SPHERE SYSTEMS, INTERSECTIONS AND THE GEOMETRY OF $\mathrm{Out}(F_n)$

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Abstract. The outer automorphism group $\mathrm{Out}(F_{2g})$ of a free group on $2g$ generators naturally contains the mapping class group of a punctured surface as a subgroup. Using topological methods in a suitable 3–manifold we show that this subgroup is undistorted. We also use topological techniques to give a new proof of a result of Handel and Mosher [HM10] stating that stabilizers of conjugacy classes of free splittings and corank 1 free factors in a free group F_n are undistorted in $\mathrm{Out}(F_n)$.

1. Introduction

The mapping class group $\mathrm{Map}(S_g)$ of a closed surface S_g of genus g is defined in topological terms: it is the quotient of the group of homeomorphisms of S_g by the connected component of the identity. The classical Dehn-Nielsen-Baer theorem identifies $\mathrm{Map}(S_g)$ with a purely algebraic object, namely the outer automorphism group $\mathrm{Out}(\pi_1(S_g, p))$ of the fundamental group of the surface S_g.

The mapping class group is finitely presented and hence it admits a family of left invariant metrics which are unique up to quasi-isometry. Such a metric can be investigated using simple topological objects as the main tool. In [MM00] the authors construct explicit families of quasi-geodesics in $\mathrm{Map}(S_g)$ using the combinatorics of isotopy classes of simple closed curves on S_g. This approach leads to a geometric understanding of the mapping class group and of many of its natural subgroups.

The outer automorphism group $\mathrm{Out}(F_n)$ of the free group with $n \geq 2$ generators is a finitely presented group which also has a topological description. To this end, let M_n be the connected sum of n copies of $S^1 \times S^2$. By a theorem of Laudenbach [L74], $\mathrm{Out}(F_n)$ is a cofinite

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quotient of the group of all isotopy classes of orientation preserving homeomorphism of M_n.

As in the case of surface mapping class groups, the geometry of $\mathrm{Out}(F_n)$ can be investigated using as a tool the simplest essential submanifolds of M_n, namely embedded spheres. This idea was used by Hatcher in [Ha95] to show homological stability for $\mathrm{Out}(F_n)$. A geometric application of this approach includes an upper bound for the growth rate of the Dehn function of $\mathrm{Out}(F_n)$ (see [HV96]).

The main goal of this note is to initiate an investigation of the large-scale geometry of $\mathrm{Out}(F_n)$ from this topological point of view. Explicitly, we analyze the extrinsic geometry of two families of subgroups of $\mathrm{Out}(F_n)$ which can be described as follows.

The fundamental group of a surface $S_{g,1}$ of genus $g \geq 1$ with one puncture is the free group F_{2g}. A version of the Dehn-Nielsen-Baer theorem for the mapping class group $\mathrm{Mod}(S_{g,1})$ of $S_{g,1}$ states that there is a group isomorphism ι of $\mathrm{Mod}(S_{g,1})$ onto the subgroup of $\mathrm{Out}(F_n)$ of all outer automorphisms which preserve the conjugacy class defined by a puncture parallel simple closed curve in $S_{g,1}$. We show

Theorem 3.2. *The homomorphism ι is a quasi-isometric embedding.*

In fact, for any number $m \geq 1$, the mapping class group of a surface S of genus $g \geq 0$ with $m \geq 0$ punctures and fundamental group F_n embeds onto a subgroup of $\mathrm{Out}(F_n)$. However, we do not investigate such subgroups in the case $m \geq 2$ here.

There is an analog of Theorem 3.2 for graphs which admit cofinite actions of $\mathrm{Mod}(S_{g,1})$ and $\mathrm{Out}(F_{2g})$, respectively. Namely, let $\mathcal{AG}(S_{g,1})$ be the arc graph of $S_{g,1}$. The vertex set of $\mathcal{AG}(S_{g,1})$ is the set of isotopy classes of essential embedded arcs connecting the puncture of $S_{g,1}$ to itself. Two such vertices are connected by an edge if the corresponding arcs are disjoint up to homotopy. The mapping class group $\mathrm{Map}(S_{g,1})$ of a once-punctured surface acts on $\mathcal{AG}(S_{g,1})$. We define the sphere graph $\mathcal{SG}(M_{2g})$ of M_{2g} as the graph whose vertex set is the set of isotopy classes of embedded essential spheres in M_{2g}. Two such vertices are connected by an edge if the corresponding spheres are disjoint up to homotopy. The tools developed for the proof of Theorem 3.2 also yield

Proposition 3.9. *There is a* $\mathrm{Map}(S_{g,1})$*–equivariant quasi-isometric embedding of the arc graph $\mathcal{AG}(S_{g,1})$ into the sphere graph $\mathcal{SG}(M_{2g})$.*

The main idea for the proof of Theorem 3.2 is as follows. The mapping class group of the 3-manifold M_{2g} acts properly and cocompactly on the graph $\mathcal{S}_0(M_{2g})$ whose vertices are *reduced sphere systems*, i.e.

systems of $2g$ pairwise non-isotopic essential spheres which cut M_{2g} into a single simply connected region. Consider an embedding $\varphi : S_g^1 \to M$ of a surface S_g^1 of genus g with one boundary component into M_{2g} which induces an isomorphism on the level of fundamental groups. The intersection of a simple sphere system with the image of φ (where both surfaces are supposed to be in general position) defines an embedded system of arcs on S_g^1 which decomposes S_g^1 into simply connected regions. We can use the set of isotopy classes of such arc systems as a vertex set for a $\mathrm{Map}(S_{g,1})$–complex on which $\mathrm{Map}(S_{g,1})$ acts properly and cocompactly. The main task is now to show that edge paths in $\mathcal{S}_0(M_{2g})$ can be arranged to trace out edge paths of the same length in this complex. We also have to establish a topological characterization of those edge-paths in $\mathcal{S}_0(M_{2g})$ which connect two points in a fixed orbit of $\iota(\mathrm{Mod}(S_{2g}))$.

Investigating $\mathrm{Out}(F_n)$ via sphere systems and intersections can also be used to give a short proof of a recent result of Handel and Mosher [HM10]. We define a procedure which makes a simple sphere system disjoint from a given essential 2–sphere σ in M_n. This procedure allows us to show that stabilizers of homotopy classes of essential spheres in the mapping class group of M_n are undistorted. Recall that a finitely generated subgroup H of a finitely generated group G is said to be undistorted if the inclusion map of H into G is a quasi-isometric embedding.

Namely, the stabilizer of the homotopy class of sphere σ in the mapping class group of M_n is equivariantly quasi-isometric to the complete subgraph $\mathcal{S}(M_n, \sigma)$ of $\mathcal{S}(M_n)$ whose vertices correspond to simple sphere systems containing σ. Let Σ, Σ' be two simple sphere systems containing σ and let $\Sigma_1, \ldots, \Sigma_N$ is a shortest path in $\mathcal{S}(M_n)$ connecting Σ to Σ'. Applying the intersection procedure to each Σ_i we obtain a path of length N in $\mathcal{S}(M_n, \sigma)$ connecting Σ to Σ'. Thus, the subgraph $\mathcal{S}(M_n, \sigma)$ is undistorted in $\mathcal{S}(M_n)$ and therefore the stabilizer of σ is undistorted in the mapping class group of M_n. By rephrasing this result in group theoretic terms, we obtain the following result of [HM10].

Theorem 2.1. *i) The stabilizer of the conjugacy class of a free splitting $F_n = G * H$ is undistorted in $\mathrm{Out}(F_n)$.*

ii) Let $G < F_n$ be a free factor of corank 1. Then the stabilizer of the conjugacy class of G is undistorted in $\mathrm{Out}(F_n)$.

The article is organized as follows. In Section 2, we first give some background on the manifold M_n and sphere systems. This section also contains the proof of Theorem 2.1. Section 3 is devoted to the proof of Theorem 3.2 and Proposition 3.9. Appendix A contains a topological lemma about stabilizers of spheres in M_n which is used in Section 2.

2. Stabilizers of spheres

Let F_n be the free group on n generators. By $\mathrm{Out}(F_n)$ we denote the outer automorphism group of F_n. Explicitly, $\mathrm{Out}(F_n)$ is the quotient of the group $\mathrm{Aut}(F_n)$ of all automorphisms of F_n by the subgroup of inner automorphisms.

The purpose of this section is to give a short topological proof of a theorem of Handel and Mosher [HM10]. For its formulation, we use the following definitions. A *free splitting* of the free group F_n consists of two subgroups $G, H < F_n$ such that $F_n = G * H$. By this we mean the following: the inclusions of G and H into F_n induce a natural homomorphism $G * H \to F_n$, where $*$ denotes the free product of groups. By stating that $F_n = G*H$ we require that this homomorphism is an isomorphism.

We say that an automorphism φ of F_n preserves the free splitting $F_n = G * H$, if φ preserves the groups G and H. It is possible to define free splittings in a more general way using actions of F_n on trees (see [HM10, Section 1.4]) but in this article we use the definition given above.

