THE GEOMETRY OF THE HANDLEBODY GROUPS I: DISTORTION

URSULA HAMENSTÄDT AND SEBASTIAN HENSEL

Abstract. We show that the mapping class group of a handlebody V of genus at least 2 (with any number of marked points or spots) is exponentially distorted in the mapping class group of its boundary surface ∂V. The same holds true for solid tori V with at least two marked points or spots.

1. Introduction

A handlebody V_g of genus g is a 3-manifold bounded by a closed orientable surface $\partial V_g = S_g$ of genus g. Explicitly, V_g can be constructed by attaching g one-handles to a 3-ball. Handlebodies are basic building blocks for closed 3-manifolds, since any such manifold can be obtained by gluing two handlebodies along their boundaries.

The handlebody group $\mathrm{Map}(V_g)$ is the subgroup of the mapping class group $\mathrm{Map}(\partial V_g)$ of the boundary surface defined by isotopy classes of those orientation preserving homeomorphisms of ∂V_g which can be extended to homeomorphisms of V_g. It turns out that $\mathrm{Map}(V_g)$ can be identified with the group of orientation preserving homeomorphisms of V_g up to isotopy.

The handlebody group is a finitely presented subgroup of the mapping class group (compare [Wa98] and [S77]), and hence it can be equipped with a word norm. The goal of this article is to initiate an investigation of the coarse geometry of the handlebody group induced by this word norm.

The geometry of mapping class groups of surfaces is quite well understood. Therefore, understanding the geometry of the inclusion homomorphism $\mathrm{Map}(V_g) \to \mathrm{Map}(\partial V_g)$ may allow to deduce geometric properties of the handlebody group from geometric properties of the

Date: April 11, 2011.
AMS subject classification: 57M99, 20E36.
Both authors are partially supported by the Hausdorff Center Bonn and the Hausdorff Institut Bonn. The second author is supported by the Max-Planck Institut für Mathematik Bonn.

mapping class group. This task would be particularly easy if the handlebody group was undistorted in the ambient mapping class group (i.e. if the inclusion was a quasi-isometric embedding).

Many natural subgroups of the mapping class group are known to be undistorted. One example is given by groups generated by Dehn twists about disjoint curves (studied by Farb, Lubotzky and Minsky in [FLM01]) where undistortion can be proved by considering the subsurface projections onto annuli around the core curves of the Dehn twists.

Another example of undistorted subgroups are mapping class groups of subsurfaces (compare [MM00] or [H09a]). In this case, the proof of undistortion relies on the construction of quasi-geodesics in the mapping class group – either train track splitting sequences as in [H09a] or hierarchy paths defined by Masur and Minsky in [MM00].

Other important subgroups of the mapping class group are known to be distorted. As one example we mention the Torelli group, which is exponentially distorted by [BFP07]. A finitely generated subgroup H of a finitely generated group G is called exponentially distorted in G if the following holds. On the one hand, the word norm in H of every element $h \in H$ is coarsely bounded from above by an exponential of the word norm of h in G. On the other hand, there is a sequence of elements $h_i \in H$ such that the word norm of h_i in G grows linearly, while the word norm of h_i in H grows exponentially.

The argument from [BFP07] can be used to show exponential distortion for other normal subgroups of the mapping class group as well. Since the handlebody group is not normal, it cannot be used to analyze the handlebody group.

Answering a question raised in [BFP07], we show that nevertheless the same conclusion holds true for handlebody groups in almost all cases.

Theorem. *The handlebody group for genus $g \geq 2$ is exponentially distorted in the mapping class group.*

Our result is also valid for handlebodies with marked points or spots; allowing to lower the genus to 1 if there are at least two marked points or spots. In the case of genus 0 and the solid torus with one marked point the handlebody group is obviously undistorted and hence we obtain a complete classification of distorted handlebody groups.

Apart from the mapping class group, the handlebody group is naturally related to another important group. Namely, the action of $\mathrm{Map}(V_g)$ on the fundamental group of the handlebody defines a projection homomorphism onto the outer automorphism group $\mathrm{Out}(F_g)$ of a free group with g generators. However, by a theorem of McCullough [Mc85], the kernel of this projection homomorphism is infinitely generated and there are no known tools for transferring properties from $\mathrm{Out}(F_g)$ to the handlebody group.

A guiding question for future work is to compare the geometry of the handlebody group to both mapping class groups and outer automorphism groups of free groups. In particular, in a forthcoming article we shall identify quasi-geodesics in the handlebody group and use this description to shed more light on the geometric nature of the projection to $\mathrm{Out}(F_g)$ and the inclusion into the mapping class group.

The basic idea for the proof of the main theorem can be sketched in the special case of a solid torus $V_{1,2}$ with two marked points. The handlebody group of a solid torus with one marked point is infinite cyclic, generated by the Dehn twist T about the unique essential simple diskbounding curve. Since point-pushing maps are contained in the handlebody group, the Birman exact sequence yields that $\mathrm{Map}(V_{1,2})$ is equal to the fundamental group of the mapping torus of the once-punctured torus defined by T. The Dehn twist T acts on the fiber $\pi_1(T_{1,1}) = F_2$ of the Birman exact sequence as a Nielsen twist, therefore in particular as an element of linear growth type. This implies that the fiber is undistorted in the handlebody group. As this fiber is exponentially distorted in the mapping class group by [BFP07], the handlebody group of a torus with two marked points is at least exponentially distorted in the corresponding mapping class group.

In the general case, the argument is more involved since we have no explicit description of the handlebody group. However, the basic idea remains to show that parts of the fiber of some suitable Birman exact sequence are undistorted in the handlebody group.

The upper distortion bound uses a geometric model for the handlebody group. This model, the *graph of rigid racks*, is similar in spirit to the train track graph which was used in [H09a] to study the mapping class group. We construct a family of distinguished paths connecting any pair of points in this graph to each other. The length of these paths can be bounded using intersection numbers. The geometric control obtained this way allows to show the exponential upper bound on distortion.

The paper is organized as follows. In Section 2 we recall basic facts about handlebody groups of genus 0 and 1. Section 3 contains the lower distortion bound for handlebodies with at least one marked point or spot. In Section 4 we show the lower distortion bound for closed surfaces. Section 5 introduces a surgery procedure for disk systems which is important for the construction of paths in the handlebody group.

Section 6 is devoted to the construction of racks, and demonstrates some of their similarities (and differences) to train tracks on surfaces. Section 7 contains the construction of the geometric model for the handlebody group and a distinguished family of paths establishing the upper bound on distortion.

ACKNOWLEDGMENT: The authors thank Karen Vogtmann for useful discussions. The authors are also grateful to Lee Mosher for pointing out the reference [A02].

2. Low-complexity cases

As a first step, we analyze the cases of those genus 0 and 1 handlebody groups which turn out to be undistorted. The results in this section are easy and well-known, and we record them here for completeness.

