0)$. Since geodesics have parallel velocity field, we have

$$0 = \frac{\nabla}{ds}\frac{\nabla}{dt}\frac{\partial f}{\partial t} = \frac{\nabla}{dt}\frac{\nabla}{ds}\frac{\partial f}{\partial t} - R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t}$$

With symmetry of R and covariant/usual partials this yields

$$0 = \frac{\nabla}{dt}\frac{\nabla}{dt}\frac{\partial f}{\partial s} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}$$

Hence, the vector field J along γ satisfies the *Jacobi equation*

$$\frac{\nabla^2}{dt^2}J(t) + R(\gamma'(t), J(t))\gamma'(t) = 0.$$

Definition 2.20. Let γ be a geodesic. A vector field J is called a *Jacobi field* if it satisfies the Jacobi equation above.

Lemma 2.21. *Given a geodesic $\gamma : [a,b] \to M$ and $v, w \in T_{\gamma(0)}M$ there is a unique Jacobi field J along γ with*

$$J(0) = v, \quad \frac{\nabla}{dt}J(0) = w.$$

Proof. Choose parallel orthonormal fields $b_i : [a,b] \to TM$ along γ. We can write any vector field along γ uniquely as

$$J(t) = \sum f^i(t) b_i(t)$$

By parallelity, we have

$$\frac{\nabla^2}{dt^2}J(t) = \sum \ddot{f}^i(t) b_i(t).$$

By linearity of R and the fact that the b_i are orthonormal, we have

$$R(\gamma'(t), J(t))\gamma'(t) = \sum_i R(\gamma'(t), J(t), \gamma'(t), b_i(t)) b_i(t) = \sum_{i,j} f^j(t) R(\gamma'(t), b_j(t), \gamma'(t), b_i(t)) b_i(t).$$

Hence, the Jacobi equation is equivalent to the system of equations

$$\ddot{f}^i(t) + \sum_j f^j(t) R(\gamma'(t), b_j(t), \gamma'(t), b_i(t)) = 0.$$

This is a linear system of ODEs. Hence, given initial conditions there exist a unique solution as claimed. □

As a consequence of the uniqueness, we see that our first examples are all the Jacobi fields with initial value 0:

Corollary 2.22. *Let $\gamma(t) = \exp_p tv$ be a geodesic, and let J be a Jacobi field along γ with $J(0) = 0$.*
Let $w = \frac{\nabla}{dt}J(0)$, and let $v(s)$ be a curve with $v(0) = v, v'(0) = w$. Then,

$$J(t) = \frac{\partial}{\partial s} \exp_p tv(s).$$

Proof. We have already shown that $I(t) = \frac{\partial}{\partial s} \exp_p tv(s)$ is a Jacobi field. Hence, by the uniqueness, it suffices to show that it has the correct initial conditions. $I(0) = J(0) = 0$ is clear. Furthermore,

$$\frac{\nabla}{dt}I(t) = \frac{\nabla}{dt}d_{tv} \exp_p(tw) = \frac{\nabla}{dt}td_{tv} \exp_p(w) = (d_{tv} \exp_p(w) + t\frac{\nabla}{dt}d_{tv} \exp_p(w)$$

hence, $\frac{\nabla}{dt}I(0) = w$. ☐

As a consequence, we should remember that *Jacobi fields compute the derivative of* exp. In the setup of the previous corollary, we have

$$J(t) = \frac{\partial}{\partial s} \exp_p tv(s) = d_{tv} \exp_p tw$$

and hence, assuming that the geodesic is defined up to time 1,

$$d_v \exp_p w = J(1).$$

This is sometimes a useful way to compute this derivative (geometrically). This also immediately implies: if we have a totally geodesic, isometric embedding $\phi : M \to N$, then ϕ maps Jacobi fields to Jacobi fields.
Another consequence is: Jacobi fields that start orthogonally to the geodesic direction stay orthogonal to the geodesic direction. This also follows from a direct computation:

$$\langle J, \gamma' \rangle' = \langle J', \gamma' \rangle, \quad \langle J', \gamma' \rangle' = \langle -R(\gamma', J)\gamma', \gamma' \rangle.$$

Some examples

Example 2.23 (Curvature of spheres, geometrically)**.** We start with S^2. Take a geodesic (great circle) γ, and let X be a parallel vector field along γ orthogonal to γ'. Then,

$$R(\gamma'(t), X(t))\gamma'(t) = K(T_{\gamma(t)}S^2)X(t).$$

Observe that $O(3)$ acts transitively and by isometries on S^2, and so $K(T_{\gamma(t)}S^2) = K$ is constant. Jacobi fields along γ starting orthogonally have the form $J(t) = f(t)X(t)$. Further, by parallelity of X we have

$$\frac{\nabla^2}{dt}J = f''(t)X(t)$$

and

$$R(\gamma', J)\gamma' = f(t)KX(t).$$

Hence, J being a Jacobi field exactly means $f''(t) = -Kf(t)$.

Now, let us specify to a specific geodesic and field. Take the geodesic

$$\gamma(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \end{pmatrix}.$$

In this case, the field $X(t) = (0, 0, 1)$ is easy to guess. To get a Jacobi field, we next just guess a family of geodesics

$$f(s,t) = \begin{pmatrix} \cos(t) \\ \sin(t)\cos(s) \\ \sin(t)\sin(s) \end{pmatrix}$$

which satisfies $f(0,t) = \gamma(t)$ and $t \to f(s,t)$ is a geodesic. Hence,

$$J(t) = \frac{\partial f}{\partial s}_{(0,t)} = \begin{pmatrix} 0 \\ 0 \\ \sin(t) \end{pmatrix} = \sin(t)X(t)$$

Hence, $f(t) = sin(t)$, and $K = -1$.
If we consider the sphere S^n with the round metric, then the group $G = \mathrm{O}(n+1)$ acts transitively on two-dimensional subspaces in TS^n, and therefore sectional curvature is again constant. Further, there is an totally geodesic, isometrically embedded S^2, and so the curvature is also constant $+1$.

Example 2.24 (Curvature of hyperbolic space, geometrically)**.** We try something similar. One geodesic is easy

$$\gamma(t) = e^t i.$$

Next, we need a parallel, orthogonal vector field. This is

$$X(t) = e^t$$

Why does this work? Because it is a constant (hyperbolic) length, constantly oriented vector field along γ.
The first part of the discussion of the round sphere above extends verbatim. Hence, Jacobi fields are of the form $J(t) = f(t)X(t)$, and $f''(t) = -Kf(t)$, where K is the (constant) curvature of $\mathbb{H}^2$.
Finally, we need a family of geodesics. We can use the maps f_θ from before

$$\psi(s,t) = \frac{\cos(s)e^t i + \sin(s)}{-\sin(s)e^t i + \cos(s)},$$

and compute

$$J(t) = \frac{\partial \psi}{\partial s}_{(0,t)} = 1 - e^t i(-i)e^t = 1 - e^{2t} = (e^{-t} - e^t)X(t)$$

Thus, $f(t) = e^{-t} - e^t$, and thus $K = -1$.

Lecture 11 (June 3)

For details on this class, compare 3.2 and 3.6 of the script available at `http://people.mpim-bonn.mpg.de/hwbllmnn/archiv/energyfun1501.pdf`

Lecture 12 (June 6)

Last time, we learned about *variation formulas* for length and energy. Namely, we have:

Lemma 2.25. *Let F be a 1–parameter variation of a curve c, piecewise smooth with breakpoints t_i and variation field V. Then we have*

$$E'(0) = -\int_a^b g(c'', V)dt + g(c', V')|_a^b - \sum g(\Delta c'(t_i), V(t_i))$$

Observe for example that this implies: minimal geodesics joining points to submanifolds meet them orthogonally.

Lemma 2.26. *Let F be a 2–parameter variation of a geodesic γ, with variation fields V, W. Then we have*

$$\partial_r\partial_s E(0,0) = \int_a^b g(V', W') - g(R(\gamma', V)\gamma', W)dt + g(c'(t), \frac{\nabla}{dr}\partial_s F(0,t))|_a^b$$

For a 1-parameter variation this is:

$$E''(0) = \int_a^b g(V', V') - g(R(\gamma', V)\gamma', V)dt + g(c'(t), \frac{\nabla}{ds}V)|_a^b$$

Observe three things. One: for proper variations this only depends on V, W. Two:

$$\frac{d}{dt}g(V, V') = g(V, V'') + g(V', V')$$

and thus, for a proper variation, we can also write

$$E''(0) = -\int_a^b g(V'' + R(\gamma', V)\gamma', V)dt$$

Three: the curvature term for a 1–parameter variation is a sectional curvature.

This allows us to use these to great effect if we have curvature bounds. To see how, recall that the *Ricci curvature* is the average

$$\mathrm{Ric}_p(w) = \frac{1}{n-1}\sum_i g(R(x, z_i)x, z_i).$$

where $z_1, \dots, z_n$ is an orthonormal basis of T_pM. Note that if w has norm 1, and we extend w to an orthonormal basis $w, z_2, \dots, z_n$, then

$$\mathrm{Ric}_p(w) = \frac{1}{n-1}\sum_i K(w, z_i)$$

is the avarage of sectional curvatures of planes through w.

