HANDLEBODY BUNDLES AND POLYTOPES

SEBASTIAN HENSEL AND DAWID KIELAK

Abstract. We construct examples of fibered three-manifolds with fibered faces all of whose monodromies extend to a handlebody.

1. Introduction

Suppose that M is an orientable three-manifold which fibers over the circle with fiber a closed connected surface; let $\omega\colon \pi_1(M) \to \mathbb{Z}$ denote the induced homomorphism (we will say that ω is a *fibered class*). Thurston [Thu86] developed a theory which describes all possible ways in which M can fiber. Namely, he constructed a convex polytope P_M in $H^1(M;\mathbb{R})$ such that the fibered classes of M are exactly those integral classes in cones over certain "fibered" faces of P_M.

In particular, all the integral classes ω' in the cone C_F containing the class ω are also fibered. For each such ω', there is an associated monodromy in *some* mapping class group. It is an interesting, albeit extremely hard, problem to investigate common properties of these monodromies.

In this article we present a construction of three-manifolds in which all of these monodromies extend to handlebodies. Namely, we show:

Theorem 1.1. *There exists infinitely many pairwise non-diffeomorphic, closed three-manifolds M with the following property: the Thurston polytope P_M of M contains a fibered face F such that every integral class in the cone C_F over F is fibered, and its associated monodromy extends from the closed surface on which it is defined to a handlebody.*

The proof of this theorem relies on a connection of handlebody bundles to free-by-cyclic groups; the latter have recently been studied in analogy to fibered three-manifolds, see e.g. [DKL15, DKL17a, DKL17b, FK18, Kie]. Formally, Theorem 1.1 follows from Theorems 1.2 and 1.3 below.

To elucidate the connection, we need the following definition. We say that a class ω is *compatible with a handlebody bundle* if ω is induced by M fibering over the circle with monodromy a mapping class φ of some closed surface S_g, such that φ extends to a mapping class of a handlebody V_g. We say that ω is *fully compatible with a handlebody bundle* if in addition the inclusion map $M \hookrightarrow W$ induces an isomorphism $H_1(M;\mathbb{Z}) \cong H_1(W;\mathbb{Z})$, where W denotes the fibered four-manifold whose monodromy is the extension of φ to V_g.

Date: April 18, 2019.

The fundamental group of W is a free-by-cyclic group $\Gamma = \pi_1(W)$ fitting into the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(S_g) & \longrightarrow & \pi_1(M) & \xrightarrow{\omega} & \mathbb{Z} & \longrightarrow & 1 \\ & & \downarrow \iota & & \downarrow \hat{\iota} & & \downarrow = & & \\ 1 & \longrightarrow & \pi_1(V_g) & \longrightarrow & \Gamma & \xrightarrow{\omega_\Gamma} & \mathbb{Z} & \longrightarrow & 1 \end{array}$$

where ω_Γ is induced by ω, and where ι and $\hat{\iota}$ are epimorphisms induced by the embeddings $S_g \hookrightarrow V_g$ and $M \hookrightarrow W$.

In recent work, the second author [Kie] constructed a convex polytope P_Γ which serves as an analogue of the Thurston polytope P_M, classifying fiberings of Γ, i.e. maps $\Gamma \to \mathbb{Z}$ with finitely generated kernel.

With this terminology, our main result is:

Theorem 1.2. *Let M be a closed three-manifold, and let $\omega \in H^1(M;\mathbb{Z})$ be fully compatible with a handlebody bundle. If F denotes the fibered face whose cone C_F contains ω, then every integral class $\omega' \in C_F$ is fully compatible with some handlebody bundle.*

The condition that the inclusion $M \hookrightarrow W$ should induce an isomorphism on H_1 (required by the definition of full compatibility) is easy to check, and allows us the flexibility to prove the following application.

Theorem 1.3. *Suppose that Γ is a free-by-cyclic group. Then there are infinitely many pairwise non-diffeomorphic, hyperbolic three-manifolds admitting fibered classes fully compatible with handlebody bundles with fundamental group Γ.*

The above theorem gives a new way of associating mapping classes of surfaces to (outer) automorphisms of free groups; it works for *every* automorphism, but there are infinitely many different mapping classes associated to a single automorphism.

Acknowledgements. The authors would like to thank the organizers of the “Moduli Spaces” conference on Ventotene in 2017, where most of this work was conducted.

The second author was supported by the grant KI 1853/3-1 within the Priority Programme 2026 ‘Geometry at infinity’ of the German Science Foundation (DFG).