To formulate the results in full generality, we need to introduce the notion of handlebodies with marked points and spots. A handlebody of genus g with k marked points and s spots $V_{g,k}^s$ is a handlebody of genus g, together with s pairwise disjoint disks $D_1, \dots, D_s$ on its boundary surface S_g, and k pairwise distinct points $p_1, \dots, p_k$ in $\partial V_g \setminus (D_1 \cup \dots \cup D_s)$.

The mapping class group $\mathrm{Map}(\partial V_{g,k}^s, p_1, \dots, p_k, D_1, \dots, D_s)$ of the boundary surface (with the same marked points and disks) consists of homeomorphisms of ∂V_g which fix the set $\{p_1, \dots, p_k\}$ and restrict to the identity on each of the D_i up to isotopy respecting the same data. Note that this group agrees with the mapping class group of the bordered surface obtained by removing the interior of the marked disks, as these mapping classes have to fix each boundary component (following the definition in [FM11, Section 2.1]). In the same way as for the case without marked points or spots, the handlebody group $\mathrm{Map}(V_{g,p}^s, p_1, \dots, p_k, D_1, \dots, D_s)$ is defined as the subgroup of those isotopy classes of homeomorphisms that extend to the interior of $V_{g,p}^s$.

All curves and disks are required not to meet any of the marked points. A simple closed curve on ∂V is *essential* if it is neither contractible nor freely homotopic to a marked point. A disk D in V is called *essential*, if $\partial D \subset \partial V$ is an essential simple closed curve.

Proposition 2.1. *Let $V = V_{0,k}^s$ be a handlebody of genus 0, with any number of marked points and spots. Then the handlebody group of V is equal to the mapping class group of its boundary.*

Proof. Let $f : S^2 \to S^2$ be any homeomorphism of the standard 2-sphere $S^2 \subset \mathbb{R}^3$ onto itself. We can explicitly construct a radial extension $F : D^3 \to D^3$ to the standard 3-ball $D^3 \subset \mathbb{R}^3$ by setting $F(t \cdot x) = t \cdot f(x)$ for $x \in S^2, t \in [0,1]$. Therefore every mapping class group element is contained in the handlebody group. □

In particular, the handlebody groups of genus 0 are undistorted in the corresponding mapping class groups. Similarly, for a solid torus with at most one marked point or spot, the handlebody group can be explicitly identified and turns out to be undistorted.

To this end, suppose V is a solid torus with at most one marked point ($V = V_{1,0}$ or $V = V_{1,1}$) or with one marked spot ($V = V_1^1$). Let δ be an essential simple closed curve on the boundary torus of V that bounds a disk in V. The curve δ is uniquely determined up to isotopy.

Proposition 2.2. *The handlebody group of V is the stabilizer of δ in the mapping class group. In particular, it is undistorted in the mapping class group.*

Thus, if $V = V_{1,0}$ or $V = V_{1,1}$, then the handlebody group is cyclic and generated by the Dehn twist about δ.

If $V = V_1^1$, then the handlebody group is the free abelian group of rank 2 which is generated by the Dehn twist about δ and the Dehn twist about the spot.

Proof. The handlebody group fixes the set of isotopy classes of essential disks in V. Since δ is the unique diskbounding curve up to isotopy, $\mathrm{Map}(V)$ therefore is contained in the stabilizer of δ. On the other hand, the disk bounded by δ cuts V into a spotted ball. Hence, by Proposition 2.1 the handlebody group $\mathrm{Map}(V)$ contains the stabilizer of δ.

If $V = V_{1,0}$ or $V_{1,1}$, the complement of δ in ∂V is an annulus (possibly with a puncture). From this, it is immediate that the handlebody group is generated by the Dehn twist about δ.

If $V = V_1^1$, the same argument shows that then the handlebody group is generated by the Dehn twist about δ and the spot. It is clear that these mapping classes commute.

Since stabilizers of simple closed curves are known to be undistorted subgroups of the mapping class group (compare [MM00] or [H09b]), the handlebody group of a solid torus with at most one spot or marked point is undistorted. □

3. Handlebodies with marked points

In this section we describe the lower bound for distortion of handlebody groups with marked points. We begin with the case of genus $g \geq 2$ with a single marked point. The case of several marked points or spots will be an easy consequence of this result. The case of a torus with several marked points requires a different argument which will be given at the end of this section.

Theorem 3.1. *Let $V = V_{g,1}$ be a handlebody of genus $g \geq 2$ with one marked point, and let $\partial V = S_{g,1}$ be its boundary surface. Then the handlebody group $\mathrm{Map}(V) < \mathrm{Map}(\partial V)$ is at least exponentially distorted.*

The proof is based on the relation between the mapping class group of a closed surface S_g and the mapping class group of a once-punctured surface $S_{g,1}$. We denote the marked point of $\partial V = S_{g,1}$ by p, and we will often denote the mapping class group of $S_{g,1}$ by $\mathrm{Map}(S_g, p)$.

Recall the definition of the *point-pushing map* $\mathcal{P} : \pi_1(S, p) \to \mathrm{Map}(S, p)$. Namely, let $\gamma : [0, 1] \to S$ be a loop in S based at p. Then there is an isotopy $f_t : S \to S$ supported in a small neighborhood of the loop $\gamma[0, 1]$ such that $f_0 = \mathrm{id}$, and $f_t(p) = \gamma(t)$. To see this, note that locally around $\gamma(t_0)$ such an isotopy certainly exists (for example, since any orientation preserving homeomorphism of the disk is isotopic to the identity). The image of γ is compact, and hence the desired isotopy can be pieced together from finitely many such local isotopies. The endpoint f_1 of such an isotopy is a homeomorphism of (S, p). We call its isotopy class the point pushing map $\mathcal{P}(\gamma)$ along γ. It depends only on the homotopy class of γ.

The image of the point pushing map is contained in the handlebody group $\mathrm{Map}(V, p)$ – to see this, simply define the local version by pushing a small half-ball instead of a disk.

By construction, the image of the point pushing map lies in the kernel of the forgetful homomorphism $\mathrm{Map}(S, p) \to \mathrm{Map}(S)$ induced by the puncture forgetting map $(S, p) \to (S, S)$. In fact this is all of the kernel, compare [Bi74].

Theorem 3.2 (Birman exact sequence)**.** *Let S be a closed oriented surface of genus $g \geq 2$ and $p \in S$ any point. The sequence*

$$1 \longrightarrow \pi_1(S, p) \xrightarrow{\mathcal{P}} \mathrm{Map}(S, p) \longrightarrow \mathrm{Map}(S) \longrightarrow 1$$

is exact.

The point pushing map is natural in the sense that

$$\mathcal{P}(f\alpha) = f \circ \mathcal{P}(\alpha) \circ f^{-1} \tag{1}$$

for each $f \in \mathrm{Map}(S, p)$ (see [Bi74] for a proof of this fact).