Theorem 2.27 (Bonnet-Myers)**.** *Suppose M is a complete Riemannian manifold, and suppose that*

$$\mathrm{Ric}_p(v) \geq \frac{1}{r^2}$$

for some $r > 0$ and all points p, all unit vectors v. Then M is compact, and

$$\mathrm{diam}(M) \leq \pi r$$

Proof. We need to show that any length-minimising geodesic has length $\leq \pi r$. So, suppose this is not the case, and let $\gamma : [0, l] \to M$ be a length-minising geodesic, parametrised by arclength, so that $l > \pi r$. Take $e_1, \ldots, e_{n-1}$ parallel orthonormal vector fields along γ, orthogonal to γ'. Put $e_n = \gamma'$.
Now, define

$$V_j(t) = \sin(\pi t/l)e_j(t)$$

These define proper variations of γ. Compute the derivatives:

$$V_j'(t) = \frac{\pi}{l}\cos(\pi t/l)e_j(t), \quad V_j''(t) = -\frac{\pi^2}{l^2}\sin(\pi t/l)e_j(t)$$

Let E_j be the energy of the variation defined by V_j. We have $E_j'(0) = 0$, and (using the second form of the variation formula)

$$\begin{aligned} E_j''(0) &= -\int_0^l -\frac{\pi^2}{l^2}\sin(\pi t/l)^2 + \sin(\pi t/l)^2 K(e_n, e_j) \\ &= \int_0^l \sin(\pi t/l)^2 \left(\frac{\pi^2}{l^2} - K(e_n, e_j)\right). \end{aligned}$$

If at this stage $K \geq \frac{1}{r^2} > \frac{\pi^2}{l^2}$, then we would know that E can be decreased by the variation V_j, violating the fact that γ is length-minimising. In our case, we just sum over all j to get

$$\sum_j E_j''(0) = \int_0^l \sin(\pi t/l)^2 \left((n-1)\frac{\pi^2}{l^2} - (n-1)\mathrm{Ric}_{\gamma(t)}(e_n(t))\right),$$

to see that some $E_j''(0) < 0$ and we are done as above. □

Corollary 2.28. *Complete manifolds with Ricci or sectional curvature bounded away from 0 from below are compact.*

Corollary 2.29. *Compact manifolds with positive Ricci or sectional curvatures have finite fundamental groups.*

Proof. There is a positive lower bound $\delta > 0$ for Ricci by compactness of M. Consider the universal cover $\widetilde{M}$. It satisfies the same lower curvature bound δ, so by the previous corollary $\widetilde{M}$ is also compact. Since the deck group acts discretely on $\widetilde{M}$ this is only possible if the deck group is finite. □

Theorem 2.30 (Weinstein-Synge)**.** *Let f be an isometry of a compact oriented Riemannian manifold M. Suppose that M has positive sectional curvature. If the dimension n on M is even, assume that f preserves orientation, and that f reverses orientation otherwise.*
Then f has a fixed point.

Proof. Suppose not. Then $d(f(p), p) > 0$ for all p, and by compactness there is a point p where this quantity is minimal. By completeness, there is a minimising geodesic $\gamma : [0, l] \to M$ joining p to $f(p)$.

Step 1: $\gamma'(l) = df\gamma'(0)$.
Take $p' = \gamma(t)$ for some $t > 0$. By the triangle inequality,

$$d(p', f(p')) \leq d(p', f(p)) + d(f(p), f(p')).$$

On the other hand, the path $\gamma|_{[t,l]} * f\gamma_{[0,t]}$ is a path of that length joining p' to $f(p')$, and therefore length-minimising. In particular, it is smooth, which implies the claim.

Step 2: Consider the map $\hat{A} = P_{\gamma(l)}^{\gamma(0)} df$ obtained as the composition of the differential of f with parallel transport back along γ. Then $\hat{A}$ fixes $\gamma'(0)$. Namely,

$$\hat{A}(\gamma'(0)) = P_{\gamma(l)}^{\gamma(0)}\gamma'(l) = \gamma'(0)$$

by the previous step and the fact that γ' is parallel along γ.

Step 3: The restriction A of $\hat{A}$ to the orthogonal complement of $\gamma'(0)$ fixes a vector.
Namely, we have

$$\det A = \det \hat{A} = \det df.$$

We have $\det df = 1$ if n is even, and $\det df = -1$ if n is odd. Let $m = n - 1$. A is an orthogonal matrix of a m-dimensional vector space. If n is even, m is odd, and $\det A = 1$. Hence, the characteristic polynomial of A has a real root. All real roots are ± 1 by orthogonality of A. Since $\det A > 0$ there needs to be a positive real root.
If n is odd, m is even, and $\det A = -1$. Hence, there needs to be a real root since pairs of complex conjugate roots multiply to positive numbers. If all roots would be negative, their product would be positive since m is even.

Step 4: Choose $e_1(0)$ a unit vector invariant by A, and extend to a parallel vector field e_1. It stays orthogonal to γ'.
Now, build the variation $h(s, t) = \exp_{\gamma(t)}(se_1(t))$. The variational field of this is e_1, and for any fixed t these are geodesics in s. Hence, we have

$$E''(0) = \int_a^b g(V', V') - g(R(\gamma', V)\gamma', V)dt + g(c'(t), \frac{\nabla}{ds}V)|_a^b = -\int_a^b K(\gamma', V) < 0.$$

Step 5: Thus, there is a path c_s in the variation of stricly smaller length. Take β the geodesic starting in p with direction $e_1(0)$. The geodesic $f\beta$ starts in $f(p)$ with direction $e_1(l)$, since $Pdf(e_1(0)) = e_1(0)$. Hence, c_s connects $\beta(s)$ to $f\beta(s)$. In particular,

$$d(\beta(s), f(\beta(s))) < l,$$

violating minimality of γ. □

Lecture 13 (June 13)

Corollary 2.31 (Synge)**.** *Suppose M is compact with positive sectional curvature.*

- *If n is even and M is orientable, then M is simply connected.*
- *If n is odd, then M is orientable.*

Proof. For the first part, consider the universal cover. It now has positive curvature, even dimension, and the deck group acts by orientation preserving isometries. Hence, any deck group element has a fixed point. Since this is only possible for the identity, the claim follows.
For the second part, suppose not. Then there is a nontrivial orientation cover, for which the nontrivial deck group element satisfies the theorem again. Hence, it would fix a point, which is impossible. □

Another way to say this is: nonorientable compact manifolds of odd dimension cannot carry metrics of positive curvature.

Conjugate points. We now continue with a discussion of the interaction of Jacobi fields with the exponential map. First, a definition

Definition 2.32. Let $\gamma : [0, a] \to M$ be a geodesic. We say that $q = \gamma(t)$ is *conjugate to p along γ* if there is a nonzero Jacobi field J along γ with $J(0) = 0 = J(t)$.
We say that the conjugate point has *multiplicity k*, if there are k such linearly independent fields.

Observe that the space of Jacobi fields which vanish at $\gamma(0)$ has dimension n (as they are determined by their initial values and derivatives). The field $t\gamma'(t)$ is one of them, and never vanishes. Hence, the multiplicity is at most $n-1$. This is actually achieved for the sphere S^n.

Lemma 2.33. *Let $\gamma : [0, l] \to M$ be a geodesic starting in $p = \gamma(0)$. Then $q = \gamma(t)$ is conjugate to p along γ if and only if $v = t\gamma'(0)$ is a critical point of $\exp_p$. The multiplicity of q is the dimension of the kernel of $d_v \exp_p$.*

Proof. Recall that Jacobi fields compute the derivative of the exponential map, in the sense that

$$J(t) = d_{tv} \exp(tw)$$

are exactly the Jacobi fields with $J(0) = 0, J'(0) = w$. This shows the lemma. □

Definition 2.34. Given a point $p \in M$, the *conjugate locus* $C(p)$ is the set of all points $q = \gamma(t)$ where γ is a geodesic starting in p, and so that q is the first conjugate point along γ.

On the sphere, the conjugate locus of any point is the antipodal point. This is also the point after which geodesics stop being minimising. Is this always the case? In general, no. To see this, first note the following:

Lemma 2.35. *Suppose M is a complete Riemannian manifold with $K \leq 0$. Then, $C(p) = \emptyset$ for all $p \in M$.*

Proof. Consider a nonzero Jacobi field J with $J(0) = 0$. Consider $f(t) = g(J(t), J(t)$ and compute

$$f'(t) = 2g(J(t), J'(t)),$$

$$f''(t) = 2g(J'(t), J'(t)) + 2g(J(t), J''(t)) = 2\|J'(t)\|^2 - 2g(R(\gamma'(t), J(t))\gamma'(t), J(t)) > 0$$

Thus, f' is strictly increasing, hence positive except at $t = 0$. As a consequence, f has no zero except at $t = 0$. This shows the lemma. $\square$

So, on a flat torus, there are no conjugate points, but geodesics stop being minimising anyway. We will understand the precise relation later. Also note

Corollary 2.36. *On a complete manifold with $K \leq 0$, the exponential map is a local diffeomorphism.*

In fact, one can show that the exponential map is a covering map, and therefore any simply connected, nonpositively curved, complete Riemannian manifold is diffeomorphic to $\mathbb{R}^n$. This is called the Hadamard theorem. We don't prove this here, as we don't have covering space theory. If you are interested, compare e.g. Chapter 7, Section 3 of do Carmo.

To really understand what conjugate points mean, we use the following fundamental object. Let $\gamma : [0, l] \to M$ be a geodesic. Let $\mathcal{V}$ be the space of piecewise differentiable vector fields along γ which vanish at $0, l$.

Definition 2.37. We define a symmetric bilinear form

$$I(V, W) = \int_0^l g(V', W') - g(R(\gamma', V)\gamma', W)dt,$$

and let the *index form* be the associated quadratic form ($I(V) = I(V, V)$).

Recall that for a bilinear form B the *null space* is the space of all vectors V so that $B(V, W) = 0$ for all W. The *nullity* is the dimension of the null space. A form is *degenerate* if the null space is nontrivial. The *index* of the form is the maximal dimension of a subspace on which the quadratic form is negative definite (this is sometimes done differently in linear algebra).

The following gives a first indication why we might care about this form. Namely, we have

$$I(V, W) = -\int_0^l g(V'' + R(\gamma', V)\gamma', W)dt - \sum g(\frac{\nabla}{dt}(t_j^+) - \frac{\nabla}{dt}(t_j^-), W(t_j))$$

where t_j are the points outside which V is differentiable. This follows by considering the derivative $\frac{d}{dt}g(V', W) = g(V'', W) + g(V', W')$ on every interval where V, W are differentiable.

Lemma 2.38. *An element $V \in \mathcal{V}$ is in the null space of I if and only if V is a Jacobi field.*

Proof. One direction is clear. For the other, take V in the null space, and let t_i be the times between which V is differentiable. If W_i is a vector field supported in $[t_i, t_{i+1}]$ then we have

$$0 = I(V, W_i) = -\int_0^l g(V'' + R(\gamma', V)\gamma', W)$$

Since this is supposed to be true for all such W_i, we conclude that

$$V'' + R(\gamma', V)\gamma' = 0$$

at all times where V is differentiable. As a consequence we see that for any W we have

$$0 = I(V, W) = -\sum g(\frac{\nabla}{dt}(t_j^+) - \frac{\nabla}{dt}(t_j^-), W(t_j))$$

In particular, by choosing W with $W(t_j) = \frac{\nabla}{dt}(t_j^+) - \frac{\nabla}{dt}(t_j^-)$ we see that V is differentiable, and thus a Jacobi field. $\square$

As a consequence, the nullity of I is the multiplicity of the conjugate point. Next, we need to study what happens with I in the absence of conjugate points. For this we need the notation I_t for the index form on the restriction of the geodesic to $[0, t]$.

