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Differentiable Manifolds

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Warnings about this document

This "script" is initially not much more than a typed version of my lecture notes. What this means is that problems this document may have include (but are not limited to): typos, mistakes, sketches where arguments are "standard", gaps when the presentation depends on the audience, missing content if I get sidetracked in the lecture, complete lack of pictures etc.

If a mistake is found, I welcome a quick email, and I will try to fix it as soon as possible.

What the document will contain is the "canon" versions of definitions and theorems for this course[1]

1. Lecture 1: Motivation and Topological Spaces

1.1. **Motivation.** Basic idea of manifolds: study spaces which "look like" the familiar $\mathbb{R}^n$ *locally* (i.e. around any point) but are not necessarily subsets of them, and develop a workable theory of *calculus* on such objects.
Before we start, let's briefly address the basic question: why do we want that?

Manifolds are everywhere. Many useful, familiar, and natural objects fall into this category.
Since they are so prevalent one goal of this class is to give lots of questions to think about more freely (as opposed to problem sets which have to be solved), and indicate other areas of maths which might be interesting.
Some examples, before we even know the definition.

Example 1.1 (Basic (sub)manifolds)**.** *(1) Planes, lines, ...*
(2) Surfaces in $\mathbb{R}^3$
(3) The sphere:

$$S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^n,\ x_0^2 + \cdots + x_n^2 = 1\}$$

Example 1.2 (Moduli spaces)**.** *Define*

$$\mathcal{L} = \{L \subset \mathbb{R}^2,\ L \text{ line through } 0\}$$

[1]What I mean by this is the following: for many objects and theorems, there are many different versions varying in flavour and strength, but sharing the same name. In case of doubt, the one in this document is the one you are able to refer to when you write something was done "in class".

The set *of lines $\mathcal{L}$ can be given more structure. Namely, such a line L is determined by its slope, or alternatively the angle $\alpha_x(L)$ it makes with the x–axis. Caveat: lines of angle α and $-\alpha$ are actually the same.*
However, if L is a line with $\alpha_x(L) = a$, then angles $(a-\epsilon, a+\epsilon)$ parametrise in a one-to-one fashion those lines which make an angle $< \epsilon$ to L.
Hence, the space *$\mathbb{R}P^1$ of such lines "locally" looks like an interval (or the real line). Globally, the structure is different: if we keep increasing the angle, at some point we return to the same line L.*
It might be instructive to think about the case of lines in $\mathbb{R}^3$. What about planes in $\mathbb{R}^3$?

Example 1.3 (Configuration spaces)**.** *Suppose we have two different (point) particles in a circle that cannot occupy the same position simultaneously. What is the space of configurations? A guess is:*

$$S^1 \times S^1 \setminus \Delta \subset \mathbb{R}^4$$

where Δ is the diagonal: $\Delta = \{(x, x) \in \mathbb{R}^2, x \in \mathbb{R}^2\}$. This is not quite correct, as the particles are not labelled! So, we should also identify (x, y) with (y, x) to get the desired space. Locally, the result still looks like $\mathbb{R}^2$ (both particles can freely move a bit in two directions), but globally it is different (as particles cannot collide, and we return to the same point if we exchange the positions of the two particles).
Try to imagine how this space "looks"! (Hint: $S^1 \times S^1$ is a familiar surface that can be glued from a square)

Play around with the configuration space of two (three, four...) points in the plane. Can you spot a connection to polynomials?

Relatedly, suppose we have a double pendulum. Consider the space of all configurations: we get a subset of $\mathbb{R}^4$ of all possible positions, which is quite complicated...

Local definition is natural, global is not. Often, these spaces are not naturally subsets of an $\mathbb{R}^n$, or if they are, the point off of the subset have no "meaning" in the problem.

Example 1.4. *$\mathbb{R}P^1$ could be embedded as a circle in $\mathbb{R}^2$, but the points off the circle then have no meaning for the question of characterising lines.*
The same is true for the configuration space of points on a circle.

On the other hand, thinking of the space abstractly might actually be an advantage, since it suggests to us that we can choose different, more useful coordinate systems:

Example 1.5. *For the double pendulum, thinking of the points as pairs in $\mathbb{R}^2$ (as for the configuration space), makes the problem seem mysterious – but we could also parametrise by two angles, giving a much more useful coordinate system.*

Calculus has meaning. Calculus will mean: develop theory of differentiation and integration for various objects. While this is interesting in its own right, the most striking reason are application to physics/geometry: volume, curvature...

1.2. **Brief Reminder 1: Basic Topology of $\mathbb{R}^n$.** For "spaces that look like $\mathbb{R}^n$", we first need spaces. Describing what a space is, and their fundamental properties is topology. We will use some ideas of point-set topology throughout, and they are crucial to understand manifolds as well. Make sure you are familiar with them, otherwise ask in your problem session!
Recall the following concepts from point-set topology in $\mathbb{R}^n$:

- i) Open sets, closed sets in $\mathbb{R}^n$: a set $U \subset \mathbb{R}^n$ is *open* if for every point $x \in U$ some ball $B_\epsilon(x)$ is contained in U. A set C is *closed*, if its complement $\mathbb{R}^n - C$ is open.
- ii) Open/Closed in a subset: suppose that $M \subset \mathbb{R}^n$ is a set. A set $U \subset M$ is *open in* M if there is an open set $U' \subset \mathbb{R}^n$ so that $U = M \cap U'$. A set $C \subset M$ is *closed in* M if $M - C$ is open in M.
- iii) Convergence: $a_i \to a$ if for any open set U containing a, eventually $a_i \in U$ for all large i. (Show that this is equivalent. Observe how Hausdorff interacts with uniqueness of limits)
- iv) Continuity via open/closed: a function $f : \mathbb{R}^n \to \mathbb{R}^m$ (or $f : M \to \mathbb{R}^m$) is *continuous* if and only if the preimage of every open set is open (or open in M). Show that this is equivalent to the familiar ϵ/δ-definition, if you have never done so before!
- v) Compact Sets: A set $K \subset \mathbb{R}^n$ is compact if every cover of K by open sets has a finite subcover. Equivalently: every sequence in K has a convergent subsequence with limit in K. The Heine-Borel theorem states that in $\mathbb{R}^n$ a set is compact if and only if it is closed and bounded.
- vi) Homeomorphisms are maps that are continuous, bijective, and so that their inverses are also continuous. Try to think of an example of a bijective continuous map whose inverse is not continuous!

$\mathbb{R}^n$ has two other important topological properties, which we will encounter again later.

Definition 1.6 (Hausdorff)**.** For any two points $x \neq y$, there are open sets U, V which are disjoint and so that $x \in U, y \in V$.

In other words, open sets separate points. If you think this is trivial or obvious for $\mathbb{R}^n$ – it is, but read on a bit to see why it is nevertheless important.

Definition 1.7 (Second Countability)**.** There is a countable family $U_i, i \in \mathbb{N}$ of open sets so that any open set U can be written as the union of some subsequence of the U_i.

To get the U_i, take balls with rational centers and rational radii (show that this works!).

1.3. **Topological Spaces for Very Impatient People.** Let S be a set. A *topology on* S is a set $\mathcal{T}$ of subsets of S with the following properties

(1) $\emptyset \in \mathcal{T}, S \in \mathcal{T}$,
(2) If $U_i \in \mathcal{T}, i \in I$ (*any* index set), then $\cup_{i\in I} U_i \in \mathcal{T}$.
(3) If $U_1, \dots, U_k \in \mathcal{T}$, then $\cap_{i=1}^k U_i \in \mathcal{T}$.

We then call (!) the elements of $\mathcal{T}$ the *open sets*. Show (convince) yourself that the usual open sets of $\mathbb{R}^n$ (and open sets in M) satisfy this!
The *pair* $(S, \mathcal{T})$ is called a topological space. I cannot stress enough that the set S itself does not determine the space. When we say: the space $\mathbb{R}^n$ we should really say: "$\mathbb{R}^n$ with the usual topology".

The definitions from the brief reminder all make sense for topological spaces, and allow to talk about convergence, compactness, continuity etc. for these general objects. Familiarise yourself with the way of thinking/proving like this by trying to re-prove some results on continuous functions from analysis.

2. Lecture 2: Defining Manifolds (first badly, then correctly)

2.1. **Towards the correct definition of manifolds.** Some examples of topological spaces: the first two to caution you that a topological space can be much stranger than $\mathbb{R}^n$, the last one to give a very large class of reasonable spaces. To get a feel for how the spaces behave: figure out when sequences converge in these topologies?

(1) ("The stupid examples") $\mathcal{T} = \mathbb{P}(S)$ (the discrete topology) makes every set open. Every function is continuous, only eventually constant sequences converge.
 $\mathcal{T} = \{\emptyset, S\}$ makes the minimal number of sets open. Only constant functions to $\mathbb{R}$ are continuous, every sequence converges to every point.
(2) On $\mathbb{C}^2$, let $\mathcal{T}$ contain all sets of the form $\mathbb{C}^2 \setminus C$, where C is the solution set of a finite number of algebraic equations
$$C = \{(x, y) \in \mathbb{C}^2 | f_1(x, y) = f_2(x, y) = \cdots = f_n(x, y) = 0\}$$
for polynomials $f_1, \dots, f_n$. This is called the Zariski topology. Showing that it is a topology is not trivial (infinite unions), but can be done. This is very much not Hausdorff: any two open sets intersect.
(3) Suppose (M, d) is a metric space. Define *open* as in $\mathbb{R}^n$. This gives a topology. As for $\mathbb{R}^n$, the topologists' notions of continuity, convergence etc. agree with the familiar ones from analysis. See problem set 1. This topology is automatically Hausdorff.

We want "space that looks like $\mathbb{R}^n$", so let's try to say that around every point there are coordinates:

Definition 2.1 (Manifolds, badly)**.** A manifold is a topological space M, so that for every point $p \in M$ there is a neighbourhood U, and a homeomorphism $\varphi : U \to V$, $V \subset \mathbb{R}^n$ open.

We can then think of the component functions $\varphi^i(u) = x^i$ as "coordinates" around the point. If $f : M \to \mathbb{R}$ is a function, and $\varphi : U \to V$ is a chart, then we can consider $f \circ \varphi^{-1} : V \to \mathbb{R}$. This is then a function in the coordinates x^i.

This definition has two problems, and we will explore them. First, it still allows some strange behaviours:

Example 2.2 (The line with two origins)**.** *Let*

$$X = \mathbb{R} \cup \{0'\}$$

and define a topology as follows: a set $U \subset X$ is open if

(1) *For any $u \in U, u \neq 0'$ there is some $\epsilon > 0$ and $(u-\epsilon, u+\epsilon) \subset X$.*
(2) *If $0' \in U$, then there is some $\epsilon > 0$ so that $(-\epsilon, 0) \cup (0, \epsilon) \subset X$.*

Intuitively, we glue two copies of $\mathbb{R}$ at every point except the origin.
This would satisfy our definition: small intervals around any point are homeomorphic to intervals in $\mathbb{R}$. However, the result is not Hausdorff, and so it is a fairly unpleasant space (does not admit a metric, has no unique limits of sequences...)

One could also build a space that is locally homeomorphic to $\mathbb{R}$, but not second countable, but we will skip the construction. Look up "the long line" if you are interested.

The second, more serious problem with Definition 2.1 is there is no way to define what a differentiable function is with this definition. One might try: $f : M \to \mathbb{R}$ is a function, then it is differentiable if in coordinates, $f \circ \varphi^{-1}$ are differentiable. The problem is that there is no reason why this is independent of the choice of coordinates.
So, we really want to require some kind of compatibility between the different coordinates. This is done by the following definition.

Definition 2.3. Let M be a topological space. A *smooth (n–dimensional) atlas for M* is a collection

$$\mathcal{A} = \{(U_i, V_i, \varphi_i), i \in I\}$$

where

(1) For any i, the set U_i is open in M, and V_i is open in $\mathbb{R}^n$,
(2) For any i, $\varphi_i : U_i \to V_i$ is a homeomorphism,
(3) If $U_i \cap U_j \neq \emptyset$, then the composition
$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$
is a smooth map (between subsets of $\mathbb{R}^n$), and
(4) Every point $p \in M$ is contained in some $U_i, i \in I$.

The φ_i are called *charts*, and we say that φ *is a chart around* p if $\varphi = \varphi_i$ where $p \in U_i$. The functions in (3) are called *chart transitions.*

Definition 2.4. A *smooth manifold* is a topological space M, which is Hausdorff and second countable, together with a smooth atlas.

Some remarks:

(1) The atlas is part of the data of a manifold (just like the topology was part of the data of a topological space). This leads to the question of when two manifolds are "the same". One way is to only consider maximal atlases (which are a bit cumbersome, but are a standard way of handling things, compare e.g. Warner). The other way (that we will take) is to ignore this issue, since "the same" is less useful than diffeomorphic, see below.

(2) Even though the atlas is cruicial, we (and everyone else) will usually suppress it from the notation and simply say: "Let M be a manifold". Similarly we will say "chart of M".

(3) The definition of smooth atlas suggests variants: one can ask other properties instead of "smooth" for the chart transitions.

One possibility is to ask less regularity, e.g. $\mathcal{C}^k$ or just homeomorphisms. Many theorems we will prove would work in that setting as well (but we don't want to be optimal with our prerequisites...)

The exact relation between these is subtle, and part of *differential topology*. We will not deal with this, but want to remark: there are topological manifolds that do not admit a smooth structure at all, and there are manifolds that admit many essentially different smooth structures.

(4) Another possibility would be to ask for more regularity: for example we could look at even dimensions $\mathbb{R}^{2n} = \mathbb{C}^n$, and require the transitions to be holomorphic. Then we get the theory of *complex manifolds*, which is very rich and interesting – and also not part of this course. Learn about *Riemann surfaces* for a hands-on and very beautiful area, or *Complex geometry* for more abstract ideas.

(5) We could also ask the transitions to preserve our favourite notion in $\mathbb{R}^n$: length, angles, straight lines... This leads to various *geometric structures* (flat, conformal, affine manifolds). Continue in *Riemannian geometry* if this sounds interesting.

We've seen examples (but haven't proved that they are manifolds yet, bear with me for a bit), so before we return to more examples, let us see that it indeed solves the problem we encountered before.

Definition 2.5. Let M, N be a smooth manifolds. A *smooth function* $f : M \to N$ is a function so that for any point $p \in M$, for any chart φ of M around p and for any chart ψ of N around $f(p)$ the function

$$\psi \circ f \circ \varphi^{-1}$$

is smooth (where defined).

Lemma 2.6. *Instead of "for any chart" we could write "for some chart" in Definition 2.5.*

Proof. Chart transitions and there inverses are smooth. In other charts, the function would be

$$\psi' \circ f \circ (\varphi')^{-1} = (\psi' \circ \psi^{-1}) \circ \psi \circ f \circ \varphi^{-1} \circ (\varphi' \circ (\varphi')^{-1}),$$

so it is smooth. □

3. Lecture 3: Examples of Manifolds

Back to our examples. We use them to learn some basic tools to show that topological spaces are Hausdorff and second countable as well.
Recall:

Example 3.1. *Planes, Lines, Spheres in* $\mathbb{R}^n$*.*

Where does the topology come from? This is a general construction, called *subspace topology.*

Definition 3.2. Suppose that $(S, \mathcal{T})$ is a topological space, and $M \subset S$ is a set. The *subspace topology* is given by

$$\mathcal{T}_{\text{sub}}(M) = \{U \cap M | U \in \mathcal{T}\}.$$

Check (for yourself) that this is indeed a topology. Also observe that the sets in $\mathcal{T}_{\text{sub}}(M)$ are (*tautologically*!) open in M, but need not be open in S.

Lemma 3.3. *If* $(S, \mathcal{T})$ *is Hausdorff or second countable, then so is* $(M, \mathcal{T}_{\text{sub}}(M))$ *for any* $M \subset S$*.*

Proof. For Hausdorff: intersect the separating open sets with S.
For second countable: intersect the countable base with S. □

Hence, the topological spaces from Example 3.1 automatically satisfy our topological requirements for a manifold.
For subspaces, a single chart suffices: suppose $v_1, \ldots, v_k$ is a basis for a subspace, then

$$\varphi^{-1} : \mathbb{R}^k \to \mathbb{R}^n, \quad (x^1, \ldots, x^k) \mapsto \sum x^i v_i$$

defines a chart.
For spheres, one chart is not enough, so we need at least two. One standard way of producing charts is *stereographic projection.* We describe this for $S^2 \subset \mathbb{R}^3$, but it works in any dimension. Define

$$U_S = \{(x, y, z) \in S^2 | z < 1\}$$

and let

$$\varphi_S : U_S \to \mathbb{R}^2, \quad (x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z} \right).$$

Similarly, put

$$U_N = \{(x, y, z) \in S^2 | z > -1\}$$

and let

$$\varphi_N : U_N \to \mathbb{R}^2, \quad (x, y, z) \mapsto \left(\frac{x}{1+z}, \frac{y}{1+z} \right).$$

By the definition of subspace topology, U_S, U_N are open and they cover S^2. Hence, we only need to show that φ_N, φ_S are homeomorphisms, and that the (two) chart transitions are smooth. To do to first, we can simply write down an inverse and observe that it is continuous, e.g. for φ_S: Put

$$z(x,y) = \frac{x^2+y^2-1}{1+x^2+y^2},$$

and

$$\psi : \mathbb{R}^2 \to U_S, \quad (x,y) \mapsto ((1-z(x,y))x, (1-z(x,y)y, z(x,y)),$$

which one can check to be an inverse. Now, we could just compute the chart transitions and observe that they are indeed smooth.

For the second example

Example 3.4 (Projective Space)**.** *Define*

$$\mathbb{R}P^{n-1} = \{L \subset \mathbb{R}^n |\ L \textit{ line through } 0\} = \mathbb{R}^n \setminus \{0\}/\sim$$

where $x \sim y$ if and only if $y = \lambda x, \lambda \neq 0$.

we deal with *quotient topology*, which allows to equip the quotient of a topological space by some equivalence relation with a natural topology.