A *corank* 1 *free factor* is a subgroup G of F_n of rank $n-1$ such that there exists a cyclic subgroup H of F_n with $F_n = G * H$. We say that an automorphism φ of F_n preserves this corank 1 free factor, if φ preserves the group G. We emphasize that φ is not required to preserve the cyclic group H, and that the group H is not uniquely determined by G.

An element $[\varphi] \in \mathrm{Out}(F_n)$ is said to preserve the conjugacy class of the free splitting $G * H$ (or corank 1 free factor G), if there is a representative φ of $[\varphi]$ which preserves the free splitting $G * H$ (or the corank 1 free factor G).

A finite, symmetric generating set of a group G defines a word norm on G. We call the metric induced by such a norm a *word metric* on G. Two different finite generating sets of G give rise to quasi-isometric metrics. Recall that a map $f : X \to Y$ between metric spaces is called a *quasi-isometric embedding*, if there is a number $K > 0$ such that

$$\frac{1}{K} d_Y(f(x), f(x')) - K \leq d_X(x, x') \leq K d_Y(f(x), f(x')) + K$$

for all $x, x' \in X$. A finitely generated subgroup $H < G$ of a finitely generated group G is called *undistorted* if the inclusion homomorphism $H \to G$ is a quasi-isometric embedding.

We can now state the main theorem of this section.

Theorem 2.1. *i) The stabilizer of the conjugacy class of a free splitting $F_n = G * H$ is undistorted in* $\mathrm{Out}(F_n)$.
ii) Let $G < F_n$ be a free factor of corank 1. *Then the stabilizer of the conjugacy class of G is undistorted in* $\mathrm{Out}(F_n)$.

As indicated in the introduction, we will prove Theorem 2.1 using the topology of the connected sum M_n of n copies of $S^2 \times S^1$ (where S^k denotes the k–sphere). Alternatively, M_n can be obtained by doubling a handlebody of genus n along its boundary. Since $\pi_1(M_n) = F_n$, there is a natural homomorphism from the group $\mathrm{Diff}^+(M_n)$ of orientation preserving diffeomorphisms of M_n to $\mathrm{Out}(F_n)$. This homomorphism factors through the mapping class group $\mathrm{Map}(M_n) = \mathrm{Diff}^+(M_n)/\mathrm{Diff}_0(M_n)$ of M_n, where $\mathrm{Diff}_0(M_n)$ is the connected component of the identity in $\mathrm{Diff}^+(M_n)$. In fact, Laudenbach [L74, Théorème 4.3, Remarque 1)] showed that the following stronger statement is true.

Theorem 2.2. *There is a short exact sequence*

$$1 \to K \to \mathrm{Diffeo}^+(M_n)/\mathrm{Diffeo}_0(M_n) \to \mathrm{Out}(F_n) \to 1$$

where K is a finite group, and the map $\mathrm{Diffeo}^+(M_n)/\mathrm{Diffeo}_0(M_n) \to \mathrm{Out}(F_n)$ *is induced by the action on the fundamental group.*

By [L74, Théorème 4.3, part 2)], we can replace diffeomorphisms by homeomorphisms in the definition of the mapping class group of M_n.

An embedded 2-sphere in M_n is called *essential*, if it does not bound a 3-ball in M_n. Throughout the article we assume that 2-spheres are smoothly embedded, essential and are two-sided in M_n.

The following observation identifies the stabilizers in $\mathrm{Map}(M_n)$ which occur in Theorem 2.1. The statement is an immediate consequence of Corollary 21 of [HM10] and a standard topological argument which is for example presented in [AS09]. For completeness of exposition we provide a purely topological proof in Appendix A.

Lemma A.1. *i) Let σ be an essential separating sphere in M_n. Then the stabilizer of σ in* $\mathrm{Map}(M_n)$ *projects onto the stabilizer of the conjugacy class of a free splitting in* $\mathrm{Out}(F_n)$. *Furthermore, every stabilizer of a conjugacy class of a free splitting arises in this way.*
ii) Let σ be a nonseparating sphere in M_n. Then the stabilizer of σ in $\mathrm{Map}(M_n)$ *projects onto the stabilizer of the conjugacy class of a*

corank 1 *free factor in* $\mathrm{Out}(F_n)$*. Furthermore, every stabilizer of a conjugacy class of a corank* 1 *free factor arises in this way.*

To study stabilizers of essential spheres in M_n we use the following geometric model for the mapping class group of M_n (compare [Ha95] and [HV96]).

A *sphere system* is a set $\{\sigma_1, \dots, \sigma_m\}$ of essential spheres in M_n no two of which are homotopic. A sphere system is called *simple* if its complementary components in M_n are simply connected. The *sphere system graph* $\mathcal{S}(M_n)$ is the graph whose vertex set is the set of homotopy classes of simple sphere systems. Two such vertices are joined by an edge of length 1 if the corresponding sphere systems are disjoint up to homotopy.

The mapping class group of M_n acts on $\mathcal{S}(M_n)$ properly discontinuously and cocompactly (see e.g. the proof of Corollary 4.4 of [HV96] for details on this). Furthermore, the surgery procedure described in Section 3 of [HV96] shows that $\mathcal{S}(M_n)$ is connected. The finite subgroup K occurring in the statement of Theorem 2.2 of $\mathrm{Map}(M_n)$ acts trivially on isotopy classes of spheres and hence this action factors through an action of $\mathrm{Out}(F_n)$.

For an essential sphere σ, let $\mathcal{S}(M_n, \sigma)$ be the complete subgraph of $\mathcal{S}(M_n)$ whose vertex set is the set of homotopy classes of simple sphere systems containing σ. The surgery procedure described in [HV96] shows that the graph $\mathcal{S}(M_n, \sigma)$ is connected. The stabilizer of σ in $\mathrm{Out}(F_n)$ acts cocompactly on $\mathcal{S}(M_n, \sigma)$. Thus the Svarc-Milnor lemma immediately implies the following.

Lemma 2.3. *i) The sphere system graph* $\mathcal{S}(M_n)$ *is quasi-isometric to* $\mathrm{Out}(F_n)$*.*
ii) The graph $\mathcal{S}(M_n, \sigma)$ *is equivariantly quasi-isometric to the stabilizer of* σ *in* $\mathrm{Out}(F_n)$*.*

Combining Lemma A.1 and Lemma 2.3, Theorem 2.1 thus reduces to the following.

Theorem 2.4. *The inclusion of* $\mathcal{S}(M_n, \sigma)$ *into* $\mathcal{S}(M_n)$ *is a quasi-isometric embedding.*

The main tool used in the proof of this statement is a surgery procedure that makes a given simple sphere system disjoint from the sphere σ. On the one hand, this surgery procedure is inspired by the construction used in [HV96] to show that the sphere system complex is contractible. On the other hand, it is motivated by the subsurface projection methods of [MM00].

To describe this surgery procedure we fix an essential sphere σ in M_n for the rest of this section. We treat separating and nonseparating spheres in a unified manner using the following notation. If σ is separating, let M^1 and M^2 be its complementary components in M_n and put $N_i = M^i \cup \sigma$. We then let N be the disjoint union of N_1 and N_2. If σ is nonseparating, let M be its complement. There is a canonical way to add two copies of σ to M to obtain a compact three-manifold N whose boundary consists of two spheres.

In both cases, N is a compact three-manifold whose boundary consists of two copies of σ. If no confusion can occur we will often treat N as if it were a submanifold of M_n and call it *the complement of* σ. In particular, we simply speak of the intersection of a sphere system with N.

Consider now a simple sphere system Σ of M_n. By applying a homotopy, we may assume that all intersections between Σ and σ are transverse. The intersection of the spheres in Σ with N is a disjoint union of properly embedded surfaces $C_1, \dots, C_m$, possibly with boundary. Each C_i is a subsurface of a sphere in Σ, and thus it is a bordered sphere. If Σ contains spheres disjoint from σ then some of the C_i may be spheres without boundary components. We call the C_i the *sphere pieces* defined by Σ.

We say that Σ and σ *intersect minimally* if the number of connected components of $\Sigma \cap \sigma$ is minimal among all sphere systems homotopic to Σ which intersect σ transversely.

Every simple sphere system Σ can be changed by a homotopy to intersect σ minimally. Unless stated otherwise, we will assume from now on that spheres and sphere systems intersect minimally. Let $\Sigma' \supset \Sigma$ be a simple sphere system and suppose that Σ intersects σ minimally. Then Σ' can be homotoped relative to Σ to intersect σ minimally.

Details on the construction of such a homotopy can be found in [Ha95]. Hatcher also shows the existence of a unique normal form of spheres with respect to simple sphere systems which gives more information than minimal intersection. Since we do not use this normal form here, we refer the reader to [Ha95] for details.