2. The Thurston polytope for three-manifolds and free-by-cyclic groups

Throughout, we will use the notation established in the introduction: M is a closed, connected and oriented three-manifold which fibres over the circle with fiber S_g, associated class $\omega \in H^1(M;\mathbb{Z})$ and monodromy φ.

A group will be called *free-by-cyclic* if it is an extension of a finitely generated free group by $\mathbb{Z}$. This is the case for the fundamental group of every handlebody bundle W with which ω is compatible.

Thurston [Thu86, Theorem 5] proved that all the different ways in which a given three-manifold can fiber over the circle are encoded by a polytope, in a way which we will make precise in a moment. The second author gave a new proof of Thurston's theorem in [Kie, Theorem 5.34], and then extended the result to cover also free-by-cyclic groups [Kie, Theorem 5.29] – in this latter setting, 'fibering over the circle' is interpreted to mean the existence of an epimorphism to $\mathbb{Z}$ with a finitely generated kernel.

Note that, thanks to a result of Stallings [Sta62], an integral cohomology 1-class $\omega\colon \pi_1(M) \to \mathbb{Z}$ of an irreducible three-manifold M is fibered if and only if $\ker\omega$ is finitely generated. Moreover, one can remove the assumption of M being irreducible thanks to Perelman's solution of the Poincaré conjecture. Hence, the group-theoretic notion of fibering used above coincides with the topological one for three-manifold groups.

It is important to note that if the kernel of $\omega\colon G \to \mathbb{Z}$ is finitely generated (that is, if ω is *fibered*), then $\ker\omega$ is in fact a surface group or a free group if G is a three-manifold group, and a free group if G is a free-by-cyclic group (the latter by [GMSW01]). In either case, the kernel has a well-defined Euler characteristic, denoted by $\chi(\ker\omega)$.

Before proceeding, let us state some definitions: a *polytope* in a finite-dimensional $\mathbb{R}$-vector space V denotes the intersection of finitely many half-spaces, and therefore a polytope P must be convex, but need not be compact. We will also require polytopes to be *symmetric*, that is preserved by the map $v \mapsto -v$. Given a face $F \neq \{0\}$ of a polytope, we define C_F to be the cone over that face; explicitly, we set

$$C_F = \{\lambda v \mid v \in F, \lambda \in (0,\infty)\}.$$

When $P = F = \{0\}$ we define $C_F = V$.

Theorem 2.1 ([Thu86, Kie])**.** *Suppose that G is a three-manifold group or a free-by-cyclic group. There exists a polytope P in $H^1(G;\mathbb{R})$ such that for every epimorphism $\omega\colon G \to \mathbb{Z}$ with a finitely generated kernel there exists a face F (the associated* fibered face*) of P with $\omega \in C_F$ such that*

(1) C_F is open, and
(2) every primitive integral class $\omega' \in H_1(G;\mathbb{Z})$ lying in C_F has a finitely generated kernel, and
(3) the map $\omega' \mapsto \chi(\ker\omega')$ defined on the primitive integral classes in C_F extends to a linear functional defined on the whole of C_F.

Proof. Let us start from the more classical case, in which G is a three-manifold group. The polytope P above is what is denoted by B_x in [Thu86], and is the unit ball of the Thurston norm $x\colon H^1(G;\mathbb{R}) \to [0,\infty)$. The Thurston norm $x(\omega')$ of a primitive cohomology class $\omega' \in H^1(G;\mathbb{Z})$ with finitely generated kernel is equal to $-\chi(\ker \omega')$ by definition. For convenience, we define $N = \frac{1}{2}x \colon H^1(G;\mathbb{R}) \to [0,\infty)$.

It is immediate that x, and hence N, are linear on C_F.

The facts that C_F is open and that every primitive integral class therein is fibered follow from [Thu86, Theorems 3 and 5].

Now suppose that G is a free-by-cyclic group. The starting point is the L^2-torsion polytope $P_{L^2} \subseteq H_1(G;\mathbb{R})$ appearing in [Kie, Theorem 5.29], and introduced first by Friedl–Lück [FL17]. The polytope P_{L^2} induces a *thickness function* $T \colon H^1(G;\mathbb{R}) \to [0,\infty)$ by setting

$$T(\omega') = \max_{p,q \in P_{L^2}} \left|\omega'(p) - \omega'(q)\right|$$

In fact, T is a semi-norm by [FK18, Corollary 3.5], and if $\ker \omega'$ is finitely generated and ω' is primitive, then

$$T(\omega') = -\chi(\ker \omega')$$

by the proof of [FK18, Theorem 4.4] (see also [HK, Theorem 6.2 and Remark 6.5]).