Definition 3.5. Suppose that $(S, \mathcal{T})$ is a topological space, and $f : S \to M$ is a surjective map. The *quotient topology* is given by

$$\mathcal{T}_{\text{quot}}(M) = \{U \subset M | f^{-1}(U) \in \mathcal{T}\}.$$

Check (for yourself) that this is indeed a topology. Quotient spaces in general neither inherit Hausdorff (think of the line with two origins) nor second countable. The next lemma gives a useful way to check second countable; Hausdorff usually requires an ad-hoc argument.

Lemma 3.6. *Suppose that $(S, \mathcal{T})$ is a second countable topological space, and that $(M, \mathcal{T}_{\text{quot}}(M))$ is a space with the quotient topology, induced by some map $f : S \to M$. Suppose that f is open. Then M is second countable.*

Proof. Take the images of the sets U_i certifying second countable. They are open by openness of f and obviously have the desired property. □

So, let us return to $\mathbb{R}P^{n-1}$. We have a map

$$f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}P^{n-1},$$

and consider some open ball B. Then

$$f^{-1}(f(B)) = \mathbb{R}B \setminus \{0\}$$

and hence f is open. With the quotient topology, $\mathbb{R}P^{n-1}$ is thus second countable.
To see Hausdorff, observe that

$$f : S^{n-1} \to \mathbb{R}P^{n-1}$$

is already surjective, and $f^{-1}([v]) = \pm v$. Take two different points represented by v, w unit vectors. Choose $\epsilon < 1/2$ small enough so that

$$(B_\epsilon(v) \cup B_\epsilon(-v)) \cap (B_\epsilon(w) \cup B_\epsilon(-w)) = \emptyset.$$

By openness of f, the sets $U_v = f(B_\epsilon(v)), U_w = f(B_\epsilon(w))$ are open neighbourhoods of $[v], [w]$. Suppose that they would not be disjoint. Then there would be a unit vector z so that $z \in f^{-1}(U_v) \cap f^{-1}(U_w)$. But, such a unit vector would have to be contained in $(B_\epsilon(v) \cup B_\epsilon(-v)) \cap (B_\epsilon(w) \cup B_\epsilon(-w))$ which is impossible.
So, we are left with defining charts. For $n = 1$ this could be done via angles, as in the beginning. For higher n this is on problem set 2.

For the third example

Example 3.7. *Let C be the configuration space of two points in S^1, i.e.*

$$C = (S^1 \times S^1 \setminus \Delta)/\sim,$$

where $(x, y) \sim (y, x)$.

we combine both methods. First, observe that we can equip

$$S^1 \times S^1 \setminus \Delta \subset \mathbb{R}^3$$

with the subspace topology, making it Hausdorff and second countable. Then, we have a quotient map

$$f : S^1 \times S^1 \setminus \Delta \to C,$$

and we use it to equip C with the quotient topology. We have

$$f^{-1}(f(U)) = \{(x, y) \in S^1 \times S^1 | (x, y) \in U \text{ or } (y, x) \in U\}$$

and so f is open, making C second countable. Similarly to the projective space example, we can show Hausdorff: suppose that $[x, y], [x', y']$ are two different points in C (that means: $(x, y) \neq (x', y')$ and $(x, y) \neq (y', x')$). We may assume without loss of generality that $x \neq x', x \neq y'$. Hence, we can choose a number ϵ so that the ϵ–neighbourhoods around the (three of four points) $\{x, y, x', y'\}$ do not intersect.
The sets

$$f(B_\epsilon(x) \times B_\epsilon(y)), q(B_\epsilon(x') \times B_\epsilon(y'))$$

are then open and disjoint, and separate $[x, y]$ and $[x', y']$.

To define charts, we can use e.g. the following maps

$$g(\theta, \phi) = \left[\left(\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix}, \begin{pmatrix}\cos(\phi)\\ \sin(\phi)\end{pmatrix}\right)\right]$$

For any pair θ, ϕ so that $\phi \neq \theta + 2\pi k$, the image $g(\theta, \phi)$ defines a point in C. For such angles, there is an $\epsilon > 0$ so that

$$g|_{(\theta-\epsilon,\theta+\epsilon)\times(\phi-\epsilon,\phi+\epsilon)} : (\theta - \epsilon, \theta + \epsilon) \times (\phi - \epsilon, \phi + \epsilon) \to C$$

is injective, and has continuous inverse on its image. Chart transitions of these maps can be written in terms of (suitable branches of) arccos, arcsin, cos, sin and are therefore smooth.

4. Lecture 4: Three Views on Tangent Vectors

4.1. **Tangent vectors.** Want to define: directions at a point in a manifold. There are three equivalent formulations, all of which are important, in order to increasing abstraction.

4.1.1. *Tangent vectors via charts.* Key idea: we know already what a direction is in $\mathbb{R}^n$ (just any vector). Push these to manifolds via the charts.

Definition 4.1 (Tangent vector, transformation version)**.** Suppose $p \in M$ is a point. A *tangent vector at* p assigns to each chart $\varphi : U \to V$ with $p \in U$ a vector $v = (v^1, \dots, v^n)$, so that the following holds: if $w = (w^1, \dots, w^n)$ is the vector assigned to a different chart $\psi : U' \to V'$, then

$$w^k = \sum_i \frac{\partial(\psi \circ \varphi^{-1})^k}{\partial x^i} v^i.$$

In other words

$$w = D_{\varphi(p)}(\psi \circ \varphi^{-1})(v).$$

Observe how the indices match: the index we sum over appears once as sub-, once as superscript.
This definition is fairly concrete, but the resulting object is somewhat hard to carry around: a vector per chart. Being concrete, we can immediately use it to show something about the set of tangent vectors.

Lemma 4.2. *The set of tangent vectors at p is a real vector space $T_p^{\mathrm{coord}}M$ of dimension n with respect to simultaneous addition/scalar multiplication of all local vectors. In fact, the map which assigns to a tangent vector the vector assigned in a single chart φ yields an isomorphism to $\mathbb{R}^n$.*

Proof. The compatibility equation in Definition 4.1 is compatible with addition/scalar multiplication, showing that it is a vector space. Since $D_{\varphi(p)}(\psi \circ \varphi^{-1})$ is an isomorphism, for any pair of charts, a tangent vector in the sense of Definition 4.1 is uniquely determined by the vector given by a single chart, hence showing that $T_p^{\mathrm{coord}}M \simeq \mathbb{R}^n$. □

In particular, a chart around p and a vector in $\mathbb{R}^n$ determines a tangent vector at p.

4.1.2. *Tangent vectors via curves.* Key idea: tangent vectors should be directions in M, i.e derivatives of curves.

Definition 4.3. A *curve in a manifold M* is a smooth map

$$c : I \to M$$

where $I \subset \mathbb{R}$ is open and connected.

Pick a point $p \in M$ and consider the set

$$C_p = \{\gamma : (-\epsilon, \epsilon) \to M | \gamma \text{ is a curve and } \gamma(0) = p\}.$$

Lemma 4.4. *Suppose that $\gamma, \rho \in C_p$. Suppose that $\varphi : U \to V$ is chart around p and suppose that*

$$(\varphi \circ \gamma)'(0) = (\varphi \circ \rho)'(0).$$

Then, for any other chart $\psi : U' \to V'$ is chart around p we have

$$(\psi \circ \gamma)'(0) = (\psi \circ \rho)'(0).$$

Observe that the "usual" derivatives (denoted by a prime) make sense, since after postcomposition with a chart we have maps from $(-\epsilon, \epsilon)$ into $\mathbb{R}^n$.

Proof. We have

$$(\psi \circ \gamma) = (\psi \circ \varphi^{-1}) \circ (\varphi \circ \gamma)$$

and the two functions that are composed on the right hand side are real functions. Hence, we can apply the chain rule, and obtain

$$(\psi \circ \gamma)'(0) = D_{\varphi \circ \gamma(0)}(\psi \circ \varphi^{-1})((\varphi \circ \gamma)'(0)) = D_{\varphi(p)}(\psi \circ \varphi^{-1})((\varphi \circ \gamma)'(0))$$

and similarly

$$(\psi \circ \rho)'(0) = D_{\varphi \circ \rho(0)}(\psi \circ \varphi^{-1})((\varphi \circ \rho)'(0)) = D_{\varphi(p)}(\psi \circ \varphi^{-1})((\varphi \circ \rho)'(0))$$

□

Hence, define

Definition 4.5 (Tangent vector, curve version)**.**

$$T_p^{\text{curve}} M = \{\gamma \in C_p\} / \sim$$

where $\gamma \sim \rho$ if for some (hence, any) chart φ around p we have

$$(\varphi \circ \gamma)'(0) = (\varphi \circ \rho)'(0).$$

Elements of $T_p^{\text{curve}} M$ are called tangent vectors.

Lemma 4.6. *The map that assigns to $[\gamma] \in T_p^{\text{curve}} M$ and a chart φ around p the vector*

$$(\varphi \circ \gamma)'(0)$$

defines a bijection between $T_p^{\text{curve}} M$ and $T_p^{\text{coord}} M$.

Proof. Showing that the result is a tangent vector as in Definition 4.1 is the chain rule, exactly as in the proof of Lemma 4.4. Injectivity is clear from the construction. To see surjectivity, suppose that v is some vector in a chart $\varphi : U \to V$ around p. Consider

$$\gamma(t) = \varphi^{-1}(\varphi(p) + t \cdot v),$$

which is defined for $t \in (-\epsilon, \epsilon)$ for some small ϵ (V open!) and will map to the tangent vector defined by v. □

In particular, this gives $T_p^{\text{curve}}M$ also the structure of a real vector space of dimension n.
This definition of tangent space is more intrinsic, but the vector space structure is not immediately visible, and we still need charts.

4.1.3. *Tangent vectors via derivations.* In this section, we follow Warner, 1.12-19.
The third definition is completely coordinate-free, and natural. The idea is that a tangent vector can be thought of as a directional derivative, acting on functions.
Such derivatives are linear, and satisfy a product rule

$$\partial_v(\lambda f + \mu g) = \lambda \partial_v(f) + \mu \partial_v(g)$$

$$\partial_v(f \cdot g) = f(p)\partial_v(g) + g(p)\partial_v(f)$$

We reverse-engineer this into a definition of tangent vectors. First, given $p \in M$, define

$$\widetilde{F}_p = \{f : U \to \mathbb{R} | U \subset M \text{ open}, p \in U, f \text{ smooth}\} / \sim$$

where $f \sim f'$ if and only if

$$f|_V = f'|_V$$

for some open neighbourhood $V \subset U \cap U'$. Elements of $\widetilde{F}_p$ are called *germs* of functions in p[2]. Observe that this is an algebra.

Definition 4.7. A tangent vector at $p \in M$ is a *linear derivation at* p, i.e. a function $v : \widetilde{F}_p \to \mathbb{R}$ so that

$$v(\lambda f + \mu g) = \lambda v(f) + \mu v(g)$$

$$v(f \cdot g) = f(p)v(g) + g(p)v(f)$$

It is easy to check that the set $T_p^{\text{der}}M$ of linear derivations at p is a vector space.
We want to understand this space. To this end, let

$$F_p = \{f \in \widetilde{F}_p | f(p) = 0\}$$

and let F_p^k be the ideals generated by k–th powers of elements in F_p.
As a first observation, note that if f is a constant function, then $v(f) = 0$ for any derivation. Namely, $f = c1$, where 1 is the constant function with value 1, and

$$v(1) = v(1^2) = 2v(1).$$

In particular, a derivation is completely determined by its values on F_p, as

$$v(f) = v(f - f(p)) + v(f(p)) = v(f - f(p))$$

where $f(p)$ is the germ of the constant function with value $f(p)$.

[2]dt.: Funktionenkeime

Lemma 4.8.
$$T_p^{\mathrm{der}}M \simeq (F_p/F_p^2)^*$$
(The isomorphism is natural)

Proof. Observe that a derivation automatically vanishes on F_p^2:
$$v(fg) = f(p)v(g) + g(p)v(f) = 0 + 0$$
and so there is a map from $T_p^{\mathrm{der}}M \to (F_p/F_p^2)^*$, simply by restricting the derivation to F_p.
Conversely, suppose that $l \in (F_p/F_p^2)^*$. Define
$$v_l(f) = l(f - f(p)).$$
This is well-defined and linear. Checking that it is a derivation is a quick computation:
$$v_l(fg) = l(fg - f(p)g(p)) = l((f - f(p))(g - g(p)) + f(p)(g - g(p)) + g(p)(f - f(p)))$$
The first term in the latter sum is in F_p^2, so l evaluates to zero. Linearity then gives
$$v_l(fg) = f(p)l(g - g(p)) + g(p)l(f - f(p)) = g(p)v_l(f) + f(p)v_l(g).$$
Now, these operations are inverses. Given l, it is clear that restricting the derivation v_l to F_p just gives l. For the other direction, it suffices to check this on F_p, where it is also clear. □

To further understand $(F_p/F_p^2)^*$, and compare to the other definitions of tangent spaces, choose some chart $\varphi : U \to V$ with $p \in U$, and let x^i be the associated coordinate functions, i.e.
$$\varphi^{-1}(x^1(u), \ldots, x^n(u)) = u, \quad \text{for all } u \in U.$$
We then define (derivation) tangent vectors in the following way:
$$\left.\frac{\partial}{\partial x^i}\right|_p (f) = \left.\frac{\partial f \circ \varphi^{-1}}{\partial x^i}\right|_{\varphi(p)}$$
That these are indeed derivations follows from the usual product rule!

Lemma 4.9. *We have*
$$T_p^{\mathrm{der}}M \simeq T_p^{\mathrm{coord}}M$$
In fact, the $\left.\frac{\partial}{\partial x^i}\right|_p$ *form a basis for* $T_p^{\mathrm{der}}M$*.*

To prove this, we will work instead in $(F_p/F_p^2)^*$; observing that
$$x_i - x_i(p)$$
give elements of F_p/F_p^2 that are dual to $\left.\frac{\partial}{\partial x^i}\right|_p$:
$$\left.\frac{\partial}{\partial x^i}\right|_p (x_j - x_j(p)) = \delta_{ij}$$
Hence, to show the lemma, it suffices to show that

Lemma 4.10. *The germs defined by $x_i - x_i(p)$ form a basis for F_p/F_p^2.*

Proof. Recall that for any function $g : V \to \mathbb{R}$ smooth, and V convex we have

$$g(q) = g(p) + \sum \frac{\partial g}{\partial x^i}\bigg|_p (q^i - p^i) + \sum_{i,j} (q^i - p^i)(q^j - p^j) \int_0^1 (1-t) \frac{\partial^2 g}{\partial x^i \partial x^j}\bigg|_{p+t(q-p)} dt.$$

(Remainder term for Taylor's theorem)
Given any function $f : U \to \mathbb{R}$ which is smooth, and so that $f(p) = 0$ we can thus write f as

$$f = \sum \frac{\partial f \circ \varphi^{-1}}{\partial x^i}\bigg|_{\varphi(p)} (x^i - x^i(m)) + \sum_{i,j} (x_i - x_i(p))(x_j - x_j(p))h$$

for some smooth function h. In other words, in F_p/F_p^2 we have

$$f = \sum \frac{\partial f \circ \varphi^{-1}}{\partial x^i}\bigg|_{\varphi(p)} (x^i - x^i(m))$$

In other words, $x_i - x_i(p)$ are a generating set for F_p/F_p^2. Since they are dual to the $\frac{\partial}{\partial x^i}\big|_p$, they are linearly independent. $\square$

5. Lecture 5: Differentials of smooth maps

5.1. **Differentials of smooth maps.** Suppose now that $f : M \to N$ is a smooth map. We want to define a (total) derivative. This will be for each p a linear map

$$d_p f : T_p M \to T_{f(p)} N.$$

Why is this what we want? In $\mathbb{R}^n$, derivatives are *best linear approximations*, i.e. if $F : \mathbb{R}^n \to \mathbb{R}^m$, then the derivative at some point p is a linear map

$$D_p F : \mathbb{R}^n \to \mathbb{R}^m,$$

that gives a better-than-linear approximation. That is, for each ϵ there is a δ so that for $h, |h| < \delta$:

$$|F(p+h) - F(p) - D_p F(h)| < \epsilon |h|$$

Hence, the usual differential sends (tangent) directions in $\mathbb{R}^n$ to the best approximating direction after mapping through F.
In both the geometric and algebraic definition of tangent vectors we can define this. For the curves, the philosophy is: enforce the chain rule.

Definition 5.1. Suppose that M, N are smooth manifolds, and $f : M \to N$ is a smooth map. Let $p \in M$ be arbitrary. The differential $d_p^{\text{curve}} f$ is the (unique) linear function

$$d_p^{\text{curve}} f : T_p^{\text{curve}} M \to T_{f(p)}^{\text{curve}} N$$

so that for any curve $c : (-\epsilon, \epsilon) \to M$ with $c(0) = p$ we have

$$(f \circ c)'(0) = d_p^{\text{curve}} f(c'(0)).$$

Neither well-definedness (or existence) nor linearity are clear from this point of view.
Let's see the other: for coordinates, we would want to associate to each chart the Jacobian matrix of the corresponding induced map.
That is,

Definition 5.2. Suppose that M, N are smooth manifolds, and $f : M \to N$ is a smooth map. Let $p \in M$ be arbitrary. Define for each pair of charts φ of M around p and ψ of N around $f(p)$ the *Jacobian matrix*

$$(J_{\varphi,\psi,p}f)^i_j = \frac{\partial(\psi \circ f \circ \varphi^{-1})^i}{\partial x^j}$$

The *differential* is a linear map $d_p^{\mathrm{coord}} f : T_p^{\mathrm{coord}} M \to T_{f(p)}^{\mathrm{coord}} N$ which associates to $v \in T_p^{\mathrm{coord}} M$ a (coordinate) tangent vector, the tangent vector

$$(d_p^{\mathrm{coord}} f(v))_\psi = J_{\varphi,\psi,p} f v_\varphi$$

Here, linearity is clear, but well-definedness requires a quick proof (the chain rule!). Similarly to the definition of (coordinate) tangent vectors, the different Jacobian matrices produced by d_p^{coord} are related via the chain rule.