The Birman exact sequence corresponds to the relation between the inner and the outer automorphism group of $\pi_1(S, p)$:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(S,p) & \xrightarrow{\mathcal{P}} & \mathrm{Map}(S,p) & \longrightarrow & \mathrm{Map}(S) & \longrightarrow & 1 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
1 & \longrightarrow & \mathrm{Inn}(\pi_1(S,p)) & \longrightarrow & \mathrm{Aut}(\pi_1(S,p)) & \longrightarrow & \mathrm{Out}(\pi_1(S,p)) & \longrightarrow & 1
\end{array}$$

where $\pi_1(S, p)$ can be identified with its inner automorphism group because it has trivial center, and the other two isomorphisms are given by the Dehn-Nielsen-Baer theorem. In other words, we have the following.

Lemma 3.3. *Let $[\gamma], [\alpha] \in \pi_1(S, p)$ be two loops at p. Then*

$$\mathcal{P}(\alpha)(\gamma) = [\alpha] * [\gamma] * [\alpha]^{-1}$$

where $*$ denotes concatenation of loops, and takes place left-to-right.

Now we are ready to give the proof of the main theorem of this section.

Proof of Theorem 3.1. Let δ be a separating simple closed curve on S such that one component of $S \setminus \delta$ is a bordered torus T with one boundary circle, and such that δ bounds a disk $\mathcal{D}$ in the handlebody V. Without loss of generality we assume that the base point p lies on δ.

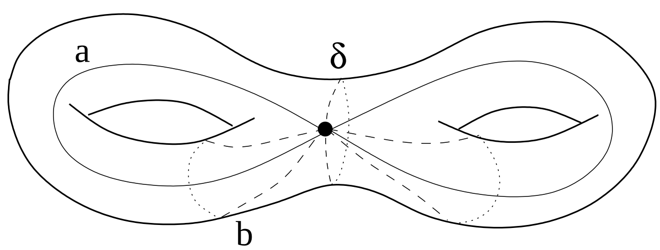


FIGURE 1. The setup in the proof of Theorem 3.1. Generators for the fundamental group of the handlebody are drawn solid, the loops extending these to a generating set of $\pi_1(S, p)$ are drawn dashed.

Choose loops a, b based at p which generate the fundamental group of T and such that b bounds a disk in V (and hence a does not). Extend

a, b to a generating set of the fundamental group of $\pi_1(S, p)$ by adding loops in the complement of T (see Figure 1). Let $f \in \mathrm{Map}(S, p)$ be a mapping class such that $f(a) = a^2 * b$ and $f(b) = a * b$ which preserves δ and acts as the identity on $S \setminus T$. Such an f can for example be obtained as the composition of suitably oriented Dehn twists along a and b.

Define $\Phi_k = \mathcal{P}(f^k a)$. By Equation (1), in the mapping class group $\mathrm{Map}(S, p)$ we have $\Phi_k = f^k \mathcal{P}(a) f^{-k}$, and hence the word norm of Φ_k in the mapping class group with respect to any generating set grows linearly in k.

On the other hand, consider the map

$$\mathrm{Map}(V, p) \xrightarrow{\pi} \mathrm{Aut}(\pi_1(V, p)) = \mathrm{Aut}(F_g)$$

defined by the action on the fundamental group. Lemma 3.3 implies that Φ_k acts on $\pi_1(S, p)$ as conjugation by $f^k(a)$. To compute the action of $\pi(\Phi_k)$ on $\pi_1(V, p)$, denote the projection of the fundamental group of the surface S to the fundamental group of the handlebody by $P : \pi_1(S, p) \to \pi_1(V, p)$.

Since b bounds a disk in V, its projection vanishes: $P(b) = 0$. The generator a of $\pi_1(S, p)$ projects to a primitive element in $\pi_1(V, p)$, $P(a) = A$. Hence $P(f^k(a)) = A^{N_k}$ for some $N_k > 0$. The choice of f guarantees that we have $N_k \geq 2^k$. Since the point pushing map is natural with respect to the projection to the handlebody, $\pi(\Phi_k)$ acts on $\pi_1(V, p)$ as conjugation by A^{N_k}.

In other words, as an element of $\mathrm{Aut}(F_g)$ the projection $\pi(\Phi_k)$ is the N_k–fold power of the conjugation by A. Since conjugation by A is an infinite order element in $\mathrm{Aut}(F_g)$ and all infinite order elements have positive translation length (compare [A02, Theorem 1.1]) this implies that the word norm of $\pi(\Phi_k)$ grows exponentially in k. As $\pi : \mathrm{Map}(V, p) \to \mathrm{Aut}(F_g)$ is a surjective homomorphism between finitely generated groups, it is Lipschitz with respect to any choice of word metrics. Therefore, the word norm of Φ_k in $\mathrm{Map}(V, p)$ also grows exponentially in k. This shows the theorem. □

Remark 3.4. The proof we gave extends verbatim to the case of the pure handlebody group of a handlebody of genus $g \geq 2$ with several marked points and any number of spots (just move everything but one marked point into the complement of T). Here, the pure handlebody group is the subgroup of those mapping classes which send each marked point to itself. Since this group has finite index in the full handlebody group, the proof also shows that handlebody groups with several marked points and any number of spots are at least exponentially distorted if the genus is at least 2.

As a next case, we consider handlebody groups of handlebodies with spots instead of marked points.

Corollary 3.5. *Let $V = V_g$ be a genus $g \geq 2$ handlebody and let $D \subset \partial V$ be a spot. Then the handlebody group* $\mathrm{Map}(V, D) < \mathrm{Map}(\partial V, D)$ *of the spotted handlebody is at least exponentially distorted.*

Proof. Note that there is a commutative diagram with surjective projection homomorphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \langle T \rangle & \longrightarrow & \mathrm{Map}(V, D) & \longrightarrow & \mathrm{Map}(V, p) & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \langle T \rangle & \longrightarrow & \mathrm{Map}(\partial V, D) & \longrightarrow & \mathrm{Map}(\partial V, p) & \longrightarrow & 0 \end{array}$$

induced by collapsing the marked spot to a point. The kernel of such a projection homomorphism is infinite cyclic and generated by the Dehn twist T about the spot. In particular, every element g in $\mathrm{Map}(\partial V, p)$ lifts to an element in $\mathrm{Map}(\partial V, D)$, and if $g \in \mathrm{Map}(V, p)$ then the lift is contained in the handlebody group $\mathrm{Map}(V, D)$. These lifts are well-defined up to the Dehn twist T which lies in the handlebody group and acts trivially on $\pi_1(V, p)$.

Choose any lift $\widetilde{f}$ of the element f used in the proof of Theorem 3.1. Let $\widetilde{\Phi}$ be a lift of the point pushing map Φ_0 defined in the proof of Theorem 3.1, and define $\widetilde{\Phi}_k = \widetilde{f}^k \widetilde{\Phi} \widetilde{f}^{-k}$. Note that these elements are lifts of the elements Φ_k and therefore contained in the handlebody group.

