Let's see what differential forms can be useful for. To make contact with the most familiar case, let us consider $\mathbb{R}^3$ with cartesian coordinates $\vec{x} = (x, y, z) \in \mathbb{R}^3$. For relativistic applications, consider also $\mathbb{R}^{1,3}$ with coordinates (t, x, y, z).

A basis for $\Lambda^1 \mathbb{R}^3$ is dx, dy and dz which we can group together as

$$d\vec{x} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

A general 1-form $A \in \Omega(\Lambda^1 \mathbb{R}^{1,3})$ can then be decomposed in coordinates as

$$A = \varphi\, dt + \vec{A} \cdot d\vec{x}$$

with $\varphi : \mathbb{R}^{1,3} \to \mathbb{R}$, $\vec{A} : \mathbb{R}^{1,3} \to \mathbb{R}^3$.

Similarly, for $\Lambda^2 \mathbb{R}^3$, we have a basis $dx \wedge dy$, $dy \wedge dz$, $dz \wedge dx$ which we write as

$$d\vec{\sigma} = \begin{pmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{pmatrix}$$

For $\Lambda^2 \mathbb{R}^{1,3}$, we have in addition

$$dt \wedge d\vec{x}$$

So, any $F \in \Gamma(\Lambda^2 \mathbb{R}^{1,3})$ can be decomposed as

$$F = \vec{E} \cdot (dt \wedge d\vec{x}) + \vec{B} \cdot d\vec{\sigma}.$$

$\vec{E}, \vec{B} : \mathbb{R}^{1,3} \to \mathbb{R}^3$ vector fields

For $\Lambda^3 \mathbb{R}^3$, the only basis elem't is

$$dVol = dx \wedge dy \wedge dz$$

while for $\Lambda^3 \mathbb{R}^{1,3}$, we in addition have

$$dt \wedge d\vec{\sigma}.$$

Let's compute some derivatives:

Start from a 0-form $\lambda \in \Lambda^0 \mathbb{R}^{1,3}$:

$$d\lambda = \partial_t \lambda \, dt + \partial_x \lambda \, dx + \partial_y \lambda \, dy + \partial_z \lambda \, dz$$

$$= \dot{\lambda} \, dt + \vec{\nabla} \lambda \cdot d\vec{x}$$

1-form:

$$dA = \partial_x \varphi \, dx \wedge dt + \partial_y \varphi \, dy \wedge dt + \partial_z \varphi \, dz \wedge dt$$

$$+ \partial_t A_x \, dt \wedge dx + \partial_y A_x \, dy \wedge dx + \partial_z A_x \, dz \wedge dx$$

$$+ \dots$$

$$+ \dots$$

$$= (\dot{\vec{A}} - \vec{\nabla}\varphi)\, dt \wedge d\vec{x}$$
$$+ (\vec{\nabla} \times \vec{A}) \cdot d\vec{\sigma}$$

2-form :

$$dF = \partial_y E_x \quad dy \wedge dt \wedge dx + \partial_z \; dz \wedge dt \wedge dz$$
$$+ \cdots$$
$$+ \cdots$$
$$+ \partial_t B_x \quad dt \wedge dy \wedge dz + \partial_x B_x \, dx \wedge dy \wedge dz$$
$$+ \cdots$$
$$+ \cdots$$
$$= (\vec{\nabla} \cdot \vec{B})\, dVol + (\partial_t \vec{B} - \vec{\nabla} \times \vec{E})\, dt \wedge d\vec{\sigma}$$

So, we found the equation

$$0 = dF$$

is equivalent to the two homogeneous Maxwell equations

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \text{(no magnetic monopoles)}$$

and

$$\vec{\nabla} \times \vec{E} = \frac{\partial \vec{B}}{\partial t}$$

This is automatically solved, if we write

F as $F = dA$ in terms of a "1-form potential" as $dF = d^2A = 0$. Comparing components of $F = dA$, we find

$$E = -\nabla\varphi + \dot{\vec{A}}, \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

Again, F does not change if we modify A by a "gauge transformation", the derivative of a 0-form

$$F = d(A + d\lambda) = dA + d^2\lambda = dA$$

For the components of $\tilde{A} = A + d\lambda$, we find

$$\tilde{\varphi} = \varphi + \dot{\lambda}, \quad \vec{\tilde{A}} = \vec{A} + \vec{\nabla}\lambda$$

What about the inhomogeneous Maxwell equations?