Lemma 5.3. *The identification $T_\cdot^{\mathrm{coord}}$ with $T_\cdot^{\mathrm{curve}}$ is compatible with the two definitions of differential.*

Proof. As anything, this is the chain rule. Given a coordinate tangent vector v at p and a chart φ around p, the identification $T_p^{\mathrm{coord}} M \to T_p^{\mathrm{curve}} M$ associates the derivative of the curve

$$c(t) = \varphi^{-1}(\varphi(p) + t v_\varphi).$$

The curve differential $d_p^{\mathrm{curve}} f(v)$ hence maps v to the (equivalence class of) the curve $(f \circ c)$. Under the identification $T_{f(p)}^{\mathrm{curve}} N \to T_{f(p)}^{\mathrm{coord}} N$ this is identified to

$$(\psi \circ f \circ c)'(0) = D_{\varphi(p)}(\psi \circ f \circ \varphi^{-1})(v_\varphi) = J_{\varphi,\psi,p} f v_\varphi$$

$\square$

Note that since the linear structure on $T_\cdot^{\mathrm{curve}}$ comes from this identification, this also proves linearity and well-defined-ness of $d_p^{\mathrm{curve}} f$.
For the derivative version, the definition is again cleanest, and most abstract.

Definition 5.4. Suppose that M, N are smooth manifolds, and $f : M \to N$ is a smooth map. Let $p \in M$ be arbitrary. The differential $d_p f$ is the linear function

$$d_p^{\mathrm{der}} f : T_p^{\mathrm{der}} M \to T_{f(p)}^{\mathrm{der}} N$$

defined by

$$(d_p^{\mathrm{der}} f(v))(g) = v(g \circ f).$$

Here, existence and linearity are clear.

Lemma 5.5. *The identification $T_\cdot^{\mathrm{coord}}$ with $T_\cdot^{\mathrm{der}}$ is compatible with the two definitions of differential.*

Proof. Since we already know linearity of the two maps in question, it suffices to show this on a basis. Recall that given a chart φ around p, we obtain a basis of $T_p^{\mathrm{der}}M$ as the (local) partial derivatives

$$\frac{\partial}{\partial x^i}(g) := \frac{\partial(g\circ\varphi^{-1})}{\partial x^i},$$

and the identification of $T_\cdot^{\mathrm{coord}}$ with $T_\cdot^{\mathrm{der}}$ works by identifying
identifies the standard basis of $\mathbb{R}^n$ with this basis.
Now, we have

$$\left(d_p^{\mathrm{der}}f\left(\frac{\partial}{\partial x^i}\right)\right)(g) = \frac{\partial}{\partial x^i}(g\circ f) = \frac{\partial(g\circ f\circ\varphi^{-1})}{\partial x^i} = \frac{\partial(g\circ\psi^{-1}\circ\psi\circ f\circ\varphi^{-1})}{\partial x^i}$$
$$= \sum_j \frac{\partial(g\circ\psi^{-1})}{\partial x^j}\frac{\partial(\psi\circ f\circ\varphi^{-1})^j}{\partial x^i} = \sum_j (J_{\psi,\varphi,p})_i^j \frac{\partial(g\circ\psi^{-1})}{\partial x^j}$$

Hence, under the identification with $T_{f(p)}^{\mathrm{coord}}N$ this is the vector with components $(J_{\psi,\varphi,p})_i^j$.
Now, on the other hand, the i–th standard basis vector maps under $d_p^{\mathrm{coord}}f$ to the i–th column of $J_{\psi,\varphi,p}$, which is the same. $\square$

6. Lecture 6: The Tangent Bundle

6.1. **The Tangent Bundle.** At each point $p\in M$ we have a vector space T_pM. Next goal: assemble all of the tangent spaces into a single object. To this end, the following will be useful:
Suppose that $\varphi: U\to V$ is a chart around p of M. We then have for all $q\in U$

$$d_qx^i = d_q\varphi^i : T_qM\to\mathbb{R},$$

the differential of φ^i, the i–th coordinate function. Explicitly, this records the i–th coordinate of the tangent vector in the chart φ. When no confusion can arise, we also simply write dx^i. Observe that

$$T_pM\to\mathbb{R}^n,\quad v\mapsto(dx^1(v),\dots,dx^n(v))$$

is the isomorphism $T_pM\to T_p^{\mathrm{coord}}M$ given by the chart.
Define

$$TM = \coprod_p T_pM,$$

first as a set. We have also an (obvious) map

$$\pi: TM\to M,\quad (p,v)\mapsto p.$$

We will equip TM with a manifold structure. To this end, let $\varphi: U\to V$ be a chart. Define

$$\widetilde{\varphi}: \coprod_{p\in U} T_pM\to V\times\mathbb{R}^n$$

by
$$\widetilde{\varphi}(p,v) = (\varphi^1(p), \dots, \varphi^n(p), dx^1(v), \dots, dx^n(v)).$$
The following is a consequence of the interpretation of the isomorphism $T_pM \to T_p^{\mathrm{coord}}M$ given above.

Lemma 6.1. *For any chart $\varphi : U \to V$, the map $\widetilde{\varphi}$ is a bijection. If ψ is any other chart, then*
$$\widetilde{\psi} \circ \widetilde{\varphi} = (\psi \circ \varphi^{-1}) \times D(\psi \circ \varphi^{-1}),$$
wherever defined.

Lemma 6.2. *Call a subset $\widetilde{U}$ of TM* open *if for each chart $\varphi : U \to V$ of M the set*
$$\widetilde{\varphi}(\widetilde{U} \cap U) \subset V \times \mathbb{R}^n \subset \mathbb{R}^{2n}$$
is open. Then these open sets form a topology, which is Hausdorff, second countable, and makes all $\widetilde{\varphi}$ homeomorphisms.

Proof. We have
$$\widetilde{\varphi}(\cup \widetilde{U}_i \cap U) = \bigcup_i \widetilde{\varphi}(\widetilde{U}_i \cap U))$$
and
$$\widetilde{\varphi}(\cap \widetilde{U}_i \cap U) = \bigcap_i \widetilde{\varphi}(\widetilde{U}_i \cap U))$$
since $\widetilde{\varphi}$ are bijections. This shows that these sets form a topology. Next, observe that for any chart φ and any open $W \subset V \times \mathbb{R}^n$, the preimage $\widetilde{\varphi}^{-1}(W)$ is open (since the transition maps discussed above are homeomorphisms). This shows that the $\widetilde{\varphi}$ are homeomorphisms (since they are clearly open). Since $\mathbb{R}^n$ is second countable, this shows that the topology is second countable.

Finally, suppose that $(p,v), (q,w)$ are two points. If $p \neq q$, then they can be separated by open sets in M, hence also in TM. Otherwise, separate v, w in $\mathbb{R}^n$. □

By the observation above, the $\widetilde{\varphi}$ in fact define a smooth atlas on TM, which makes the projection map π smooth.

Definition 6.3. A *vector field* is a smooth map $V : M \to TM$ so that $\pi(V(p)) = p$ for all $p \in M$.

Intuitively, vector fields are smooth associations of tangent vectors at each point.

6.2. **Vector Bundles.** TM is an example of a *vector bundle*:

Definition 6.4. Let M be a smooth manifold. A *smooth vector bundle* consists of the following data:

- A smooth manifold E,
- A smooth map $\pi : E \to M$, and
- the structure of an $\mathbb{R}$–vector space on each *fibre* $\pi^{-1}(p)$,

so that there are *local trivialisations*: for each point $p \in M$ there is an open neighbourhood U and a smooth diffeomorphism

$$f_U : U \times \mathbb{R}^n \to \pi^{-1}(U),$$

in such a way that

(1) $\pi(f_U(q, v)) = q$.
(2) $f_U(u, \cdot) : \mathbb{R}^n \to \pi^{-1}(u)$ is an isomorphism of vector spaces.

Often, one writes $E_p = \pi^{-1}(p)$ for the fibres.

In the case of the tangent bundle, the vector space structure on the fibres is the structure of the tangent spaces, and the trivialisations come from the charts: for any chart $\varphi : U \to V$ of M, we define

$$f_U(u, v^1, \dots, v^n) = \left(u, \sum v^i \frac{\partial}{\partial x^i}\right).$$

The compatibility condition of (coordinate) tangent spaces implies that these satisfy (2).

7. Lecture 7: More Vector Bundles

For later use, note that if f_U, f_V are two local trivialisations of a vector bundle, then the *transition functions*

$$f_V^{-1} \circ f_U : U \cap V \times \mathbb{R}^n \to U \cap V \times \mathbb{R}^n$$

has the form

$$f_V^{-1} \circ f_U(u, v) = (u, g_{UV}(u)(v))$$

for a (smooth) function

$$g_U V : U \cap V \to \mathrm{GL}(n).$$

Definition 7.1. Let M, N be two smooth manifolds. Suppose that $\pi_E : E \to M$ and $\pi_F : F \to N$ are vector bundles. A *(vector) bundle map* is a smooth map

$$\Phi : E \to F$$

so that there is a smooth map $\varphi : M \to N$ satisfying

(1) $\pi_F(\Phi(x)) = \varphi(\pi_E(x))$ for all $x \in E$, and
(2) the induced maps

$$\Phi_p = \Phi|_{E_p} : E_p \to F_{\varphi(p)}$$

are vector space homomorphisms.

Equivalently, we want that for any local trivialisations f_U, f_V of E, F we have

$$f_V^{-1} \circ \Phi \circ f_U(q, v) = (\phi(q), \Phi_{UV}(q)(v)),$$

where $\phi : M \to N$ is a smooth map, and

$$\Phi_{UV} : U \cap V \to \mathrm{GL}(\mathbb{R}^n)$$

is a family of linear maps. A good example to think of is the differential of a smooth map.
One advantage of this point of view is that we “inherit” a lot of constructions from vector spaces to vector bundles. Formally, we can apply smooth functorial constructions of vector spaces also to vector bundles. See e.g. Milnor-Stasheff “Characteristic classes” for this point of view.
We mention three explicit examples, requiring us to review a little bit of linear algebra.

(1) Direct sums. Universal property: easy to map out of.
(2) (Tensor) products. Universal property: mapping into for pairs.
(3) Dual spaces. Observe inversion of arrows.

Lemma 7.2. *Let E_1, E_2 be two vector bundles above a manifold M. Then there is a vector bundle $E_1 \oplus E_2$ over M, so that*

$$\pi^{-1}_{E_1 \oplus E_2}(u) = \pi_1^{-1}(u) \oplus \pi_2^{-1}(u),$$

and the following hold:

i) There are bundle maps $\iota_i : E_i \to E_1 \oplus E_2$, over the identity map of M, inducing the natural inclusions $(E_i)_x \to (E_1 \oplus E_2)_x$ fibrewise.
ii) Suppose N is a smooth manifold and $F \to N$ a vector bundle. Suppose Φ_i are bundle maps $\Phi_i : E_i \to F$. Then there is a unique bundle map

$$\Phi : E_1 \oplus E_2 \to F$$

which has

$$\Phi_p \circ \iota_i = (\Phi_i)_p$$

Proof. We define, as a set,

$$E = \coprod_{p \in M} \pi_1^{-1}(u) \oplus \pi_2^{-1}(u),$$

and let $\pi : E \to M$ be the obvious projection. Note that the fibres then have a natural vector space structure.
Now, let $U \subset M$ be an open set, so that there are local trivialisations $f_i : U \times \mathbb{R}^{n_i} \to \pi_i^{-1}(U)$, for $i = 1, 2$. Then, define

$$f_U : U \times \mathbb{R}^{n_1+n_2} \to \pi^{-1}(U), \quad (u, v_1, v_2) \mapsto f_1(v_1) \oplus f_2(v_2) \in \pi_1^{-1}(u) \oplus \pi_2^{-1}(u)$$

Observe that the f_U are bijections, and by construction have transitions

$$(f')^{-1} \circ f : (U \cap U') \times \mathbb{R}^{n_1+n_2} \to (U \cap U') \times \mathbb{R}^{n_1+n_2}, \quad (u, v_1, v_2) \mapsto (u, g^1_{UV}(v_1), g^2_{UV}(v_2))$$

We define

$$\mathcal{A} = \{(\varphi, \mathrm{id}) \circ f_U^{-1} : \pi_i^{-1}(U) \to V \times \mathbb{R}^n \subset \mathbb{R}^{n+m} \mid \varphi : U \to V \text{chart of } M\}$$

and just as in the definition of the tangent bundle, we check that these maps can be used to induce a topology and smooth structure on E. By construction, the maps f_U are local trivialisations.
It remains to show that the bundle E has the desired property. To construct the ι_1, we simply send an element $v \in (E_1)_p$ to $v \oplus 0 \in (E_1 \oplus E_2)_p$. We just

need to show that this map is smooth. To do so, we can use the trivialisations above, and note
$$f_U^{-1} \circ \iota_1 \circ f_1(u, v) = (u, v, 0),$$
showing i).
To construct Φ in ii) we simply define
$$\Phi \circ f_U(u, v_1, v_2) = \Phi_1 f_U^1(u, v_1) + \Phi_2 f_U^2(u, v_2)$$
where the $+$ is the vector space sum in the fibre of F. One easily checks that this is smooth and has the desired properties. □

Vector fields are examples of

Definition 7.3. Let E be a vector bundle over M. A *section* is a smooth map $s : M \to TM$ so that $\pi(s(p)) = p$ for all $p \in M$.

A section of TM^* is intuitively something which assigns to each point $p \in M$ something that takes tangent vectors at p, and outputs a number. We have seen something like this:
Namely, if $f : M \to \mathbb{R}$ is a smooth function, then df can be interpreted as a section in TM^*, $p \mapsto d_p f : T_p M \to \mathbb{R}$.

8. Lecture 8: Forms and Exterior Derivatives

Recall sections: if $E \to M$ is a vector bundle, we denote by $\Gamma(E)$ the vector space of sections of E.
Let's return briefly to dual bundles:

Lemma 8.1. *Let E be a vector bundles above a manifold M. Then there is a vector bundle E^*, called the* dual bundle*, over M, so that*
$$\pi_{E^*}^{-1}(u) = (\pi_1^{-1}(u))^*,$$
which has the following properties:

1. *If σ is a section of E, and η is a section of E^*, then*
$$x \mapsto \eta(x)(\sigma(x))$$
is a smooth function.
2. *Suppose that N is a smooth manifold and $F \to N$ a vector bundle, with dual bundle N^*. Suppose Φ is a bundle map $\Phi : E \to F$ over a diffeomorphism f. Then there is a bundle map*
$$\Phi^* : F^* \to E^*$$
over f^{-1}, which has
$$\Phi_p^* = (\Phi_p)^*$$

The *tensor product* is explicitly given by
$$V \otimes W = \operatorname{span}\{v \otimes w, v \in V, w \in W\}/$$
$$\langle (v_1 + v_2) \otimes w - v_1 \otimes w - v_2 \otimes w, v \otimes (w_1 + w_2) - v \otimes w_1 - v \otimes w_2,$$
$$(\lambda v) \otimes w - \lambda(v \otimes w), v \otimes (\lambda w) - \lambda(v \otimes w) \rangle$$

Lemma 8.2. *Let E_1, E_2 be two vector bundles above a manifold M. Then there is a vector bundle $E_1 \otimes E_2$ over M, so that*

$$\pi_{E_1 \otimes E_2}^{-1}(u) = \pi_1^{-1}(u) \otimes \pi_2^{-1}(u),$$

natural in a similar way to before, see problem set.

We now also have bundles

$$T_{r,s}M = (TM)^r \otimes (TM^*)^s$$

over any manifold M. These are called *tensor bundles of type* (r, s). A section of this is a *tensor of type* (r, s). It will be a while until we see meaningful examples of this.

8.1. **Differential forms.** A very quick reminder on a few linear algebra terms

- If V is a $\mathbb{R}$–vector space, then we have a (noncanonical) isomorphism between $(V^*)^k$ and the multilinear maps $V \times \cdots \times V \to \mathbb{R}$: a multilinear map h corresponds to the tensor

  $$\sum_{i_1,\ldots,i_k} h(e_{i_1}, \ldots, e_{i_k}) e_{i_1} \otimes \cdots \otimes e_{i_k},$$

  where e_i is a basis for V. We will swap between the two points of view regularly. Prove and understand this isomorphism if you have never seen this or don't remember the details.

  There is a product

  $$V^{\otimes n} \times V^{\otimes m} \to V^{\otimes n+m}$$

  corresponding to the product of the multilinear maps. This turns $\bigoplus (V^*)^k$ into an $\mathbb{R}$–algebra.
- A multilinear map $h : V \times \cdots \times V \to \mathbb{R}$ is called

  Symmetric: if

  $$h(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = h(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

  for all $v_1, \ldots, v_k \in V$, and all $i < j$.

  Alternating: if

  $$h(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -h(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

  for all $v_1, \ldots, v_k \in V$, and all $i < j$.

  $$\mathrm{Sym}^k(V^*), \Lambda^k(V^*) \subset (V^*)^{\otimes k}.$$

- *Exterior powers* of vector spaces V, more generally. Consider the k–th tensor power $V^{\otimes k}$ of a vector space. On it, there is the anti-symmetrisation map

  $$\mathrm{Alt} : V^{\otimes k} \to V^{\otimes k}$$

defined by

$$\mathrm{Alt}(v_{i_1} \otimes \cdots \otimes v_{i_k}) = \frac{1}{k!} \sum_\sigma \mathrm{sgn}(\sigma) v_{\sigma(i_1)} \otimes \cdots \otimes v_{\sigma(i_k)}$$

where we sum over the symmetric group on k letters. We then define

$$\Lambda^k(V) = \mathrm{im}(\mathrm{Alt}) \subset V^{\otimes k}.$$

Observe that $\Lambda^k(V^*)$ is the same as defined above, and also that Alt is a projection. If $e_1, \ldots, e_n$ is a basis of V, then the following are a basis of $\Lambda^k(V)$:

$$e_{i_1} \wedge \cdots \wedge e_{i_k} = \mathrm{Alt}(e_{i_1} \otimes \cdots \otimes e_{i_k}), \quad i_1 < \ldots < i_k$$

(If you are interested, one can also define the exterior algebra via quotients; compare Warner 2.4)

- Alternatively:

$$\Lambda^2(V) = \mathrm{span}\{v \wedge w, v, w \in V\}/$$

$$\langle (v_1 + v_2) \wedge w - v_1 \wedge w - v_2 \wedge w, v \wedge (w_1 + w_2) - v \wedge w_1 - v \wedge w_2,$$

$$(\lambda v) \wedge w - \lambda(v \wedge w), v \wedge (\lambda w) - \lambda(v \wedge w), v \wedge w + w \wedge v \rangle$$

- There is a product

$$\wedge : \Lambda^r(V) \times \Lambda^s(V) \to \Lambda^{r+s}(V)$$

defined as the restriction of the map

$$V^{\otimes r} \times V^{\otimes s} \to \Lambda^{r+s}(V), \quad v \times w \mapsto \frac{(r+s)!}{r!s!} \mathrm{Alt}(v \otimes w)$$

(where the right hand side is the action on pure tensors). Note that this is compatible with the notation in the basis given above.