The polytope P is defined to be the unit ball of the semi-norm T. This immediately implies that T is linear on the cones of the faces of P.

Since $\ker \omega$ is finitely generated, we have ω and $-\omega$ lying in the (first) Bieri–Neumann–Strebel invariant $\Sigma^1(G)$ by [BNS87, Theorem B1], and therefore [Kie, Theorem 5.29] tells us that there are unique points p and $q \in P_{L^2}$ such that ω restricted to P_{L^2} attains its minimum at p and maximum at q. But this is an open condition, and therefore T is linear on a neighbourhood of ω. This implies that the cone C_F containing ω is open.

The cone C_F consists of precisely these cohomology classes which, when restricted to P_{L^2}, attain their minimum precisely at p and their maximum precisely at q. Therefore, every integral class in C_F is fibered by [Kie, Theorem 5.29]. □

3. All Fiberings are Handlebody

In this section we assume in addition to the assumptions of the last section that ω is fully compatible with a handlebody bundle W which fibers with fiber a handlebody V_g. We also let F denote the fibered face of P_M whose cone C_F contains ω. We set $\Gamma = \pi_1(W)$ as before.

As indicated in the introduction, we have the following diagram with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(S_g) & \longrightarrow & \pi_1(M) & \xrightarrow{\omega} & \mathbb{Z} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \hat\iota} & & \downarrow{\scriptstyle =} & & \\
1 & \longrightarrow & \pi_1(V_g) & \longrightarrow & \pi_1(W) & \xrightarrow{\omega_\Gamma} & \mathbb{Z} & \longrightarrow & 1
\end{array}$$

Here $\iota, \hat{\iota}$ are the maps induced by the inclusions of the boundary. Note that since ι is surjective, so is $\hat{\iota}$.

Recall that we are also assuming that the epimorphism $\hat{\iota}$ induces an isomorphism

$$\hat{\iota}_* : H_1(M;\mathbb{Z}) \to H_1(W;\mathbb{Z}).$$

Let F_k denote the free group of rank k. We need the following ingredient:

Proposition 3.1 (Co-rank theorem for surface groups)**.** *If $f\colon \pi_1(S_g) \to F_k$ is a surjective map, then $k \leqslant g$. In the case of equality, the map f is induced by the identification of S_g with the boundary of a genus g handlebody V_g. Furthermore, if in that case ψ is any mapping class of S_g which preserves* $\ker(f)$*, then ψ has an extension to V_g.*

Proof. The fact that $k \leqslant g$ is [LR02, Lemma 2.1], while the fact on the identification with a handlebody is [LR02, Lemma 2.2]. The fact that any mapping class of ∂V_g which preserves $\ker(\pi_1(\partial V_g) \to \pi_1(V_g))$ extends to the handlebody V is standard, see e.g. [Hen17, Corollary 5.11]. □

We are now ready to prove the main theorem.

Proof of Theorem 1.2. Let $\omega'\colon \pi_1(M) \to \mathbb{Z}$ be an epimorphism lying in the cone C_F. Since we are assuming that $H_1(M;\mathbb{Z}) \cong H_1(W;\mathbb{Z})$, there is an epimorphism ω'_Γ which makes the right square in the following diagram commute:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(S_h) & \longrightarrow & \pi_1(M) & \xrightarrow{\omega'} & \mathbb{Z} & \longrightarrow & 1 \\
 & & \big\downarrow f & & \big\downarrow \hat{\iota} & & \big\downarrow = & & \\
1 & \longrightarrow & \ker(\omega'_\Gamma) & \longrightarrow & \Gamma & \xrightarrow{\omega'_\Gamma} & \mathbb{Z} & \longrightarrow & 1
\end{array}$$

By a simple diagram chase, a homomorphism f which makes the left square commute exists, and is surjective. Therefore, $H = \ker(\omega'_W)$ is finitely generated. But $\Gamma = \pi_1(W)$ is a free-by-cyclic group, and hence H is a free group by [GMSW01].