We need additional structure: Observe that for an n-dimensional manifold

$$\dim \Lambda^k T_p^* M = \binom{n}{k}$$
$$= \binom{n}{n-k} = \dim \Lambda^{n-k} T_p^* M.$$

So these are isomorphic as vector spaces. For a fixed (orthonormal) basis we can define an isomorphism ("Hodge star")

$$* : \Lambda^k T_p^* M \to \Lambda^{n-k} T_p^* M$$

by "mapping to the missing directions", i.e.

for $M = \mathbb{R}^3$ for example

$$* \, d\vec{x} = d\vec{\sigma}$$

$$* \, d\vec{\sigma} = d\vec{x}$$

$$* \, 1 = dVol$$

As it turns out, this depends on the basis chosen. It can be made independent of choices when we have a metric and an orientation (sorting of orthogonal basis elements). Recall that in the first homework, we also needed a metric to write the inhomogeneous Maxwell equation $g^{\mu\nu} \partial_\mu F_{\nu\beta} = j_\beta$.

For $\Lambda^2 \mathbb{R}^{1,3}$, one finds

$$* F = \vec{B} \cdot dt \wedge d\vec{x} - \vec{E} \cdot d\vec{\sigma}$$

Then

$$\begin{aligned} d * F = {} & \partial_y B_x \, dy \wedge dt \wedge dx + \partial_z B_x \, dz \wedge dt \wedge dx \\ & + \dots \\ & + \dots \\ & - \partial_t E_x \, dt \wedge dy \wedge dz - \partial_x E_x \, dx \wedge dy \wedge dz \\ & - \dots \end{aligned}$$

$$= \ldots$$

$$= (-\vec\nabla\times\vec B - \dot{\vec E})\, dt\wedge d\vec\sigma - \vec\nabla\cdot\vec E\, dVol$$

The inhomogeneous Maxwell equations

$$\vec\nabla\cdot\vec E = \rho \quad \text{(Coulomb's law)}, \qquad \vec\nabla\times\vec B + \partial_t \vec E = \vec j \quad \text{(Faraday's law)}$$

displacement current

are then encoded in

$$d * F = -j$$

with the "current density 3-form"

$$j = \rho\, dVol + \vec j\, dt \wedge d\vec\sigma$$

From $0 = d^2 * F = dj$ we get the continuity equation (conservation of charge)

$$-\dot\rho = \vec\nabla\cdot\vec j$$

Finally, the motion of a particle in an electromagnetic field is encoded in the variational principle for the action

① ② ③

$$S = \int \left(L_{kin} - \varphi(\vec{x}(t)) + \dot{\vec{x}} \cdot \vec{A}(\vec{x}(t)) \right) dt$$

for paths (world lines) $x : \mathbb{R} \to \mathbb{R}^{1,3}$

For example, we can use "static gauge"

$$t \mapsto (t, \vec{x}(t))$$

We can then integrate the gauge potential (a 1-form!) over this path

$$\int_x A = \int_{\mathbb{R}} x^* A$$

$$= \int_{\mathbb{R}} \left(\varphi \, dt + \vec{A} \cdot x^* d\vec{x} \right)$$

$$= \int_{\mathbb{R}} \left(\varphi \, dt + \vec{A} \cdot \frac{\partial \vec{x}}{\partial t} \, dt \right)$$

But this is exactly the electromagnetic contribution to the action! It has a geometric origin. And gauge invariance is given by Stoke's law:

$$\int \tilde{A} = \int A + \int_{\mathbb{R}} d\lambda = \int dA + \lambda \Big|_{x = \text{initial}}^{x_{\text{final}}}$$

kept fixed.

All these notions work for general Lorentzian 4-manifolds

Section on flows and Lie derivatives

Hamilton mechanics and symplectic geometry.