Maybe the most important example of the wedge product is in $\Lambda^k(V^*)$, i.e. alternating multilinear maps on V. In this case, the wedge product corresponds to the natural antisymmetrisation:

$$(\omega \wedge \eta)(e_{i_1}, \ldots, e_{i_{r+s}}) = \frac{1}{r!s!} \sum_\sigma \mathrm{sgn}(\sigma) \omega(e_{\sigma(i_1)}, \ldots, e_{\sigma(i_r)}) \eta(e_{\sigma(i_{r+1})}, \ldots, e_{\sigma(i_{r+s})})$$

Now, we can define

Definition 8.3. A *differential k–form* is a smooth section ω of $(TM^*)^{\otimes k}$ so that $\omega(p) \in \Lambda^k(T_pM^*)$ for all p. We denote by $\Omega^k(M)$ the vector space of k–forms on M.

In other words: associating a smoothly varying k–form to every tangent space.

Oberve that we can locally describe a differential k-form in a chart φ as

$$\sum a_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

as the dx^i are a basis of $(T_pM)^*$. What is the transformation rule for these? We could, just like for tangent vectors, *define* k-forms using this description and that compatibility rule.

Also observe that the wedge product makes sense for differential forms, by wedging in each fibre.

One crucial construction is the *pullback*: if $f : M \to N$ is any smooth map, and $\omega \in \Omega^k(N)$, then we can define $f^*\omega \in \Omega^k(M)$ via

$$(f^*\omega)_p(v_1, \ldots, v_k) = \omega_{f(p)}(df_p(v_1), \ldots, df_p(v_k)).$$

Observe that if we want to pull back or push forward vector fields instead. Then, in general, we need f to be a diffeomorphism!

8.2. **Exterior Derivatives.** We have seen that if f is a function, then df is a differentiable 1-form, and f is a 0–form. We now want to extend differentiation to more general forms.
Our goal is the following theorem

Satz 8.4. *There is a linear map*

$$d : \Omega^*(M) \to \Omega^*(M)$$

that is uniquely determined by the following properties:

(1) For any $f \in \Omega^0(M)$, df is the usual differential.
(2) For any $\omega \in \Omega^r(M), \eta \in \Omega^*(M)$ we have

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^r \omega \wedge (d\eta).$$

(3)

$$d^2 = 0$$

9. Lecture 11 - More exterior derivatives, and Integrals

Proof of the existence and uniqueness of exterior derivatives. Fix some chart $\varphi : U \to V$. We will first define a linear map

$$d^\varphi : \Omega^*(U) \to \Omega^*(U),$$

in the following way. Given a form $\omega \in \Omega^*(U)$, we can write it locally as

$$\omega = \sum_{i_1 < \ldots < i_k} f_{i_1,\ldots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where the $f_{i_1,\ldots,i_k}$ are smooth functions on U. We define

$$d^\varphi \omega = \sum_{i_1 < \ldots < i_k} df_{i_1,\ldots,i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

The function d^φ is linear, maps k–forms to $(k+1)$–forms, and $(d^\varphi\omega)_p$ depends only on the germ of the $f_{i_1,\ldots,i_k}$ at $\varphi(p)$. We claim that we also have

$$d^\varphi(\omega_1 \wedge \omega_2)_p = (d\omega_1)_p \wedge (\omega_2)_p + (-1)^r (\omega_1)_p \wedge (d\omega_2)_p$$

if ω_1 is an r–form. To see this, it suffices to show this for forms

$$\omega_1 = f dx_{i_1} \wedge \cdots \wedge dx_{i_r},$$

$$\omega_2 = g dx_{j_1} \wedge \cdots \wedge dx_{j_s},$$

Observe that if some pair i_u, j_v of indices is equal, the desired equation is trivially true (both sides are zero).
If $r = 0 = s$, then this is just the usual product rule. If $r = 0$ and $s = 1$, we have

$$\omega_1 \wedge \omega_2 = fg dx_{j_1},$$

and thus

$$d^\varphi(\omega_1 \wedge \omega_2) = d(fg) \wedge dx_{j_1} = (g \cdot df + f \cdot dg) \wedge dx_{j_1}$$
$$= (df \wedge g dx_{j_1}) - f dg \wedge dx_{j_1} = d^\varphi \omega_1 \wedge \omega_2 - \omega_1 \wedge d^\varphi \omega_2$$

(The $r = 1, s = 0$ case is similar). Now,

$$\omega_1 \wedge \omega_2 = \epsilon fg dx_{l_1} \wedge \cdots \wedge dx_{l_{r+s}}$$

where the l_i are in order, and $\epsilon = \pm 1$ is the sign of the permutation applied to do this. Then

$$d^\varphi(\omega_1 \wedge \omega_2)_p = \epsilon(g(p) df_p + f(p) dg_p) \wedge dx_{l_1} \wedge \cdots \wedge dx_{l_{r+s}}$$
$$= (g(p) df_p + f(p) dg_p) \wedge (dx_{i_1} \wedge \cdots \wedge dx_{i_r}) \wedge (dx_{j_1} \wedge \cdots \wedge dx_{l_{r+s}})$$
$$= df_p \wedge (dx_{i_1} \wedge \cdots \wedge dx_{i_r}) \wedge (g(p) dx_{j_1} \wedge \cdots \wedge dx_{j_s})$$
$$+ dg_p \wedge (f(p) dx_{i_1} \wedge \cdots \wedge dx_{i_r}) \wedge (dx_{j_1} \wedge \cdots \wedge dx_{j_s})$$
$$= df_p \wedge (dx_{i_1} \wedge \cdots \wedge dx_{i_r}) \wedge (g(p) dx_{j_1} \wedge \cdots \wedge dx_{j_s})$$
$$+ (-1)^r (f(p) dx_{i_1} \wedge \cdots \wedge dx_{i_r}) \wedge (dg_p \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_s}).$$
$$= (d^\varphi \omega_1 \wedge \omega_2)_p + (-1)^r (\omega_1 \wedge d^\varphi \omega_2)_p.$$

Finally, observe that if f is a function, then

$$d^\varphi f = \sum_i \frac{\partial f}{\partial x^i} dx^i$$

and thus

$$(d^\varphi)^2 f = \sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^i = 0$$

as second derivatives are symmetric. Note that if $U' \subset U$ and $\omega \in \Omega^*(U')$, then $d^\varphi \omega \in \Omega^*(U')$ is defined.
Now, suppose that $\psi : U' \to V'$ is any other chart. Then we claim that $(d^\varphi \omega)_p = (d^\psi \omega)_p$ for all $\omega \in \Omega^*(U \cap U')$ and $p \in U \cap U'$. To this end, write ω as before, in terms of the forms dx^i coming from the chart φ:

$$\omega = \sum_{i_1 < \ldots < i_k} f_{i_1, \ldots, i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Now apply d^ψ, treating the forms in the back as (exterior) deriviatives of functions:

$$(d^\psi \omega)_p = \sum_{i_1 < \ldots < i_k} df_{i_1, \ldots, i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} + f_{i_1, \ldots, i_k} \wedge d^\psi (dx_{i_1} \wedge \cdots \wedge dx_{i_k}).$$

To finish the claim, we thus need that all the superfluous terms in the sum are zero. This follows, inductively applying the product rule, since

$$d^{\psi}(dx_l) = (d^{\psi})^2(\varphi^l) = 0.$$

Hence, given a form ω, we can define the exterior derivative $d\omega$, pointwise via any chart. All properties are clear, except $d^2 = 0$. However, this follows analogously to the case of functions via symmetry of second derivatives.

It remains to show uniqueness, but this follows similarly to independence of charts. As a first step, note that if ω is a form which vanishes on U containing p, then

$$d'\omega_p = 0$$

Namely, take a function $\varphi : M \to \mathbb{R}$, identically zero near p, and identically one outside U. Then

$$d'(\omega) = d'(\varphi\omega) = d'\varphi \wedge \omega + \varphi d'\omega.$$

Evaluating at p gives the result. This in turn shows that $d'\omega_p$ only depends on the germ at p. Hence, we can extend d' to forms defined on neighbourhoods of p (by choosing any extension). The proof of independence of the chart now applies verbatim to show that $d' = d$. $\square$

Next, we want to learn what forms are for. To formulate the important applications, we need three more notions: orientations, manifolds with boundary, and integrals of forms.

First, recall the following:

Definition 9.1. A diffeomorphism $f : U \to V$ between open sets $U, V \subset \mathbb{R}^n$ is called *orientation preserving*, if

$$\det D_p f > 0 \text{ for all } p \in U$$

and

Definition 9.2.

(1) Let M be a topological space, and $\mathcal{A}$ be a smooth atlas. $\mathcal{A}$ is called a *oriented (smooth) atlas* if all chart transitions are orientation preserving.
(2) An *oriented manifold* is a manifold together with the choice of an oriented atlas.
(3) A manifold is *orientable* if it admits an oriented atlas.

Note: not all manifolds are orientable; see problem set.

Next, we need a technical tool:

Definition 9.3. Let M be a manifold. A *partition of unity* is a collection $\mathcal{P} = \{\rho_i, i \in I\}$ of smooth functions on M, so that

(1) Any point $p \in M$ is contained in finitely many supports $\mathrm{supp}\rho_i$.
(2)
$$\sum_i \rho_i(p) = 1$$
for all $p \in M$.

A partition of unity is *subordinate to an open cover* $\{U_j\}$ if each support of ρ_i is contained in some U_j.

Satz 9.4. *On a manifold M, for any open cover there is a partition of unity subordinate to that cover.*

We will prove this later. First, we note how to use this to define integrals of forms.
To this end, suppose that M is an orientable manifold of dimension n, and that ω is an n–form. Let $\{\varphi_i\}$ be an oriented atlas, and let ρ_i be a partition of unity subordinate to that atlas. Now, define

$$\int_M \omega = \sum_i \int_{V_i} (\varphi_i^{-1})^*(\rho_i \omega)(e_1, \dots, e_n) dx^1 \cdots dx^n.$$

We need to check that this is independent of the choice of charts and partitions of unity.
To this end, suppose that $\{\psi_j\}$ be a different atlas, and η_j a subordinate partition of unity. We then have

$$(\varphi_i^{-1})^*(\rho_i \omega) = \sum_j (\varphi_i^{-1})^*(\rho_i \eta_j \omega)$$

and therefore

$$\int_M \omega = \sum_{i,j} \int_{\varphi_i(V_i \cap V_j')} (\varphi_i^{-1})^*(\rho_i \eta_j \omega)(e_1, \dots, e_n) dx^1 \cdots dx^n.$$

Also, we have

$$(\varphi_i^{-1})^*(\omega)(e_1, \dots, e_n) = (\psi_j^{-1})^*(\omega)(D(\psi_j \circ \varphi^{-1})(e_1), \dots, D(\psi_j \circ \varphi^{-1})(e_n))$$

Now, observe that if $A : \mathbb{R}^n \to \mathbb{R}^n$ is any linear map, and h is any nonzero alternating n-linear map, we have

$$h(Ae_1, \dots, Ae_n) = \det(A) h(e_1, \dots, e_n).$$

Trick proof: The function $A \mapsto h(Ae_1, \dots, Ae_n)/h(e_1, \dots, e_n)$ has the properties of a determinant, hence is the determinant!
Thus, we have

$$(\varphi_i^{-1})^*(\rho_i \eta_j \omega)(e_1, \dots, e_n) = \det D(\psi_j \circ \varphi^{-1})(\psi_i^{-1})^*(\rho_i \eta_j \omega)(e_1, \dots, e_n) = |\det D|(\psi_j \circ \varphi^{-1})(\psi_i^{-1})^*(\rho_i \eta_j \omega)($$

Hence, by the transformation theorem:

$$\int_{\varphi_i(V_i \cap V_j')} (\varphi_i^{-1})^*(\rho_i \eta_j \omega)(e_1, \dots, e_n) dx^1 \cdots dx^n = \int_{\psi_i(V_i \cap V_j')} (\psi_j^{-1})^*(\rho_i \eta_j \omega)(e_1, \dots, e_n) dx^1 \cdots dx^n$$

and this shows that the integral is well-defined.

10. Lecture 12 - Using cohomology

Satz 10.1. *On a manifold M, for any open cover there is a partition of unity subordinate to that cover.*

A proof of this can be found e.g. in Warner's book, 1.7-1.11. Note that this is where we use second countable of M in a crucial way.

Next, we need *manifolds with boundary.* To this end, define

$$\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n, x_n \geq 0\}$$

A function between open subsets of $\mathbb{H}^n$ is *smooth*, if it has a smooth extension. Observe that the differential at a boundary point does not depend on the extension chosen!

Definition 10.2.
(1) A *boundary chart* on a topological space M is a homeomorphism $\varphi : U \to V$ where $U \subset M$ is open, and $V \subset \mathbb{H}^n$ is open and intersects $\partial\mathbb{H}^n$.
(2) A *manifold with boundary* is a second countable Hausdorff space together with an atlas $\mathcal{A}$ each element of which is a chart or a boundary chart, and so that all transition functions are smooth.
(3) An *oriented manifold with boundary* is a manifold with boundary whose atlas has orientation preserving transitions.

If M is a manifold with boundary, we let ∂M be the subset of all boundary points.

To see if you have absorbed the definition, show that: the set of boundary points ∂M of a manifold with boundary is a manifold of one dimension less. If M was oriented, then ∂M inherits an orientation.
Now, we can state probably the most important result on exterior derivatives:

Satz 10.3 (Stokes' theorem)**.** *Suppose that M is a manifold with boundary of dimension n, and let ω be a $(n-1)$-form. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

(on the right, the form ω should formally be the restriction $\omega|_{\partial M}$ of ω to the boundary.)

Korollar 10.4. *If M is a manifold without boundary, and ω is a $(n-1)$-form, then*

$$\int_M d\omega = 0.$$

Finally, we want:

Satz 10.5 (Poincaré lemma)**.** *Any closed form on D^n is exact.*

The proof of this requires some tools. First, define the following field on D^n:

$$X = \sum r_i \frac{\partial}{\partial x^i}.$$

Then we have

$$\mathcal{L}_X(f dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \left(kf + \sum_i r_i \frac{\partial f}{\partial x^i} \right) dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Namely:

$$\mathcal{L}_X(f dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \mathcal{L}_X(f) dx^{i_1} \wedge \cdots \wedge dx^{i_k} + f \mathcal{L}_X(dx^{i_1} \wedge \cdots \wedge dx^{i_k})$$

and

$$\mathcal{L}_X(f) = \iota_x df = \sum r_i \frac{\partial f}{\partial x^i}$$

$$\mathcal{L}_X(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \iota_X 0 + d\iota_X(dx^{i_1} \wedge \cdots \wedge dx^{i_k})$$

$$\iota_X(dx^{i_1} \wedge \cdots \wedge dx^{i_k})(X_2, \ldots, X_k) = \sum_{j=1} (-1)^j r_j dx^{i_1} \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^{i_k}$$

and thus

$$d\iota_X(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = k dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Lemma 10.6. *There are linear transformations*

$$h_k : \Omega^k(D^n) \to \Omega^{k-1}(D^n)$$

satisfying

$$h_{k+1} \circ d + d \circ h_k = \mathrm{id}$$

Proof. Define an operator $\alpha_k : \Omega^k(D^n) \to \Omega^k(D^n)$ via

$$\alpha_k(f dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \left(\int_0^1 t^{k-1} f(tp) \right) dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

For the vector field X as above we have

$$\alpha_k \circ \mathcal{L}_X = \mathrm{id}$$

since

$$\begin{aligned}
& \alpha_k \circ \mathcal{L}_X(f dx^{i_1} \wedge \cdots \wedge dx^{i_k})(p) \\
= \;& \alpha_k \left(\left(kf + \sum_i r_i \frac{\partial f}{\partial x^i} \right) dx^{i_1} \wedge \cdots \wedge dx^{i_k} \right) \\
= \;& \left(\int_0^1 kt^{k-1} f(tp) + \sum_i r_i(tp) \frac{\partial f}{\partial x^i}|_{tp} \right) dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\
= \;& \left(\int_0^1 kt^{k-1} f(tp) + \sum_i t^{k-1} r_i(tp) \frac{\partial f}{\partial x^i}|_{tp} \right) dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\
= \;& \left(\int_0^1 \frac{d}{dt} f(t^k f(tp)) \right) dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\
= \;& f(p) dx^{i_1} \wedge \cdots \wedge dx^{i_k}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \alpha_{k+1} \circ d(f dx^{i_1} \wedge \cdots \wedge dx^{i_k})(p) \\
= & \alpha_{k+1} \left(\sum \frac{\partial f}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \right)(p) \\
= & \sum \left(\int_0^1 t^k \frac{\partial f}{\partial x^i}|_{tp} \right) dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}
\end{aligned}$$

and that is the same as $d \circ \alpha_k(f dx^{i_1} \wedge \cdots \wedge dx^{i_k})$. Then we have:

$$\mathrm{id} = \alpha_k \circ \mathcal{L}_X = \alpha_k \circ \iota_X \circ d + \alpha_k \circ d \circ \iota_X = \alpha_k \circ \iota_X \circ d + d \circ \alpha_{k-1} \circ \iota_X$$

and hence $h_k = \alpha_{k-1} \circ \iota_X$ has the desired property. □

This immediately implies the Poincaré lemma.