Lemma 2.39 (Index Lemma)**.** *Suppose $\gamma : [0, l] \to M$ is a geodesic without conjugate points. Let J be a Jacobi field orthogonal to γ' and let V be piecewise differentiable which is also orthogonal to γ'. Suppose $J(0) = V(0) = 0$ and $J(t_0) = V(t_0)$. Then*

$$I_{t_0}(J, J) \leq I_{t_0}(V, V)$$

with equality if and only if $J = V$.

Proof. Let $J_1, \ldots, J_{n-1}$ be a basis for the Jacobi fields starting with 0 and being orthogonal to γ'. Write $J = \sum \alpha_i J_i$. Since we assume that there are no conjugate points, the $J_i(t)$ are a basis at all $\gamma(t)$. Hence, we can write

$$V(t) = \sum f_i(t) J_i(t),$$

for all $t \in (0, l]$.

Step 1: First, a techincal point. We need to show that the f_i are actually piecewise differentiable and continuous on $[0, l]$. To this end, write the $J_i(t) = tA_i(t)$ for a smooth A_i with $A_i(0) = J_i'(0)$ (this is possible since $J_i(0) = 0$). The A_i are linearly independent for all t, and so we can write

$V(t) = \sum g_i(t)A_i(t)$ with piecewise differentiable g_i starting at 0. Write $g_i(t) = th_i(t)$, with piecewise differentiable h_i. Then

$$V(t) = \sum th_i(t)A_i(t) = \sum h_i(t)J_i(t),$$

hence $f_i = h_i$ for $t \neq 0$.

Step 2: Where f_i is differentiable, we have:

$$g(\sum f_i J_i', \sum f_j' J_j) = g(\sum f_i J_i, \sum f_j' J_j')$$

To prove this, consider

$$h(t) = g(J_i', J_j) - g(J_i, J_j')$$

and differentiate:

$$h'(t) = g(J_i'', J_j) + g(J_i', J_j') - g(J_i', J_j') - g(J_i, J_j'')$$

and this is zero by the Jacobi identity and symmetry of R. Since $h(0) = 0$, the function h is identically zero, showing Step 2.

Step 3: Where f_i is differentiable, we have:

$$g(V', V') - g(R(\gamma', V)\gamma', V) = \| \sum f_i' J_i \|^2 + \frac{d}{dt} g(\sum f_i J_i, \sum f_j J_j').$$

To see this, note first that

$$V'(t) = \sum f_i' J_i + \sum f_i J_i',$$

and

$$R(\gamma', V)\gamma' = -\sum f_i J_i''$$

by the Jacobi identity. Then the claim follows by expanding and using Step 3.

Step 4:

$$I_{t_0}(V, V) = \int_0^{t_0} \| \sum f_i' J_i \|^2 dt + g(\sum f_i(t_0) J_i(t_0), \sum f_j(t_0) J_j'(t_0)).$$

and, writing $J = \sum_k \alpha_k J_k$,

$$I_{t_0}(J, J) = g(\sum \alpha_i J_i(t_0), \sum \alpha_j J_j'(t_0)).$$

Now, as $J(t_0) = V(t_0)$, this means

$$I_{t_0}(V, V) = I_{t_0}(J, J) + \int_0^{t_0} \| \sum f_i' J_i \|^2 dt$$

This shows that $I_{t_0}(V, V) \geq I_{t_0}(J, J)$ with equality only if $\sum f_i' J_i = 0$. As the J_i are linearly independent, this implies $f_i = 0$, and thus $V = J$. $\square$

LECTURE 15 (JUNE 24)

The Rauch comparison theorem. First, an application of the index lemma from last time. Throughout we will have two Riemannian manifolds of the same dimension m, which we denote by (M, g) and $(\hat{M}, \hat{g})$. Further, we have two geodesics parametrised by arclength, denoted by

$$\gamma : [0, l] \to M,$$

$$\hat{\gamma} : [0, l] \to \hat{M},$$

Lemma 2.40. *Suppose $J, \hat{J}$ are Jacobi fields along $\gamma, \hat{\gamma}$. Assume that*

i) $J(0) = 0 = \hat{J}(0)$,
ii) $J'(0)$ *is normal to* $\gamma'(0)$, $\hat{J}'(0)$ *is normal to* $\hat{\gamma}'(0)$.
iii) $g(J(l), J(l)) = \hat{g}(\hat{J}(l), \hat{J}(l))$,
iv) γ *has no conjugate points on* $[0, l]$, *and*
v)

$$\max\{\hat{K}(V), V \subset T_{\hat{\gamma}(t)}\hat{M}, \hat{\gamma}'(t) \in V\} \le \min\{K(V), V \subset T_{\gamma(t)}M, \gamma'(t) \in V\}.$$

Then

$$I(J, J) \le \hat{I}(\hat{J}, \hat{J}).$$

The inequality is strict if the curvature condition is strict.

Proof. Choose orthonormal parallel frames $e_i, \hat{e}_i$ so that $e_1(t) = \gamma'(t), \hat{e}_1 = \hat{\gamma}'(t)$ and

$$e_2(l) = J(l)/\|J(l)\|, \hat{e}_2(l) = \hat{J}(l)/\|\hat{J}(l)\|$$

which is well-defined by iv). Write the Jacobi fields in these bases as

$$J(t) = \sum f_i(t)e_i(t), \quad \hat{J}(t) = \sum \hat{f}_i(t)\hat{e}_i(t).$$

Now define a vector field

$$X = \sum \hat{f}_i(t)e_i(t)$$

and observe

- $X(0) = 0$, as $\hat{J}(0) = 0$.
- $X(l) = J(l)$, as $\hat{f}_i(l) = 0$ unless $i = 2$, and $\hat{f}_2(l) = \|\hat{J}(l)\| = \|J(l)\| = f_2(l)$.
- $X(t)$ is orthogonal to $\gamma'(t)$, as $\hat{f}_1(t) = 0$ for all t.

Hence, X and J satisfy the conditions of the index lemma. Thus

$$I(J, J) \le I(X, X)$$

On the other hand, we simply compute

$$
\begin{aligned}
I(X,X) &= \int_0^l g(X',X') - g(R(\gamma',X)\gamma',X)dt \\
&= \int_0^l \sum_i (\hat{f}_i')^2(t) - (\sum_i (\hat{f}_i)^2(t))K(\gamma'(t),X(t))dt \\
&\leq \int_0^l \sum_i (\hat{f}_i')^2(t) - (\sum_i (\hat{f}_i)^2(t))\hat{K}(\hat{\gamma}'(t),\hat{J}(t))dt \\
&= \int_0^l \sum_i (\hat{f}_i')^2(t) - \hat{g}(\hat{R}(\hat{\gamma}',\hat{J})\hat{\gamma}',\hat{J})dt \\
&= \int_0^l g(\hat{J}',\hat{J}') - \hat{g}(\hat{R}(\hat{\gamma}',\hat{J})\hat{\gamma}',\hat{J})dt \\
&= \hat{I}(\hat{J},\hat{J}).
\end{aligned}
$$

□

Theorem 2.41 (Rauch comparison theorem)**.** *Suppose $J, \hat{J}$ are Jacobi fields along $\gamma, \hat{\gamma}$. Assume*

i) $J(0) = 0 = \hat{J}(0)$,
ii) $\|J'(0)\| = \|\hat{J}'(0)\|$,
iii) $g(\gamma'(0), J'(0)) = \hat{g}(\hat{\gamma}'(0), \hat{J}'(0))$,
iv) *γ has no conjugate points on $[0,l]$, and*
v)

$$\max\{\hat{K}(V), V \subset T_{\hat{\gamma}(t)}\hat{M}, \hat{\gamma}'(t) \in V\} \leq \min\{K(V), V \subset T_{\gamma(t)}M, \gamma'(t) \in V\}.$$

Then $\hat{J}$ has no conjugate points on $[0,l]$ and in fact

$$\|J(t)\| \leq \|\hat{J}(t)\|$$

for all $t \in [0,l]$. The inequality is strict for $t > 0$ if the curvature condition is strict.

Proof. Write $J = J^t + J^n$ as a sum of normal and tangential components. Then $J^t(t) = at\gamma'(t)$. As the analogous is true for $\hat{J}$, the tangential components grow exactly alike, and it suffices to show the theorem for normal Jacobi fields.

In that case, define

$$u(t) = g(J(t), J(t)), \quad \hat{u}(t) = \hat{g}(\hat{J}(t), \hat{J}(t)).$$

Note that

$$u'(t) = 2g(J'(t), J(t)), \quad u''(t) = 2g(J'(t), J'(t)) + 2g(J''(t), J(t))$$

In particular, $u(0) = u'(0) = 0$ and $u''(0) \neq 0$. Analogous formulas hold for for $\hat{u}$.

For $t > 0$ the function $\hat{u}(t)/u(t)$ is well-defined as γ has no conjugate points. For $t = 0$, we can apply L'Hospital (twice), to see that $\hat{u}(t)/u(t)$ extends to $t = 0$ continously and

$$\lim_{t\to 0} \frac{\hat{u}(t)}{u(t)} = \frac{\hat{u}''(0)}{u''(0)} = 1.$$

Thus, to prove the theorem, it suffices to show that $\hat{u}/u$ is weakly increasing on $(0, l)$, which is implies by $(\hat{u}/u)' \geq 0$. In other words, we want to show

$$\hat{u}'(t)u(t) - \hat{u}(t)u'(t) \geq 0.$$

Let $a \in (0, l)$ be any number so that $\hat{u}(a) > 0$. Define

$$J_a(t) = \frac{J(t)}{\|J(a)\|}, \quad \hat{J}_a(t) = \frac{\hat{J}(t)}{\|\hat{J}(a)\|}$$

These fields satisfy the condition of the previous lemma, and so we get

$$I(J_a, J_a) \leq \hat{I}(\hat{J}, \hat{J}).$$

Recall that we can rewrite the index form as

$$I(X, X) = -\int_0^l g(X'' + R(\gamma', X)\gamma', X)dt + g(X', X)|_0^l.$$

Hence, for Jacobi fields starting with value 0, we have

$$I(J_a, J_a) = g(J_a'(a), J_a(a))$$

which in our case yields

$$g(J_a'(a), J_a(a)) \leq \hat{g}(\hat{J}_a'(a), \hat{J}_a(a)).$$

Compute

$$\frac{u'(a)}{u(a)} = \frac{2g(J'(a), J(a))}{g(J(a), J(a))} = 2g(J_a'(a), J_a(a))$$

and analogously for $\hat{J}$ yields

$$\frac{u'(a)}{u(a)} \leq \frac{\hat{u}'(a)}{\hat{u}(a)},$$

and in turn $\|J(a)\| \leq \|\hat{J}(a)\|$.