Definition A *symplectic manifold* is 2n-dim a manifold X equipped with a "symplectic form" $\omega \in \Omega(\Lambda^2 T^* X)$ that is

i) closed $d\omega = 0$

ii) non-degenerate $\underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ factors}} \neq 0$.

rem: ω non-degenerate means in coordinates

$$\omega = \sum_{i<j} \omega_{ij} \, dx^i \wedge dx^j$$

that the matrix $(\omega_{ij})_{i,j=1..2n}$ is invertible

Skew-diagonalization means that any antisymmetric matrix can be brought to a form

$$S^{-1} \Omega S = \begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & & & & \\ & \boxed{\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}} & & & & \\ & & \ddots & & & \\ & & & \boxed{\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}} & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}$$

$S \in SO(2n)$

then non-degenerate means there are no 0's in the end.

A non-degenerate ω provides us with an identification of $T_p X$ with $T_p^* X$

via
$$X \mapsto \iota_X \omega$$

non-degeneracy ensures this is invertible.

Darboux: $d\omega = 0$ implies the diagonalization can be done in an open neighbourhood. This is diff't from metric case where curvature is an obstruction.

examples:

1) Let Q be an n-dimensional manifold. Then $X = T^*Q$ is symplectic with ω define in a chart $\varphi : U \subset Q \to \mathbb{R}^n$ providing coordinates $q^i = \varphi(q)^i$

$$\omega = \sum_i dq^i \wedge dp_i$$

with $\{dp_i\}_{i=1..n}$ the dual basis of

$$dp_j\left(\frac{\partial}{\partial q^i}\right) = \delta_i^j .$$

This is independent of the chart φ since in a diff't chart $\tilde{\varphi}$ with coordinates $\tilde{q}^i$

$$dq^i = \frac{\partial q^i}{\partial \tilde{q}^j} d\tilde{q}^j$$

while $\quad \frac{\partial}{\partial \tilde{q}^i} = \sum_j \frac{\partial q^j}{\partial \tilde{q}^i} \frac{\partial}{\partial q^j}$

so the transformation of dq^i cancels the transformation of dp_i.

Physicists think of Q as "configuration space" providing "generalized coordinates" while T_q^*Q contains "canonical momenta". ω encodes the canonical pairing. $X = T^*Q$ is called "phase space".

2) $X = S^2 = \{ \vec{x} \in \mathbb{R} \mid \|x\| = 1 \}$ together with any non-vanishing 2-form (e.g. volume from above embedding) is a symplectic manifold ($d\omega = 0$ by dimension) that is not of the form $X = T^*Q$ for some Q.

This is the symplectic space appropriate to describe "spin".

Def $H : X \to \mathbb{R}$ ($\in \Omega(\Lambda^0 X)$) smooth is called a "Hamiltonian". According to the remark above, it defines a "Hamiltonian vector field" X_H defined via

$$dH = \iota_{X_H}\omega$$

The flow Φ_t of X_H is the "solution to the equations of motion".

NB: ω: kinematics H: dynamics

<u>example</u>: $X = T^*Q$ with coordinates (q^i, p_i) and $\omega = \sum_i dq^i \wedge dp_i$.

We have $dH = \sum_i \left(\frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i \right)$

Now, we need to find $X = \sum_i \left(X^i \frac{\partial}{\partial q^i} + X_i \frac{\partial}{\partial p_i} \right)$

such that $dH = i_{X_H} \omega$

Let's compute

$$i_{X_H} \omega = \sum_{i,j} \left(X^i \underbrace{dq^j\left(\frac{\partial}{\partial q^i}\right)}_{\delta^j_i} dp_j - X_i \underbrace{dp_j\left(\frac{\partial}{\partial p_i}\right)}_{\delta^i_j} dq^j \right)$$

$$= \sum \left(X^i \, dp_j - X_i \, dq^i \right)$$

comparing coefficients, we find

$$X^i = \frac{\partial H}{\partial p_i} \qquad X_i = -\frac{\partial H}{\partial q^i}$$

so $$X = \sum_i \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right)$$

Finally, write for $(x, p) \in T^*Q$

$$(x(t), p(t)) = \Phi_t((x, p))$$

Thus from $\frac{d}{dt}\Big|_{t=0} \Phi_t = X_H$, we read off

$$T_{(q,p)}X \ni \sum_i \left(\dot q^i(t) \frac{\partial}{\partial q^i} \quad \dot p_i \frac{\partial}{\partial p_i}\right) = \sum_i \left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}\right)$$

$$\Leftrightarrow \quad \dot q^i = \frac{\partial H}{\partial p_i} \wedge \dot p_i = -\frac{\partial H}{\partial q^i}$$

These are Hamilton's equations of motion. $\Phi_t((q,p))$ describes the time evolution of a state ~~that~~ is $(q,p) \in T^*Q$ initially.