We now claim that the rank of H is at least h. Suppose first that we have shown the claim. Now the co-rank theorem for surface groups (Proposition 3.1) tell us that the rank is exactly h. Let $x \in \ker f$, and let $z \in \pi_1(M)$ denote some preimage under ω' of a generator of $\mathbb{Z}$. We have

$$f(z^{-1}xz) = \hat{\iota}(z^{-1}xz) = \hat{\iota}(z^{-1})\hat{\iota}(x)\hat{\iota}(z) = \hat{\iota}(z^{-1})f(x)\hat{\iota}(z) = 1$$

and so $\ker f$ is preserved by the monodromy induced by ω' (whose action coincides with conjugation by z). The second part of the co-rank theorem now gives us a homeomorphism of the corresponding handlebody V_h with boundary S_h extending the monodromy induced by ω'.

We are left with proving the claim. For a contradiction, suppose that the rank of H is strictly smaller than h. Write $v = \omega_\Gamma - \omega'_\Gamma$; we then have

$\hat{\imath}^* v = \omega - \omega'$. Observe that by Theorem 2.1, there are nondegenerate linear functionals N (half of the Thurston norm) and T (the thickness function), such that

$$g - h = \frac{1}{2}\left(\chi(S_h) - \chi(S_g)\right) = N(\hat{\imath}^*(v))$$

and

$$\mathrm{rk}\left(\ker(\omega_\Gamma)\right) - \mathrm{rk}\left(\ker(\omega'_\Gamma)\right) = \chi\big(\ker(\omega'_\Gamma)\big) - \chi\big(\ker(\omega_\Gamma)\big) = T(v)$$

Since $g = \mathrm{rk}(\ker(\omega_\Gamma))$, and $\mathrm{rk}(\ker(\omega'_\Gamma)) < h$, this implies

$$N(\hat{\imath}^*(v)) < T(v).$$

Consider $\omega''_\Gamma = \omega_\Gamma + qv$ and $\omega'' = \omega + q\hat{\imath}^* v$ for a small rational number q. Since q is rational, the cohomology class ω'' is also rational, in the sense that $\omega'' \in H^1(M;\mathbb{Q})$. There exists a unique positive integer k such that $k\omega''$ is integral and primitive. Also, since q is small, $k\omega''$ lies in the cone of the same fibered face as ω, and hence is a fibered character. Arguing as before using Proposition 3.1 and [GMSW01], we thus have

$$T(\omega''_\Gamma) = \frac{-1}{k}\chi(\ker(k\omega''_\Gamma)) \leqslant \frac{-1}{2k}\chi(\ker(k\omega'')) = N(\omega'')$$

We also have

$$N(\omega'') = N(\omega) + qN(\hat{\imath}^* v) < T(\omega_\Gamma) + qT(v) = T(\omega''_\Gamma)$$

and so

$$T(\omega''_\Gamma) \leqslant N(\omega'') < T(\omega''_\Gamma),$$

which is a contradiction. We have therefore proven the claim. □

4. Existence of fully compatible fibered classes

In this section we show how to construct bundles with the assumption that $\hat{\imath}_*\colon H_1(M;\mathbb{Z}) \cong H_1(W;\mathbb{Z})$ compatible with any given free-by-cyclic group. More precisely, we will show the following, which is a rephrasing of Theorem 1.3.

Theorem 4.1. *Given any free group automorphism $f : F_g \to F_g$, there are mapping classes φ_i of the handlebody V_g such that*

i) φ_i induces the automorphism f on $\pi_1(V_g) = F_g$ for all i.
ii) The (four-manifolds) W_i obtained as the mapping tori of the mapping classes φ_i satisfy $\hat{\imath}_*\colon H_1(\partial W_i;\mathbb{Z}) \cong H_1(W_i;\mathbb{Z})$ for all i.
iii) The (three-manifolds) $M_i = \partial W_i$ are hyperbolic for all i and are pairwise non-diffeomorphic.

Before we can give the proof, we need some basic notation. Recall that if S_g is a closed surface of genus g, the algebraic intersection number defines a symplectic pairing

$$\sigma\colon H_1(S_g;\mathbb{Z}) \times H_1(S_g;\mathbb{Z}) \to \mathbb{Z}$$

on the first homology group. Suppose now that S_g has been identified with the boundary ∂V_g of a handlebody. Then, the inclusion of the boundary defines a map

$$\iota_*\colon H_1(S_g;\mathbb{Z}) \to H_1(V_g;\mathbb{Z})$$

whose kernel we denote by L. Explicitly, let $\alpha_1, \dots, \alpha_g$ be disjoint curves bounding disks which cut the handlebody V_g into a ball. Choose curves β_i with the property that $\sigma(\alpha_i, \beta_j)$ is 0 if $i \neq j$ and 1 otherwise. Then the homology classes a_i, b_j defined by the curves α_i, β_j, respectively, are a basis for $H_1(S_g;\mathbb{Z})$. We then have that