Let C be one of the sphere pieces of Σ, and let $\alpha_1, \dots, \alpha_k$ be its boundary components on ∂N. A *gluing datum for* C is a set of disks $D_1, \dots, D_k$ contained in ∂N such that $\partial D_i = \alpha_i$ for all $1 \leq i \leq k$. The disks D_i are called *closing disks*. Let C' be the surface obtained from C by gluing D_i along ∂D_i to α_i. Since C is a bordered sphere, the surface C' is an immersed sphere in N (which may be inessential). We say that C' is obtained from C by *capping off the boundaries according* *to the gluing datum*. By a *gluing datum* $\mathcal{D}$ *for* $\{C_i, 1 \leq i \leq m\}$ (or *gluing datum for* Σ) we mean a set of disks on σ consisting of a gluing datum for each sphere piece C_i of Σ.

We say that a gluing datum $\mathcal{D}$ for Σ is *admissible* if it satisfies the following compatibility property: if $D, D' \in \mathcal{D}$ are two disks which intersect nontrivially then $D \subset D'$ or $D' \subset D$. We say that any set of disks with this property is *properly nested*.

Note that if Σ is disjoint from σ then the empty set is the only admissible gluing datum for Σ.

Lemma 2.5. *If $\mathcal{D}$ is an admissible gluing datum for Σ then every sphere obtained by capping off the boundary components of a sphere piece according to $\mathcal{D}$ is embedded up to homotopy. Furthermore, the spheres obtained by capping off the boundary components of all sphere pieces according to $\mathcal{D}$ can be embedded disjointly.*

Proof. If $\mathcal{D}$ is empty, there is nothing to show.

Otherwise, say that a disk $D \in \mathcal{D}$ is *innermost* if $D \subset D'$ for every $D' \in \mathcal{D}$ with $D \cap D' \neq \emptyset$. Since $\mathcal{D}$ is admissible, there is at least one innermost disk D_1 bounded by a curve α_1.

In N, the curve α_1 occurs twice as the boundary of a sphere piece, once on each boundary component of N. Let C^1 and C^2 be the two sphere pieces having a copy of α_1 contained in their boundary.

The disk D_1 also occurs on both boundary components of N, and both of these disks have the property that they only intersect a single sphere piece in N, namely one of the C^j. Let D^j be the copy of D_1 intersecting C^j.

We glue D^j to the corresponding sphere piece C^j and then slightly push D^j inside N with a homotopy to obtain a properly embedded bordered sphere C'^j in N. Since D is innermost, this sphere is disjoint from all sphere pieces $C_k \neq C^j$, and has one less boundary component than C^j. We replace the sphere piece C^j by the bordered sphere C'^j for $j = 1, 2$.

The collection of bordered spheres obtained in this way is a collection of disjointly embedded sphere pieces, and $\mathcal{D} \setminus \{D_1\}$ is an admissible gluing datum for this collection. The lemma now follows by induction on the number of elements in $\mathcal{D}$. ∎

For an admissible gluing datum $\mathcal{D}$ for Σ, let $\mathcal{S}(\mathcal{D})$ be the collection of disjointly embedded spheres obtained by capping off the boundaries of each sphere piece according to $\mathcal{D}$. The set $\mathcal{S}(\mathcal{D})$ may contain inessential spheres and parallel spheres in the same homotopy class. We denote by $\pi_\sigma(\Sigma, \mathcal{D})$ the union of σ with one representative for each essential homotopy class of spheres occurring in $\mathcal{S}(\mathcal{D})$.

To show that the sphere system obtained in this way from a simple sphere system Σ is again simple, we require the following topological lemma.

Lemma 2.6. *Let C be a sphere piece in N intersecting the boundary of N in at least one curve α. Let $D \subset \partial N$ be a disk with $\partial D = \alpha$. Let C' be the sphere piece obtained by gluing D to C and slightly pushing D into N (which might be a sphere without boundary components).*

Then every closed curve in N which can be homotoped to be disjoint from C' can also be homotoped to be disjoint from C.

Proof. Pushing the disk D slightly inside of N with a homotopy traces out a three-dimensional cylinder Q in N. The boundary of Q consists of two disks (the disk D, and the image of D under the homotopy) and an annulus A which can be chosen to lie in C (see Figure 1 for an example).

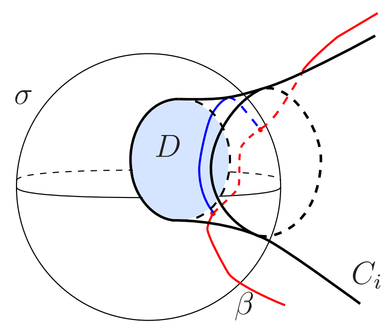

FIGURE 1. Reducing the number of boundary components of a sphere piece.

Suppose that β is a closed curve in N which is disjoint from C' but not from C. Then any intersection point between β and C is contained in the annulus A. Up to homotopy, the intersection between β and Q is a disjoint union of arcs connecting A to itself. Since Q is simply connected, each of these arcs can be moved by a homotopy relative to its endpoints to be contained entirely in A. Slightly pushing each of these arcs off A then yields the desired homotopy that makes β disjoint from C. □

Lemma 2.7. *Let Σ be a simple sphere system, and let $\mathcal{D}$ be an admissible gluing datum for Σ. Then $\pi_\sigma(\Sigma, \mathcal{D})$ is a simple sphere system.*

Proof. Let Σ be a simple sphere system, and let $\mathcal{D}$ be an admissible gluing datum. As $\pi_\sigma(\Sigma, \mathcal{D})$ contains σ by construction, it suffices to show that the spheres $S \in \pi_\sigma(\Sigma, \mathcal{D})$ which are distinct from σ decompose N into simply connected regions.

Since the fundamental group of N injects into the fundamental group of M and Σ is a simple sphere system, no essential simple closed curve in N is disjoint from $\Sigma \cap N$. In other words, no essential simple closed curve in N is disjoint from all sphere pieces defined by Σ.

By Lemma 2.6, this property is preserved under capping off one boundary component on a sphere piece. By induction, no essential simple closed curve in N is disjoint from all spheres $S \in \mathcal{S}(\mathcal{D})$. Removing inessential spheres and parallel copies of the same sphere from $\mathcal{S}(\mathcal{D})$ does not affect this property.

This implies that $\pi_\sigma(\Sigma, \mathcal{D})$ is a simple sphere system as claimed. □

It is not hard to show that for each Σ there is an admissible gluing datum (e.g by considering the dual graph to the intersection of Σ with σ as in the upcoming proof of Lemma 2.8). Since we do not need this statement in the sequel, we do not give a proof here.

We do however need the following relative version of this statement, which is the central ingredient for the proof of Theorem 2.4.

Lemma 2.8. *Let Σ be a simple sphere system, and let $\mathcal{D}$ be an admissible gluing datum for Σ. Suppose that Σ' is a simple sphere system which is disjoint from Σ up to homotopy. Then there is an admissible gluing datum $\mathcal{D}'$ for Σ' such that $\pi_\sigma(\Sigma, \mathcal{D})$ and $\pi_\sigma(\Sigma', \mathcal{D}')$ are disjoint up to homotopy.*

Proof. As a first step, note that if Σ' is obtained from Σ by removing some spheres then the claim is immediate – one can simply take $\mathcal{D}'$ as a subset of $\mathcal{D}$.

Since Σ' is disjoint from Σ, the union $\Sigma \cup \Sigma'$ is a simple sphere system (where we discard multiple copies of the same homotopy class of a sphere). The sphere system Σ' is then obtained from $\Sigma \cup \Sigma'$ by removing some number of spheres. Therefore, to show the lemma it suffices to consider the case that $\Sigma' \supset \Sigma$.

Let $\Sigma = \{\sigma_1, \dots, \sigma_r\}$ and let $\Sigma' = \{\sigma_1, \dots, \sigma_r, \sigma_1', \dots, \sigma_s'\}$. We call the sphere pieces defined by one of the σ_i *old sphere pieces* and those defined by one of the σ_j' *new sphere pieces*.

The gluing datum $\mathcal{D}$ contains a disk for each boundary component of every old sphere piece. We will construct $\mathcal{D}'$ inductively as an extension of $\mathcal{D}$. The boundary components of new sphere pieces fall into two different classes: those that are contained in some disk from the collection $\mathcal{D}$ and those that are disjoint from any disk in $\mathcal{D}$.

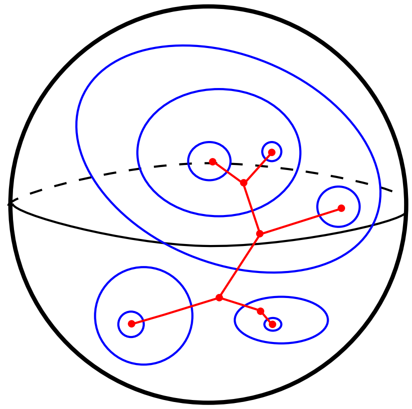

FIGURE 2. An example of a collection of boundaries of sphere pieces and the corresponding dual tree.