Now $\widetilde{\Phi}_k$ has word norm in $\mathrm{Map}(S, D)$ again bounded linearly in k. As elements of the spotted handlebody group the word norm of $\widetilde{\Phi}_k$ grows exponentially in k, as this is true for the Φ_k. □

Remark 3.6. Again, the same proof works for handlebodies with more than one spot and any number of marked points.

As a last case, we consider the handlebody of a torus with more than one marked point.

Theorem 3.7. *Let $V = V_{1,n}$ be a solid torus with $n \geq 2$ marked points. Then the handlebody group* $\mathrm{Map}(V)$ *is at least exponentially distorted in* $\mathrm{Map}(\partial V)$.

Proof. The strategy of this proof is similar to the preceding ones. We consider the Birman exact sequence for pure mapping class groups and pure handlebody groups.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1\mathcal{C}_n & \xrightarrow{\mathcal{P}} & P\mathrm{Map}(\partial V, p_0, p_1, \dots, p_n) & \longrightarrow & \mathrm{Map}(\partial V, p_0) & \longrightarrow & 1 \\
 & & \uparrow = & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \pi_1\mathcal{C}_n & \longrightarrow & P\mathrm{Map}(V, p_0, p_1, \dots, p_n) & \longrightarrow & \mathbb{Z} = \langle T \rangle & \longrightarrow & 1
\end{array}$$

where $\mathcal{C}_n$ denotes the configuration space of n points in $\partial V \setminus \{p_0\}$, and T the Dehn twist along the (unique) disk δ on $\partial V \setminus \{p_0\}$. An element of $\pi_1\mathcal{C}_n$ can be viewed as an n-tuple of parametrized loops γ_i, where γ_i is based at p_i (subject to the condition that at each point in time, the values of all these loops are distinct). Note that the pure mapping class group $P\mathrm{Map}(\partial V, p_0, p_1, \dots, p_n)$ acts on $\mathcal{C}_n$ by acting on all component loops. The map $\mathcal{P}$ is the generalized point pushing map, pushing all marked points simultaneously along the loops γ_i. The map $\mathcal{P}$ is natural with respect to the action of $P\mathrm{Map}(\partial V, p_0, p_1, \dots, p_n)$ in the sense that $\mathcal{P}(f\gamma) = f \circ \mathcal{P}(\gamma) \circ f^{-1}$.

Every element of $P\mathrm{Map}(V, p_0, p_1, \dots, p_n)$ can be written in the form $\mathcal{P}(\gamma) \cdot \widetilde{T}^l$, where γ denotes an n-tuple of loops, and $\widetilde{T}$ is some (fixed) lift of the Dehn twist T. In this description, the multiplicity l and the homotopy class of the n-tuple of loops γ is well-defined. Now note that

$$\left(\mathcal{P}(\gamma) \cdot \widetilde{T}^l\right) \cdot \left(\mathcal{P}(\gamma') \cdot \widetilde{T}^{l'}\right) = \mathcal{P}(\gamma) \cdot \mathcal{P}\left(\widetilde{T}^{l'}(\gamma')\right) \widetilde{T}^{l+l'} \tag{2}$$

$$= \mathcal{P}\left(\widetilde{T}^{l'}(\gamma') * \gamma\right) \widetilde{T}^{l+l'}$$

by the naturality of $\mathcal{P}$ and the fact that $\mathcal{P}$ is a homomorphism (note that concatenation of loops is executed left-to-right, while composition of maps is done right-to-left).

Choose an element $\beta \in \pi_1(\partial V, p_0)$ which extends δ to a basis of $\pi_1(\partial V, p_0) = F_2$. Note that then β is a generator of the fundamental group $\pi_1(V, p_0) = \mathbb{Z}$ of the solid torus V_1. We also choose loops $\beta_i \in \pi_1(\partial V, p_i)$ for all $i = 1, \dots, n$ which are freely homotopic to β. These loops give an identification of $\pi_1(V, p_i)$ with $\mathbb{Z}$.

Define a map $b : P\mathrm{Map}(V, p_0, p_1, \dots, p_n) \to \mathbb{Z}$ as follows. Let $\varphi = \mathcal{P}(\gamma) \cdot \widetilde{T}^l$ be any element of the pure handlebody group. Each component loop γ_i of γ defines a loop in $\pi_1(V, p_i)$ (which might be trivial). This loop is homotopic to the k_i-th power of β_i for some number k_i. Associate to φ the sum of the k_i.

Now choose any generating set $\gamma^1, \dots, \gamma^N$ of $\pi_1\mathcal{C}_n$. Then the pure handlebody group $P\mathrm{Map}(V, p_0, p_1, \dots, p_n)$ is generated by $\mathcal{P}(\gamma^j)$ and

$\widetilde{T}$. We claim that there is a constant k_0, such that

$$b\left(\varphi \cdot \mathcal{P}(\gamma^i)\right) \geq b(\varphi) - k_0 \tag{3}$$

Namely, by equation (2), we have to compare the projections of the components of

$$\gamma \qquad \text{and} \qquad \widetilde{T}^l(\gamma^j) * \gamma$$

to each of the $\pi_1(V, p_i)$. However, applying $\widetilde{T}$ does not change this projection. Since γ^j is one of finitely many generators, there is a maximal number of occurrences of the projection of β_i which can be canceled by adding the projection of γ^j. This shows inequality (3).

Now we can finish the proof using a similar argument as in the proof of Theorem 3.1. Namely, choose again f a pseudo-Anosov element with the property that applying f multiplies the number of occurrences of β_i by 2 in all $\pi_1(\partial V, p_i)$. Then $\mathcal{P}(f^k\beta)$ has length growing linearly in the mapping class group, while $b(f^k\beta)$ grows exponentially. By inequality (3) this implies that the word norm in the pure handlebody group also grows exponentially. Since the pure handlebody group has finite index in the full handlebody group the theorem follows. □

Remark 3.8. The same argument that extends Theorem 3.1 to Corollary 3.5 applies in this case and shows that also all torus handlebody groups with at least two spots or marked points are exponentially distorted.

4. Handlebodies without marked points

In this section we complete the proof of the exponential lower bound on the distortion of the handlebody groups by showing that the handlebody group of a handlebody of genus $g \geq 2$ without marked points or spots is distorted in the mapping class group.

For genus $g \geq 3$, the idea is to replace the point pushing used in the proofs above by pushing a subsurface around the handlebody. The resulting handlebody group element does not induce a conjugation on $\pi_1(V, p)$, but instead induces a partial conjugation on the fundamental group of the complement of the pushed subsurface. Since $g \geq 3$, such an element projects to a nontrivial element in the outer automorphism group of F_g. Then a similar reasoning as in Section 3 applies. The case of genus 2 requires a different argument and will be given at the end of this section.