$$L = \ker(\iota_*) = \langle a_1, \dots, a_g \rangle.$$

Furthermore, if we define

$$D = \langle b_1, \dots, b_g \rangle,$$

then the restriction $\iota_*\colon D \to H_1(V_g;\mathbb{Z})$ is an isomorphism. Furthermore, σ vanishes identically on L and D. In other words, we have

$$H_1(S_g;\mathbb{Z}) = L \oplus D,$$

and both L, D are Lagrangian subspaces. With respect to this decomposition, σ corresponds to the matrix

$$J = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}.$$

Denote by $\mathcal{H}_g < \mathrm{Mcg}(S_g)$ the *handlebody group*, i.e. the subgroup of those mapping classes of S_g which extend to V_g. If ϕ is an element of the handlebody group, then $\phi_*(L) = L$. This gives the following obstruction for how the handlebody group acts on homology.

Lemma 4.2 (e.g. [Bir75, Lemma 2.2])**.** *For a symplectic basis as above, every handlebody group element ϕ acts on $H_1(S_g;\mathbb{Z})$ as a matrix of the form*

$$\phi_* = \begin{pmatrix} A & B \\ 0 & (A^t)^{-1} \end{pmatrix},$$

where A is invertible and B satisfies $B^t(A^t)^{-1} = A^{-1}B$. Conversely, any such matrix is realised as the action of a suitable handlebody group element ϕ.

We also need the following variant, which is likely well-known to experts.

Lemma 4.3. *For a basis of $H_1(S_g;\mathbb{Z})$ as above, every symplectic matrix of the form*

$$\begin{pmatrix} \mathrm{Id} & B \\ 0 & \mathrm{Id} \end{pmatrix}$$

can be realised as the homology action of a handlebody mapping class which acts trivially on the fundamental group of the handlebody.

Proof. The condition that the matrix is symplectic implies that B has to be symmetric. First, let i be given. The twist about α_i acts as the matrix

$$\begin{pmatrix} \mathrm{Id} & E_i \\ 0 & \mathrm{Id} \end{pmatrix}$$

where E_i is the matrix which is zero, except a single diagonal entry 1 in column i.

Next, let $i \neq j$ be given. Let δ be a diskbounding curve which intersects each of β_i, β_j in a single point, and defines the homology class $a_i + a_j$. The twist about δ acts as

$$\begin{pmatrix} \mathrm{Id} & E_i + E_j + E_{ij} + E_{ji} \\ 0 & \mathrm{Id} \end{pmatrix}$$

where E_{ij} is the elementary matrix with entry 1 in row i, column j. Since Dehn twists about diskbounding curves extend to the handlebody, and their extensions act trivially on the fundamental group of the handlebody, the lemma is proved for matrices of the form $B = E_i$ and $B = E_{ij} + E_{ji}$. Since these (additively) generate the group of symmetric matrices, the lemma follows. □

To certify that $\hat{\iota}_*\colon H_1(M_i;\mathbb{Z}) \cong H_1(W_i;\mathbb{Z})$ in the proof of Theorem 4.1, we will use the following criterion.

Lemma 4.4. *Suppose that ϕ is a handlebody group element, and let A, B be as in Lemma 4.2. If*

$$\mathrm{im}(A - \mathrm{Id}) + B(\ker((A^t)^{-1} - \mathrm{Id})) = L$$

then the handlebody bundle W obtained as the mapping torus of ϕ satisfies $\hat{\iota}_*\colon H_1(M;\mathbb{Z}) \cong H_1(W;\mathbb{Z})$ where $M = \partial W$.

Proof. We have

$$H_1(M;\mathbb{Z}) = \mathbb{Z} \oplus (H_1(S_g;\mathbb{Z}))/(\mathrm{Id} - \phi_*) = \mathbb{Z} \oplus (L \oplus D)/(\mathrm{Id} - \phi_*).$$

The assumption of the lemma implies that the natural map

$$\mathbb{Z} \oplus D/(\mathrm{Id} - (A^t)^{-1}) \to \mathbb{Z} \oplus (L \oplus D)/(\mathrm{Id} - \phi_*)$$

is surjective, and it is clearly injective. On the other hand, we have

$$H_1(W;\mathbb{Z}) = \mathbb{Z} \oplus \big(H_1(V_g;\mathbb{Z})/(\mathrm{Id} - \phi_*)\big) \cong \mathbb{Z} \oplus D/(\mathrm{Id}_D - (A^t)^{-1}),$$

which completes the proof. □

We are now ready to begin the proof of Theorem 4.1 in earnest.