Finally, suppose that there would be a point $a \leq l$ with $\hat{u}(a) = 0$. By taking a limit to the left and arguing as above we would conclude $J(a) = 0$, contradicting that γ has no conjugate points. Hence, the estimate $\|J\| \leq \|\hat{J}\|$ holds on $[0, l]$, and γ' has no conjugate points. □

Often it is useful to rewrite the Rauch comparison theorem in terms of geodesic variations and the exponential map. Namely, suppose that $w \in T_pM$ is a vector. The function

$$f(s, t) = \exp_p(t(\gamma'(0) + sw))$$

is a geodesic variation and hence its variation vector field

$$J(t) = \frac{\partial f}{\partial s}_0(t) = d_{t\gamma'(0)} \exp_p(tw)$$

is the unique Jacobi field with $J(0) = 0, J'(0) = w$.

Corollary 2.42 (Rauch comparison, exponential form)**.** *Suppose that*

i) γ *has no conjugate points on* $[0, l]$, *and*

ii)

$$\max\{\hat{K}(V), V \subset T_{\hat{\gamma}(t)}\hat{M}, \hat{\gamma}'(t) \in V\} \leq \min\{K(V), V \subset T_{\gamma(t)}M, \gamma'(t) \in V\}.$$

Then, if $w \in T_{\gamma(0)}M$ *and* $\hat{w} \in T_{\hat{\gamma}(0)}\hat{M}$ *are such that*

$$g(w, \gamma'(0) = \hat{g}(\hat{w}, \hat{\gamma}'(0)), \quad \|w\| = \|\hat{w}\|$$

we have

$$\|d_{t\gamma(0)} \exp_p(w)\| \leq \|d_{t\hat{\gamma}(0)} \hat{\exp}_{\hat{p}}(\hat{w})\|.$$

The inequality is strict if the curvature condition is strict.

Lecture 16 (June 27)

At this point, two applications. One on the location of conjugate points under positive curvature pinching.

Lemma 2.43. *Let* L, H *be numbers, and suppose that* M *is a Riemannian manifold whose sectional curvatures satisfy*

$$0 < L \leq K \leq H.$$

Let γ *be any geodesic in* M*. Denote by* d *the distance between two consecutive conjugate points (i.e. there is a Jacobi field* J *along* γ *so that* $J(a) = 0 = J(a + d)$, *and* $J(t) \neq 0, a < t < a + d$*). Then*

$$\frac{\pi}{\sqrt{H}} \leq d \leq \frac{\pi}{\sqrt{L}}.$$

Proof. Consider the sphere $(S^n, r^2 g_{\text{round}})$ with the round metric of diameter r. Under rescaling the metric tensor by r^2, curvatures scale by $1/r^2$.
First, the lower bound. A sphere N of diameter $\frac{\pi}{\sqrt{H}}$ has constant curvature H by the comment above. Take any geodesic ρ on N, and note that it doesn't have conjugate points before $\frac{\pi}{\sqrt{H}}$. Also, M is less positively curved than this sphere by assumption, so we can apply the Rauch comparison theorem for a suitable Jacobi field on the sphere.
The upper bound is essentially the same: assuming that $d > \frac{\pi}{\sqrt{L}}$ compare M with a sphere of the that diameter to contradict that Jacobi fields on the sphere are zero at the antipodal points. □

The other important application of Rauch concerns the length of curves.

Lemma 2.44 (Length Comparison)**.** *Suppose that* $M, \hat{M}$ *are two Riemannian manifolds. Suppose that* $\hat{K} \geq K$ *(for all points, all tangent planes). Let* $p \in M, \hat{p} \in \hat{M}$ *be any two points, and* $I : T_pM \to T_{\hat{p}}\hat{M}$ *be any (linear) isometry. Choose a radius* r *so that* $\exp_p$ *is a diffeomorphism on* $B_r(0)$ *and that* $\hat{\exp}_{\hat{p}}$ *is a local diffeomorphism on* $B_r(0)$.

Now suppose $c : [0, l) \to \exp_p(B_r(0))$ *is any differentiable curve, and put*

$$\hat{c}(t) = \hat{\exp}_{\hat{p}} \circ I \circ \exp_p^{-1}(c(s))$$

Then $l(c) \geq l(\hat{c})$*.*

Proof. Write

$$c(s) = \exp_p e(s)$$

and define

$$f(s, t) = \exp_p te(s).$$

Observe that for any fixed s the curves $t \mapsto f(s, t)$ are geodesics, and so $J_s(t) = \frac{\partial}{\partial s} f$ is a Jacobi field along $f(s, t)$. Further $J_s(0) = 0$, and $J_s(1) = c'(s)$. Finally,

$$J_s'(t) = \frac{\nabla}{dt}(d_{te(s)} \exp_p te'(s)) = d_{te(s)} \exp_p e'(s) + t\frac{\nabla}{dt}(d_{te(s)} \exp_p e'(s))$$

and thus $J_s'(0) = d_0 \exp_p e'(s) = e'(s)$.

Now consider

$$g(s, t) = \hat{\exp}_{\hat{p}} tI(e(s)).$$

This again has $t \mapsto g(s, t)$ geodesics, so $\hat{J}_s(t) = \frac{\partial}{\partial s} g$ is a Jacobi field, and $\hat{J}_s(0) = 0$, $\hat{J}_s(1) = \hat{c}'(s)$, $\hat{J}_s'(0) = Ie(s)$. Since I is an isometry, we can therefore apply Rauch's theorem to conclude that $\|\hat{c}'(s)\| \leq \|c'(s)\|$ for all s. □

We will now study what *upper curvature bounds* mean metrically. By a *geodesic triangle* we mean a triangle formed by distance realising geodesics.

Lemma 2.45 (Comparison Hinges)**.** *Suppose* M *is a Riemannian manifold,* $K \leq \kappa$*, and* T *is a geodesic triangle with sides* a, b, c *and angles* α, β, γ*. Assume that* T *is contained in a geodesic ball around the vertex where* a, b *meet.*

Consider a comparison hinge *in a model space* N *of constant curvature* κ*, with sides* a, b *and angle* γ*. Then the geodesic closing the hinge in* N *has length* $\leq c$*.*

Proof. We want to apply the length comparison lemma. Let p be the point where the sides a, b meet. Choose an isometry $I : T_pM \to T_qN$ and let v, w be the directions of the sides a, b leaving p. Consider now the side c, and write it as

$$c(t) = \exp_p(e(t))$$

where $c(0), c(1)$ correspond to the points on the radial geodesics a, b. This uniquely possible by the assumption on the triangle.

Now consider $\hat{c}(t) = \hat{\exp}_q Ie(t)$. By the length comparison lemma, we have $c \geq l(\hat{c})$. Finally, the geodesic closing the hinge in the model space joins the same points, so it is again shorter.

□

Lecture 17 (July 1)

Corollary 2.46 (Comparison Triangle Angles)**.** *Suppose M is a Riemannian manifold, $K \leq \kappa$, and T is a geodesic triangle with sides a, b, c and angles α, β, γ. Assume that T is contained in a geodesic ball around the vertex where a, b meet.*
Consider a comparison triangle Δ *in a model space N of constant curvature κ, with sides a, b, c. Then the angles in Δ are bigger than the angles in T.*

Proof. Consider a comparion hinge. Its closing edge is too short, by the previous lemma. Since in the model geometries, for geodesic triangles the length of the opposite side is monotonic in the angle (e.g. by the law of cosines), this shows the claim. □

Lemma 2.47 (Comparison Triangle Across)**.** *Suppose M is a Riemannian manifold, $K \leq \kappa$, and T is a geodesic triangle with sides a, b, c and angles α, β, γ. Assume that T is contained in a geodesic ball around the vertex p where a, b meet. If $\kappa > 0$ also assume that the diameter of T is at most $\frac{\pi}{2\sqrt{\kappa}}$.*
Consider a comparison triangle Δ *in a model space N of constant curvature κ, with sides a, b, c. Now, consider a minimising geodesic d joining p to a point q on c. Consider a minimising geodesic d' in the model space joining the corresponding vertex to a point with the same distance along c'. Then*

$$l(d) \leq l(d')$$

Proof. Cut the triangle in two along the geodesic d, and consider comparison triangles for the two pieces. Arrange them in the model space so they share the side d'. Then the angle of the union at the endpoint of d' is $\geq \pi$. Pulling tight so the angle becomes $= \pi$ increases the length of d' (in the positive curvature case, this is where we use the diameter bound). □

Lemma 2.48 (Comparison Triangle Secants)**.** *Suppose M is a Riemannian manifold, $K \leq \kappa$, and T is a geodesic triangle with sides a, b, c and angles α, β, γ. Assume that T is contained in a geodesic ball around the vertex p where a, b meet. If $\kappa > 0$ also assume that the diameter of T is at most $\frac{\pi}{2\sqrt{\kappa}}$.*
Consider a comparison triangle Δ *in a model space N of constant curvature κ, with sides a, b, c. Now, consider a minimising geodesic s joining points on, say a and b, and consider a minimising geodesic s' in the model space joining the corresponding points with the same distances along a', b'. Then*

$$l(s) \leq l(s')$$

Proof. This follows from a comparison figure. □

In fact, the converse is also true. One way to see this is the following: Suppose γ_0, γ_1 are two distinct geodesics emanating from p with directions v, w. Define

$$L(\epsilon) = d(\gamma_0(\epsilon), \gamma_1(\epsilon)).$$

Lemma 2.49. *One has*

$$L(\epsilon) = \epsilon\|v-w\| - \frac{1}{6}\frac{g(R(v,w)v,w)}{\|v-w\|}\epsilon^3 + O(\epsilon^4).$$

Assuming this for the moment, we can consider orthogonal geodesics and see that if $K \geq \kappa$, the geodesic closing a hinge is longer than in the comparison situation. E.g. it is easy to see that hinge closing is faster than linear in negative curvature and slower than linear in positive curvature.

Definition 2.50. A geodesic metric space is called *locally CAT(κ)* if every point has a convex neighbourhood U with the property that every triangle in U is thinner than its comparison triangles in N_κ.