Theorem 4.1. *For a handlebody $V = V_g$ of genus $g \geq 3$, the handlebody group* $\mathrm{Map}(V)$ *is at least exponentially distorted in the mapping class group* $\mathrm{Map}(\partial V)$.

Proof. Choose a curve δ which bounds a disk $\mathcal{D}$, such that $V \setminus \mathcal{D}$ is the union of a once-spotted genus 2 handlebody V_1 and a once-spotted genus $g-2$ handlebody V_2. Denote the boundary of V_i by S_i, and choose a basepoint $p \in \delta$. This defines a free decomposition of the fundamental group of the handlebody

$$F_g = \pi_1(V,p) = \pi_1(V_1,p) * \pi_1(V_2,p) = F_2 * F_{g-2}.$$

We denote by $\mathrm{Map}(S_i,\delta)$ the mapping class group of the bordered surface S_i, emphasizing that each such mapping class has to fix δ pointwise. The stabilizer of δ in the mapping class group of S is of the form

$$G_S = \mathrm{Map}(S_1,\delta) \times \mathrm{Map}(S_2,\delta)/\sim$$

where the equivalence relation $\sim$ identifies the Dehn twist about δ in $\mathrm{Map}(S_1,\delta)$ and $\mathrm{Map}(S_2,\delta)$. Note that the Dehn twist about δ lies in the handlebody group and acts trivially on $\pi_1(V,p)$. Therefore, the stabilizer of δ in the handlebody group is of the form

$$G_V = \mathrm{Map}(V_1,\mathcal{D}) \times \mathrm{Map}(V_2,\mathcal{D})/\sim.$$

In particular, the handlebody group $\mathrm{Map}(V_1,\mathcal{D})$ injects into G_V. There is a homomorphism $G_V \to \mathrm{Aut}(F_2) \times \mathrm{Aut}(F_{g-2})$ induced by the actions of $\mathrm{Map}(V_i,p)$ on $\pi_1(V_i,p)$. This homomorphism is natural with respect to the inclusion $\mathrm{Aut}(F_2) \times \mathrm{Aut}(F_{g-2}) \to \mathrm{Aut}(F_g)$ defined by the free decomposition of $\pi_1(V,p)$ given above. It is also natural with respect to the inclusion $\mathrm{Aut}(F_2) \to \mathrm{Aut}(F_2) \times \mathrm{Aut}(F_{g-2})$ defined by $\mathrm{Map}(V_1,\mathcal{D}) \to G_V$. Summarizing, we have the following commutative diagram.

$$\begin{array}{ccccccc}
\mathrm{Map}(S_1,\delta) & \longrightarrow & G_S & \longrightarrow & \mathrm{Map}(S,p) & \longrightarrow & \mathrm{Map}(S) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{Map}(V_1,\mathcal{D}) & \longrightarrow & G_V & \longrightarrow & \mathrm{Map}(V,p) & \longrightarrow & \mathrm{Map}(V) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Aut}(F_2) & \longrightarrow & \mathrm{Aut}(F_2) \times \mathrm{Aut}(F_{g-2}) & \longrightarrow & \mathrm{Aut}(F_g) & \longrightarrow & \mathrm{Out}(F_g)
\end{array}$$

Let $\widetilde{\Phi}_k \in \mathrm{Map}(V_1,\mathcal{D})$ be the elements constructed in the proof of Corollary 3.5. The image of $\widetilde{\Phi}_k$ in $\mathrm{Aut}(F_2) \times \mathrm{Aut}(F_{g-2})$ is the N_k–th power of a conjugation in the free factor F_2 defined by V_1, and the identity on the free factor F_{g-2} defined by V_2, where $N_k \geq 2^k$. In other words, this projection is a N_k–th iterate of a partial conjugation. Therefore, it projects to a nontrivial element of infinite order in $\mathrm{Out}(F_g)$. From there, one can finish the proof using the argument in the proof of Theorem 3.1. □

The last case is that of a genus 2 handlebody without marked points or spots. In this case, the strategy is to use the distortion of the handlebody group of a solid torus with two spots to produce distorted elements in the stabilizer of a nonseparating disk in the genus 2 handlebody.

To make this precise, we use the following construction. Let V be a genus 2 handlebody and S its boundary surface. Choose a nonseparating essential simple closed curve δ that bounds a disk $\mathcal{D}$ in V. Cutting S at δ yields a torus S_1^2 with two boundary components δ_1 and δ_2. Choose once and for all a continuous map $S_1^2 \to S$ which maps both δ_1 and δ_2 to δ and which restricts to a homeomorphism

$$S_1^2 \setminus (\delta_1 \cup \delta_2) \to S \setminus \delta.$$

The isotopy class of such a map depends on choices, but we fix one such map for the rest of this section. This map induces induces a homomorphism

$$\mathrm{Map}(S_1^2) \to \mathrm{Stab}_{\mathrm{Map}(S)}(\delta)$$

since the homeomorphisms and isotopies used to define the mapping class group $\mathrm{Map}(S_1^2)$ of the torus S_1^2 have to fix δ_1 and δ_2 pointwise and therefore extend to S.

Since δ bounds a disk, an analogous construction works for the handlebody groups, and we obtain

$$\mathrm{Map}(V_1^2) \to \mathrm{Stab}_{\mathrm{Map}(V)}(\mathcal{D}).$$

Let $p \in \delta$ be a base point, and let a, b be smooth embedded loops in S with the following properties (compare Figure 2).

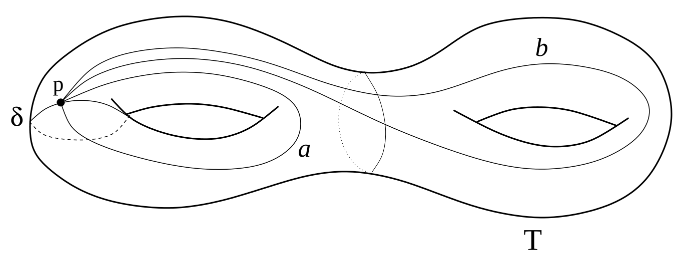


FIGURE 2. The setting for a genus 2 handlebody.

i) The projections A and B of a and b to $\pi_1(V, p)$ form a free basis of $\pi_1(V, p) = F_2$.
ii) The loops a and b intersect δ exactly in the basepoint p.
iii) The loop a hits δ from different sides at its endpoints, while b returns to the same side.

On the surface S_1^2 obtained by cutting S at δ, the loop a defines an arc from one boundary component to the other, while b defines a loop. By slight abuse of notation we will denote these objects by the same symbols. We choose the initial point of the loop b as base point of this cut-open surface, and call it again p. Then the projection B of b to the spotted solid torus V_1^2 is a generator of its fundamental group $\pi_1(V_1^2, p) = \mathbb{Z}$.