Lemma 10.7. *Consider*

$$\omega = \sum (-1)^{j-1} x^j dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n$$

Then the restriction to S^{n-1} is closed but not exact.

Proof. Closed is clear for dimension reasons. For nonexactness, note that

$$\int_{S^{n-1}} \omega = \int_{D^n} d\omega = n\mathrm{vol}(D^n) > 0.$$

□

Lemma 10.8. *There is no smooth retraction $r : D^n \to S^{n-1}$.*

Proof. If we had this, we could write the identity as $r \circ i$, and then pullback induces

$$H^{n-1}(S^{n-1}) \to H^{n-1}(D^n) \to H^{n-1}(S^{n-1})$$

But, the middle group is zero, while the outer ones aren't. □

Korollar 10.9. *Any smooth map $F : D^n \to D^n$ has a fixed point.*

Proof. If not, consider the map which sends a point $x \in D^n$ to the first intersection point of the ray from x to $F(x)$ with S^{n-1}. □

11. (Half-)Lecture 14 - Computing some cohomology groups

Now, we want to compute some cohomology groups of some basic manifolds. We begin by discussing $\mathbb{R}^n$. First observe that

$$H^0(\mathbb{R}^n) = \mathbb{R}$$

Namely, 0–forms are functions. The zeroth cohomology is therefore the space of all functions f with $df = 0$. In other words, (locally) constant functions. On $\mathbb{R}^n$ such a function is globally constant, and therefore determined by its value at any point. More generally, this discussion shows:

Lemma 11.1. *Let M be any smooth manifold. Then*

$$\mathrm{dim} H^0(M) = \#\{\textit{connected components of } M\}$$

Also, note that since there are no nonzero alternating k–forms on a vector space of dimension $< k$, we have:

Lemma 11.2. *Let M be n-dimensional. Then*

$$H^i(M) = 0, \quad \forall i > n$$

11.1. **The Circle.** Now, we consider the manifold $M = S^1$. By the two general lemmas above, we only have to compute $H^1(S^1)$. Observe that we have an integration map

$$I : \Omega^1(S^1) \to \mathbb{R}, \quad \omega \mapsto \int_{S^1} \omega$$

By Stokes theorem, the integral of an exact ω is zero, and therefore the integration map I descends to a map on $H^1(S^1)$.

Lemma 11.3. *The map*

$$I : H^1(S^1) \to \mathbb{R}$$

is an isomorphism.

Proof. Linearity of I is clear. Recall from a homework set that there is a 1-form η_0 so that $\int \eta_0 \neq 0$. This implies that I is surjective (by integrating multiples of η_0). Hence, we are left with showing that I is injective. In other words, suppose we are given a 1–form ω so that

$$\int_{S^1} \omega = 0,$$

and we need to show that $\omega = df$ for some function $f : S^1 \to \mathbb{R}$.
Let $\gamma : \mathbb{R} \to \S^1$ be the map

$$t \mapsto \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{pmatrix},$$

and define

$$f(\gamma(t)) = \int_{\gamma|_{[0,t]}} \omega$$

Observe that this f actually defines a function $f : S^1 \to \mathbb{R}$ since $\int_\gamma \omega = 0$. Also, note that

$$f(\gamma(t_0 + s)) - f(\gamma(t_0)) = \int_{\gamma|_{[t_0, t_0+s]}} \omega$$

Writing this out in coordinates, coming from restricting γ to a small interval, we see that

$$df = \omega$$

which shows the lemma. $\square$

We will see two different proofs of the same result later.

12. Lecture 15 - Computing more cohomology (badly, unfortunately)

Note: Many things did not go optimally during this lecture – the script only recalls some the results that we will use later.

12.1. **The Annulus.**

Next, we want to look at the manifold $M = S^1 \times \mathbb{R}$, and prove the following:

Lemma 12.1. *The integration map*

$$P : H^1(M) \to \mathbb{R}, \quad \omega \mapsto \int_{S^1 \times \{0\}} \omega$$

is an isomorphism.

Proof. Linearity and well-definedness is just as last time for the circle. To show surjectivity, consider the map

$$p : S^1 \times \mathbb{R} \to S^1$$

projecting on the first coordinate. Let η_0 be a 1-form on S^1 so that $\int_{S^1} \eta_0 = 1$. Then,

$$\int_{S^1 \times \{0\}} p^* \eta_0 = \int_{S^1} \eta_0 = 1,$$

and thus we get surjectivity.
To prove injectivity, we could argue similarly to next time, but we will give a different argument. Let ω be a closed 1-form so that $\int_{S^1 \times \{0\}} \omega = 0$. Define

$$F^{(k)} : M \times [0,1] \to M, \quad ((s,x),t) \mapsto (s, x + kt).$$

Note that this satisfies the prerequisites of exercise 1 of sheet 1. Hence, we have for any x:

$$0 = \int_{S^1 \times \{0\}} \omega = \int_{S^1 \times \{0\}} (F_1^{(x)})^* \omega = \int_{S^1 \times \{x\}} \omega$$

where the first equality follows since $[\omega] = [(F_1^{(x)})^* \omega]$, and the second follows since $F_1^{(x)}(S^1 \times \{0\}) = S^1 \times \{x\}$.
Now, recall the curve $\gamma : \mathbb{R} \to \S^1$ from last time, and define

$$I^+ = \gamma(-\epsilon, \frac{1}{2} + \epsilon), \quad I^- = \gamma(\frac{1}{2} - \epsilon, 1 + \epsilon).$$

Then

$$S^1 = I^- \cup I^+$$

and

$$I^- \cap I^- = J_1 \cup J_2$$

with

$$J_1 = \gamma(-\epsilon, \epsilon), \quad J_2 = \gamma(\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon).$$

Observe that I^+, I^-, J_1, J_2 are all diffeomorphic to open intervals.

Note that then

$$R^- = I^- \times \mathbb{R}, \quad R^+ = I^+ \times \mathbb{R}$$

are two open sets in M, both of which are diffeomorphic to $\mathbb{R}^2$. Hence, by the Poincaré lemma, there are functions $f^\pm : R^\pm \to \mathbb{R}$ so that

$$d_u f^\pm = \omega_u \quad \forall u \in R^\pm.$$

Consider the rectangle $J_1 \times \mathbb{R} \subset R^+ \cap R^-$, and note that f^+ and f^- are defined on this rectangle and have the same derivative (namely, ω). Hence, by adding a constant to f^- we may assume that

$$f^+(p) = f^-(p) \quad \forall p \in J_1 \times \mathbb{R}.$$

Finally, note that

$$f^+((\gamma(t), x)) = \int_{\gamma \times \{x\}|_{[0,t]}} \omega + f^+((\gamma(0), x))$$

for any $t \in (-\epsilon, \frac{1}{2} + \epsilon)$ and

$$f^-((\gamma(t), x)) = \int_{\gamma \times \{x\}|_{[t,0]}} \omega + f^-((\gamma(0), x))$$

for any $t \in (\frac{1}{2} - \epsilon, \epsilon)$. Hence, we have for any $t \in J_2$:

$$f^+((\gamma(t), x)) - f^-((\gamma(t), x)) = \int_{S^1 \times \{x\} \omega} + f^+((\gamma(0), x)) - f^-((\gamma(0), x)) = 0.$$

Hence, f^+, f^- are equal at all points where they are both defined, yielding a smooth function $f : M \to \mathbb{R}$ with $df = \omega$. $\square$

13. Lecture 16 - Computing cohomology (correctly)

13.1. Top-dimensional cohomology.

We will prove the following theorem:

Satz 13.1. *Suppose that M is a compact oriented connected manifold. Then the integration map*

$$H^n(M) \to \mathbb{R}, \quad \omega \mapsto \int_M \omega$$

is an isomorphism.

Before proving this, we need the following tool. Briefly discuss compactly supported, and mention why necessary below.

Lemma 13.2 (Compactly supported Poincaré lemma for families)**.** *Suppose that*

$$\omega_u = f(x, u) dx^1 \wedge \cdots \wedge dx^n$$

is a smooth family of n-forms on $(0,1)^n$ (i.e. $f : (0,1)^n \times U \to \mathbb{R}$ is smooth, for some open $U \subset \mathbb{R}^k$), so that for each u, the form ω_u is compactly supported and

$$\int_{(0,1)^n} \omega_u = 0$$

Then there is a family of $(n-1)$-forms β_u so that $d\beta_u = \omega_u$.

Proof. We perform induction on n. For $n = 1$, and forms $\omega_u = f_u(x)dx$, simply put

$$g_u(x) = \int_0^x f(t,u)dt.$$

For any u, this has $dg_u = f_u dx = \omega_u$. Furthermore, since each $f(t,u)$ is compactly supported, and integrates to zero, each g_u is compactly supported.

Now, suppose that the lemma is proven in dimension $n-1$. Take

$$\omega_u = f(x,u)dx^1 \wedge \cdots \wedge dx^n = \beta_{(x^n,u)} \wedge dx^n$$

where now $\beta_{(x^n,u)}$ is a family of $(n-1)$–forms on $\mathbb{R}^{n-1}$.
Fix, once and for all, a form $\sigma \in \Omega^{n-1}((0,1)^{n-1})$ which is compactly supported and $\int_{(0,1)^{n-1}} \sigma = 1$. Define

$$F(x^n,u) = \int_{(0,1)^{n-1}} f(x^1,\ldots,x^n,u)dx^1 \cdots dx^{n-1}$$

and

$$\overline{\beta}_{(x^n,u)} = F(x^n,u)\sigma.$$

Then, by the induction step, we have

$$\beta_{(x^n,u)} - \overline{\beta}_{(x^n,u)} = d\gamma_{(x^n,u)}$$

for a family of $(n-1)$–forms $\gamma_{(x^n,u)}$ on $(0,1)^{n-1}$. We can interpret this as a family of $(n-1)$–forms $\hat{\gamma}_u$ on $(0,1)^n$. Observe

$$d\hat{\gamma}_u \wedge dx^n = d\gamma_{(x^n,u)} \wedge dx^n.$$

Now we have

$$\omega_u = \beta_u \wedge dx^n = \overline{\beta}_{(x^n,u)} \wedge dx^n + d\gamma_{(x^n,u)} \wedge dx^n$$
$$= (F(x^n,u)\sigma) \wedge dx^n + d\hat{\gamma}_u \wedge dx^n = \sigma \wedge (F(x^n,u)dx^n) + d\hat{\gamma}_u \wedge dx^n$$

Since $F(\cdot,u)$ integrates to zero, by the $n = 1$ case, there is a family G_u of compactly supported functions so that

$$dG_u = F(x^n,u)dx^n$$

We then have

$$d(\sigma \wedge G_u) = d\sigma \wedge G_u \pm \sigma \wedge dG_u = \pm\sigma \wedge F(x^n,u)dx^n$$
$$d(\hat{\gamma}_u \wedge dx^n) = d\hat{\gamma}_u \wedge dx^n \pm \hat{\gamma}_u \wedge d^2x^n = d\hat{\gamma}_u \wedge dx^n$$

Observe that both $\sigma \wedge G_u$ and $\hat{\gamma}_u \wedge dx^n$ are compactly supported, and thus we are done. $\square$

Now, we can prove Theorem 13.1. To this end, cover M with open sets $U_1,\ldots,U_N$ which are all diffeomorphic to open balls. Define

$$M_k = U_1 \cup \cdots \cup U_k$$

and we may assume that $U_{k+1} \cap M_k \neq \emptyset$ for all k.

Lemma 13.3. *In the setup as above, if $\omega \in \Omega^n(M_k)$ is compactly supported and $\int_{M_k} \omega = 0$. Then there is $\eta \in \Omega^{n-1}(M_k)$ compactly supported so that $d\eta = \omega$.*

Proof. We induct on k. The case $k = 1$ is just the lemma above. Next, suppose the lemma is true for M_k. Take σ compactly supported in $M_k \cap U_{k+1}$ with $\int \sigma = 1$.
Let φ, ψ be a partition of unity subordinate to (M_k, U_{k+1}). Define

$$c = \int_{M_{k+1}} \varphi\omega$$

and observe that $\varphi\omega - c\sigma$ is then a compactly supported form in M_k with zero integral by definition. Thus, there is a compactly supported α in M_k with $d\alpha = \varphi\omega - c\sigma$. Since α is compactly supported, it extends (by 0) to a form on all of M_{k+1}.
Next, look at $\psi\omega + c\sigma$ and note that this is compactly supported in U_{k+1}. Also, it has zero integral as well:

$$0 = \int_{M_{k+1}} \omega = \int_{M_{k+1}} \varphi\omega - c\sigma + c\sigma + \psi\omega = \int_{M_k} \varphi\omega - c\sigma + \int_{U_{k+1}} c\sigma + \psi\omega$$

Thus, there is a β so that $d\beta = c\sigma + \psi\omega$, compactly supported in U_{k+1}. As above, β extends.
Then, the form $\alpha + \beta$ is compactly supported and has ω as its exterior derivative. $\square$

Korollar 13.4. *Let M be connected, oriented and compact. Then the kernel of the integration map*

$$H^n(M) \to \mathbb{R}$$

consists exactly of the exact n–forms.

Since on any compact oriented n–manifold there is a closed n-form with nonvanishing integral, this finishes the proof of Theorem 13.1.

13.2. **The torus.** Now, consider $M = S^1 \times S^1$. The only cohomology group we have to compute is $H^1(M)$. To this end, use the two projections

$$p_i : S^1 \times S^1 \to S^1$$

which project to the first or second circle. Also, let η be a closed 1-form on S^1 with integral 1. Finally, let

$$a = S^1 \times \{x\}, b = \{x\} \times S^1$$

be the two standard curves. Then we have

$$\int_a p_1^*\eta = 1, \int_b p_1^*\eta = 0,$$

and

$$\int_a p_2^*\eta = 0, \int_b p_2^*\eta = 1,$$

which shows that $p_1^*\eta, p_2^*\eta$ are linearly independent. In fact, we have the following

Lemma 13.5. *The period map*

$$P : H^1(M) \to \mathbb{R}^2, \quad \omega \mapsto (\int_a \omega, \int_b \omega)$$

is an isomorphism.

Proof. In light of what we did before it suffices to show that P is injective. Hence, suppose that ω is contained in the kernel. Similar to the case of the annulus, we then have

$$\int_{S^1\times\{y\}} \omega = \int_{\{y\}\times S^1} \omega = 0$$

for all $y \in S^1$.
Now, cover the torus by four rectangles $R_1, \dots, R_4$. On each R_i there is a function $f_i : R_i \to \mathbb{R}$ so that $df_i = \omega$, where defined.
Consider R_1 and R_2 (assuming that $\{0\} \times S^1 \subset R_1 \cup R_2$) These intersect in two subrectangles. By adding a constant to f_2, we may assume that $f_1 = f_2$ on one of the components of $R_1 \cap R_2$. However, since $\int_{\{y\}\times S^1} \omega = 0$, as in the annulus case, we then get that $f_1 = f_2$ on $R_1 \cup R_2$.
The same argument applies to R_3, R_4. Now, $(R_1 \cup R_2) \cap (R_3 \cup R_3)$ again consists of two connected components. On one of them, we may assume that $f_1 = f_3$. On the other, we then get the same equality since $\int_{S^1\times\{y\}} \omega = 0$. $\square$

14. Lecture 17 - Connections (Motivations, Definitions, Examples)

We have defined the derivative of functions

$$f : M \to N$$

as a map

$$df : TM \to TN,$$

and this was actually easy: we just compute in charts, and observe that the result transforms correctly.
Suppose we wanted to define a derivative of *vector fields*. Let's start with an example of why this is harder. Consider the manifold

$$M = \mathbb{R}$$

and the vector field

$$X(t) = \frac{d}{ds}|_0(t+s).$$

In the (global) chart for M given by the identity, we thus have

$$X(t) = 1\frac{\partial}{\partial x^1}.$$

In other words, X seems to be a "constant vector field", and the only reasonable derivative of such a thing should be 0.
Now consider another chart for M, namely

$$\varphi : (0, \infty) \to (0, \infty), \quad \varphi(x) = x^2.$$

In this chart, we have

$$X(t) = \frac{1}{2t} \frac{\partial}{\partial x^1},$$

and so the vector field seems nonconstant. *No* linear transformation rule as we used for *df* can transform zero to nonzero.
On a general manifold, the situation is worse: we don't know what the correct chart is.

A small aside: why aren't we content with the Lie derivative, which can differentiate vector fields? The problem is that $\mathcal{L}_X Y$ at a point p depends on X in a *neighbourhood* of p – not just the value $X(p)$. In particular, Lie derivatives cannot be used to define directional derivatives of Y in the direction of a tangent *vector* $v \in T_pM$.

Slightly more general, and much more geometric, this problem is due to the following underlying issue. Vector fields are sections σ in the tangent bundle. A derivative should capture how $\sigma(p)$ changes as p changes. Consider a trivialisation

$$\pi^{-1}(U) \simeq U \times \mathbb{R}^k,$$

but note that this identification is *not unique*. In other words, how

$$\sigma(p) \sim (p, v(p))$$

changes in the second coordinate depends on the choice of the trivialisation. What is going on here? The issue is that there is no natural way to compare the different fibers $\pi^{-1}(p), \pi^{-1}(q)$ even locally – the *choice* of trivialisation influences this. Compare this to the example above. The two charts (identity versus square) re-identify the fibers of $T\mathbb{R}$, smoothly rescaling. Each gives a trivialisation, but fibers are sheared against each other.
Yet another perspective: If $\pi : E \to M$ is a vector bundle, then it is well-defined to say that a vector in TE is *vertical* (i.e. in the kernel of $d\pi : TE \to TM$), but it is impossible to define *horizontal* vectors in an intrinsic way.

To get around this issue, we will follow two paths. One is to consider "derivatives" of vector fields etc. which have the properties we want, and the other is to study the geometric problem of horizontal transport of vectors in TE. Both will turn out to be equivalent, but very different in flavor.