Together these prove

Theorem 2.51. *A Riemannian manifold is locally CAT(κ) if and only if its sectional curvatures are bounded from above by κ.*

This is somewhat remarkable: a purely analytic condition is completely equivalent to a purely metric one. Also, the CAT condition applies to much more general spaces than Riemannian manifolds, and leads into the field of *metric geometry.*
To show the theorem, we will instead prove a *global* comparison result for triangles under lower curvature bounds later (Toponogov's theorem)

Lecture 18 (July 4)

Isometric immersions, briefly. In this section, we are always concerned with the following situation. We have two Riemannian manifolds M, N and an isometric immersion $f : M \to N$. We want to (eventually) relate the curvatures of M and N. First, observe that for every point $p \in M$, the map f is a local diffeomorphism (inverse function theorem), and as curvature is a local property we can actually assume that f is an isometric embedding. Then, we can (via df) identify T_pM with $d_pf(T_pM) \subset T_{f(p)}M$, and we will do so without mention. We can then write things like:

$$T_pN = T_pM \oplus (T_pN)^\perp$$

Given a tangent vector v, we can write it as

$$v = v^t + v^n$$

in the decomposition (tangent and normal component). We have already seen that the Levi-Civita connection of M satisfies

$$\nabla^M_X Y = (\nabla^N_{X'} Y')^t$$

for X', Y' any extensions of X, Y to open neighbourhoods. The normal component

$$B(X,Y) = (\nabla^N_{X'}Y')^n = \nabla^N_{X'}Y' - \nabla^M_X Y$$

will also be important. This is a vector field on M normal to M, and it also does not depend on the choices of X', Y'. To see this, first observe that for a different X'' we have

$$(\nabla^N_{X'} Y')^n - (\nabla^N_{X''} Y')^n = \nabla^N_{X'-X''} Y' = 0$$

where for the first equality we used that the tangential part is independent, and for the second one we used that for all points on M, the difference $X' - X'' = 0$. Also

$$(\nabla^N_{X'} Y')^n - (\nabla^N_{X'} Y'')^n = \nabla^N_{X'} Y' - Y'' = 0$$

for the same reason. In fact, B is a $\mathcal{C}^\infty$–bilinear symmetric form. The linearity follows from the Leibniz rules of the connections, and the symmetry from the symmetry of the two Levi-Civity connections as well as $[X', Y'] = [X, Y]$ on M. Just as for connections, the $\mathcal{C}^\infty$–bilinearity implies that we can interpret

$$B : T_pM \times T_pM \to (T_pM)^\perp$$

Given a normal vector $\eta \in (T_pM)^\perp$, we then define the *second fundamental form*

$$H_\eta(X, Y) = g(B(X, Y), \eta)$$

Associated we have the *shape operator* (or *Weingarten map*) S_η defined by

$$-H_\eta(X, Y) = g(S_\eta(X), Y).$$

The shape operator can be computed in the following way: let N be a (local) extension of η to a normal vector field. Then

$$S_\eta(X) = (\nabla^N_X N)^t$$

This follows since

$$g(S_\eta(X), Y) = g(B(X, Y), \eta) = g(\nabla^N_{X'} Y' - \nabla^M_X Y, N) = g(\nabla^N_{X'} Y', N) = -g(\nabla^N_{X'} N, Y)$$

where the last equality follows since $g(Y', N) = 0$ along M.

Theorem 2.52 (Theorema Egregium (modern form), Gaußformula)**.**

$$K^M(x, y) - K^N(x, y) = g(B(x, x), B(y, y)) - g(B(x, y), B(x, y)).$$

For a proof, compare Theorem 2.5 in Chapter 6 of do Carmo, or wait for the general Gaußequation below.

In the sequel, we will be mostly interested in *hypersurfaces*, i.e. the case where M has codimension 1 in N. Further, we will be dealing with the case that M, N are oriented, in which case there is a global normal unit vector field η, which is unique up to sign. In this circumstance we will drop the explicit mention of η for the shape operator etc. In this case, we can choose a orthonormal basis x_i diagonalising S (with eigenvalues λ_i), and the Gaußformula takes the form

$$K^M(x_i, x_j) - K^N(x_i, x_j) = \lambda_i \lambda_j.$$

There are three more fundamental equations for immersions. All require the notion of *normal connection* and *normal curvature*. The first is

$$\nabla^{\perp}_X \eta = (\nabla^N_X \eta)^n = \nabla^N_X \eta - S_\eta(X),$$

which is a connection on the normal bundle of M. Now, define the normal curvature

$$R^{\perp}(X,Y)\eta = \nabla^{\perp}_Y \nabla^{\perp}_X \eta - \nabla^{\perp}_X \nabla^{\perp}_Y \eta + \nabla^{\perp}_{[X,Y]}\eta.$$

Lemma 2.53 (Gauß equation)**.**

$$g(R^N(X,Y)Z,T) = g(R^M(X,Y)Z,T) - g(B(Y,T),B(X,Z)) + g(B(X,T),B(Y,Z)).$$

Lemma 2.54 (Ricci equation)**.**

$$g(R^N(X,Y)\eta,\zeta) - g(R^{\perp}(X,Y)\eta,\zeta) = g([S_\eta,S_\zeta]X,Y).$$

Proofs of these are Proposition 3.1 of Chapter 6 of do Carmo. In the case of a hypersurface, the Ricci equation is empty. The final equation (and the one we really want) uses

$$B(X,Y,\eta) = g(B(X,Y),\eta)$$

and is

Lemma 2.55 (Codazzi equation)**.**

$$g(R^N(X,Y)Z,\eta) = (\nabla^N_Y B)(X,Z,\eta) - (\nabla^N_X B)(Y,Z,\eta)$$

where we differentiate the tensor B with the induced connection.

Lecture 19 (July 8)

2.1. **Shape operators of local distance functions.** The first thing we need to understand are local distance functions. Let $U \subset M$ be a (convex) open subset, and $p \in U$ a point. We define

$$r(q) = d_M(p,q), \quad r_U(q) = d_U(p,q).$$

These are not everywhere differentiable, but for example on $B - p$ they are (where B is a geodesic ball). Recall that the *gradient* of a function f is the vector field $\mathrm{grad} f$ with

$$Xf = g(\mathrm{grad} f, X).$$

At points where r is differentiable, we have $\|\mathrm{grad} r\| = 1$. Namely, we have

$$|Xr| \leq 1$$

for all X by the triangle inequality, and moving along a geodesic joining to p the value 1 can be realised.
In fact, if f is any function with $\|\mathrm{grad} f\| = 1$, then the gradient flow lines of f are unit speed geodesics. This follows from

$$g(\nabla_{\mathrm{grad} f}\mathrm{grad} f, X) = g(\nabla_X \mathrm{grad} f, \mathrm{grad} f) = 1/2 X g(\mathrm{grad} f, \mathrm{grad} f) = 0$$

where the first equality is the fact that the Hessian is selfadjoint (see problem set).

As a consequence, the level sets of f are equidistant: the distance from any point on $f^{-1}(a)$ to $f^{-1}(a+\epsilon)$ is ϵ.
Next, observe that for a (local) distance function, the vector field $N = \mathrm{grad} f$ is a unit normal vector field on all level sets. The shape operator of these submanifolds is therefore

$$S(X) = \nabla_X \mathrm{grad} f = \mathrm{Hess} f(X),$$

which is tangent to the level sets (We don't need to take the tangential part here). We define the (covariant) derivative of the shape operator in the normal direction as

$$S'(X) = \nabla_N SX - S(\nabla_N X)$$

for X tangent to the levels of f. One can check that $S'(X)$ is well-defined, and again tangent to the level sets.
Now suppose that $M_0 = f^{-1}(0)$ is a fixed level (the normalisation to 0 is not relevant), and that $M_t = f^{-1}(t)$ is another level. Then, for small t, the function

$$E_t(p) = \exp_p(tN(p))$$

defines a diffeomorphism from M_0 to M_t. We want to compute the derivative of this function.
To this end, consider a curve $p(t)$ in M_0 with initial velocity v and the geodesic

$$t \mapsto E_t(p(0))$$

and the geodesic variation

$$V(s,t) = E_t \circ p(s).$$

Let J be the Jacobi field defined by this variation:

$$J(t) = \frac{\partial V}{\partial s} V = \frac{\partial V}{\partial s}_0 \exp_{p(s)}(tN(p(s)))$$

Its initial conditions are

$$J(0) = \frac{\partial V}{\partial s} V = dE_0(p'(0)) = v.$$

$$J'(0) = \nabla_v N = Sv$$

Now, for any s, the geodesics $\gamma_s(t) = V(s,t)$ are flow lines of $\mathrm{grad} f$. Therefore, we have $\frac{\partial}{\partial t} \exp_{p(s)}(tN(p(s))) = \gamma_s'(t) = N \circ V$, and computing as above we get $J'(t) = SJ(t)$ for all t.
We covariantly differentiate this (see above) to obtain

$$J''(t) = S'J(t) + SJ'(t) = S'J(t) + S^2 J(t)$$

Recall that J is a Jacobi field, and we therefore also have

$$J'' + R(N,J)N = 0$$

to get

$$-R(N,J)N = S'J + S^2 J$$

Now, observe that the initial direction $v = J(0)$ of J was arbitrary, and so this actually leads to the *Ricatti equation*:

$$S' = R_N - S^2$$

where $R_N(X) = R(X, N)N$. We summarise this in the following statement

Proposition 2.56. *Suppose that f is differentiable and that* $\mathrm{grad} f$ *is a unit vector field where defined. Then the shape operator on any level set is*

$$S(X) = \mathrm{Hess} f(X),$$

and it is defined by the equation

$$J'(t) = SJ(t)$$

for Jacobi fields defined by geodesics joining levels.
Further, it satisfies the Ricatti equation

$$S' = R_N - S^2$$

for $R_N = R(N, J)N$, $N = \mathrm{grad}(f)$.

2.2. **Jacobi fields in the model spaces, unified formulas.** Let's study this in model spaces explicitly. As before, N^n_κ is the dimension n model space of constant curvature κ. Jacobi fields have the form

$$J = fY$$

where Y is parallel, and f satisfies

$$f'' + \kappa f = 0.$$

Now, for the model spaces we have the solutions

Example 2.57.

$$\mathrm{sn}_\kappa(t) = \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} t$$

$$\mathrm{cs}_\kappa(t) = \cos \sqrt{\kappa} t$$

These solve with initial conditions $0, 1$ and $1, 0$.

Example 2.58.

$$\mathrm{sn}_0(t) = t$$

$$\mathrm{cs}_0(t) = 1$$

These solve with initial conditions $0, 1$ and $1, 0$.

Example 2.59.

$$\mathrm{sn}_\kappa(t) = \frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa} t$$

$$\mathrm{cs}_\kappa(t) = \cosh \sqrt{-\kappa} t$$

These solve with initial conditions $0, 1$ and $1, 0$.