Now consider the torus $T' \subset S$ with one boundary component obtained as the tubular neighborhood of $a \cup \delta$ in S (compare Figure 2 for the situation). The complement of T' in S again is a torus with one boundary component which we denote by T. Choose a reducible homeomorphism f of V_1^2 which preserves T and restricts to a pseudo-Anosov homeomorphism f on the torus $T \subset S$ with the property that the projection of the loop $f^k(b)$ to $\pi_1(V_1^2)$ is B^{N_k}, for $N_k \geq 2^k$. Such an element can be constructed explicitly as in the proof of Theorem 3.5. In particular, we may assume that f fixes the arc a pointwise.

Consider now as in the proof of Theorem 3.5 the map that collapses the boundary components of V_1^2 to marked points. On this solid torus $V_{1,2}$ with two marked points, a defines an arc from marked point to marked point, and b defines a based loop at one of the marked points which we again use as base point for this surface. Let $P = \mathcal{P}(b)$ be the point pushing map on $V_{1,2}$ defined by b, and let $\widetilde{P}$ be any lift of this point-pushing map to the surface S_1^2 with boundary. As before, $\widetilde{P}$ is an element of the handlebody group. We define

$$\Phi_k = f^k \circ \widetilde{P} \circ f^{-k}.$$

Lemma 4.2. Φ_k *is an element of the handlebody group of* V_1^2. $\Phi_k(B)$ *is homotopic to* B *as a loop based at* p *in the handlebody* V_1^2*, and* $\Phi_k(A)$ *is homotopic, as an arc relative to its endpoints, to* $A * B^{N_k}$ *in* V_1^2.

Proof. Φ_k projects to the point-pushing map along $f^k(b)$ on the solid torus with two marked points $V_{1,2}$ obtained by collapsing the boundary components of V_1^2. Hence, Φ_k is the lift of a handlebody group element and therefore lies in the handlebody group itself (see the discussion in the proof of Theorem 3.5). This yields the first claim.

To see the other claims, we can work in the solid torus $V_{1,2}$ with two marked points, as the projection from V_1^2 to $V_{1,2}$ that collapses the spots to marked points induces a isomorphism on fundamental groups.

Here by construction Φ_k projects to the point-pushing map along $f^k(b)$. Lemma 3.3 now implies that this projection acts as conjugation by B^{N_k} on the fundamental group, giving the second claim.

By construction of f, the arc a and the loop $b_k = f^k(b)$ only intersect at the base point. The loop b_k is a simple curve and thus there is an embedded tubular neighborhood of b_k on $V_{1,2}$ which is orientation preserving homeomorphic to $[0,1]/(0 \sim 1) \times [-1,1] = S^1 \times [-1,1]$ and such that $S^1 \times \{0\}$ is the loop b_k. After perhaps reversing the orientation of b_k and performing an isotopy, we may assume that the intersection of a with this tubular neighborhood equals $\{0\} \times [-1,0]$.

Since b_k is simple, the point pushing map along b_k is isotopic to the map supported on the tubular neighborhood which is defined by

$$(x,t) \mapsto (x+(t+1),t) \quad \text{for} \quad t \in [-1,0]$$
$$(x,t) \mapsto (x-t,t) \quad \text{for} \quad t \in [0,1]$$

This implies that the point pushing map acts on the homotopy class of a by concatenating a with the loop $f^k(b)$ (up to possibly changing the orientation of a). Since $f^k(b)$ projects to B^{N_k} in the handlebody, this implies the last claim of the lemma. □

Theorem 4.3. *The handlebody group of a genus* 2 *handlebody is at least exponentially distorted.*

Proof. We use the notation from the construction described above. Consider the image Ψ_k of Φ_k in the stabilizer of $\mathcal{D}$ in the handlebody group $\mathrm{Map}(V_2)$. By construction, Ψ_k fixes the curve δ pointwise and therefore acts on $\pi_1(V,p)$. By the preceding lemma, this action is given by

$$\begin{aligned} A &\mapsto A * B^{N_k} \\ B &\mapsto B \end{aligned}$$

Therefore, Ψ_k acts as the N_k-th power of a simple Nielsen twist on F_2. In particular, it projects to the N_k-th power of a nontrivial element in $\mathrm{Out}(F_2)$. From here, one can finish the proof as for the preceding distortion theorems. □

5. Disk exchanges and surgery paths

In this section we study disk systems in handlebodies and introduce certain types of surgery operations for disk systems. These surgery operations form the basis for the construction of distinguished paths in the handlebody group (see Lemma 7.8).

In the sequel we always consider a handlebody V of genus $g \geq 2$ with a finite number m of marked points on its boundary ∂V. The discussion remains valid if some of the marked points are replaced by spots.

Definition 5.1. A *disk system* for V is a set of essential disks in V which are pairwise disjoint and non-homotopic. A disk system is called *simple* if all of its complementary components are simply connected. It is called *reduced* if it is simple and has a single complementary component.

We usually consider disk systems only up to isotopy. For a handlebody of genus g, a reduced disk system consists of precisely g non-separating disks. The complement of a reduced disk system in V is a ball with $2g$ spots (and possibly some marked points). The boundary of a reduced disk system is a multicurve in ∂V with g components which cuts ∂V into a $2g$-holed sphere (with some number of marked points). The handlebody group acts transitively on the set of isotopy classes of reduced disk systems.

We say that two disk systems $\mathcal{D}_1, \mathcal{D}_2$ are in *minimal position* if their boundary multicurves intersect in the minimal number of points and if every component of $\mathcal{D}_1 \cap \mathcal{D}_2$ is an embedded arc in $\mathcal{D}_1 \cap \mathcal{D}_2$ with endpoints in $\partial\mathcal{D}_1 \cap \partial\mathcal{D}_2$. Disk systems can always be put in minimal position by applying suitable isotopies. In the sequel we always assume that disk systems are in minimal position.

Note that the minimal position of disks behaves differently than the normal position of sphere systems as defined in [Ha95]. Explicitly, let Σ be a reduced disk system and D an arbitrary disk. Suppose D is in minimal position with respect to Σ. Then a component of $D \setminus \Sigma$ may have several boundary components on the same side of a disk in Σ. In addition, the collection of components of $D \setminus \Sigma$ does not determine the disk D uniquely.

Let $\mathcal{D}$ be a disk system. An *arc relative to* $\mathcal{D}$ is a continuous embedding $\rho : [0, 1] \to \partial V$ such that its endpoints $\rho(0)$ and $\rho(1)$ are contained in $\partial\mathcal{D}$. An arc ρ is called *essential* if it cannot be homotoped into $\partial\mathcal{D}$ with fixed endpoints and if the number of intersections of ρ with $\partial\mathcal{D}$ is minimal in its isotopy class.