Let $D_1, \dots, D_k$ be the maximal disks in $\mathcal{D}$ with respect to the partial order defined by inclusion (this makes sense as $\mathcal{D}$ is admissible). Let α be a boundary component of a new sphere piece C' such that $\alpha \subset D_l$ for some l. We then choose the closing disk $D(\alpha)$ to be the unique embedded disk in σ bounded by α which is contained in D_l. Let $\mathcal{D}_1$ be the union of $\mathcal{D}$ with the set of all disks obtained in this way. By construction, $\mathcal{D}_1$ is properly nested in the sense defined before Lemma 2.5.

If $\mathcal{D}_1$ is a gluing datum for Σ' then we are done. Otherwise, consider the set of those boundary components $\alpha_1, \dots, \alpha_k$ of new sphere pieces which are disjoint from every disk in $\mathcal{D}$ and hence also from every disk in $\mathcal{D}_1$. If $k = 0$ there is nothing to show, so we may assume $k \geq 1$.

Let $I = \bigcup_{i=1}^k \alpha_i \subset \sigma$ and let T be the dual graph to I. Explicitly, T is the graph whose vertex set is the set of connected components of $\sigma \setminus I$. Two such vertices corresponding to components U_1, U_2 are joined by an edge if there is a component of I contained in the closure of both U_i. As every circle on a sphere is separating, the graph T is in fact a tree (see Figure 2).

Let v be a leaf of T, corresponding to a complementary component whose closure is a disk $D(v)$. This disk $D(v)$ intersects I in a single component $\alpha(v)$.

If $k \geq 2$ then $D(v)$ is the unique disk on σ bounded by $\alpha(v)$ which is disjoint from all $\alpha_i \neq \alpha(v)$. If $k = 1$ then $D(v)$ is one of the two embedded disks in σ bounded by α_1.

In both cases, if $D(v)$ intersects a disk $D \in \mathcal{D}_1$, then $D \subset D(v)$ since otherwise $\alpha(v) \subset D$.

Hence, the set of disks $\mathcal{D}_2 = \mathcal{D}_1 \cup \{D(v)\}$ is properly nested. Furthermore, the set of boundary components of new sphere pieces that are not contained in any disk of $\mathcal{D}_2$ is $\{\alpha_1, \dots, \alpha_k\} \setminus \{\alpha(v)\}$.

The lemma now follows by repeating this procedure, assigning a closing disk to each α_i. $\square$

Proof of Theorem 2.4. Let Σ, Σ' be two simple sphere systems containing σ. Choose an edge-path $\Sigma = \Sigma_1, \dots, \Sigma_L = \Sigma'$ of shortest length connecting Σ to Σ' in the sphere system graph $\mathcal{S}(M_n)$.

Since Σ_1 is disjoint from σ by assumption, $\mathcal{D}_1 = \emptyset$ is an admissible gluing datum for Σ_1.

By Lemma 2.8 there is an admissible gluing datum $\mathcal{D}_2$ for Σ_2 such that $\Sigma_1 = \pi_\sigma(\Sigma_1, \mathcal{D}_1)$ is disjoint from $\pi_\sigma(\Sigma_2, \mathcal{D}_2)$.

Inductively applying Lemma 2.8, one obtains admissible gluing data $\mathcal{D}_i$ for Σ_i such that $\pi_\sigma(\Sigma_i, \mathcal{D}_i)$ is disjoint from $\pi_\sigma(\Sigma_{i+1}, \mathcal{D}_{i+1})$ for all $i = 2, \dots, L-1$.

As Σ_L is disjoint from σ, the only admissible gluing datum is the empty set, and hence $\pi_\sigma(\Sigma_L, \mathcal{D}_L) = \Sigma_L$.

By construction, the sequence $\pi_\sigma(\Sigma_i, \mathcal{D}_i)$ for $1 \leq 1 \leq L$ defines an edge-path in $\mathcal{S}(M_n, \sigma)$ connecting Σ to Σ'. Thus the distance between Σ and Σ' in $\mathcal{S}(M_n)$ equals the distance between Σ and Σ' in $\mathcal{S}(M_n, \sigma)$ and the theorem is shown. $\square$

3. Mapping class groups in $\mathrm{Out}(F_n)$

In this section we study an embedding of a surface mapping class group into $\mathrm{Out}(F_n)$. Let S_g^1 be a surface of genus g with one boundary component, and let $S_{g,1}$ be the surface obtained by collapsing the boundary component of S_g^1 to a marked point. We often view the marked point as a puncture of the surface, so that the fundamental group of $S_{g,1}$ is the free group F_{2g} on $2g$ generators.

A simple closed curve on $S_{g,1}$ which bounds a disk containing the marked point defines a distinguished conjugacy class in $\pi_1(S_{g,1})$ called the *cusp class*. The mapping class group of $S_{g,1}$ preserves the cusp class.

The following analog of the Dehn-Nielsen-Baer theorem for punctured surfaces is well-known (see e.g. Theorem 8.8 of [FM11]).

Theorem 3.1. *The homomorphism*

$$\iota : \mathrm{Map}(S_{g,1}) \to \mathrm{Out}(F_{2g})$$

induced by the action on the fundamental group of $S_{g,1}$ is injective. Its image consists of those outer automorphisms which preserve the cusp class.

The goal of this section is to prove

Theorem 3.2. *The homomorphism ι is a quasi-isometric embedding.*

We employ the following geometric model for the mapping class group of $S_{g,1}$. A *binding loop system for $S_{g,1}$* is defined to be a collection of embedded loops $\{a_1, \dots, a_n\}$ based at the marked point of $S_{g,1}$ which intersect only at the marked point and which decompose $S_{g,1}$ into a disjoint union of disks.

Let $\mathcal{BL}(S_{g,1})$ be the graph whose vertex set is the set of isotopy classes of binding loop systems. Here isotopies are required to fix the marked point. Two such systems are connected by an edge if they intersect in at most K points. As the mapping class group of $S_{g,1}$ acts with finite quotient on the set of isotopy classes of binding loop systems, we can choose the number $K > 0$ such that the following lemma is true.

Lemma 3.3. *The graph $\mathcal{BL}(S_{g,1})$ is connected. The mapping class group of $S_{g,1}$ acts on $\mathcal{BL}(S_{g,1})$ with finite quotient and finite point stabilizers.*

Instead of working with binding loop systems of $S_{g,1}$ directly we will frequently use binding arc systems of S_g^1. By this we mean a collection A of disjointly embedded arcs $\{a_1, \dots, a_n\}$ connecting the boundary component of S_g^1 to itself which decompose S_g^1 into simply connected regions. We will consider such binding arc systems up to isotopy of properly embedded arcs. A binding arc system for S_g^1 defines a binding loop system for $S_{g,1}$ by collapsing the boundary component of S_g^1 to the marked point. Note that if A_1, A_2 are two disjoint binding arc systems for S_g^1 then the corresponding binding loop systems for $S_{g,1}$ are uniformly close in $\mathcal{BL}(S_{g,1})$. By this we mean that there is a number $K > 0$ depending only on g such that the distance between the two binding loop systems in $\mathcal{BL}(S_{g,1})$ is at most K. The Dehn twist about the boundary component of S_g^1 acts trivially on the isotopy class of any arc system. Thus the action of the mapping class group of S_g^1 on binding arc systems factors through an action of $\mathrm{Map}(S_{g,1})$.

We can now describe the strategy of the proof of Theorem 3.2; details for each step will be given below. As in Section 2, we use simple sphere systems in M_{2g} to build a graph $\mathcal{S}_0(M_{2g})$ which is quasi-isometric to $\mathrm{Out}(F_{2g})$ (for technical reasons we choose a subgraph of $\mathcal{S}(M_{2g})$ in this section, see below for the definition). Let $\phi \in \mathrm{Map}(S_{g,1})$ be given and let F be a diffeomorphism of M_{2g} which represents the outer automorphism $\iota(\phi)$ of the fundamental group F_{2g}. We may choose F in such a way that it preserves an embedded surface $S_g^1 \subset M_{2g}$ and such that F restricts to a representative of ϕ on S_g^1. Now consider a shortest path $\Sigma_0, \Sigma_1, \dots, \Sigma_N = F(\Sigma_0)$ connecting a base sphere system to its image under F in $\mathcal{S}_0(M_{2g})$. The intersections of Σ_i with the surface $S_g^1 \subset M_{2g}$ then yield a sequence of binding arc systems $A_0, \dots, A_N$ on the surface S_g^1 and therefore a path of length coarsely bounded by N in the graph of binding loop systems of $S_{g,1}$. Each of the Σ_i is only determined up to homotopy and therefore the arc systems A_i are not defined canonically. The main technical difficulty now consists in obtaining enough control on the representatives of the homotopy classes to ensure that $A_N = \Sigma_N \cap S_g^1$ defines the homotopy class $\phi(A_0)$. To this end we also have to successively modify the surface S_g^1 by homotopies. Once this is done, the word norm of ϕ is coarsely bounded by N, and hence by the word norm of $\iota(\phi)$ in $\mathrm{Out}(F_{2g})$, showing Theorem 3.2.