All of them have

$$\mathrm{sn}_\kappa(t)' = \mathrm{cs}_\kappa(t), \mathrm{cs}_\kappa(t)' = -\kappa \mathrm{sn}_\kappa(t)$$

With these, a basis for Jacobi fields in N_κ orthogonal to the geodesic is

$$\mathrm{sn}_\kappa(t)Y(t), \mathrm{cs}_\kappa(t)Y(t)$$

for Y a orthogonal parallel field.

Lecture 20 (July 11)

Given that we already know that S is determined by $SJ = J'$ for Jacobi fields corresponding to variations of geodesics connecting levels, we can use these to compute the shape operator for distance spheres like this: consider a distance sphere centered at p, and a geodesic starting in p (which then hits the distance sphere orthogonally). The Jacobi field

$$J(r) = \mathrm{sn}_\kappa(t)Y(t)$$

corresponds to a variation of geodesics starting in p, which thus also consists of geodesics connecting levels orthogonally, and is of the form considered before. We have

$$J'(r) = \mathrm{cs}_\kappa(t)Y(t) = \mathrm{ct}_\kappa(t)J(r)$$

for

$$\mathrm{ct}_\kappa(t) = \frac{\mathrm{cs}_\kappa(t)}{\mathrm{sn}_\kappa(t)}.$$

Hence, the shape operator $S_{c(t)}$ is simply multiplication by $\mathrm{ct}_\kappa(t)$ (in other words, all principal curvatures are $\mathrm{ct}_\kappa(t)$).
Also observe that

$$\mathrm{ct}_\kappa'(t) = \frac{\mathrm{cs}_\kappa'(t)\mathrm{sn}_\kappa(t) - \mathrm{cs}_\kappa(t)\mathrm{sn}_\kappa'(t)}{\mathrm{sn}_\kappa^2(t)} = \frac{-\kappa \mathrm{sn}_\kappa^2(t) - \mathrm{cs}_\kappa^2(t)}{\mathrm{sn}_\kappa^2(t)} = -\kappa - \mathrm{ct}_\kappa^2(t).$$

2.3. **The Ricatti comparison argument.** Why is any of this useful for us? Suppose that Y is a parallel unit vector field along γ tangent to the levels of f on an arbitrary manifold M. Then

$$\begin{aligned} g(SY, Y)' &= g((SY)', Y) + g(SY, Y') = g(S'Y + SY', Y) = g(S'Y, Y) \\ &= g(R_N Y - S^2 Y, Y) = -K(N, Y) - \|SY\|^2 \end{aligned}$$

Under a *lower* curvature bound $K \geq \kappa$ we thus get

$$g(SY, Y)' \leq -\kappa - g(SY, Y)^2$$

which we call a *Ricatti inequality.* Why is this useful? Here's a basic calculus lemma

Lemma 2.60. *Suppose that g, G are two differentiable functions so that*

$$g' \leq -\kappa - g^2$$
$$G' \geq -\kappa - G^2$$

Then,

i) If $g(r_0) \geq G(r_0)$ then $g(r) \geq G(r)$ for all $r \leq r_0$.
ii) If $g(r_0) \leq G(r_0)$ then $g(r) \leq G(r)$ for all $t \geq r_0$.

Proof. Simply compute

$$\frac{d}{dt}[(g-G)e^{\int(g+G)}] = (g'-G')e^{\int(g+G)} + (g+G)e^{\int(g+G)}(g+G)$$
$$= e^{\int(g+G)}[(g'-G') + (g-G)(g+G)] \leq 0.$$

□

So: under curvature bounds, we will be able to conclude growth bounds for the Hessians of distance spheres. This allows us comparisons to the functions appearing in the model spaces:

Lemma 2.61. *Suppose κ is arbitrary, and $a \leq \pi/\sqrt{\kappa}$ if $\kappa > 0$. Suppose $g : (0,a) \to \mathbb{R}$ is differentiable and $g' \leq -\kappa - g^2$. Then*

$$g(r) \leq \mathrm{ct}_\kappa(r)$$

for all r.

Proof. Suppose not, and $g(r_0) \geq \mathrm{ct}_\kappa(r_0 - \epsilon)$. Put $G(r) = \mathrm{ct}_\kappa(r-\epsilon)$, and observe that

$$G' = \mathrm{ct}_\kappa'(r-\epsilon) = -\kappa - \mathrm{ct}_\kappa^2(r-\epsilon) = -\kappa - G^2$$

and so G satisfies the Ricatti equation. Hence we can apply the comparison argument and conclude $g(r) \geq G(r)$ on (ϵ, r_0). But, then $g(\epsilon) = \lim_{r\to\epsilon} g(r) \geq \lim_{r\to\epsilon} G(r) = \infty$ which is absurd. □

Here's how we can use this. Consider a normal geodesic segment c with starting point p, which does not have any conjugate points. Then, in a small neighbourhood of c we have a local distance function

$$f(q) = d_U(p,q).$$

Denote by $\tau_i(q)$ the principal curvatures of the distance sphere $q \in f^{-1}(r)$. Now, apply the Lemma to $t \mapsto g(SY(t), Y(t))$ for any Y parallel ending in the i–th eigenvector to obtain

$$\tau_i(q) \leq \mathrm{ct}_\kappa(r)$$

Now, recall that the shape operator is simply the Hessian of the distance to a point (restricted to the orthogonal complement of the gradient) in this setup. The Hessian is the covariant derivative of the gradient, and thus (as the gradient lines are geodesic) the final eigenvalue of the Hessian is 0 (independent of the manifold).
To make things easier later, we therefore define a rescaled distance function using

$$\mathrm{md}_\kappa(r) = \int_0^r \mathrm{sn}_\kappa(t)dt.$$

This is
$$\frac{1}{\kappa}(1-\mathrm{cs}_\kappa(r))$$
for $\kappa \neq 0$ and $1/2r^2$ otherwise. In both cases we have
$$\mathrm{cs}_\kappa + \kappa \mathrm{md} = 1.$$
Now consider the *modified distance function* $\mathrm{md}_\kappa \circ f$, where f is a distance function as before. We have the chain rule
$$\mathrm{grad}(\rho \circ f) = (\rho' \circ f)\mathrm{grad}(f),$$
and as a consequence the chain rule
$$\mathrm{Hess}(\rho \circ f)v = (\rho' \circ f)\mathrm{Hess}f(v) + (\rho'' \circ f)\langle \mathrm{grad}f, v\rangle \mathrm{grad}f$$
Using this on $\mathrm{md}_\kappa \circ f$ we see that
$$\mathrm{Hess}(\mathrm{md}_\kappa \circ f)v = (\mathrm{sn}_\kappa \circ f)\mathrm{Hess}f(v) + (\mathrm{cs}_\kappa \circ f)\langle \mathrm{grad}f, v\rangle \mathrm{grad}f$$
Hence, in the manifold M the Hessian of $\mathrm{md}_\kappa \circ f$ has the eigenvalues
$$\mathrm{sn}_\kappa(f(q))\tau_i(q) \quad \text{and} \quad \mathrm{cs}_\kappa(f(q))$$
We thus have the operator inequality
$$\mathrm{Hess}(\mathrm{md}_\kappa \circ f) \leq (\mathrm{cs}_\kappa \circ f)\mathrm{Id}.$$
for any point q until the first conjugate point. In the model space we have equality.
Suppose we use $g = f + \eta$ for a constant η instead. This has the same Hessian, so Hess$g$ has eigenvalues $\mathrm{sn}_\kappa(g(q))\tau_i(q), \mathrm{cs}_\kappa(g(q))$ by the same argument as before. We also get
$$(\mathrm{sn}_\kappa \circ g)\tau_i(q) \leq (\mathrm{sn}_\kappa \circ g)\mathrm{ct}_\kappa(r) = (\mathrm{sn}_\kappa \circ g)\mathrm{ct}_\kappa(g-\eta)$$
which we can compute to be equal to
$$= \mathrm{cs}_\kappa \circ g + \frac{\mathrm{sn}_\kappa(\eta)}{\mathrm{sn}_\kappa(g-\eta)}$$
We thus have for small η with $0 < g - \eta < \pi/\sqrt{\kappa}$ the operator inequality
$$\mathrm{Hess}(\mathrm{md}_\kappa \circ g) \leq \left(\mathrm{cs}_\kappa \circ g + \frac{\mathrm{sn}_\kappa(\eta)}{\mathrm{sn}_\kappa(g-\eta)}\right)\mathrm{Id}.$$

2.4. **The Toponogov theorem.**

Theorem 2.62. *Suppose that M is a complete Riemannian manifold with $K \geq \kappa$. Let c be a geodesic connecting p_0 to p_1, and let c_i be minimising geodesics from q to p_i. Suppose we have $l(c) \leq l(c_1) + l(c_2)$ and (if $\kappa > 0$) $|c| \leq \pi/\sqrt{\kappa}$. Let $\overline{c}, \overline{c}_i$ be the sides of a comparison triangle in N_κ. Then*
$$d(\overline{q}, \overline{c}(t)) \leq d(q, c(t))$$
Intuitively: the triangle in M is "fatter" than the one in N_κ.
Before we prove the theorem, note a few variants:

Bemerkung 2.63. *With similar arguments as in the case of lower curvature bounds, the conclusion of Topogonov's theorem can be used to show that secants in M are longer than in the comparison triangle (if we assume that c is also minimising). We skip this, but details can be found in Meyer, "Topogonov's theorem and applications".*

Bemerkung 2.64. *Using monotonicity of angles and closing sides in the model spaces, we could also use Topogonov to show that the angles in M are larger than in the comparison triangle.*

Bemerkung 2.65. *There is a version for hinges: if we compare hinges with the same side lengths and angle, then the closing side is longer in the model space.*

Lecture 21 (July 15)