Choose an orientation of the curves in $\partial\mathcal{D}$. Since ∂V is oriented, this choice determines a left and a right side of a component α of $\partial\mathcal{D}$ in a small annular neighborhood of α in ∂V. We then say that an endpoint $\rho(0)$ (or $\rho(1)$) of an arc ρ *lies to the right (or to the left) of* α, if a small neighborhood $\rho([0, \epsilon])$ (or $\rho([1 - \epsilon, 1])$) of this endpoint is contained in the right (or left) side of α in a small annulus around α. A *returning arc relative to* $\mathcal{D}$ is an arc both of whose endpoints lie on the same side of some boundary ∂D of a disk D in $\mathcal{D}$, and whose interior is disjoint from $\partial\mathcal{D}$.

Let E be a disk which is not disjoint from $\mathcal{D}$. An *outermost arc* of ∂E relative to $\mathcal{D}$ is a returning arc ρ relative to $\mathcal{D}$ such that there is a component E' of $E \setminus \mathcal{D}$ whose boundary is composed of ρ and an arc $\beta \subset D$. The interior of β is contained in the interior of D. We call such a disk E' an *outermost component* of $E \setminus \mathcal{D}$.

For every disk E which is not disjoint from $\mathcal{D}$ there are at least two distinct outermost components E', E'' of $E \setminus \mathcal{D}$. Every outermost arc of a disk is a returning arc. However, there may also be components of $\partial E \setminus \mathcal{D}$ which are returning arcs, but not outermost arcs. For example, a component of $E \setminus \mathcal{D}$ may be a rectangle bounded by two arcs contained in $\mathcal{D}$ and two subarcs of ∂E with endpoints on $\partial \mathcal{D}$ which are homotopic to a returning arc relative to $\partial \mathcal{D}$.

Let now $\mathcal{D}$ be a simple disk system and let ρ be a returning arc whose endpoints are contained in the boundary of some disk $D \in \mathcal{D}$. Then $\partial D \setminus \{\rho(0), \rho(1)\}$ is the union of two (open) intervals γ_1 and γ_2. Put $\alpha_i = \gamma_i \cup \rho$. Up to isotopy, α_1 and α_2 are simple closed curves which are disjoint from $\mathcal{D}$ (compare [St99] and [M86] for this construction). Therefore both α_1 and α_2 bound disks in the handlebody which we denote by Q_1 and Q_2. We say that Q_1 and Q_2 are obtained from D by *simple surgery along the returning arc* ρ.

The following observation is well-known (compare [M86, Lemma 3.2], or [St99]).

Lemma 5.2. *If Σ is a reduced disk system and ρ is a returning arc with endpoints on $D \in \Sigma$, then for exactly one choice of the disks Q_1, Q_2 defined as above, say the disk Q_1, the disk system obtained from Σ by replacing D by Q_1 is reduced.*

Proof. A reduced disk system equipped with an orientation defines a basis over $\mathbb{Z}$ for the relative homology group $H_2(V, \partial V; \mathbb{Z}) = \mathbb{Z}^n$. The homology class of the oriented disk D is the sum of the homology classes of the suitably oriented disks Q_1 and Q_2. Since D is a generator of $H_2(V, \partial V; \mathbb{Z})$, there is exactly one of the disks Q_1, Q_2, say the disk Q_1, so that the disk system $\mathcal{D}'$ obtained from $\mathcal{D}$ by replacing D by Q_1 defines a basis for $H_2(V, \partial V; \mathbb{Z})$. Then this disk system is reduced. $\square$

Note that the disk Q_1 is characterized by the requirement that the two spots in the boundary of $V \setminus \Sigma$ corresponding to the two copies of D are contained in distinct connected components of $V \setminus (\Sigma \cup Q_1)$. It only depends on Σ and the returning arc ρ.

Definition 5.3. Let Σ be a reduced disk system. A *disk exchange move* is the replacement of a disk $D \in \Sigma$ by a disk D' which is disjoint from Σ and such that $(\Sigma \setminus D) \cup D'$ is a reduced disk system. If D'

is determined as in Lemma 5.2 by a returning arc of a disk in a disk system $\mathcal{D}$ then the modification is called a *disk exchange move of* Σ *in direction of* $\mathcal{D}$ or simply a *directed disk exchange move*.

A sequence (Σ_i) of reduced disk systems is called a *disk exchange sequence in direction of* $\mathcal{D}$ (or *directed disk exchange sequence*) if each Σ_{i+1} is obtained from Σ_i by a disk exchange move in direction of $\mathcal{D}$.

Lemma 5.4. *Let* Σ_1 *be a reduced disk system and let* $\mathcal{D}$ *be any other disk system. Then there is a disk exchange sequence* $\Sigma_1, \dots, \Sigma_n$ *in direction of* $\mathcal{D}$ *such that* Σ_n *is disjoint from* $\mathcal{D}$.

Proof. We define the sequence Σ_i inductively. Suppose Σ_i is already defined and not yet disjoint from $\mathcal{D}$. Then there is a outermost arc ρ of $\mathcal{D}$ with respect to Σ_i. By Lemma 5.2, there is a disk system Σ_{i+1} obtained by a disk exchange move along this returning arc. As a result of this surgery, the geometric intersection number between Σ_{i+1} and $\mathcal{D}$ is strictly smaller than the geometric intersection number between Σ_i and $\mathcal{D}$. Now the lemma follows by induction on the geometric intersection number between $\partial\Sigma_1$ and $\partial\mathcal{D}$. □

6. Racks

In this section we define and investigate combinatorial objects which serve as analogs of train tracks for handlebodies. Let again V be a handlebody of genus $g \geq 2$, perhaps with marked points on the boundary.

Definition 6.1. A *rack* R in V is given by a reduced disk system $\Sigma(R)$, called the *support system* of the rack R, and a collection of pairwise disjoint essential embedded arcs in $\partial V \setminus \partial\Sigma(R)$ with endpoints on $\partial\Sigma(R)$, called *ropes*, which are pairwise non-homotopic relative to $\partial\Sigma(R)$. At each side of a support disk $D \in \Sigma(R)$, there is at least one rope which ends at the disk and approaches the disk from this side.

A rack R is called *large*, if the union of $\partial\Sigma(R)$ and the set of ropes decompose ∂V into disks.

Note that the number of ropes of a rack is uniformly bounded. In the sequel we often consider isotopy classes of racks.

Explicitly, we say that two racks R, R' are *(weakly) isotopic* if their support systems $\Sigma(R), \Sigma(R')$ are isotopic and if after an identification of $\Sigma(R)$ with $\Sigma(R')$, each rope of R is freely homotopic relative to $\partial\Sigma(R)$ to a rope of R'. In Section 7 we will introduce a more restrictive notion of equivalence of racks.