<u>Lemma</u> (Liouville's theorem) For $t \in \mathbb{R}$

$$\Phi_t^* \omega = \omega$$

<u>pf</u> (infinitesimally)

$$\frac{d}{dt}\Big|_{t=0} \Phi^* \omega = \mathcal{L}_{X_H} \omega$$

$$= \pm i_{X_H} \underset{\overset{\shortparallel}{0}}{d\omega} \pm d\, i_{X_H} \omega$$

$$= \qquad \pm d\, dH$$

$$= 0.$$

$\square$

Def $f : X \to \mathbb{R}$ is called an <u>observable</u>.

NB: This is formally the same as Hamiltonian. So, there is also X_f.
For f, g observables, we can define the <u>Poisson bracket</u>

$$\{f, g\} := -\omega(X_f, X_g) = -df[X_g]$$

Lemma $(\Gamma(\Lambda^0 X), \{\,,\,\})$ forms a Lie algebra, i.e. $\{\,,\,\}$ is antisymmetric and obeys a Jacobi identity.

pf direct calculation.

Lemma $H : X \to \mathbb{R}$ induces a derivation $\{H, \cdot\}$ on $\Gamma(\Lambda^0 X)$:
For $f : X \to \mathbb{R}$

$$\begin{aligned} \dot{f} &= \{H, f\} \\ &= df[X_H] \\ &= X_H[f] \\ &= \tfrac{d}{dt} \Phi_t^* f \big|_{t=0} \qquad \text{(Heisenberg com)} \end{aligned}$$

<u>Noether's Theorem</u>: let G be a Lie group with a symplectic action $G \curvearrowright X$ on X, i.e. $\forall g \in G$: $g^*\omega = \omega$ that leaves a Hamiltonian $H: X \to \mathbb{R}$ invariant, i.e. $\forall g \in G$: $g^* H = H$. Then there is a linear map from $\mathfrak{g} = T_e G$

$$\lambda : \mathfrak{g} \longrightarrow \Gamma(T^* X)$$

such that $\lambda(v)$ is closed $d\lambda(v) = 0$ and conserved $\Phi_t^* \lambda(v) = \lambda(v)$.

(locally $\lambda(v) = dc(v)$ for $c(v) : X \to \mathbb{R}$ (or globally if e.g. $H^1(X) = \{0\}$) is called a conserved charge.

By reversing the arrows in

$$\mathfrak{g} \longrightarrow C^\infty(X)$$

we can view this as a map

$$X \to \mathfrak{g}^*$$

called a <u>moment map</u> by mathematicians.

examples: Conserved momentum, angular momentum
energy, see homework.

Pf From the action $G \times X \to X$
$(g, x) \mapsto g \cdot x$

take the derivative at $e \in G$

$$D: \mathfrak{g} \times X \to TX, \text{ ie } \mathfrak{g} \to \Gamma(TX)$$

call this vectorfield V.

From $g^* \omega = \omega$ (as derivation) it follows by differentiation

$$0 = \mathcal{L}_V \omega = \pm i_V \underbrace{d\omega}_{0} \pm d i_V \omega$$

$$= d\, i_V \omega$$

So, set $\lambda(V) := i_V \omega$. This is clearly linear.