We now define the geometric model of $\mathrm{Out}(F_{2g})$ used in this section. Let $M = M_{2g}$ be the connected sum of $2g$ copies of $S^2 \times S^1$. Say that a simple sphere system Σ for M is *reduced* if $M \setminus \Sigma$ is connected. Let $\mathcal{S}_0(M)$ be the complete subgraph of $\mathcal{S}(M)$ whose vertices correspond to reduced sphere systems. We call $\mathcal{S}_0(M)$ the *reduced sphere system graph.*

Lemma 3.4. *The graph $\mathcal{S}_0(M)$ is connected.*

This lemma can be shown using a surgery argument which is well-known in the analogous case of reduced disk systems for handlebodies (see e.g. [HH11, Lemma 5.2], [St99] or [M86, Lemma 3.2]) For convenience of the reader we sketch a proof in the sphere system case here.

Proof. Let Σ, Σ' be two reduced sphere systems. We may assume without loss of generality that Σ and Σ' are in general position and hence the intersection is a disjoint union of finitely many circles. We prove the lemma by induction on the number of such intersection circles.

Let $\sigma' \in \Sigma'$ be a sphere which intersects Σ. The intersection $\sigma' \cap \Sigma$ is a disjoint union of finitely many circles $\alpha_1, \dots, \alpha_k$. There is at least one such circle $\alpha = \alpha_i$ which bounds a disk $D' \subset \sigma'$ whose interior contains no other intersection circle $\alpha_j, j \neq i$. Suppose that α is contained in the sphere $\sigma \in \Sigma$. Denote by D_1, D_2 the two embedded disks in σ bounded by α and put $\sigma_j = D_j \cup D'$ for $j = 1, 2$. Up to homotopy, the surface σ_j is a sphere which is disjoint from Σ.

We claim that the sphere system $\Sigma_j = \Sigma \cup \{\sigma_j\} \setminus \{\sigma\}$ is reduced for a suitable j. To prove the claim, note that a sphere system with $2g$ components is reduced if it defines a basis of $H_2(M, \mathbb{Z})$. Since Σ is reduced, for exactly one choice of $j = 1, 2$ the system Σ_j defines a basis of $H_2(M, \mathbb{Z})$ (the corresponding sphere σ_j has to separate the two sides of σ in the complement of Σ). This shows the lemma. □

The graph $\mathcal{S}_0(M)$ is Out(F_{2g})-invariant. Moreover, the action of Out(F_{2g}) on $\mathcal{S}_0(M)$ is proper and thus $\mathcal{S}_0(M)$ is equivariantly quasi-isometric to Out(F_{2g}).

The advantage of using reduced sphere systems is that they make it easy to encode free homotopy classes of curves in M. Namely, let $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ be a reduced sphere system. We choose a transverse orientation for each sphere σ_i so we may speak of a positive and a negative side of σ_i.

Let $p \in M$ be a base point in the complement of Σ. A *basis dual* to Σ is a set of loops $\gamma_1, \ldots, \gamma_n$ in M based at p such that the loop γ_i is disjoint from σ_j for all $j \neq i$ and intersects σ_i in a single point. We orient γ_i such that it approaches σ_i from the positive side. Since the complement of Σ is simply connected, the loops γ_i define a basis of $\pi_1(M, p)$.

Now let α be a closed curve in M. Choose an orientation of α. Up to applying a homotopy to α we may assume that α and Σ are in general position and thus intersect in a finite set of points. Apply a homotopy to α in the complement of Σ such that α passes through the basepoint p. The resulting based loop $\hat{\alpha}$ is a representative of the free homotopy class defined by α. Since the complement of Σ is simply connected, the sequence of (oriented) spheres in Σ which are consecutively hit by $\hat{\alpha}$ (and hence α) defines a word in the $\gamma_i^{\pm}$ representing $\hat{\alpha}$. In other words, the free homotopy class defined by α is determined by the sequence of sides of spheres in Σ that α intersects.

We next put α in tight position with respect to Σ as follows. Let M_Σ be the complement of Σ in the sense described for a single sphere in Section 2 – that is, M_Σ is a compact connected three-manifold whose boundary consists of $2n$ boundary spheres $\sigma_1^+, \sigma_1^-, \ldots, \sigma_n^+, \sigma_n^-$. The boundary spheres σ_i^+ and σ_i^- correspond to the two sides of σ_i. If α is not disjoint from Σ then the intersection of α with M_Σ is a disjoint union of arcs connecting the boundary components of M_Σ. We call these arcs the Σ*–arcs of* α. An orientation of α induces a cyclic order on the Σ–arcs of α.

We say that α *intersects* Σ *minimally* if no Σ-arc of α connects a boundary component of M_Σ to itself.

Lemma 3.5. *Every closed curve α in M can be modified by a homotopy to intersects Σ minimally. Let α and α' be two simple closed curves which are freely homotopic and which intersect Σ minimally. Then there is a bijection f between the Σ–arcs of α and the Σ–arcs of α'*

such that $f(a)$ is homotopic to a for each Σ–arc a of α. If orientations of α and α' are chosen appropriately, f may be chosen to respect the cyclic orders on the Σ–arcs.

Proof. Since M_Σ is simply connected, an arc in M_Σ connecting a boundary component to itself is homotopic into that boundary component. This shows the first claim.

To see the other claims, let p be a base point in the complement of Σ and let γ_i be a basis of $\pi_1(M,p)$ dual to Σ. The sequence of oriented spheres from Σ determined by the consecutive intersections of α defines a word in the γ_i representing the conjugacy class of α.

If α intersects Σ minimally, this word representing the conjugacy class of α is reduced and cyclically reduced. The analogous statements are also true for α'. Since α and α' are freely homotopic, they define the same conjugacy class in $\pi_1(M,p)$. Up to cyclic permutation, a conjugacy class in a free group contains a unique cyclically reduced word. Therefore, the words in γ_i defined by α and α' are equal up to cyclic permutation and possibly reversing the orientation of α'. This implies the lemma. □

Now let $V = S_g^1 \times [0,1]$ be the trivial oriented interval bundle over S_g^1. We identify $M = M_{2g}$ with the three-manifold obtained by doubling V along its boundary. To simplify notation, we put $n = 2g$.

As can be seen from the description of M as the double of V, the surface $S_g^1 \times \{\frac{1}{2}\}$ is incompressible in M. Let $\varphi_0 : S_g^1 \to M$ be the thus defined embedding of S_g^1 into M. Let β be the boundary curve of S_g^1. The image $\varphi_0(\beta)$ is an embedded closed curve in M which maps to the cusp class in $\pi_1(S_{g,1}) = \pi_1(M)$.

Next we put $\varphi_0(S_g^1)$ and $\varphi_0(\beta)$ in good position with respect to a given reduced sphere system. Since for surfaces and curves in M homotopy is in general not the same as isotopy, we need to take some care in defining these notions.

Consider the more general case of a surface $\varphi(S_g^1)$, where $\varphi : S_g^1 \to M$ is any embedding of S_g^1 into M which is homotopic to φ_0 (note that such an embedding need not be isotopic to φ_0). Up to modifying φ with a small isotopy, we may assume that Σ intersects the surface $\varphi(S_g^1)$ transversely. Then the preimage $\varphi^{-1}(\Sigma)$ is a one-dimensional submanifold of S_g^1, and hence it is a disjoint union of simple closed curves and properly embedded arcs.

Definition 3.6. We say that φ is in *ribbon position with respect to* Σ if each component of $\varphi^{-1}(\Sigma)$ is a properly embedded arc. It is said to be in *minimal position* if in addition $\varphi(\beta)$ is in minimal position with respect to Σ. In either case, we call the preimage $\varphi^{-1}(\Sigma)$ the *arc system induced by Σ and φ.*

Note that a priori the homotopy class of the arc system induced by Σ and φ need not be determined by the isotopy class of Σ even if φ is in minimal position with respect to Σ.

A main step towards the proof of Theorem 3.2 consists in establishing some control over the homotopy class of the arc system induced by φ and a sphere system associated to the an element in the image of $\mathrm{Map}(S_{g,1})$. To make this precise, fix an embedded binding arc system $A_0 = \{a_1^0, \dots, a_{2g}^0\}$ of S_g^1 consisting of precisely $2g$ arcs which cut $S_{g,1}$ into a single disc. We require that β intersects each arc a_i^0 in two points, and that a subarc of β defined by the intersection points with a_i^0 approaches a_i^0 from the same side at both of its endpoints. Such a binding arc system can easily be obtained from the standard description of S_g^1 as a one-holed $4g$–gon with opposite sides identified.