Proof of Toponogov's theorem. **Step 1: Assume that** $\operatorname{diam}(M) < \pi/\sqrt{\kappa}$ **or that** $\kappa \leq 0$**, and that** $l(c) + l(c_1) + l(c_2) < 2\pi/\sqrt{\kappa}$.
In this case, note that we have

$$l(c) < \pi/\sqrt{\kappa}$$

(in the positive curvature case), since $l(c) \leq l(c_1) + l(c_2)$. Choose an $\epsilon > 0$ so that $\operatorname{diam} M < \pi/\sqrt{\kappa} - 2\epsilon$, $l(c) < \pi/\sqrt{\kappa} - 2\epsilon$.
If $q \in c$, then $l(c) \geq l(c_1) + l(c_2)$ since c_1, c_2 are minimising, and therefore $l(c) = l(c_1) + l(c_2)$. Hence, we have equality in the claim.
Otherwise, consider the distance function r from q in M and $\overline{r}$ from $\overline{q}$ in N_κ. Define

$$h(t) = \operatorname{md}_\kappa \circ r \circ c(t)$$
$$\overline{h}(t) = \operatorname{md}_\kappa \circ \overline{r} \circ \overline{c}(t)$$

and we define

$$\lambda(t) = h(t) - \overline{h}(t).$$

We aim to show that λ cannot have a negative minimum. We want to use the Hessian estimate for this, but the problem is that λ is in general not smooth.
To get around that, we do the following. Suppose that γ is a minimising geodesic joining q to $c(t_0)$. For a (small) number $\eta \in (0, l(\gamma))$, and a neighbourhood U of $\gamma(\eta, l(\gamma))$ (not containing $\gamma(\eta)$) so that any two points in U are joined by a unique geodesic in U, we define the "superdistance function"

$$r_\eta(x) = \eta + d_U(\gamma(\eta), x) \geq r(x).$$

The advantage is that r_η is smooth on U. Hence $r_\eta \circ c$ is smooth in a neighbourhood of t_0, and we define

$$h_\eta(t) = \operatorname{md}_\kappa \circ r_\eta \circ c(t)$$

Observe that

$$h_\eta(t_0) = h(t_0), h_\eta \geq h$$

and where h_η is smooth we have

$$h''_\eta = \langle \mathrm{Hess}(\mathrm{md}_\kappa \circ r_\eta)c', c' \rangle$$

and we can estimate using the operator inequality from above to get

$$h''_\eta \leq \mathrm{cs}_\kappa \circ r_\eta \circ c + \frac{\mathrm{sn}_\kappa(\eta)}{\mathrm{sn}_\kappa(r_\eta \circ c - \eta)}$$

Observe further, that for $\eta < \eta_0$ (some small constant), we have

$$\delta \leq r_\eta \circ c(t) - \eta \leq \frac{\pi}{\sqrt{\kappa}} - 2\epsilon$$

where $\delta = \delta(\eta_0)$ is some constant. Remembering $\mathrm{cs}_\kappa + \kappa \mathrm{md}_\kappa = 1$, we get

$$\begin{aligned} h''_\eta + \kappa h_\eta &\leq \mathrm{cs}_\kappa \circ r_\eta \circ c + \frac{\mathrm{sn}_\kappa(\eta)}{\mathrm{sn}_\kappa(r_\eta \circ c - \eta)} + \kappa \mathrm{md}_\kappa \circ r_\eta \circ c(t) \\ &= 1 + \frac{\mathrm{sn}_\kappa(\eta)}{\mathrm{sn}_\kappa(r_\eta \circ c - \eta)} \leq 1 + K\mathrm{sn}_\kappa(\eta) \end{aligned}$$

where $K = K(\epsilon_0)$ is some constant. Observing $\overline{h}'' + \kappa\overline{h} = 1$, we thus get for the difference $\lambda_\eta = h_\eta - \overline{h}$:

$$\lambda''_\eta + \kappa\lambda_\eta \leq K\mathrm{sn}_\kappa(\eta)$$

Also

$$\lambda_\eta(t_0) = \lambda(t_0), \lambda_\eta \geq \lambda.$$

Case A: $\kappa < 0$ Suppose λ has a negative minimum $-\mu$ at t_0. Then λ_η has a local negative minimum $-\mu$ at t_0 as well.
In this case, we have

$$\lambda''_\eta(t_0) \leq -\kappa\lambda(t_0) + K\mathrm{sn}_\kappa(\eta) = \kappa\mu + K\mathrm{sn}_\kappa(\eta).$$

Since $\kappa\mu < 0$ this is impossible for small η.
Case B: $\kappa = 0$ In this case, we define

$$\overline{\lambda}(t) = \lambda(t) + \eta\frac{t(l(c) - t)}{l(c)^2}$$

If λ has a negative minimum -2μ somewhere in $(0, l(c))$, then $\overline{\lambda}$ has a negative minimum $< -\mu$ somewhere in $(0, l(c))$. Now put

$$\overline{\lambda}_\eta(t) = \lambda_\eta(t) + \mu\frac{t(l(c) - t)}{l(c)^2}$$

and observe that

$$\overline{\lambda}_\eta \geq \overline{\lambda}, \overline{\lambda}_\eta(t_0) = \overline{\lambda}(t_0).$$

Thus, $\overline{\lambda}_\eta$ also has a negative local minimum at t_0. But we have

$$\overline{\lambda}''_\eta \leq K\mathrm{sn}_\kappa(\eta) - \frac{2\mu}{l(c)^2},$$

which again yields a contradiction for small η.

Case C: $\kappa > 0$ In this case, we define

$$\sigma_\epsilon(t) = \mathrm{sn}_\kappa(t+\epsilon) - \mathrm{sn}_\kappa(\epsilon/2).$$

We have $l(c) \leq \pi/\sqrt{\kappa} - 2\epsilon$ and therefore σ_ϵ is positive on the interval $[0, l(c)]$, increasing at the beginning and decreasing at the end. Hence, the function

$$\hat{\lambda} = \frac{\lambda}{\sigma_\epsilon}$$

has a negative minimum in $(0, l(c))$, say at t_0. As before, look at

$$\hat{\lambda}_\eta = \frac{\lambda_\eta}{\sigma_\epsilon}$$

which has

$$\hat{\lambda}_\eta \geq \hat{\lambda}, \hat{\lambda}_\eta(t_0) = \hat{\lambda}(t_0).$$

Thus, as before, $\hat{\lambda}_\eta$ has a negative minimum $-\mu_0$ at t_0.
We differentiate to get

$$0 = \hat{\lambda}_\eta'(t_0) = \frac{\lambda_\eta'\sigma_\epsilon - \lambda_\eta\sigma_\epsilon'}{\sigma_\epsilon^2}(t_0),$$

from which we get

$$\hat{\lambda}''(t_0) = \frac{1}{\sigma_\epsilon^2}(\sigma_\epsilon\lambda_\eta'' - \sigma_\epsilon''\lambda_\eta)(t_0).$$

$$= \frac{1}{\sigma_\epsilon^2}(\sigma_\epsilon\lambda_\eta'' + \kappa\mathrm{sn}_\kappa(t+\epsilon)\lambda_\eta'') = \frac{1}{\sigma_\epsilon^2}((\lambda_\eta'' + \kappa\lambda_\eta)\sigma_\epsilon + \kappa\lambda_\eta\mathrm{sn}_\kappa(\epsilon/2))$$

$$\leq \frac{1}{\sigma_\epsilon(t_0)}K\mathrm{sn}_\kappa(\eta) - \kappa\mu_0\mathrm{sn}_\kappa(\epsilon/2)\frac{1}{\sigma_\epsilon(t_0)^2}$$

which, again, is < 0 for small η.

Step 2: Assume that $\kappa > 0$**, and that** $l(c) + l(c_1) + l(c_2) \leq 2\pi/\sqrt{\kappa}$**.** First observe that $\mathrm{diam}(M) \leq \pi/\sqrt{\kappa}$ by Bonnet-Myers (from the curvature bound). Now, choose κ_i with $\kappa_i \to \kappa$ from below. By Step 1, we can perform comparisons with spheres N_{κ_i} and get the desired comparsion. Now, the claim follows from continuity.

Step 2: Assume that $\kappa > 0$ **and that** $l(c) + l(c_1) + l(c_2) > 2\pi/\sqrt{\kappa}$**.** Choose δ so that $l(c) + l(c_1) + l(c_2) = 2\pi/\sqrt{\delta}$. We have $\delta < \kappa$, and so we can perform comparison with N_δ. There, the comparison triangle is now a great circle, and the antipode $-\overline{q}$ is on $\overline{c}$. Thus,

$$\pi/\sqrt{\delta} = d(-\overline{q}, \overline{q}) \leq d(q, c(t_0))$$

for some t_0. But then

$$d(q, c(t_0)) \leq \mathrm{diam} M \leq \pi/\sqrt{\kappa} < \pi/\sqrt{\delta}$$

which is a contradiction. $\square$

LECTURE 22 (JULY 18)

Sphere diameter theorem.

Theorem 2.66. *Suppose that M^n is complete, $K \geq \delta > 0$ and* $\mathrm{diam} M > \pi/2\sqrt{\delta}$. *Then M is homeomorphic to S^n.*

The idea of the proof is simple: we find the two "poles" and then construct a vector field corresponding to a flow from north to south pole. This flow will then yield the desired homeomorphism.
In order to carry out this strategy, we need to study distance functions and vector fields a bit more.

Definition 2.67. Let $A \subset M$ be closed. Then define d_A to be the distance to the set A: $d_A(q) = \inf d(a, q)$.
We say that $q \in M$ is *critical for d_A (or A)* if for any $v \in T_qM$ there is a distance minimising geodesic c from q to A with $g(v, c'(0)) \geq 0$. Otherwise, say q is a *regular point*.

A way to rephrase this is: q is regular if the velocities $c'(0)$ of all minimisers are contained in an open halfspace $g(c'(0), v) < 0$.
Some examples:

Example 2.68.
1. $A = \{p\}$ in $\mathbb{R}^2$. Then there are no critical points.
2. $A = \{p\}$ in S^2. Then the antipode is the only critical point.
3. A the equator in S^2. Then the critical points are the two poles of maximal distance.
4. A a point on the flat cylinder of diameter one. Then q is critical for p if it is opposite. This shows in particular that even if there are two minimisers a point need not be critical.
5. A the center of the square torus. Then the critical points are the origin and the middle points of the sides.

Lemma 2.69. *Let M be complete and $A \subset M$ be closed. Then for any regular point q for A there is an open neighbourhood U of q and a unit vector field X on U so that*

$$g(X(r), c'(0)) < 0$$

for any $r \in U$ and any minimising geodesic c from r to A.

Proof. Take v as above, a tangent vector in T_qM which has negative scalar product with minimisers at A. Extend v to a local unit vector field X. Suppose that there would be $q_i \to q$ and c_i minimisers from q_i to A with $g(c_i'(0), X(q_i)) \geq 0$. Then, as c_i converge to a minimiser from q to A we get a contradiction. This shows that X has the desired property in a neighbourhood. $\square$

We call a vector field like in the lemma *gradient-like*.

Lemma 2.70. *Let M be complete and $A \subset M$ be closed. Then*

a) The set of regular points is open.
b) There is a gradient-like vector field defined on the set of regular points.