From $g^* H = H$, it follows infinitesimally

as derivation: $0 = V(H) = dH(V)$

$$= \omega(X_H, V)$$

$$= -\lambda(V)(X_H)$$

$$= -\iota_{X_H} \lambda(v)$$
$$= \frac{d}{dt}\Big|_{t=0} \Phi_t^* \lambda(v)$$
$\square$

Flows & Lie derivative

Given a vector field $V \in \Gamma(TX)$, we can integrate it to a flow

A family of diffeomorphisms

$$\Phi : \mathbb{R} \times X \to X$$
$$(t, x) \mapsto \Phi_t(x)$$

with
$$\Phi_0 = id_X$$
$$\frac{d}{dt}\Big|_{t=0} \Phi_t = V$$
$$\Phi_{s+t} = \Phi_s \circ \Phi_t \quad \forall s,t \in \mathbb{R}$$
(group hom) $\mathbb{R} \to \mathrm{Diff}(X)$

is called a "Hamiltonian flow". At least for compact X, its existence is guaranteed by standard ODE-arguments. (finite atlas, in each chart ODE, smoothness $\Rightarrow$ Lipschitz).

rmk:

- $$\frac{d}{dt}\Big|_{t=t_0} \Phi_t = \frac{d}{dt}\left(\Phi_{t_0} \circ \Phi_{t-t_0}\right)$$

$$= \Phi_{t_0} \circ V$$

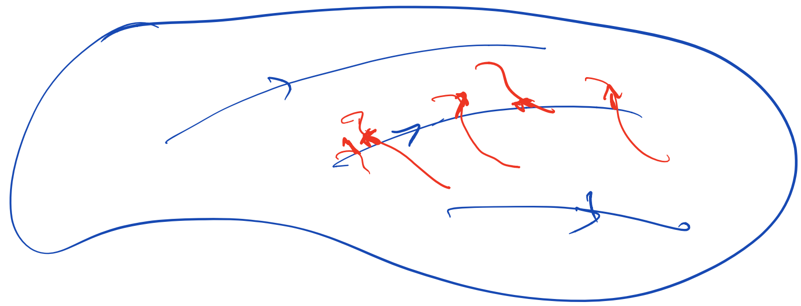
$$\Phi_{t_0}^{-1} \frac{d}{dt}\Big|_{t=t_0} \Phi_t = V \qquad \text{so ODE for all } t.$$

- In coordinates u^i, $(v \circ \varphi^{-1})^i$

$$\dot{u}^i = V^i(u)$$

is an ODE, $C^\infty \Rightarrow$ Lipschitz $\Rightarrow$ local existence and uniqueness: $\forall p \in X: \exists \varepsilon > 0, U \ni p$: flow exists for $t \in (-\varepsilon, \varepsilon)$.
X cpt $\Rightarrow$ finite number of ε's $\Rightarrow$ $\varepsilon_{min} > 0$ for global flow, then use group law to extend to all $t \in \mathbb{R}$.

We can now use this flow to transport tensors along it



This way, we have means to compare tensors at different points.

This then leads us to

Def <u>Lie derivative</u>: $V \in \Gamma(TM)$
Φ_t its flow. Now, e.g. $W \in \Gamma(TM)$
another vector field

$$\mathcal{L}_V W|_p := \frac{d}{dt}\Big|_{t=0} d\Phi_t(W(\Phi_t(p)))$$

$$(= [V, W])$$

Similar for other tensors. In particular, we have

- $\mathcal{L}_V f = \iota_V df$
- $\mathcal{L}_V (T_1 \otimes T_2) = \mathcal{L}_V T_1 \otimes T_2 + T_1 \otimes \mathcal{L}_V T_2$
- $\mathcal{L}_V (\iota_X \alpha) = \iota_{\mathcal{L}_V X} \alpha + \iota_X \mathcal{L}_V \alpha$

- $d \mathcal{L}_V = \mathcal{L}_V d$

Turns out, these properties define $\mathcal{L}_V$.

<u>NB</u>: Besides the exterior derivative (which does not have a direction but increases the form degree), this is another derivative which leaves the tensor type intact.
However, it depends not only on a vector (at p) but on a vector field (or its germ). i.e. it is

<u>not</u> $C^\infty(M)$- linear:

$$\mathcal{L}_{(fV)} W = f \mathcal{L}_V W + (\iota_W df) V$$

<u>Lemma</u> (Cartan's formula) For $\alpha \in \Omega^q(M)$, $V \in \Gamma(TM)$:

$$\mathcal{L}_V \alpha = d \iota_V \alpha + \iota_V d \alpha$$

<u>Pf</u> in HW.