14.1. **Linear Connections, Definitions and Examples.** Let M be a manifold, and let $E \to M$ be a vector bundle. The following is the core definition.

Definition 14.1. A *linear connection* on E is a map

$$\nabla : \Gamma(TM) \otimes \Gamma(E) \to \Gamma(E),$$

usually written as $\nabla_V \sigma$ for V a vector field on M and σ a section on E. We require that the following rules hold for V, W vector fields on M, f a smooth function on M, and σ, η sections of E:

Additivity:

$$\nabla_{V+W}\sigma = \nabla_V \sigma + \nabla_W \sigma$$
$$\nabla_V(\sigma + \eta) = \nabla_V \sigma + \nabla_V \eta$$

Tensorality:

$$\nabla_{fV}\sigma = f\nabla_V \sigma$$

Leibniz rule:

$$\nabla_V(f\sigma) = df(V)\sigma + f\nabla_V \sigma$$

Let us consider a few examples.

Example 14.2 (The flat connection on Euclidean space)**.** *Consider the manifold $M = \mathbb{R}^n$, the tangent bundle $E = TM$, and the map*

$$(\nabla_V W)_p = D_p W(V(p)).$$

We need to check that this has the desired properties. The Leibniz rule is just the usual product rule, maybe easiest to see when remembering that $D_p F(v) = (F \circ c)'(0)$ where c is a curve through p with derivative v at 0..

Example 14.3 (The round connection on the sphere)**.** *Consider the manifold $M = S^2 \subset \mathbb{R}^3$, and again the tangent bundle $TM = E$. First, observe that for any $p \in S^2$ there is a natural identification*

$$T_p S^2 = p^{|}$$

as curve tangent spaces. Now, denote for any $p \in S^2$ by

$$\Pi_p : \mathbb{R}^3 \to p^{\perp}$$

the orthogonal projection. Suppose that X is a vector field on S^2. We can interpret this as a smooth function

$$X : S^2 \to \mathbb{R}^3, \quad X(p) \perp p \forall p.$$

Let U be an open neighbourhood of S^2, e.g.

$$U = \{q \in \mathbb{R}^3, \frac{1}{2} < \|q\| < 2\},$$

and let $\hat{X} : U \to \mathbb{R}^3$ be a smooth extension of X to U, e.g.

$$\hat{X}(v) = X\left(\frac{v}{\|v\|}\right).$$

We then define, with our identifications,

$$(\nabla_V W)_p = \Pi_p(D_p \hat{W}(V(p))).$$

As a first step, observe that this does not depend on the choice of U or the extensions (this can again be seen by interpreting as the derivative along a

curve which is completely contained in S^2). The desired properties follow from the product rule etc.
If this seems familiar from GTF, that is not a coincidene.

Example 14.4 (New from old)**.** *If ∇ is a linear connection, and ω is a 1–form, consider*

$$\nabla'_V\sigma = \nabla_V W + \omega(V)\sigma.$$

This is a connection again.

Example 14.5 (The flat connection on the torus)**.** *Let $M = S^1 \times S^1$ the torus, and E the tangent bundle. Recall that we defined a (universal cover) map*

$$\pi : \mathbb{R}^2 \to M$$

Suppose that W is a vector field on M. Then there is a (unique) vector field $\widetilde{W}$, called a lift, *on $\mathbb{R}^2$ so that*

$$d_p\pi(W(p)) = V(p)$$

since all $d_p\pi$ are invertible. Now, suppose that V, W are vector fields on M and $\widetilde{V}, \widetilde{W}$ are lifts.
Define

$$(\nabla_V W)_p = d\pi_q(D_q\widetilde{W}(\widetilde{V}(q)))$$

where q is a point with $p = \pi(q)$. Once we would know well-definedness, the connection properties would follow immediately. But why is this well-defined? If q' is another choice, then $q' = q + v$ for some vector $v \in \mathbb{Z}^2$. Furthermore, we have

$$\pi(z) = \pi(z+v) \quad \forall z,$$

and therefore

$$d_z\pi(w) = d_{z+v}\pi(w)$$

for any $w \in \mathbb{R}^2 = T_z\mathbb{R}^2 = T_{z+v}\mathbb{R}^2$. Hence, we have

$$d\pi_q(D_q\widetilde{W}(\widetilde{V}(q))) = d\pi_{q'}(D_q\widetilde{W}(\widetilde{V}(q)))$$

Similarly, since $\widetilde{W}$ is a lift of V, we have

$$\widetilde{W}(z+v) = \widetilde{W}(z)$$

and therefore

$$D_q\widetilde{W} = D_{q'}\widetilde{W}$$

15. Lecture 18 - Connections (Local Properties)

As a first step, we want to show that linear connections do not have the "problem" that Lie derivatives have. To this end, let ∇ be a linear connection on some bundle E over M. Let $p \in M$ be a point, and consider a "bump function" $\rho : M \to \mathbb{R}$ with the properties:

- ρ is identically equal to 1 on an open neighbourhood U_1 of p,
- ρ is identically equal to 0 outside an open neighbourhood U_2 of p.

Such functions can easily constructed in charts. In particular, U_2 can be chosen to be as small as we want.
Now compute, for some $p \in U_1$,

$$(\nabla_{\rho V}\sigma)_p = \rho(p)(\nabla_V\sigma)_p = (\nabla_V\sigma)_p,$$

using tensorality. Similarly, we have

$$(\nabla_V \rho\sigma)_p = d\rho(p)\sigma + \rho(p)(\nabla_V\sigma)_p = (\nabla_V\sigma)_p,$$

This already shows the following

Lemma 15.1. *The value of $(\nabla_V\sigma)_p$ depends only on the germs of V and σ. In particular, $(\nabla_V\sigma)_p$ is defined for vector fields V and sections σ defined on open neighbourhoods of p rather than all of M.*

Proof. Suppose that V_1, V_2 are two vector fields with the same germ at p. That is, there is a neighbourhood U of p so that $V_1(p) = V_2(p)$ for all p in U. We choose a bump function ρ which is zero outside U. Observe that then $\rho V_1 = \rho V_2$ globally, and hence $\nabla_{\rho V_1}\sigma = \nabla_{\rho V_2}\sigma$ for all σ. Using the above equality then shows that $\nabla_V\sigma$ depends only on the germ of V.
Now, even if V is a vector field defined only on U, the vector field ρV extends by 0 to all of M, and so $\nabla_{\rho V}\sigma$ is defined. By the above, the value does not depend on the choice of ρ.
The case of varying σ works exactly the same, using the other equality. $\square$

In particular, we can therefore take a vector field V, and using a chart write it *locally* as

$$V(p) = \sum a^i(p)\frac{\partial}{\partial x^i}$$

and then compute, using tensorality,

$$(\nabla_V\sigma)_p = \sum a^i(p)\nabla_{\frac{\partial}{\partial x^i}}\sigma.$$

In particular, the dependence is only on the values $a^i(p)$ *at the point* p. We emphasise this:

Korollar 15.2. *The value $(\nabla_V\sigma)_p$ depends on the value $V(p)$ of V at p, and the germ of σ at p. In particular, $\nabla_v\sigma$ is already defined for tangent vectors $v \in T_pM$.*

Taking this idea a little bit further, we can exactly characterise the data a linear connection comprises. Namely, around any point p the bundle E admits sections $\mu_1, \dots, \mu_k : U \to E|_U$ so that for each $p \in U$, the values $\mu_i(p)$ are a basis of $\pi^{-1}(p)$. Then we can write

$$\nabla_{\frac{\partial}{\partial x^i}}\mu_j = \sum_k \Gamma^k_{ij}\mu_k$$

for smooth functions $\Gamma^k_{ij} : U \to \mathbb{R}$. These are called the *Christoffel symbols* of the connection. It is clear, but important, that

Korollar 15.3. *The Christoffel symbols at p determine $(\nabla_V\sigma)_p$ for all V,σ. In fact, if we locally have*

$$\sigma(p)=\sum s^i(p)\mu_i(p)$$

and

$$V(p)=\sum v^i(p)\frac{\partial}{\partial x^i},$$

then we have

$$(\nabla_V\sigma)_p=\sum_k\left(\sum_i v^i\left(\frac{\partial s^k}{\partial x^i}+\sum_j s^j\Gamma^k_{ij}\right)\right)\mu_k$$

The key thing to remember from this formula is: the Christoffel symbols measure how much the linear connection differs (in charts) from just taking the usual derivative!
Also, we can strengthen our dependence corollary one last time, by noting that

$$\sum_i v^i\frac{\partial s^k}{\partial x^i}=(s^k\circ\gamma)'(0)$$

for γ any curve through p with derivative $V(p)$.

Korollar 15.4. *The value $(\nabla_V\sigma)_p$ depends on the value $V(p)$ of V at p, and the values of σ on any choice of a curve $\gamma:(-\epsilon,\epsilon)\to M$ with $\gamma(0)=p,\gamma'(0)=V(p)$.*
In particular, $\nabla_v\sigma$ is already defined for tangent vectors $v\in T_pM$.

Example time!

Example 15.5. *The flat connection of $\mathbb{R}^n$ is just the usual derivative, so by the intuition above we would expect that the Christoffel symbols all vanish. In fact, this is true. Namely, for any pair i,j take the* constant *vector fields X_i,X_j equal to the i,j-th standard basis vector. Then, since they are constant, we have*

$$D_pX_i(X_j)=0$$

and so indeed all Christoffel symbols vanish.

Example 15.6. *The round connection on the sphere has more interesting Christoffel symbols. We compute them only at one point, say $p=(0,0,1)$. First, we need a chart (which then also gives a trivilisation of the tangent bundle). A choice that makes computations not so hard comes from polar coordinates. Namely, consider*

$$F(\theta,\phi)=\begin{pmatrix}\cos(\theta)\cos(\phi)\\ \sin(\theta)\cos(\phi)\\ \sin(\phi)\end{pmatrix}$$

in a small neighbourhood of $(0,\pi/2)$. The tangent vector fields are

$$\frac{\partial F}{\partial\theta}=\begin{pmatrix}-\sin(\theta)\cos(\phi)\\ \cos(\theta)\cos(\phi)\\ 0\end{pmatrix},\frac{\partial F}{\partial\phi}=\begin{pmatrix}-\cos(\theta)\sin(\phi)\\ -\sin(\theta)\sin(\phi)\\ \cos(\phi)\end{pmatrix}$$

Strictly speaking, we get local (trivialising) vector fields in the following way:

$$X_\theta(q)=\left.\frac{\partial F}{\partial\theta}\right|_{F^{-1}(q)},X_\phi(q)=\left.\frac{\partial F}{\partial\phi}\right|_{F^{-1}(q)}$$

and so we get e.g. the basis of T_pM:

$$X_\theta(p)=\begin{pmatrix}0\\1\\0\end{pmatrix},X_\phi(p)=\begin{pmatrix}-1\\0\\0\end{pmatrix}.$$

For later use, we note that hence the orthogonal projection onto T_pM in this basis is just $(x,y,z)\mapsto(x,-y)$.
Now, to compute the Christoffel symbols, we have to first remember the definition:

$$(\nabla_VW)_p=\Pi_p(D_p\hat{W}(V(p))).$$

We can make this a bit more managable. Namely, let c be a curve on S^2 through p with derivative $V(p)$. Then we have

$$D_p\hat{W}(V(p))=(\hat{W}\circ c)'(0)=(W\circ c)'(0).$$

In our case, this is particularly easy. Namely, we need to compute

$$D_pX_i(X_j)$$

and hence we would need curves c on S^2 whose derivatives are the X_j. These are given by $c_\theta=F(\cdot,\phi),c_\phi=F(\theta,\cdot)$. Hence, what we actually want to compute is:

$$D_pX_i(X_j)=\left(\left.\frac{\partial F}{\partial\theta}\right|_{F^{-1}(c_j(t))}\right)'(0)=\frac{\partial^2F}{\partial x^i\partial x^j},$$

in other words the second partial derivatives of F. This we can do:

$$\frac{\partial^2F}{\partial\theta\partial\phi}=\begin{pmatrix}\sin(\theta)\sin(\phi)\\ -\cos(\theta)\sin(\phi)\\ 0\end{pmatrix},\frac{\partial^2F}{\partial\theta\partial\theta}=\begin{pmatrix}-\cos(\theta)\cos(\phi)\\ -\sin(\theta)\cos(\phi)\\ 0\end{pmatrix},$$

$$\frac{\partial^2F}{\partial\phi\partial\phi}=\begin{pmatrix}-\cos(\theta)\cos(\phi)\\ -\sin(\theta)\cos(\phi)\\ -\sin(\phi)\end{pmatrix},\frac{\partial F}{\partial\phi\partial\theta}=\begin{pmatrix}\sin(\theta)\sin(\phi)\\ -\cos(\theta)\sin(\phi)\\ \cos(\phi)\end{pmatrix}$$

and evaluating at $p=F(0,\pi/2)$ yields

$$\frac{\partial^2F}{\partial\theta\partial\phi}=\begin{pmatrix}0\\-1\\0\end{pmatrix},\frac{\partial^2F}{\partial\theta\partial\theta}=\begin{pmatrix}-1\\0\\0\end{pmatrix},$$

$$\frac{\partial^2 F}{\partial\phi\partial\phi} = \begin{pmatrix} -1 \\ -0 \\ -1 \end{pmatrix}, \frac{\partial F}{\partial\phi\partial\theta} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

To get the Christoffel symbols, we now project onto T_pM in our chosen basis, and obtain

$$\Gamma^\theta_{\theta\phi} = \Gamma^\theta_{\phi\theta} = 0, \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = 1.$$

$$\Gamma^\theta_{\theta\theta} = -1, \Gamma^\phi_{\theta\theta} = 0.$$

$$\Gamma^\theta_{\phi\phi} = -1, \Gamma^\phi_{\phi\phi} = 0.$$

There seems to be symmetry. Also, these are not *all zero. Hence, in the polar coordinate chart, the round connection is* not *just the usual deriviative.*

16. Lecture 20 - Riemannian Metrics

Our next goal is twofold: we will start with geometry properly, and at the same time discover one of the main sources of interesting connections on manifolds.
The key definition is the following

Definition 16.1. Let M be a smooth manifold. A *(Riemannian) metric* is a smooth section

$$g : M \to (T^*M)^{\otimes 2}$$

so that for any p, the biliniear form $g(p)$ is symmetric and positive definite. A pair (M, g) of a smooth manifold with Riemannian metric is called a *Riemannian manifold.*

In other words, a Riemannian metric is the choice of a scalar product on each tangent space, varying smoothly.

Let us discuss a few basics first. To start: why is this called a "metric"? Immediately, this allows us to measure lengths of tangent vectors in the usual way:

$$\|v\| = \sqrt{g(v, v)}.$$

Using this, we can then measure the length of curves.

Definition 16.2. Let (M, g) be a Riemannian manifold, and let $\gamma : [a, b] \to M$ be a piecewise continuously differentiable curve. Define

$$l(\gamma) = \int_a^b \|\gamma'(t)\| dt.$$

We have the following basic properties:

Lemma 16.3. *i)* $l(\gamma) \geq 0$.
ii) $l(\gamma) = 0$ *exactly if* γ *is a constant curve.*
iii) $l(\gamma * \gamma') = l(\gamma) + l(\gamma')$.

and these allow the following definition to make sense.

Definition 16.4. Let (M, g) be a connected Riemannian manifold. We define the corresponding metric by

$$d(x, y) = \inf\{l(\gamma) \mid \gamma \text{piecewise continuously differentiable, connects} x, y\}$$

To show that d is indeed a metric, the only really nontrival part is that $d(x, y) = 0$ implies $x = y$. In other words, we need to show

Lemma 16.5. *Suppose that $x \neq y$. Then $d(x, y) > 0$.*

Proof. (Will be filled in shortly). □

Time for some examples:

(1) If U is an open subset of $\mathbb{R}^n$, let $g(p)$ be the standard scalar product for all p. The metric d one gets is the *induced path metric* on the set U.
(2) Consider $M = S^n \subset \mathbb{R}^{n+1}$. We have already seen that there is a natural identification $T_pM = p^\perp$. Hence, we can define $g(p)$ to be the restriction of the standard scalar product. Again, the metric is the induced one.
(3) Consider the torus $M = S^1 \times S^1$, and identify $S^1 \subset \mathbb{C}$. Then, we have an identification
$$T_{(s,t)}M = s^\perp \times t^\perp$$
We can define a metric by setting
$$g_{(s,t)}(x \oplus y, x' \oplus y') = xx' + yy'$$
There is an equivalent description via the universal cover.

Is being a Riemannian manifold a restriction on M? No, as the following shows

Lemma 16.6. *Let M be a smooth manifold. Then there is a Riemannian metric g on M.*

Proof. Cover M with chart neighbourhoods U_i, and choose a partition of unity ρ_i subordinate to that cover. On each U_i define a metric by pulling back the standard scalar product from $\mathbb{R}^n$.

$$g_i = \varphi_i^* \langle ., . \rangle$$

Observe that $\rho_i g_i$ defines a global section of $(T^*M)^{\otimes 2}$. Define

$$g = \sum_i \rho_i g_i$$

This is symmetric as a sum of symmetric forms. Furthermore, is is positive definite:

$$g(v, v) = \sum_i \rho_i g_i(v, v) \geq 0$$

as a sum of non-negative (and at least one positive!) terms. □

Bemerkung 16.7. *We are using here that non-negative sums of positive definite forms are positive definite. Hence, if we would require other signatures, such a proof would not work.*

17. Lecture 20 - Levi-Civita connections, curvature basics

Our next goal is that for any Riemannian metric, there is a corresponding connection on TM that is compatible with it. What is compatible supposed to mean? One, simple-minded approach is the following

Definition 17.1. Let g be a Riemannian metric on M. A connection ∇ on TM is called *compatible with the metric* if for all vector fields X, Y, Z we have:

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

Before discussing existence, note that this is satisfied by the flat connection on $\mathbb{R}^n$ and the round connection on S^n. For the second one, write the equation down for the ambient connection, and note that since X, Y are tangent to the sphere we can replace by the round connection.