Proof. Part a) is an immediate consequence of the previous lemma. Part b) follows by taking a sum of the local gradient-like vector fields with a partition of unity. Observe that the defining angle property is stable under convex combinations. Hence, the (norm 1 rescaling) has the desired property. □

Lemma 2.71. *Let M be complete, $A \subset M$ be closed, U open, and X gradient-like on U for A. Let Φ be the flow for X and Ψ the flow for $-X$. Then:*

a) d_A is strictly decreasing along integral curves of $-X$ (the flow Ψ strictly decreases d_A).

b) For any compact $C \subset U$ there is a constant L so that

$$d_A\Phi(q, t_0 + \tau) \leq d_A\Phi(q, t_0) - \tau L$$

as long as $\Phi(q, t_0 + \sigma) \in C$ for all $0 \leq \sigma \leq \tau$.

c) For any compact $C \subset U$ there is a constant L so that

$$d_A\Psi(q, t_0 + \tau) \geq d_A\Phi(q, t_0) + \tau L$$

as long as $\Phi(q, t_0 + \sigma) \in C$ for all $0 \leq \sigma \leq \tau$.

Proof. It is clear that b) and c) are equivalent, and that a) is implied by b). To show b), observe that there is a constant L so that

$$g(-X(q), c'(0)) \geq 2L > 0$$

for all $q \in C$ and all minimising geodesics c from q to A. Namely, otherwise, as above, we can take a limit of mimimisers and contradict the defining property of X (here, we use that C is compact, and so the limiting point is in U).
Now define

$$h(t) = d_A\Phi(q, t)$$

Suppose q and t_0 are given, and choose some $p \in A$ with

$$h(t_0) = d_A\Phi(q, t_0) = d(p, q).$$

Choose c a minimising geodesic from $\Phi(q, t_0)$ to p. For a (small) η define

$$\overline{h}(t) = \eta + d(c(l(c) - \eta), \Phi(q, t)),$$

and observe that $\overline{h}(t)$ is differentiable close to t_0. Further, we have

$$\overline{h}(t_0) = h(t_0), \quad \overline{h}(t) \geq h(t)$$

and we can compute

$$\overline{h}'(t_0) = g(\mathrm{grad} d_{c(l(c)-\eta)}, \partial_t\Phi) = g(-c'(0), -X\Phi(q, t_0)) \leq -2L$$

and thus we have

$$h(t_0 - \tau) \leq \overline{h}(t_0 - \tau) \leq \overline{h}(t_0) - L\tau = h(t_0) - L\tau$$

for small τ. Since we can do this approximation for all t_0, the desired claim follows. $\square$

In particular, we have

Corollary 2.72. *A local maximum point of d_A is critical for A.*

LECTURE 23 (JULY 19)

We can now prove the diameter sphere theorem. By rescaling, we may assume $K \geq 1$ and $\mathrm{diam} M > \pi/2$. Now, as M is compact (by the curvature assumption) we can choose two points p, q maximising distance. We then know by the corollary that q is critical for p.

Lemma 2.73. *q is the unique point maximising distance to p.*

Proof. Suppose q_1, q_2 are two such points, and let c_1, c_2 be minimising geodesics from q_i to p and c from q_1 to q_2. By criticality of p for q_1, we may choose c_1 so that

$$\alpha_1 = \angle(c_1'(0), c'(0)) \leq \pi/2.$$

Put

$$l_1 = l(c_1) = l(c_2) = \mathrm{diam} M > \pi/2,$$

and

$$l = l(c) \leq \mathrm{diam} M = l_1$$

Hence, we can apply Toponogov, and consider the comparison triangle on S^n with sidelengths $l(c_1), l(c_2), l(c)$ and angle $\overline{\alpha}_1$. We then have

$$\overline{\alpha}_1 \leq \alpha_1 \leq \pi/2.$$

Apply the law of cosines in the sphere to get

$$0 \leq \sin(l_1)\sin(l)\cos(\overline{\alpha}_1) = \cos(l_1) - \cos(l_1)\cos(l) = (1 - \cos(l))\cos l_1 \leq 0$$

(since $\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos\alpha$). Therefore, $l = 0$, hence $q_1 = q_2$. $\square$

In fact,

Lemma 2.74. *Suppose q_2 is any point $q_2 \neq q, q_2 \neq p$ and let c be a minimal geodesic from q to q_2. Then for any minimal geodesic c_2 from q_2 to p. Then*

$$g(c'(1), c_2(0)) > 0,$$

in other words: q_2 is regular for p.

Proof. Let c_1 be a minimal geodesic from $q_1 = q$ to p. Since q is the unique distance maximiser to p, we have

$$l = l(c) < l(c_1) = l_1.$$

Similarly,

$$l_2 < l_1.$$

Consider the geodesic triangle c, c_1, c_2 with angle $\alpha_2 = \angle(c_2'(0), -c'(1))$ and consider the comparison triangle with angle $\overline{\alpha}_2$. Then we have

$$g(c_2(0), -c'(1)) = \cos\alpha_2 \le \cos\overline{\alpha}_2.$$

Using the spherical law of cosines again we see

$$\cos\overline{\alpha}_2 = \frac{\cos l_1 - \cos l \cos l_2}{\sin l \sin l_2} < 0$$

Since $\cos l_2 \ge \cos l_1$ and $\cos l_1 < 0$ we have $\cos\overline{\alpha}_2 < 0$ which shows the claim. $\square$

Now, let $\epsilon > 0$ be small enough so that the ϵ–ball around p, q are geodesic. Then

$$X_1 = \mathrm{grad}(d_p|_{B(p,\epsilon)-p})$$

is local gradient-like (for p), and by the lemma so is

$$X_2 = \mathrm{grad}(d_p|_{B(q,\epsilon)-q}).$$

Hence, we can find a gradient-like vector field X on $M - \{p, q\}$ which agrees with X_1, X_2 on $B(p, \epsilon/2) - p, B(q, \epsilon/2) - q$.
The flow for X changes distance by a fixed Lipschitz constant on $M \setminus B_p(\epsilon/2) \cup B_q(\epsilon/2)$ by a previous lemma. This implies that all flow lines of X have finite length, and additionally extend continuously into p, q at their endpoints.
Define

$$\varphi_v(t) = \Psi(\exp(\epsilon/2v), t - \epsilon/2)$$

for $v \in T_pM$ unit and let l_v be the length of the flowline. Define

$$F(t, v) = \varphi_v(tl_v)$$

for v unit and $t \in (0, 1)$. This extends by $F(0, v) = p, F(1, v) = q$ to a continous map sending the unit ball in T_pM to M, and its boundary to q. Hence, F induces a homeomorphism between the quotient space (which is S^n) to M. $\square$
Next, we will show a rigidity for the round sphere. To prove it, we need a geometric way to detect the sphere. This is done in the following way:
Suppose $M, \overline{M}$ are two Riemannian manifolds of the same dimensions and points $p, \overline{p}$. Choose an isometry

$$I : T_pM \to T_{\overline{p}}\overline{M}$$

and suppose that $V \subset M$ is a normal neighbourhood of p so that $\exp_{\overline{p}}$ is defined on $I \circ \exp_p^{-1}(V)$. Define

$$f = \exp_{\overline{p}} \circ I \circ \exp_p^{-1}$$

on V. Given any $q \in V$, by normality of V there is a unique geodesic γ joining p to q. Let P_t parallel transport along γ and $\overline{P}_t$ parallel transport along the geodesic with initial data $\overline{p}, I\gamma'(0)$. Define

$$\phi_t(v) = \overline{P}_t \circ I \circ P_t^{-1}$$

Lemma 2.75. *Suppose that for all q, γ as above, and all $x, y, u, v \in T_qM$ we have*

$$R(x, y, u, v) = \overline{R}(\phi_t x, \phi_t y, \phi_t u, \phi_t v)$$

Then f is a local isometry and $d_q f = I$.

Proof. Take $q, \gamma : [0, l] \to V$ as above. Let $v \in T_qM$ be given, and let J be a Jacobi field with $J(0) = 0, J(l) = v$. Take an orthonormal basis $e_1, \ldots, e_n$ of T_pM with $e_n = \gamma'(0)$, and parallel transport along γ. Write $J(t) = \sum y_i(t)e_i(t)$ and compute

$$y_j'' + \sum R(e_n, e_i, e_n, e_j)y_i = 0$$

Put

$$\overline{J}(t) = \phi_t(J(t)), \quad \overline{e}_i(t) = \phi_t(e_j(t))$$

Observe that the $\overline{e}_i(t)$ are parallel, and therefore

$$\overline{J}'' = \sum y_i''(t)\overline{e}_i(t) = -\sum R(e_n, e_i, e_n, e_j)y_i\overline{e}_i(t) = -\sum \overline{R}(\overline{e}_n, \overline{e}_i, \overline{e}_n, \overline{e}_j)y_i\overline{e}_i(t)$$

and thus $\overline{J}$ is a Jacobi field. Since parallel transport is an isometry, $\|\overline{J}(l)\| = \|J(l)\|$.
Next, observe that $\overline{J}(t) = \phi_t(J(t))$ implies that $\overline{J}'(0) = IJ'(0)$. Thus, we have

$$J(l) = d_{l\gamma'(0)} \exp_p(lJ'(0))$$

$$\overline{J}(l) = d_{l\overline{\gamma}'(0)} \exp_{\overline{p}}(l\overline{J}'(0))$$

which implies

$$\overline{J}(l) = d_q f(J(l))$$

Thus, $d_q f$ is norm preserving, hence an isometry. This shows that f is a local isometry as claimed. □

In fact, there is a global version of this:

Theorem 2.76 (Cartan-Ambrose-Hicks)**.** *Assume in addition to above that $M, \overline{M}$ are simply connected, and that the curvature condition holds along all piecewise geodesics.*
Then the map which sends an endpoint of the broken geodesic to the endpoint of the corresponding broken geodesic is an isometry.

The proof uses a refinement of the previous argument (to broken geodesics) and also some topology (covering space theory), and so we don't give the proof. Compare Cheeger-Ebin, Theorem 1.42, for a full proof.
A simple corollary is:

Corollary 2.77. *Suppose M is simply connected and has constant curvature κ. Then M is isometric to N_κ.*

Proof. This follows from the observation that if M is a manifold with constant curvature κ, then

$$R(X, Y, W, Z) = \kappa(g(X, W)g(Y, Z) - g(Y, W)g(X, Z)).$$

Namely, two tensors with the symmetry of the Riemann curvature tensors and the same section curvatures are equal (we proved this last semester). Now, if M is constant curvature, then by the above and the fact that parallel transport is an isometry, the curvature condition of Cartan-Ambrose-Hicks is satisfied. □