The interval bundle over A_0 is a disk system in the handlebody V which cuts V into a ball. Doubling this disk system across the boundary of V, we obtain a reduced simple sphere system Σ_0 in M_{2g} such that the arc system induced by φ_0 and Σ_0 is the arc system A_0. Similarly, any diffeomorphism f of $S_{g,1}$ extends in this way to a diffeomorphism $I(f)$ of M by first extending f to a product map of $S_{g,1}$ and then extending further by doubling. The action of $I(f)$ on homotopy classes of sphere systems then coincides with the action of the image of the projection of f to $\mathrm{Map}(S_g^1)$ under the inclusion $\iota : \mathrm{Map}(S_g^1) \to \mathrm{Out}(F_n)$. This observation is used in the following lemma which gives control on some classes of arc systems.

Lemma 3.7. *Let f be an orientation preserving diffeomorphism of S_g^1 and let Σ be a simple sphere system which is homotopic to $I(f)(\Sigma_0)$. Suppose that $\varphi : S_g^1 \to M$ is homotopic to φ_0 and in minimal position with respect to Σ. Then the arc system induced by φ and Σ is homotopic to $f(A_0)$.*

Proof. Let f be an orientation preserving diffeomorphism of S_g^1 and let $F = I(f)$. The sphere system $F(\Sigma_0)$ is then in minimal position with respect to φ_0, and $\varphi_0^{-1}(F(\Sigma_0)) = f(A_0)$ by construction. By applying an isotopy to M which maps Σ to $F(\Sigma_0)$ we may assume without loss of generality that $\Sigma = F(\Sigma_0)$. If we replace φ by its composition with this isotopy then this does not change $\varphi^{-1}(\Sigma)$.

With respect to the sphere system $\Sigma = F(\Sigma_0)$, the curve $\varphi_0(\beta)$ is in minimal position. Furthermore, by choice of the arc system A_0, the curve $\varphi_0(\beta)$ intersects each sphere σ_j in Σ in exactly two points x_j'

and y_j'. Let $\beta_j'^1$ and $\beta_j'^2$ be the two subarcs of $\varphi_0(\beta)$ defined by these intersection points.

By Lemma 3.5, minimal position of curves is unique and only depends on the homotopy class of the curve and of the sphere system. Therefore the curve $\varphi(\beta)$ intersects each sphere $\sigma_j \in \Sigma$ also in two points, say x_j and y_j. Denote by β_j^1 and β_j^2 the two subarcs of $\varphi(\beta)$ defined by these intersection points. Again by uniqueness of minimal position of curves, the arc β_j^r is homotopic to $\beta_j'^r$ with endpoints sliding on σ_j for $r = 1, 2$ (after possibly exchanging β_j^1 and β_j^2).

Let $a_j \subset S_g^1$ be the preimage of σ_j under φ and let a_i' be the preimage of σ_j under φ_0. The boundary of a regular neighborhood of $\beta \cup a_j$ in S_g^1 is the union of two simple closed curves d_j^1, d_j^2 and the boundary curve β, and the boundary of a regular neighborhood of $\beta \cup a_j'$ consists of two simple closed curves $d_j'^1, d_j'^2$ and the boundary curve β.

Up to exchanging d_j^1 and d_j^2, the curve $\varphi(d_j^k) \subset M$ is freely homotopic to a curve $\delta_j^k = \beta_j^k * \alpha_j$ obtained by concatenating β_j^k and an embedded arc α_j on σ_j. Similarly, the curve $\varphi_0(d_j'^k)$ is freely homotopic to a curve $\delta_j'^k = \beta_j'^k * \alpha_j'$ obtained by concatenating $\beta_j'^k$ and an embedded arc α_j' on σ_i.

Since σ_j is simply connected and $\beta_j'^k$ is homotopic to β_j^k relative to σ_j, the curves $\delta_j'^k$ and δ_j^k are freely homotopic. Since φ and φ_0 induce the same isomorphism on the level of fundamental groups, this implies that also the simple closed curves d_j^k and $d_j'^k$ in S_g^1 are freely homotopic.

The curves β, d_j^1 and d_j^2 bound a pair of pants P_i on S_g^1. The arc a_j is up to isotopy the unique essential embedded arc in P_j connecting β to itself. Similarly, β, $d_j'^1$ and $d_j'^2$ bound a pair of pants P_j', which is isotopic to P_j. As a_j' is the unique essential embedded arc in P_j' connecting β to itself, it is therefore isotopic to a_j.

Hence we have shown that the arc system induced by a map φ and a sphere system Σ as in the statement is isotopic to the arc system induced by φ_0 and Σ, hence isotopic to $f(A_0)$. $\square$

To apply Lemma 3.7 we have to keep φ in minimal position when changing the sphere system. For this we use an inductive method which is described in the next lemma. For its proof, we need the following observation, which also motivates the terminology "ribbon position".

Suppose that φ is in ribbon position with respect to the reduced sphere system Σ. Since φ is homotopic to φ_0, it induces an isomorphism between the fundamental groups of S_g^1 and M. The intersection of $\varphi(S_g^1)$ with M_Σ is a union of surfaces $P_1, \dots, P_k$. As Σ is a simple sphere system, the arc system $\varphi^{-1}(\Sigma)$ on S_g^1 is binding and hence each of the surfaces P_i is a disk whose boundary is not completely contained in a boundary component of M_Σ.

Pick one such disk, say P_i, and consider its boundary curve δ_i. We can write this curve in the form

$$\delta_i = a_1 * b_1 * \cdots * a_r * b_r$$

where each a_i is an arc contained in one of the boundary spheres of M_Σ, and each b_i is a properly embedded arc in M_Σ. Let $\Gamma_i \subset P_i$ be an embedded graph in P_i defined in the following way. The graph Γ_i has one distinguished vertex v_0 contained in the interior of P_i and one vertex v_r contained in each arc a_r. Each vertex v_r $(r \geq 1)$ is connected by an edge to the vertex v_0. The oriented surface P_i determines a *ribbon structure* on Γ_i. Here a ribbon structure on Γ_i is simply a cyclic order of the half-edges at v_0.

To reconstruct $\varphi(S^1_g)$ from the ribbon graphs Γ_i we equip Γ_i with a *twisting datum*. Namely, fix the arcs a_r and an orientation of each of the arcs a_r, so that we can refer to the left and right endpoint of each a_r. A twisting datum on Γ_i associates to each edge of Γ_i a sign $+$ or $-$. We call the graph Γ_i equipped with a twisting datum a *decorated ribbon graph*.

The *surface associated to a decorated graph* Γ_i is defined in the following way. Put a small embedded oriented disk D at the central vertex v_0 of Γ_i containing a neighborhood of v_0 so that the cyclic order of the edges at v_0 corresponds to the counterclockwise order on D. Connect each arc a_r to the disk D with a *band*, i.e. an embedded product of two intervals $[0,1] \times [0,1]$ in M, as follows. One of the sides of B_r is the arc a_r, and the opposite side is contained in ∂D. We call these sides the *horizontal sides*. Correspondingly, the *vertical sides* are properly embedded arcs in M. The orientation of ∂D determines a left and right endpoint of each of these intervals. Up to homotopy, there are two ways to glue a band between two prescribed horizontal sides which correspond to the two ways of pairing the endpoints of these intervals. If the edge corresponding to the band B_r is decorated with a $+$, we match the left endpoint of a_r with the left endpoint of the interval on ∂D, otherwise we pair the left with the right endpoint. If the twisting data on Γ_i is chosen appropriately, the surface associated to Γ_i is homotopic to U_i relative to ∂M_Σ to U_i.

Lemma 3.8. *Suppose that φ is in minimal position with respect to Σ. Let σ' be an embedded sphere disjoint from Σ. Suppose that there is a sphere $\sigma \in \Sigma$ such that $\Sigma' = \Sigma \cup \{\sigma'\} \setminus \{\sigma\}$ is a reduced sphere system.*

Then there is an embedding $\varphi' : S^1_g \to M$ with the following properties.

i) φ' *is homotopic to* φ.
ii) φ' *is in minimal position with respect to* Σ'.
iii) *The arc system induced by* φ' *and* Σ *is the same as the arc system induced by* φ *and* Σ.

Proof. By assumption, φ is in ribbon position with respect to Σ. Let $P_1, \ldots, P_k$ be the components of $\varphi(S^1_g) \cap M_\Sigma$. By applying an isotopy to φ that does not change $\varphi^{-1}(\Sigma)$, we may assume that each P_i is the surface associated to a decorated ribbon graph Γ_i as described after Definition 3.6. We may choose Γ_i in such a way that no intersection point of σ' with Γ_i is a vertex of Γ_i and that σ' intersects each Γ_i transversely. Hence the intersection between σ' and Γ_i consists of a finite union of points, and the intersection between P_i and σ' consists of a disjoint union of arcs. Namely, the surface associated to a decorated ribbon graph may be chosen to lie in an arbitrarily small neighborhood of the graph.