Satz 17.2. *Let (M, g) be a Riemannian manifold. Then there is a unique connection which is symmetric:*

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

and compatible with the metric.

The connection guaranteed by the theorem is called the *Levi-Civita connection.*

Proof. We begin with uniqueness. Suppose we have such a connection, and suppose that X, Y, Z are three vector fields. Using compatibility, we get

$$\begin{aligned} Xg(Y,Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ Yg(X,Z) &= g(\nabla_Y X, Z) + g(X, \nabla_Y Z) \\ Zg(X,Y) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \end{aligned}$$

Subtracting the first two from each other yields

$$\begin{aligned} Xg(Y,Z) - Yg(X,Z) &= g(\nabla_X Y - \nabla_Y X, Z) + g(Y, \nabla_X Z) - g(X, \nabla_Y Z) \\ &= g([X,Y], Z) + g(Y, \nabla_X Z) - g(X, \nabla_Y Z) \end{aligned}$$

Subtracting the third from this gives

$$\begin{aligned} Xg(Y,Z) - Yg(X,Z) - Zg(X,Y) &= g([X,Y],Z) + g(Y,[X,Z]) - g(X,\nabla_Y Z) - g(X,\nabla_Z Y) \\ &= g([X,Y],Z) + g(Y,[X,Z]) + g(X,\nabla_Z Y) - g(X,\nabla_Y Z) - 2g(X,\nabla_Z Y) \\ &= g([X,Y],Z) + g(Y,[X,Z]) + g(X,[Z,Y]) - 2g(X,\nabla_Z Y) \end{aligned}$$

In other words, $g(X, \nabla_Z Y)$ is determined by other data. Since g is non-degenerate, this means that $\nabla_Z Y$ is determined. This shows uniqueness. Conversely, we can define $\nabla_Z Y$ by this formula, and check that this is actually a connection. $\square$

Korollar 17.3 (Koszul formula)**.** *The Levi-Civita connection satisfies:*

$$2g(\nabla_X Y, Z) = Xg(Y,Z)+Yg(X,Z)-Zg(X,Y)+g([X,Y],Z)-g([X,Z],Y)-g([Y,Z],X).$$

Recall the curvature of a connection:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

For the Levi-Civita connection, this is called the *Riemann curvature tensor*. Alternatively, the name is sometimes used for

$$R(X,Y,Z,W) = g(R(X,Y)Z,W)$$

which carries the same information.
This has a few symmetries. The first is obvious from the definition:

$$R(X,Y,Z,W) = -R(Y,X,Z,W)$$

The second one is more involved:

$$R(X,Y,Z,W) = -R(X,Y,W,Z)$$

This follows from:

$$R(X,Y,Z,Z) = g(R(X,Y)Z,Z) = g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, Z)$$

To compute the first and second terms, note that

$$Yg(\nabla_X Z, Z) = g(\nabla_Y\nabla_X Z, Z) + g(\nabla_X Z, \nabla_Y Z).$$
$$Xg(\nabla_Y Z, Z) = g(\nabla_X\nabla_Y Z, Z) + g(\nabla_Y Z, \nabla_X Z)$$

and thus

$$g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z, Z) = Xg(\nabla_Y Z, Z) - Yg(\nabla_X Z, Z)$$
$$= \frac{1}{2}\left(X(Yg(Z,Z)) - Y(Xg(Z,Z))\right) = \frac{1}{2}[X,Y]g(Z,Z)$$

Also, we have

$$g(\nabla_{[X,Y]}Z, Z)) = \frac{1}{2}[X,Y]g(Z,Z).$$

and thus $R(X,Y,Z,Z) = 0$, from which the claim follows.

18. Lecture 21 - More about the curvature tensor

18.1. **More symmetries of R.** We continue with symmetries of the curvature tensor. First note the Bianchi identity

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0,$$

(which was done in much bigger generality on the problem set). In our case, this is a strightforward computation, using the Jacobi identity for Lie brackets.
This in turn gives

$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0.$$

Sum four copies of these with cyclic permutations, to get

$$R(X,Y,Z,W) = R(Z,W,X,Y).$$

Next, we come to a crucial question: what does the curvature tensor actually measure? We will answer this from two perspectives – analytical (non-commuting second derivatives) and geometric (non-flatness).
Hence, we take a small digression on other things one can do with connections.

18.2. **Connections along curves.** Suppose that M is a manifold and E is a vector bundle over M. Also consider a (piecewise) smooth curve

$$\gamma : [0,1] \to M.$$

For a bundle E, a *section of E along γ* is a smooth map

$$\mu : [0,1] \to E$$

so that $\pi\mu(t) = \gamma(t), \forall t \in [0,1]$. This also defines vector fields along curves. A central example is $\dot{c}$, which is a vector field along c for any curve c. We observe

Lemma 18.1. *Suppose that ∇ is a connection on E. Then there is an operator $\frac{\nabla}{dt}$ acting on sections along γ, which is uniquely determined by the requirements*

1. *$\frac{\nabla}{dt}$ is $\mathbb{R}$–linear.*
2. *$\frac{\nabla}{dt}$ satisfies a product rule:*
$$\frac{\nabla}{dt}(f\mu)(t) = f'(t)\mu(t) + f(t)\frac{\nabla}{dt}(t)$$
3. *If a section μ along γ is induced by a (local) section σ of E, i.e.*
$$\mu(t) = \sigma(\gamma(t)),$$
then we have
$$\frac{\nabla}{dt}\mu = \nabla_{\dot{c}}\mu$$
(which is well-defined by our discussion of dependence)

Proof. Locally write

$$\mu(t) = \sum_i a^i(t)\eta_i(t)$$

where η_i are a local basis of sections. Then, the rules completely determine $\frac{\nabla}{dt}$:

$$\frac{\nabla}{dt}(t) = \sum_i \dot{a}(t)\eta_i(t) + a(t)\nabla_{\dot{c}}\eta_i$$

, showing uniqueness. Existence follows from the same formula. □

In other words, connections induce actual derivative operators for sections along paths.

18.3. R **measures non-symmetry of second derivatives.** Suppose now that we have (M,g) a Riemannian manifold. Let $A \subset \mathbb{R}^2$ an open set. A *parametrised surface* is a smooth map

$$f : A \to M.$$

We denote by (s,t) the usual coordinates on A. Then, as above, we have vector fields along f, and induced connections $\nabla/ds, \nabla/dt$.

Lemma 18.2. *In the setting as above*

$$\frac{\nabla}{dt}\frac{\nabla}{ds}V - \frac{\nabla}{ds}\frac{\nabla}{dt}V = R\left(\frac{\partial f}{ds}, \frac{\partial f}{dt}\right)V$$

In other words, $R(x,y)V$ measures how non-symmetric in a (sub)surface tangent to $\langle x,y\rangle$, a vector field with value V is.

Proof. This is only a computation, using local description and using that coordinate vector fields commute. See do Carmo, "Riemannian geometry", Lemma 4.1 (p. 98), for the details. □

18.4. **Relatives of the curvature tensor.** Even with the symmetries, R is a fairly complicated object. To try and make it more managable, one possibility is to discard information by taking traces. The first yields the following

Definition 18.3. Let (M,g) be a Riemannian manifold. The *Ricci tensor* is defined by

$$\mathrm{Ric}_p(x,y) = \mathrm{tr}(z \mapsto R(x,z)y)$$

The other sign, and a factor of $1/(n-1)$ are also common. Sometimes $\mathrm{Ric}_p(x) = \mathrm{Ric}_p(x,x)$ is used.

Concretely,

$$\mathrm{Ric}_p(x,y) = \sum_i R(x,z_i,y,z_i)$$

for z_i an orthonormal basis for T_pM. The latter description shows that Ricci is symmetric:

$$\mathrm{Ric}_p(x,y) = \mathrm{Ric}_p(y,x),$$

and therefore $\mathrm{Ric}_p(x,x)$ actually determines all values by polarization

$$Q(x+y,x+y) = Q(x,x) + 2Q(x,y) + Q(y,y)$$

Ricci curvature measures *volume growth of (small) geodesic cones* (see next semester).
A tiny tiny aside. The Ricci tensor is of the same type as the metric itself. Hence, they can be compared, and this is (surprisingly?) meaningful. In fact, we call (M,g) an *Einstein manifold*, if there is some constant k so that

$$\mathrm{Ric} = kg.$$

Flowing the metric in the direction of Ricci yields Ricci flow, a very powerful tool in topology.

We can also take traces again to get

Definition 18.4. Let (M, g) be a Riemannian manifold. The *scalar curvature* is defined as

$$K(p) = \operatorname{tr}(x \mapsto \operatorname{Ric}_p(x, x))$$

Scalar curvature measures *volume growth of (small) balls.*

Examples. On the flat plane, $R = 0$, so all other curvatures are also zero. What about the sphere S^n? On the exercise sheet you compute that R has one interesting component $R(X_i, X_j)X_i, X_j$ up to symmetries, and that one is $+1$. On the "flat torus" R is also constantly 0 – explaining the name (a bit).

Last, but not least, one other notion of curvature.

Lemma 18.5. *Given $V \subset T_pM$ a two-dimensional subspace, and x, y a basis of V. The quantity*

$$\frac{R(x, y, x, y)}{g(x, x)g(y, y) - g(x, y)^2}$$

does not depend on x, y and is called the sectional curvature $\kappa(\sigma)$.

Proof. $GL(\sigma)$ is generated by elementary transformations of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

and by the symmetries of R and properties of a scalar product these do not change $\kappa(x, y)$. Hence, all of $GL(\sigma)$ doesn't, and the lemma follows. ☐

Lemma 18.6. *Sectional curvature κ determines R.*

Proof. Suppose R, R' are two curvature tensors defining the same sectional curvature. In other words, we have

$$R(x, y, x, y) = R'(x, y, x, y)$$

for all x, y. But

$$R(x+z, y, x+z, y) = R(x, y, x, y) + R(x, y, z, y) + R(z, y, x, y) + R(z, y, z, y)$$

and hence $R(x, y, z, y)$ can be written in terms of $R(a, b, a, b)$ type terms. Thus we get

$$R(x, y, z, y) = R'(x, y, z, y)$$

Play this game again, with

$$R(x, y+t, z, y+t) = R(x, y, z, y) + R(x, y, z, t) + R(x, t, z, y) + R(x, t, z, t)$$

and hence $R(x, y, z, t) + R(x, t, z, y)$ can be written in terms of $R(a, b, c, b)$ type terms. Such sums are therefore the same for R, R':

$$R(x, y, z, t) + R(x, t, z, y) = R'(x, y, z, t) + R'(x, t, z, y)$$

$$R(x, y, z, t) - R'(x, y, z, t) = -R(x, t, z, y) + R'(x, t, z, y) = R(y, z, x, t) - R'(y, z, x, t)$$

Hence, the difference $R - R'$ is invariant under cyclic permutations. Since both R and R' sum to zero under cyclic permutations, we get $3(R - R') = 0$. □

19. Lecture 22 - Classical Differential Geometry

Next, we want to briefly connect the Riemann curvature tensor to more classical notions of curvature.

19.1. **Curvature of Curves.** First, we define the curvature of space curves. Let $c : [a, b] \to \mathbb{R}^3$ be a smooth curve which is parametrised by arclength, i.e. $\|c'(t)\| = 1$, for all t. We define the curvature of the curve as

$$\kappa(t) = \|c''(t)\|$$

It's instructive to consider circles. Namely, the following parametrises a circle of radius r:

$$c(t) = r \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}, c : [0, 2\pi] \to \mathbb{R}^3.$$

However, this is not by arclength (the norm of the derivative is r), hence we should take

$$c(t) = r \begin{pmatrix} \cos(t/r) \\ \sin(t/r) \end{pmatrix}, c : [0, 2\pi r] \to \mathbb{R}^3.$$

We then have $\kappa(t) = 1/r$.
In fact, this leads to the following geometric interpretation of curvature.

Lemma 19.1. *Let c be a smooth curve parametrised by arclength. Then $\kappa(t) = 1/r$, where r is the radius of the circle which approximates c in the best possible way at $c(t)$.*

Proof. By Taylor's theorem, the best approximating circle is the one which agrees with c in value, first and second derivative at $c(t)$. By the computation above, this shows the claim. □

19.2. **Curvature of Surfaces.** Throughout, we will consider an embedded surface in three-space. That is, we fix a smooth map

$$f : S \to \mathbb{R}^3$$

where S is a two-dimensional manifold, and we assume that $d_q f$ has full rank (i.e. 2) for every $q \in S$. We can assume that f is a topological embedding in addition (but this is not strictly necessary).
Recall from before that we can define a metric and connection on S in the following way. For the metric, simply define

$$g_p(v, w) = \langle d_p f(v), d_p f(w) \rangle$$

The connection is a little bit more involved. Suppose that X, Y are vector fields on S, and that $p \in S$ is given. We claim that there is a neighbourhood

U of p and W of $f(p)$ and smooth vector fields $\widetilde{X}, \widetilde{Y}$ on W so that for all $q \in V$:

$$\widetilde{X}(f(q)) = d_q f(X(q)),$$

and similarly for Y. To see that such a vector field exists, let e_1, e_2 be a basis of T_pS, and N be a vector extending $d_pf(e_1), d_pf(e_2)$ to a basis of $\mathbb{R}^3$ (this exists by the rank condition). Consider the map

$$F : S \times \mathbb{R} \to \mathbb{R}^3$$

defined by

$$F(s,t) = f(s) + tN.$$

This is clearly smooth, and $D_{(p,0)}F$ now has full rank (i.e. 3). By the inverse function theorem, F is therefore a diffeomorphism onto its image in a small neighbourhood $U \times (-\epsilon, \epsilon)$ of $(p, 0)$. Let W be the image of that neighbourhood. Then we can define

$$\widetilde{X}(w) = d_{\pi_1 F^{-1}(w)} f(X(\pi_1 F^{-1}(w)))$$

where π_1 denotes projection to the first coordinate of a tuple $(s,t) \in S \times \mathbb{R}$. In other words, we have

$$\widetilde{X}(F(q,t)) = d_q f(X(q)).$$

We call any such $\widetilde{X}$ an *extension* of X. To define the connection we now put

$$(\nabla_X Y)_p = d_p f^{-1} \left(\Pi_{\mathrm{im} d_p f} D_{f(p)} \widetilde{Y}(\widetilde{X}(f(p))) \right)$$

where $\Pi_{\mathrm{im} d_p f}$ denotes orthogonal projection of $\mathbb{R}^3$ to the image of d_pf.

To simplify exposition and notation, we will from now on assume that f is an embedding (in particular injective), using f to identify p to $f(p)$ and always using d_pf to identify T_pS with the subspace $\mathrm{im} d_p f$ of $\mathbb{R}^3$. In other words, we restrict to the case where S is a submanifold of $\mathbb{R}^3$. With this identification, the formula above becomes

$$(\nabla_X Y)_p = \Pi_{T_pS} D_p \widetilde{Y}(\widetilde{X}(p))$$

This actually does not depend on the choice of extensions. Namely, we have

$$D_p \widetilde{Y}(\widetilde{X}(p)) = (\widetilde{Y} \circ c)'(0),$$

where c is any curve with $c(0) = p, c'(0) = \widetilde{X}(p)$. Now, since $p \in S$, we have $\widetilde{X}(p) = X(p)$, and we can choose the curve c to lie completely inside S (by the definition of tangent vectors). However, if c is contained in S, we have

$$(\widetilde{Y} \circ c)(t) = (Y \circ c)(t)$$

for all t. This shows that

$$D_p \widetilde{Y}(\widetilde{X}(p)) = (Y \circ c)'(0),$$

which does not depend on the extensions.

It is easy to check (do this!) that ∇ is the Levi-Civita connection on S for the metric g defined above (Hint: you only have to check that it satisfies the properties, by the uniqueness of the Levi-Civita connection).

19.3. **A Review of Orientations.** At this point, we want to give a few more details on orientations on manifolds. First, we recall the following notion from linear algebra.

Definition 19.2. An *orientation* on a real vector space V is an equivalence class $\mathcal{O}$ of bases with respect to the equivalence relation

$$(b_1, \dots, b_n) \sim (b_1', \dots, b_n')$$

if $\det A > 0$ for the linear map with $Ab_i = b_i'$ for all i.

It is clear that there are exactly two orientations on a vector space. If we have chosen an orientation $\mathcal{O}$, we often say that a basis is positively oriented, if it is contained in $\mathcal{O}$. On $\mathbb{R}^n$, there is a *standard orientation*, defined by the standard basis vectors, which we denote by $\mathcal{O}^{\mathrm{std}}$. Also note that if $\mathcal{O}$ is an orientation, and A is an isomorphism of vector spaces, then $A\mathcal{O}$ is an orientation.
Now, we define orientations on manifolds as compatible choices of orientations on all of the tangent spaces. Namely:

Definition 19.3. Let M be a smooth manifold. An *orientation* on M is a choice of an orientation $\mathcal{O}_p$ on each T_pM which is locally constant in the following sense: for each $p \in M$ there is a neighbourhood U of p, a diffeomorphism $\varphi : U \to V$ with $V \subset \mathbb{R}^n$, and an orientation $\mathcal{O}$ on $\mathbb{R}^n$ so that

$$d_q\varphi\mathcal{O}_q = \mathcal{O}$$

for all $q \in U$.

