Exercise 1. Rays
Let M be a complete, noncompact Riemannian manifold.

(a) Show that there is a geodesic $\gamma : [0, \infty) \to M$ so that

$$d(\gamma(r), \gamma(s)) = |r - s|$$

for all r, s.

(b) Let γ be a geodesic as in (a). Show

$$\liminf_{t\to\infty} \min\{K(V), V \subset T_{\gamma(t)}M \text{ 2-dimensional}\} \le 0$$

where K denotes sectional curvature.

(c) Assume now in addition that M has non-negative sectional curvature everywhere. Suppose γ is as in (a), and σ is any geodesic with $\sigma(0) = \gamma(0)$ and $\angle(\sigma'(0), \gamma'(0)) < \pi/2$. Show that

$$\lim_{t\to\infty} d(\sigma(0), \sigma(t)) = \infty.$$

(d) By giving an example, show that the angle bound in (c) cannot be relaxed.

a) Since M is complete and noncompact, it has infinite diameter (otherwise, it would be contained in the image of a ball under the exponential map, hence compact). Thus, we can find length-realising unit speed geodesics $\gamma_i : [0, i] \to M$ with $\gamma_i(0) = p$ for some fixed p. Let $v_i = \gamma_i'(0)$. Up to subsequence, we have $v_i \to v$. Let γ be the geodesic starting in p with initial velocity v. We claim that it has the desired property.

   Namely, suppose not, i.e.

$$d(\gamma(r), \gamma(s)) = |r - s| - \epsilon$$

   for some $\epsilon > 0$. Since $v_i \to v$, and the exponential map is continuous, for large enough i we have

$$d(\gamma(r), \gamma_i(r)) < \epsilon/4, \quad d(\gamma(s), \gamma_i(s)) < \epsilon/4,$$

   and therefore

$$d(\gamma_i(r), \gamma_i(s)) < |r - s|.$$

   This is a contradiction to the fact that γ_i is length-realising.

b) Suppose this would be false, which would mean that there is some $\epsilon > 0$ and t_0 so that

$$K(V) \ge \epsilon > 0 \quad \text{for all } V \subset T_{\gamma(t)}M, t \ge t_0.$$

   As γ is length-minimising, it does not have any conjugate points. Thus, we can apply the Rauch comparison theorem to compare an arbitrary segment of γ to a segment of the same length on the sphere of curvature ϵ and conclude that it does not have conjugate points either. This is clearly absurd.

c) We will be considering a comparison hinge defined by $\sigma[0, s]$ and $\gamma([0, t])$ in $\mathbb{R}^2$. First observe that for any s there is a t_s so that the closed hinge has a right angle at $\widetilde{\sigma(s)}$ (since the angle α between σ, γ is strictly less than $\pi/2$). Also note that $t_s \to \infty$ as $s \to \infty$ (This is easier to explain with a figure).

   Now, the closing segment in $\mathbb{R}^2$ has length $\sin(\alpha)t_s$, and by Toponogov's theorem a (minimal) closing segment in M is shorter. Thus, we have

$$d(\sigma(0), \sigma(s)) \ge d(\sigma(0), \gamma(t_s)) - \sin(\alpha)t_s = t_s - \sin(\alpha)t_s \to \infty$$

   as $s \to \infty$.

d) This can be seen by a flat two-dimensional cylinder. A core circle is orthogonal to a ray.

Exercise 2. Homogeneous manifolds
Let M be a connected Riemannian manifold, and suppose that for any $p, q \in M$ there is an isometry $\phi_{p,q} : M \to M$ so that $\phi_{p,q}(p) = q$.

(a) Show that M is complete (*if you want to use a result from the pracice exam here, you need to reprove it!*)

(b) Does M have constant sectional curvature? Give a proof or a counterexample.

a) By the Hopf-Rinow theorem, it suffices to show that any geodesic can be extended indefinitely. So, suppose that $\gamma : [0,t] \to M$ is a geodesic. Take ϵ so that the exponential map is defined on $B_\epsilon(0)$ in $T_{\gamma(0)}M$.

   By assumption there is an isometry ϕ so that $\phi(\gamma(0)) = \gamma(t - \epsilon/2)$. Since isometries are diffeomorphisms, we may choose a $v \in T_{\gamma(0)}M$ so that $d\phi(v) = \gamma'(t - \epsilon/2)$, and let ρ be the geodesic starting in $\gamma(0)$ with velocity v.

   Since geodesics are determined by their initial point and velocity, we have that

$$\phi(\rho(s)) = \gamma(t - \epsilon/2 + s), \quad \forall 0 < s < \epsilon/2.$$

   Thus, the concatenation of γ and $\phi \circ \rho|_{[\epsilon/2,\epsilon]}$ is an extension of γ to $[0, t + \epsilon/2]$, showing the result.

b) No. We know e.g. from the exercises that $S^2 \times S^2$ does not have constant curvature. But, $O(2) \times O(2)$ is contained in the isometry group and acts transitively on points.

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Exercise 3. Jacobi fields
Let M be a Riemannian manifold, $\gamma : [0,l] \to M$ a geodesic, and J a Jacobi field along γ. Show that

$$g(J(t), \gamma'(t)) = g(J'(0), \gamma'(0))t + g(J(0), \gamma'(0)).$$

a) We compute

$$g(J(t), \gamma'(t))' = g(J'(t), \gamma'(t))$$

by the product rule and the fact that $\gamma'' = 0$ (since it is a geodesic). Similarly,

$$g(J(t), \gamma'(t))'' = g(J''(t), \gamma'(t)) = g(-R(\gamma'(t), J(t))\gamma'(t), \gamma'(t)) = 0$$

by the Jacobi equation and symmetries of the Riemann curvature tensor. Thus, $g(J(t), \gamma'(t))'$ is constant, which shows the claim.

Exercise 4. Geodesics

Suppose that M is a complete Riemannian manifold and suppose that γ is a unit speed curve. Assume that there is an isometry $F : M \to M$ so that $F(\gamma(t)) = \gamma(t)$ for all t. Additionally assume that

$$\ker(\mathrm{id} - d_{\gamma(t)}F) = \mathbb{R}\gamma'(t)$$

for all t. Show that γ is a geodesic. (*Hint: Consider a geodesic starting in $\gamma(t)$ with initial velocity $\gamma'(t)$ and compare them in a small geodesic ball)*

a) Suppose that γ is not geodesic. Then there is a time t_0 and σ be a geodesic starting in $\gamma(t_0)$ with initial velocity $\gamma'(t_0)$, so that $\sigma(\epsilon) \neq \gamma(t_0 + \epsilon)$ for some small ϵ. Let U be a small neighbourhood of $\gamma(t_0)$ so that any two points in U can be joined by a unique length-minimising geodesic (we know this exists from class). We may assume that $\gamma(t_0+\epsilon), \sigma(\epsilon) \in U$.

   Let $\rho \subset U$ be the unique length-minimising geodesic joining $\gamma(t_0)$ to $\gamma(t_0 + \epsilon)$. Note that $\sigma'(0)$ and $\rho'(0)$ are not proportional, as $\sigma(\epsilon) \neq \rho(\epsilon)$ and geodesics are determined by their initial data.

   On the other hand, since $F(\gamma(t)) = \gamma(t)$ for all t, and ρ is the unique length minimising geodesic joining its endpoint, we have $F \circ \rho = \rho$. Thus, we also have that $dF\rho'(0) = \rho'(0)$. This contradicts the assumption, as $\sigma'(0) = \gamma'(t)$.