As a consequence, the sphere σ' intersects each component of $\varphi(S^1_g) \cap M_\Sigma$ in a disjoint union of arcs. Each component of $\varphi(S^1_g) \cap M_{\Sigma \cup \{\sigma'\}}$ is a disk whose boundary contains a subarc of β and hence φ is in ribbon position with respect to $\Sigma \cup \{\sigma'\}$ and thus also with respect to Σ'.

It remains to show that φ can be changed by a homotopy as claimed in the lemma.

Let b be a Σ'–arc of β. Assume first that b also is a Σ–arc. Then b has both endpoints on a sphere distinct from σ. By assumption on Σ, the arc b does not connect the same boundary component of M_Σ to itself. This then also holds true for b viewed as a Σ'-arc.

If b is not of this form, at least one of its endpoints is contained in the sphere σ'. Suppose that both endpoints of b are contained on the same side of σ' (alternatively, on the same boundary component of $M_{\Sigma'}$). We call such subarcs of β *problematic*. A problematic subarc b does not intersect the sphere σ. Namely, we observed in the proof of Lemma 3.4 that in M_Σ, the sphere σ' separates the two boundary components corresponding to σ. Thus if b intersected σ, a subarc of b would return to the same side of σ. By assumption on Σ, this is not the case.

Let P_i be the component of $\varphi(S^1_g) \cap M_\Sigma$ containing b in its boundary. Choose small open tubular neighborhoods $\mathcal{U}_1 \subset \mathcal{U}_2$ of Σ so that $\overline{\mathcal{U}}_1 \subset \mathcal{U}_2$ and that $\overline{\mathcal{U}}_2$ is disjoint from σ'. We also assume that $\varphi(S^1_g)$ intersects $\overline{\mathcal{U}}_2$ in a union of disjoint embedded rectangles with two opposite sides on two different boundary components of $\overline{\mathcal{U}}_2$. Choose a homotopy H supported in the complement of $\mathcal{U}_1$ such that in the complement of $\mathcal{U}_2$ the image of P_i under this homotopy is Γ_i.

We compose this homotopy with φ such that the resulting map collapses P_i to the graph Γ_i. Explicitly, this means that we modify φ with a homotopy to obtain a map $\varphi_1 : S^1_g \to M$ in the following way. On the set $\varphi^{-1}((M \setminus P_i) \cup \mathcal{U}_1)$, the maps φ and φ_1 coincide. On $\varphi^{-1}(P_i \setminus \mathcal{U}_2)$ we set φ_1 to be the postcomposition of φ with the endpoint of the homotopy H.

The map φ_1 is not an embedding of S^1_g into M since it collapses the region $\varphi^{-1}(P_i \cap \mathcal{U}_2)$ to the graph $\Gamma_i \cap \mathcal{U}_2$. However, by construction, φ_1 is homotopic to φ_0 and the preimage $\varphi_1^{-1}(\Sigma)$ is the same as $\varphi^{-1}(\Sigma)$.

Since b is contained in a boundary arc of P_i and connects a side of σ' to itself, the same is true for the graph Γ_i. Since each complementary component of σ' in M_Σ is simply connected, the graph Γ_i is therefore homotopic with fixed endpoints to a graph Γ'_i in M_Σ which intersects σ' in fewer points than Γ_i and which is disjoint from all other $\Gamma_j, j \neq i$. The graph Γ'_i inherits the structure of a decorated ribbon graph from Γ_i.

We now modify φ_1 using this homotopy (in the same way that we constructed φ_1) to obtain a map $\varphi_2 : S^1_g \to M$ which maps $\varphi^{-1}(P_i \cap \mathcal{U}_2)$ to $\Gamma'_i \cap \mathcal{U}_2$ and still agrees with φ on $S^1_g \setminus \varphi^{-1}(P_i)$.

As a last step, we modify φ_2 by a homotopy to make it again an embedding. Namely, let P'_i be the surface defined by the decorated graph Γ'_i as described above. Then P'_i is homeomorphic to the disk P_i with a homeomorphism that induces an isomorphism of the decorated graphs Γ'_i and Γ_i and restricts to the identity on each component $P_i \cap \partial M_\Sigma$.

Hence we can apply a homotopy to the map φ_2 (supported on $\varphi^{-1}(P_i \setminus \mathcal{U}_1)$) to obtain a embedding $\varphi_3 : S^1_g \to M$ with the following properties. On $\varphi^{-1}(M_\Sigma \setminus P_i)$, the maps φ_3 and φ agree. Furthermore, the set $\varphi^{-1}(P_i \cap \mathcal{U}_2)$ is mapped to the surface P'_i which can be chosen to be contained in a small regular neighborhood of Γ'_i in M. Finally, $\varphi^{-1}(\Sigma) = \varphi_3^{-1}(\Sigma)$. We can choose this homotopy such that φ_3 is in ribbon position with respect to Σ' by the same argument as before.

By construction, the image of β under φ_3 has fewer problematic arcs than the image of β under φ. The existence of the desired φ' follows then by inductively applying this procedure (with φ_3 in the place of φ). $\square$

We now have collected all the necessary tools to prove the main theorem of this section.

Proof of Theorem 3.2. Let $f \in \mathrm{Map}(S_{g,1})$ be given. To prove the theorem, we need to show that the word norm of f as an element of the surface mapping class group is coarsely bounded by the word norm of $\iota(f)$ in $\mathrm{Out}(F_{2g})$.

The word norm of $\iota(f)$ in $\mathrm{Out}(F_{2g})$ is coarsely equal to the distance between Σ_0 and $\iota(f)(\Sigma_0)$ in the reduced sphere system graph.

Choose a shortest path connecting Σ_0 to $\iota(f)(\Sigma_0)$ in the reduced sphere system graph, and denote the corresponding sphere systems by $\Sigma_0, \Sigma_1, \ldots, \Sigma_N$.

We now inductively define a sequence of binding arc systems. By construction, φ_0 is in minimal position with respect to Σ_0. As Σ_1 is connected to Σ_0 by an edge in the reduced sphere system graph, Σ_1 is obtained from Σ_0 by replacing a single sphere.

Thus Lemma 3.8 applies, and yields a reduced sphere system Σ_1' which is homotopic to Σ_1 and disjoint from Σ_0, and furthermore an embedding φ_1. This embedding is homotopic to φ_0, in minimal position with respect to Σ_1 and such that $\varphi_1^{-1}(\Sigma_0) = \varphi_0^{-1}(\Sigma_0)$. Put $A_1 = \varphi_1^{-1}(\Sigma_1')$. By the choice of φ_1, the arc system A_1 is binding and disjoint from A_0.

Inductively applying Lemma 3.8, we obtain a sequence of sphere systems Σ_i' and embeddings $\varphi_i : S_g^1 \to M$ such that the following holds. Each Σ_i' is homotopic to Σ_i and each φ_i is homotopic to φ_0. Furthermore, the arc systems A_i induced by Σ_i' and φ_i define a path in the graph $\mathcal{BL}(S_{g,1})$ whose length is coarsely bounded by N.

By Lemma 3.7 the arc system A_N is homotopic to $f(A_0)$. Hence, as the binding loop system graph is quasi-isometric to $\mathrm{Map}(S_{g,1})$, the theorem follows. $\square$

The method employed in the proof of Theorem 3.2 has another application. For its formulation, recall that the *arc graph of* S_g^1 is the graph whose vertex set is the set of isotopy classes of embedded essential arcs connecting the boundary of S_g^1 to itself. Again, isotopies are only required to fix the boundary component setwise. Two such vertices are joined by an edge if the corresponding arcs can be embedded disjointly. Similarly, define the *sphere graph of* M to be the graph whose vertex set is the set of isotopy classes of essential 2-spheres in M. Two such vertices are connected by an edge if the corresponding spheres can be realized disjointly.

Let a be an arc representing a vertex of the arc graph of S_g^1. The interval bundle over a is a disk $D(a)$ in the handlebody $V = S_g^1 \times [0,1]$. The isotopy class of this disk only depends on the isotopy class of a, since the Dehn twist about the boundary of S_g^1 is contained in the kernel of the map $\mathrm{Map}(S_g^1) \to \mathrm{Map}(V)$. We let $\sigma(a)$ be the essential sphere in M which is obtained by doubling $D(a)$ along ∂V.

Proposition 3.9. *The map sending a to $\sigma(a)$ induces a quasi-isometric embedding of the arc graph of S_g^1 into the sphere graph of M.*

In particular, this theorem immediately implies the following.

Corollary 3.10. *For each $g \geq 1$ the sphere graph of M_{2g} has infinite diameter.*

Proof of Proposition 3.9. Let a, a' be two essential arcs in S_g^1. Since the mapping class group of S_g^1 acts transitively on the set of isotopy classes of essential arcs in S_g^1, there is a mapping class f such that $f(a) = a'$. Furthermore, we may assume that a is contained in the standard arc system A_0.