This definition is connected to the one we gave before:

Lemma 19.4. *A manifold admits an orientation exactly if it admits an oriented atlas.*

Proof. Suppose that M admits an oriented atlas $\mathcal{A}$. Then, define

$$\mathcal{O}_p = d_q\varphi^{-1}\mathcal{O}^{\mathrm{std}}$$

where $\varphi \in \mathcal{A}$ is a chart around p. Note that since $\mathcal{A}$ is an oriented atlas, this is well-defined (different choices of chart φ yield the same orientation on T_pM). This is obviously locally constant (the charts φ can be used as the diffeomorphisms).
Conversely, suppose that an orientation is given. Observe that then for each $p \in M$ there is a neighbourhood U of p, a diffeomorphism $\varphi : U \to V$ with $V \subset \mathbb{R}^n$ so that

$$d_q\varphi\mathcal{O}_q = \mathcal{O}^{\mathrm{std}}$$

for all $q \in U$. This φ can be obtained from the definition of locally constant by possibly postcomposing with a linear isomorphism of determinant -1. Now, simply define the atlas $\mathcal{A}$ to comprise exactly those φ. □

We also mention the following, which was a homework problem:

Proof. A manifold M of dimension n admits an orientation exactly if it admits a n-form ω which is nowhere vanishing. □

The connection to orientations is here very explicit: if $\mathcal{O}$ is the orientation, then the form ω has the property that

$$\omega_p(b_1, \ldots, b_n) > 0 \Leftrightarrow (b_1, \ldots, b_n) \in \mathcal{O}_p.$$

The form ω is not determined by the orientation, only up to a multiple by a positive smooth function on the manifold. If we have a metric, we can make this choice canonical:

Lemma 19.5. *Suppose that (M, g) is a Riemannian manifold of dimension n, and $\mathcal{O}$ is an orientation on M. Then there is a unique n-form with the properties:*

i) $\omega_p(b_1, \ldots, b_n) > 0 \Leftrightarrow (b_1, \ldots, b_n) \in \mathcal{O}_p$.
ii) $|\omega_p(b_1, \ldots, b_n)| = 1$ *if* $b_1, \ldots, b_n$ *is a* g_p*–orthonormal basis.*

Proof. Suppose that ω_0 is a n–form compatible with the orientation in the sense of i). This exists by the previous lemma. Now, we claim that the value

$$|\omega_0(b_1, \ldots, b_n)|$$

is the same for all orthonormal bases $b_1, \ldots, b_n$ of T_pM. This follows from the fact that

$$\omega_0(Ab_1, \ldots, Ab_n) = \det A \omega_0(b_1, \ldots, b_n)$$

for linear maps A, and the fact that any two orthonormal bases differ by a linear map of determinant ± 1. Hence, we can define a function $f(p) = |\omega_0(b_1, \ldots, b_n)|$, where $b_1, \ldots, b_n$ is a orthonormal basis of T_pM. Then $\omega = \omega_0/f$ is the desired form. □

The form guaranteed by the lemma is usually called the *Riemannian volume form.*
Finally, we mention the following, which follows from a homework problem:

Lemma 19.6. *Suppose that M is a manifold of dimension n, and that $\iota : M \to \mathbb{R}^{n+1}$ is a smooth immersion of M into $\mathbb{R}^{n+1}$. Then M is orientable exactly if there is a smooth unit normal vector field, i.e. a smooth map $N : M \to \mathbb{R}^{n+1}$ so that*

i) $\|N(p)\| = 1$ *for all* p,
ii) $N(p)$ *is orthogonal to* $\operatorname{im} d_p\iota(T_pM)$ *for all* p.

19.4. **Gaussian curvature and sectional curvature.** Let S be an immersed surface in $\mathbb{R}^3$, and let N be a unit normal field. Consider now a curve $c : [a, b] \to S$ in a surface S in $\mathbb{R}^3$. The curvature of c itself does not tell us anything about the surface (a flat plane contains arbitrarily curved circles). To avoid this, we make the following defintion: the *normal curvature* is defined by

$$\kappa_{\mathrm{norm}}(t) = \langle c''(t), N(c(t))\rangle.$$

In other words, normal curvature measures how much c is curving away from the surface. Now, since c is a curve in S, its derivative is tangent to S and therefore

$$0 = \langle c'(t), N(c(t))\rangle$$

Differentiating yields

$$0 = \langle c''(t), N(c(t))\rangle + \langle c'(t), dN_{c(t)}(c'(t))\rangle,$$

in other words

$$\kappa_{\mathrm{norm}}(t) = \langle c'(t), -dN_{c(t)}(c'(t))\rangle.$$

Observe that this implies that the normal curvature of c only depends on the tangent vector $c'(t)$. The term

$$H(x, y) = \langle x, -dN(y)\rangle, \quad H : T_pM \times T_pM \to \mathbb{R}$$

is classically knows as the *second fundamental form.* Note that it uses not just intrinsic data about S, but also how it is embedded in $\mathbb{R}^3$ (via N).
Next, we want to connect H to the connection ∇ on S. Recall that $\nabla_X Y$ is the tangential part of $D\widetilde{Y}(X)$. To make the notation a bit more uniform, we denote $D\widetilde{Y}(X)$ by $\overline{\nabla}_X\widetilde{Y}$ (i.e. the flat connection on $\mathbb{R}^n$). Put

$$B(X, Y) = \overline{\nabla}_X\widetilde{Y} - \nabla_X Y,$$

in other words, the normal component of the connection on $\mathbb{R}^n$.

Lemma 19.7. *B is symmetric and $\mathcal{C}^\infty$–bilinear.*

Proof. Bilinearity follows directly from properties of connections. To see symmetry, compute (using the fact that $\nabla, \overline{\nabla}$ are both Levi-Civita connections)

$$B(Y, X) = \overline{\nabla}_{\widetilde{Y}}\widetilde{X} - \nabla_Y X = \overline{\nabla}_{\widetilde{X}}\widetilde{Y} + [\widetilde{Y}, \widetilde{X}] - \nabla_X Y - [Y, X]$$

Now, we have that $[\widetilde{Y}, \widetilde{X}] = [Y, X]$ at points on S (either compute this in local coordinates, or observe that when it acts on functions it only depends on the values on S). This shows the claim. $\square$

Lemma 19.8.

$$\langle B(x, y), N\rangle = \langle -dN(x), y\rangle = H(x, y)$$

Proof.
$$\langle B(x,y),N\rangle = \langle \overline{\nabla}_X \widetilde{Y} - \nabla_X Y, N\rangle = \langle \overline{\nabla}_X \widetilde{Y}, N\rangle$$
where the second equality holds since $\nabla_X Y$ is tangent to S, while N is normal. Now, since Y is tangent to S, we have $\langle Y,N\rangle = 0$ on S, and therefore when we differentiate in the direction of the tangent vector X we get
$$0 = X\langle Y,N\rangle = \widetilde{X}\langle \widetilde{Y},\widetilde{N}\rangle = \langle \overline{\nabla}_{\widetilde{X}}\widetilde{Y}, N\rangle + \langle \widetilde{Y}, \overline{\nabla}_{\widetilde{X}} N\rangle$$
Putting this together, we get
$$\langle B(x,y),N\rangle = \langle \overline{\nabla}_X \widetilde{Y}, N\rangle = -\langle \widetilde{Y}, \overline{\nabla}_X \widetilde{N}\rangle = H(X,Y).$$
□

In other words, we have $B(X,Y) = H(X,Y)N$. As a consequence, we also see that H is symmetric. In particular, there is an orthonormal basis b_1, b_2 and numbers $\lambda_1 \le \lambda_2$ so that
$$H(b_1,b_2) = 0, H(b_i,b_i) = \lambda_i.$$
Since $H(c',c')$ computes the normal curvature of the curve c, this implies that the possible normal curvatures of curves on S through a point are exactly the interval $[\lambda_1,\lambda_2]$. Classically, the λ_i are called *principal curvatures*, and the product $\lambda_1\lambda_2$ is called *Gaussian curvature*.
We have already connected H to the connection ∇. The next lemma interprets curvature in this way.

Lemma 19.9. *In the setup as above,*
$$R(X,Y,Z,W) = H(X,W)H(Y,Z) - H(X,Z)H(Y,W).$$

Proof. By definition, we have
$$\overline{\nabla}_{\widetilde{X}}\widetilde{Y} = \nabla_X Y + H(X,Y)N.$$
Thus, we get
$$\overline{\nabla}_{\widetilde{X}}\overline{\nabla}_{\widetilde{Y}}\widetilde{Z} = \overline{\nabla}_{\widetilde{X}}(\nabla_Y Z + H(Y,Z)N) = \overline{\nabla}_{\widetilde{X}}(\nabla_Y Z) + \overline{\nabla}_{\widetilde{X}}(H(Y,Z)N)$$
$$= \nabla_X\nabla_Y Z + H(X,\nabla_Y Z)N + XH(Y,Z)N + H(Y,Z)\overline{\nabla}_{\widetilde{X}}N$$
Note that the second and third summand are orthogonal to S, and therefore
$$\langle \overline{\nabla}_{\widetilde{X}}\overline{\nabla}_{\widetilde{Y}}\widetilde{Z}, W\rangle = \langle \nabla_X\nabla_Y Z + H(Y,Z)\overline{\nabla}_{\widetilde{X}}N, W\rangle = \langle \nabla_X\nabla_Y Z, W\rangle + H(Y,Z)H(X,W).$$
In the same way, we get
$$\langle \overline{\nabla}_{\widetilde{Y}}\overline{\nabla}_{\widetilde{X}}\widetilde{Z}, W\rangle = \langle \nabla_Y\nabla_X Z, W\rangle + H(X,Z)H(Y,W).$$
Also note that by definition,
$$\langle \nabla_{[X,Y]}Y, Z\rangle = \langle \overline{\nabla}_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}, W\rangle.$$
Since the curvature of $\overline{\nabla}$ is zero, this implies the claim. □

Use this formula for the orthonormal basis b_1, b_2 above to obtain

$$\kappa(T_pM) = R(b_1, b_2, b_1, b_2) = -H(b_1, b_1)H(b_2, b_2) + H(b_1, b_2)^2 = -\lambda_1\lambda_2.$$

In other words, the sectional curvature of an immersed surface in $\mathbb{R}^3$ is the negative of the Gaussian curvature (with our sign conventions). This yields a geometric intuition for sectional curvature in general.

20. Last lecture 1: Parallel Transport

Let us now connect the linear connections to the problem of identifying nearby fibers.

20.1. **Parallel Transport.** First, we need the following basic statement about bundles.

Lemma 20.1. *Suppose that E is a vector bundle over M, and suppose that $c : [a, b] \to M$ is a curve in M. Then the bundle is trivial over c: there are sections $\mu_1, \dots, \mu_n : [a, b] \to E$ over c, so that $\mu_1(t), \dots, \mu_n(t)$ are a basis of $E_{c(t)}$ for every t.*

Proof. As a first step, we will construct μ_i as desired, which are just continuous, not smooth. To this end, choose numbers $\epsilon > 0$ and

$$a = t_1 < t_2 < \cdots < t_k = b$$

so that $c[t_i, t_{i+1}]$ lies completely inside a trivialising neighbourhood for E. By induction we can now find the desired continuous functions μ_i. Now, we can slightly deform the μ_i to make them smooth. Since being a basis is an open condition, the resulting smooth μ_i still form a basis of $E_{c(t)}$ for every t. □

Lemma 20.2. *Suppose that E is a vector bundle over M, and that γ is a curve in M. Let $v \in \pi^{-1}(\gamma(0))$ be arbitrary. Then there is a unique section σ of E along γ so that*

$$\sigma(0) = v,$$

and

$$\nabla_{\dot{c}(t)}\sigma(t) = 0 \forall t$$

Proof. By the previous lemma, we can choose μ_i which are a basis of sections along c. We can then make the ansatz $\sigma = \sum s^i \mu_i$, and compute

$$\frac{\nabla}{dt}\sigma = \sum \dot{s}^i \mu_i + s^i(t) \frac{\nabla}{dt}\mu_i(t).$$

Hence, $\frac{\nabla}{dt}\sigma = 0$ is a system of linear ODEs for the coefficient functions s^i. Since the μ_i, $\frac{\nabla}{dt}\mu_i(t)$ are smooth, Picard-Lindelöf applies, and guarantees a unique solution given the initial value v. □

The final value $\sigma(1)$ of the solution is called the *parallel transport of v along γ*.

Where does this name come from? Example time!

Example 20.3. *For the flat connection of $\mathbb{R}^n$, ∇ is the usual derivative, and so a section σ is parallel exactly if it is constant. Hence, parallel transport actually consists of "shifting the vector to a parallel one at a different basepoint".*

Example 20.4. *For the round connection on the sphere, parallel transport means that the change of the vector is* orthogonal *to the sphere.*
So, paths with constant tangent vectors are fine (but not always possible), but so are paths which rotate "in the direction of the center".
Observe that the path matters!

Parallel transport is important in part since it allows us to completely reconstruct the connection. Namely, we have the following.

Lemma 20.5. *Let M be a smooth manifold, and E be a vector bundle over M. Let $c : (-\epsilon, \epsilon) \to M$ be a smooth curve, and σ a section of E along c. Denote by $P_t : E_{c(0)} \to E_{c(t)}$ parallel transport. Then*

$$\frac{\nabla}{dt}\sigma(0) = \lim_{t\to 0} \frac{P_t^{-1}\sigma(c(t)) - \sigma(c(0))}{t}.$$

Proof. Choose μ_i parallel bases along c. Write

$$\sigma(t) = \sum s^i(t)\mu_i(t).$$

By uniqueness of parallel transport, we then have

$$P_t^{-1}(\sigma(t)) = \sum s^i(t)\mu_i(0).$$

Since the μ_i are parallel, we can also compute

$$\frac{\nabla}{dt}\sigma = \sum \dot{s}^i(t)\mu(t),$$

and thus

$$\frac{\nabla}{dt}\sigma(0) = \sum \dot{s}^i(0)\mu(0),$$

which implies the claim. □

As a consequence, we have

$$(\nabla_X Y)_p = \lim_{t\to 0} \frac{P_t^{-1}(Y(c(t))) - Y(p)}{dt},$$

where $c : (-\epsilon, \epsilon) \to M$ is any curve with $c(0) = p, c'(0) = X(p)$, and P_t denotes parallel transport along c as above.
Note that this is similar to the definition of the Lie derivative, except that we use parallel transport and not a flow to identify nearby fibres.

This point of view also leads to another interpretation of curvature. To describe it, we consider a parametrised surface

$$f : [0, a] \times [0, b] \to M.$$

Let X, Y be the partial derivatives of f, seen as vector fields along f. Put $p = f(0, 0)$, and let $z \in E_p$ be any vector. We let $z_{s,t}$ be the result of parallel transport of z along the boundary of the rectangle $[0,s]\times[0,t]$, starting with the horizontal edge.

Lemma 20.6. *In the setup as above,*

$$R(X_p, Y_p)Z_p = \lim_{s,t\to 0} \frac{z_{s,t} - z}{st}$$

Proof. Recall that we have

$$R(X,Y)z = \frac{\nabla}{ds}\frac{\nabla}{dt}Z - \frac{\nabla}{dt}\frac{\nabla}{ds}Z,$$

where Z is any section along f with $Z(0,0) = z$. We choose a particular such section. Namely, let

$$Z(s,t) = P^{(s,t)}_{(s,0)} P^{(s,0)}_{(0,0)} z,$$

where $P^{(s,0)}_{(0,0)} : E_{f(0,0)} \to E_{f(s,0)}$ is parallel transport along a horizontal path, and for every s, $P^{(s,t)}_{(s,0)} : E_{f(s,0)} \to E_{f(s,t)}$ is parallel transport along a vertical path. Observe that therefore $\frac{\nabla}{ds}Z = 0$, since dependence on s is parallel. On the other hand, we can apply the previous lemma twice to compute

$$\frac{\nabla}{dt}Z(s_0,t_0) = \lim_{t\to 0} \frac{(P^{(s_0,t_0+t)}_{(s_0,t_0)})^{-1}Z(s_0,t_0+t) - Z(s_0,t_0)}{t}.$$

$$\frac{\nabla}{ds}\frac{\nabla}{dt}Z(0,0) = \lim_{s\to 0} \frac{(P^{(s,0)}_{(0,0)})^{-1}\frac{\nabla}{dt}Z(s,0) - \frac{\nabla}{dt}Z(0,0)}{s}$$

$$= \lim_{s,t\to 0} \frac{(P^{(s,0)}_{(0,0)})^{-1}((P^{(s,t)}_{(s,0)})^{-1}Z(s,t) - Z(s,0)) - (P^{(0,t)}_{(0,0)})^{-1}Z(0,t) - z)}{st}$$

which shows the result. □

In other words, the curvature tensor measures how parallel transport around infintesimal rectangles changes vectors.

21. Last lecture 2: A wide outlook

A brief outlook on the connection of geometry and topology.
Here's a classical result:

Satz 21.1. *Suppose that (M,g) is a complete Riemannian manifold, and suppose that $\kappa_p \geq c > 0$ for all p. Then M is compact, and its diameter can be bounded in terms of c.*

Another theorem.

Satz 21.2. *Suppose that (M,g) is a Riemannian surface, and κ is the sectional curvature. Then*

$$\int_M \kappa = -2\pi\left(\dim H^0(M) - \dim H^1(M) + H^2(M)\right) = -2\pi\chi(M)$$

where $\chi(M) = v - e + f$ *with* v, e, f *the number of vertices, edges and faces in a triangulation.*

We will not prove this, but note: it shows a connection between a purely topological, and a geometric quantity.
Such theorems abound. Geometric information constrains topology, and vice versa. For example, *any* sphere (S^2, g) has total curvature positive, *any* torus $(S^1 \times S^1, g)$ has total curvature 0, and *any* higher genus surface has negative curvature.
Thus, if curvature is > 0 or < 0 everywhere, we know something about the shape of the surface.
More to the point, there are actually metrics that distribute the (fixed!) total curvature homogeneously over the surface. For $S^2, S^1 \times S^1$ we know these. For higher genus surfaces, these are so called *hyperbolic* metrics. A small outlook: these are not unique, and there are *moduli spaces* of such metrics: manifolds whose points correspond to the different metrics on a surface. These manifolds carry metrics again, turning them into geometric objects in their own right.
To study the *topology* of a surface, we can therefore use the *geometry* of the surface as a tool. This is particularly useful when studying the symmetries of the surface, i.e. their self-diffeomorphisms. These act (by pulling back metrics) on the moduli spaces by isometries. One can use these connections

What about higher dimensions? In dimension 3, in theory there is still a geometric classification, but it is much more involved: one has to cut M at embedded spheres, and then embedded tori to get "geometric pieces", each of which carries one of eight possible geometries. Out of these, again the hyperbolic one is the most mysterious, but in a sense the most typical one.