Proof of Theorem 4.1. Let $f\colon F_g \to F_g$ be given. Up to replacing f by a conjugate, we may assume that f acts on homology as

$$f_* = \begin{pmatrix} \mathrm{Id} & U \\ 0 & V \end{pmatrix}$$

where V does not have any eigenvalue 1. This follows since $\mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$ is surjective, and integral matrices can be (integrally) conjugated to have this form.

Since the map $\mathcal{H}_g \to \mathrm{Out}(F_g)$ is also surjective (e.g. [Gri63]) and the claims in Theorem 4.1 are invariant under replacing (φ_i) by $(\psi\varphi_i\psi^{-1})$, it suffices to show the theorem under this assumption on f_*.

Now, take a handlebody group element ϕ which acts as f on $\pi_1(V)$. Let A be the matrix satisfying $(A^t)^{-1} = f_*$.

Lemma 4.5. *Under the assumptions above, there is a matrix B such that*

$$\mathrm{im}(A - \mathrm{Id}) + B(\ker((A^t)^{-1} - \mathrm{Id})) = L$$

and $B^t(A^t)^{-1} = A^{-1}B$.

Proof. Under the assumptions, we have

$$A^t = \begin{pmatrix} \mathrm{Id} & Y \\ 0 & Z \end{pmatrix}$$

where Z is a $k \times k$ matrix such that $Z - \mathrm{Id}$ is injective, and Y is a $(g-k) \times k$ matrix. We then have

$$A = \begin{pmatrix} \mathrm{Id} & 0 \\ Y^t & Z^t \end{pmatrix}.$$

Put

$$B = \begin{pmatrix} \mathrm{Id} & 0 \\ Y^t & 0 \end{pmatrix}.$$

Observe that $\ker((A^t)^{-1} - \mathrm{Id}) = \ker(A^t - \mathrm{Id})$, and therefore it is the subspace spanned by the first $g-k$ basis vectors. Hence, B satisfies

$$\mathrm{im}(A - \mathrm{Id}) + B(\ker((A^t)^{-1} - \mathrm{Id})) = L.$$

Furthermore, we have

$$AB^t = \begin{pmatrix} \mathrm{Id} & Y \\ Y^t & Y^tY \end{pmatrix} = BA^t.$$ □

Now, let B be a matrix as given by Lemma 4.5. Since $\mathcal{H}_g \to \mathrm{Out}(F_g)$ is surjective, we can find a handlebody group element ϕ mapping to f. It then acts on $H_1(S;\mathbb{Z})$ as

$$\begin{pmatrix} A & B' \\ 0 & (A^t)^{-1} \end{pmatrix},$$

since the lower right block describes the action on the first homology of the handlebody. Applying Lemma 4.3, we can therefore find a handlebody group element φ_0 which acts as

$$\begin{pmatrix} A & B \\ 0 & (A^t)^{-1} \end{pmatrix}.$$

By construction of B and Lemma 4.4, the mapping torus W_0 defined by φ_0 satisfies conditions i) and ii) of Theorem 4.1. Now let ψ be an element of the kernel of the map

$$\mathcal{H}_g \to \mathrm{Out}(F_g)$$

such that $\psi|_{\partial V_g}$ is pseudo-Anosov and such that ψ acts as the identity on $H_1(\partial V_g;\mathbb{Z})$. Such a mapping class can for example be constructed as the product of two Dehn twists about separating curves bounding disks.

Observe that for any n, the mapping tori defined by the elements $\psi^n\varphi_0$ then also satisfy i) and ii), since they act on $H_1(S;\mathbb{Z})$ in the same way as φ_0. On the other hand, for large n, the elements $\psi^n\varphi_0|_{\partial V_g}$ are pseudo-Anosov with diverging Weil-Petersson translation length. Thus the mapping tori defined by the boundary maps of $\psi^n\varphi_0$ are hyperbolic manifolds, and by the main theorem of [Bro03] their volumes diverge. By Mostow rigidity this implies in particular that there are infinitely many distinct diffeomorphism classes in the sequence.

This shows Theorem 4.1. □

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Sebastian Hensel
hensel@math.lmu.de
Mathematisches Institut
der Universität München
Theresienstraße 39
80333 München
Germany

Dawid Kielak
dkielak@math.uni-bielefeld.de
Fakultät für Mathematik
Universität Bielefeld
Postfach 100031
33501 Bielefeld
Germany