A single arc in S_g^1 does not separate the surface S_g^1. Thus the sphere $\sigma(a)$ is a nonseparating essential sphere in M.

Let $\sigma(a) = \sigma_1, \sigma_2, \ldots, \sigma_N = \sigma(a')$ be a shortest path in the sphere graph of M. We may assume without loss of generality that each σ_i is a nonseparating sphere. Namely, suppose that σ_i is separating and let M_1, M_2 be its two complementary components. If σ_{i-1} and σ_{i+1} are contained in different components, then they are connected by an edge in the sphere graph. In this case, the sphere σ_i can be removed from the edge-path. If σ_{i-1} and σ_{i+1} are contained in the same component, say M_1, then one can replace σ_i by a nonseparating sphere σ_i' contained in M_2.

Choose reduced sphere systems Σ_i containing σ_i. Let $\Sigma_i^{(1)}, \ldots, \Sigma_i^{(N_i)}$ be a path in the reduced sphere system graph connecting Σ_i to Σ_{i+1} such that each $\Sigma_i^{(j)}$ contains σ_i for each $1 \leq j \leq N_i - 1$.

We now argue as in the proof of Theorem 3.2. Applying Lemma 3.8 inductively, we change the sphere systems $\Sigma_i^{(j)}$ by isotopy and obtain a sequence of embeddings $\varphi_i^{(j)}$ which intersect $\Sigma_i^{(j)}$ minimally. Let $A_i^{(j)}$ be the arc systems induced by $\varphi_i^{(j)}$ and $\Sigma_i^{(j)}$.

By construction, for $1 \leq j \leq N_i - 1$ the arc systems $A_i^{(j)}$ contain a common arc a_i. The sequence a_i defines an edge-path in the arc graph of length at most $2N$. Furthermore, by Lemma 3.5, the arc a_N is contained in $f(A_0)$ and thus is adjacent to a'. This proves the theorem. ☐

Appendix A. Stabilizers of spheres

In this appendix we identify the stabilizers of conjugacy classes of free splittings and corank one free factors of a free group topologically. To this end, let M_n be the connected sum of n copies of $S^1 \times S^2$.

As explained in Section 2, the mapping class group of M_n projects onto $\mathrm{Out}(F_n)$ with a finite kernel. Our goal is to give an elementary topological proof of the following

Lemma A.1. *i) Let σ be an essential separating sphere in M_n. Then the stabilizer of σ in $\mathrm{Map}(M_n)$ projects onto the stabilizer of the conjugacy class of a free splitting in $\mathrm{Out}(F_n)$. Furthermore, every stabilizer of a conjugacy class of a free splitting arises in this way.*

ii) Let σ be an essential nonseparating sphere in M_n. Then the stabilizer of σ in $\mathrm{Map}(M_n)$ projects onto the stabilizer of the conjugacy class of a corank 1 free factor in $\mathrm{Out}(F_n)$. Furthermore, every stabilizer of a conjugacy class of a corank 1 free factor arises in this way.

Proof. Let σ be as in *i)*, and denote by M^1 and M^2 the two complementary components of σ in M_n. We let $N^i = M^i \cup \sigma$. Since σ is simply connected, the van-Kampen theorem yields that the fundamental group of M_n can be written as a free product $\pi_1(M_n) = \pi_1(N^1) * \pi_1(N^2)$. The fundamental groups of N^1 and N^2 are thus free groups of rank $n-i$ and i, respectively. A mapping class of M_n that stabilizes σ (up to homotopy) induces an outer automorphism of $\pi_1(M_n)$ that stabilizes the free splitting $\pi_1(M_n, x) = \pi_1(N^1, x) * \pi_1(N^2, x)$ up to conjugation (here, x is an arbitrary basepoint on σ).

Conversely, let $[\varphi] \in \mathrm{Out}(F_n)$ be an outer automorphism fixing the conjugacy class of the free splitting $F_n = \pi_1(N^1, x) * \pi_1(N^2, x)$. We can choose a representative φ which fixes the free splitting itself. Such an automorphism φ induces automorphisms of the groups $\pi_1(N^1, x)$ and $\pi_1(N^2, x)$. By the pointed version of Theorem 2.2 ([L74, Théorème 4.3, part 1)]), there are homeomorphisms f_i of N_i which induce $\varphi|_{\pi_1(N_i,x)}$ on the respective fundamental groups. By gluing f_1 and f_2 across σ we obtain a homeomorphism of M_n which fixes S and which induces $[\varphi]$ as desired. This shows that the stabilizer of σ maps onto the stabilizer of the conjugacy class of the free splitting $\pi_1(M_n, x) = \pi_1(N_1, x) * \pi_1(N_2, x)$.

Let now $F_n = G * H$ be an arbitrary free splitting, where G has rank i and H has rank $n-i$. Choose a sphere σ_i separating M_n into N^1 and N^2 as above, such that the rank of $\pi_1(N^1, x)$ is i (and thus the rank of $\pi_1(N^2, x)$ is $n-i$). Since the automorphism group of F_n acts transitively on the set of free splittings with fixed ranks, the last sentence of part *i)* follows from Theorem 2.2.

To prove part *ii)*, let σ be a non-separating sphere. Choose a basepoint $p \in M \setminus \sigma$. Then the subgroup $G < \pi_1(M, p) = F_n$ of all homotopy classes of loops which do not intersect σ is a free factor of corank one. Any diffeomorphism of M which preserves σ also preserves the conjugacy class of G. Therefore the stabilizer of σ in $\mathrm{Out}(F_n)$ injects into the stabilizer of the conjugacy class of G.

To show that it is equal to this stabilizer, let $\varphi \in \mathrm{Out}(F_n)$ be an outer automorphism which preserves the conjugacy class of G. We may choose a diffeomorphism f of M which fixes p and such that the induced isomorphism f_* of the fundamental group is contained in the conjugacy class defined by φ and fixes G.

Let $\sigma' = f(\sigma)$ be the image of σ under f. Since f_* preserves the group G, the subgroup of all homotopy classes of loops which do not intersect σ' is equal to G.

By Lemma 2.2 of [HV98] the group G is thus the subgroup of $\pi_1(M, p)$ defined by all homotopy classes of loops which do not intersect both σ and σ' simultaneously. We now argue by contradiction, supposing that σ and σ' are not homotopic.

Suppose first that σ' and σ are disjoint up to homotopy. Then the fundamental group of the complement of $\sigma \cup \sigma'$ has rank at most $n-2$. This is a contradiction since G has rank $n - 1$.

If σ and σ' intersect, we argue similarly. Namely, at least one connected component of $\sigma' \setminus \sigma$ is an open disk. Let D be the closure of this component in σ'. The surface D is a closed disk whose boundary curve ∂D is contained in σ. Let D' be a complementary component of ∂D on σ. The union $S = D \cup D'$ is an essential sphere which, up to homotopy, is disjoint from σ.

Furthermore, every loop which is disjoint from both σ and σ' is also disjoint from S'. Thus G can be identified with the subgroup of $\pi_1(M, p)$ of those loops which are disjoint from σ, σ' and S.

Since σ and S are disjoint, the fundamental group of the complement of $\sigma \cup S$ has rank at most $n - 2$. Since removing σ' as well decreases the rank of the fundamental group further, this again contradicts the fact that G has rank $n - 1$.

Thus, f preserves the homotopy class of σ. $\square$

References

[AS09] J. Aramayona, J. Souto, *Automorphisms of the graph of free splittings*, arXiv:0909.3660.

[FM11] B. Farb, D. Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, Princeton University Press, Princeton N.J. 2011.

[HH11] U. Hamenstädt, S. Hensel, *The geometry of the handlebody groups I: Distortion*, arXiv:1101.1838.

[HM10] M. Handel, L. Mosher, *Lipschitz retraction and distortion for subgroups of* $\mathrm{Out}(F_n)$, arXiv:1009.5018.

[Ha95] A. Hatcher, *Homological stability for automorphism groups of free groups*, Comm. Math. Helv. 70 (1995), 39–62.

[HV96] A. Hatcher, K. Vogtmann, *Isoperimetric inequalities for automorphism groups of free groups*, Pacific Journal of Mathematics, Vol. 173, No. 2, 1996.

[HV98] A. Hatcher, K. Vogtmann, *The Complex of Free Factors of a Free Group*, Quart. J. Math. Oxford Ser. (2) 49 (1998), no. 196, 459–468.

[M86] H. Masur, *Measured foliations and handlebodies*, Erg. Th. & Dynam. Sys. 6 (1986), 99–116.

[L74] F. Laudenbach, *Topologie de la dimension trois. Homotopie et isotopie*, Asterisque 12 (1974)

[MM00] H. Masur, Y. Minsky, *Geometry of the complex of curves II: Hierarchical structure*, Geom. Funct. Anal. 10 (2000), 902–974.

[St99] J. Stallings, *Whitehead graphs on handlebodies*, Geometric group theory down under (Canberra, 1996), 317–330, de Gruyter, Berlin, 1999.

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