The handlebody group $\mathrm{Map}(V)$ acts transitively on the set of reduced disk systems, and it acts on the set of weak isotopy classes of racks. For every reduced disk system Σ the stabilizer of $\partial\Sigma$ in $\mathrm{Mod}(\partial V)$ is contained in Map(V) (compare Proposition 2.1). This implies that there are only finitely many orbits for the action of Map(V) on the set of weak isotopy classes of racks. The stabilizer in Map(V) of a weak isotopy class of a rack R with support system $\Sigma(R)$ contains the group $\mathbb{Z}^n$ of Dehn twists about the components of $\partial\Sigma(R)$. In particular, this stabilizer is infinite.

Definition 6.2. (1) A disk system $\mathcal{D}$ (or an arbitrary geodesic lamination λ on ∂V) is *carried* by a rack R if it is in minimal position with respect to the support system $\Sigma(R)$ of R and if each component of $\partial\mathcal{D}\setminus\partial\Sigma(R)$ (or of $\lambda\setminus\partial\Sigma(R)$) is homotopic relative to $\partial\Sigma(R)$ to a rope of R.

(2) An embedded essential arc ρ in ∂V with endpoints in $\partial\Sigma(R)$ is *carried* by R if each component of $\rho\setminus\partial\Sigma(R)$ is homotopic relative to $\partial\Sigma(R)$ to a rope of R.

(3) A *returning rope* of a rack R is a rope which begins and ends at the same side of some fixed support disk D (i.e. defines a returning arc relative to $\partial\Sigma(R)$).

Remark 6.3. i) A disk system $\mathcal{D}$ is carried by a rack R if and only if each individual disk $D\in\mathcal{D}$ is carried by R.

ii) Every disk which does not intersect the support system $\Sigma(R)$ of a rack R is not carried by R. In particular, the support system itself is not carried by R.

Let R be a rack with support system $\Sigma(R)$ and let α be a returning rope of R with endpoints on a support disk $D\in\Sigma(R)$. By Lemma 5.2, for one of the components γ_1,γ_2 of $\partial D\setminus\alpha$, say the component γ_1, the simple closed curve $\alpha\cup\gamma_1$ is the boundary of an embedded disk $D'\subset H$ with the property that the disk system $(\Sigma\setminus D)\cup D'$ is reduced.

A *split* of the rack R at the returning rope α is any rack R' with support system $\Sigma'=(\Sigma(R)\setminus D)\cup D'$ whose ropes are given as follows.

(1) Up to isotopy, each rope ρ' of R' has its endpoints in $(\partial\Sigma(R)\setminus\partial D)\cup\gamma_1\subset\partial\Sigma(R)$ and is an arc carried by R.
(2) For every rope ρ of R there is a rope ρ' of R' such that ρ is a component of $\rho'\setminus\partial\Sigma(R)$.

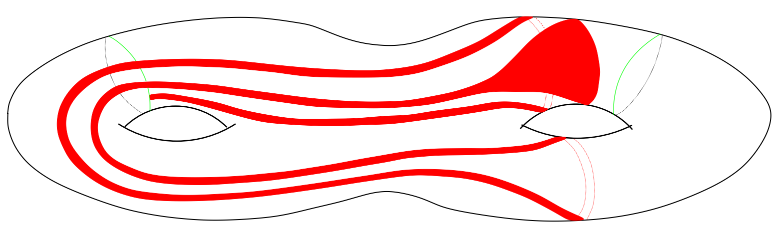
The above definition implies in particular that a rope of R which does not have an endpoint on ∂D is also a rope of R'. Moreover, there is a map $\Phi: R'\to R$ which maps a rope of R' to an arc carried by R, and which maps the boundary of a support disk of R' to a simple closed curve γ of the form $\gamma_1\circ\gamma_2$ where γ_1 either is a rope of R or trivial, and where γ_2 is a subarc of the boundary of a support disk of

R (which may be the entire boundary circle). The image of Φ contains every rope of R.

Splits of racks behave differently from splits of train tracks. Although this distinction is not explicitly needed for the rest of this work, we note some important differences in the remainder of this section. For these considerations we always consider racks up to weak isotopy.

A split of a rack R at a returning rope is not unique. If R' is a split of R and if φ is a Dehn-twist about the boundary of a support disk of R then $\varphi(R')$ is a split of R as well. Moreover the following example shows that even up to the action of the group of Dehn twists about the boundaries of the support system of R, there may be infinitely many racks which can be obtained from R by a split.

Example: Let V be the handlebody of genus 2 and let Σ be a reduced disk system consisting of two disks. Let R be a rack with support system Σ which contains two distinct returning ropes α, β approaching the same support disk $D \in \Sigma$ from two distinct sides. Let $E \subset V$ be an essential disk carried by R with the following property. There is an outermost component E' of $E \setminus \Sigma$ which contains an arc homotopic to α in its boundary. Attached to $E' \subset E$ is a rectangle component $R_\beta \subset E$ of $E \setminus \Sigma$ with two opposite sides on D which is a thickening of the returning rope β. The rectangle R_β is attached to a rectangle R_α with two sides on D which is a thickening of α looping about the half-disk E'. R_α in turn is attached to a second copy of R_β etc (see the figure). A rack R' whose support system is obtained from Σ by a single disk



exchange in direction of E and which carries ∂E contains a returning rope ρ which is carried by R and so that $\rho \setminus \Sigma$ has an arbitrarily large number of components.

Another important difference between racks and train tracks concerns the relation between carrying and splitting. One the one hand, there are splits R' of R which carry disks which are not carried by R. Namely, let R be a rack and R' be a split of R. Denote the support disk of R' which is not a support disk of R by D. In particular, if D is a disk carried by both R and R', then images of D under arbitrary powers of the Dehn twist about ∂D are still carried by R', but not necessarily by R.

On the other hand, let D be a disk carried by a rack R. Then there may be no split R' of R which still carries D. Namely, R may have a single returning rope ρ and thus every split of R has the same support system Σ'. If Σ' is disjoint from D, no rack with support system Σ' carries D.

7. The graph of rigid racks

In this section we construct a geometric model for the handlebody group. By a geometric model we mean a connected locally finite graph on which the handlebody group acts properly and cocompactly as a group of automorphisms. The construction is similar in spirit to the construction of the train track graph in [H09a], which is a geometric model for the mapping class group. The model we construct admits a family of distinguished paths which are used for a coarse geometric control of the handlebody group. These paths are constructed below in Lemmas 7.6 and 7.8.

As a first step one can define a *graph of racks* $\mathcal{R}(V)$ in direct analogy to the definition of the train track graph in [H09a]. The vertex set of $\mathcal{R}(V)$ is the set of weak isotopy classes of large racks (satisfying a suitable completeness condition which is not important for the current work). Two such vertices are connected by an edge of length one if the corresponding racks are related by a single split. By construction, the handlebody group acts on $\mathcal{R}(V)$ as a group of automorphisms. Imitating the proof of connectivity for the train track graph from [H09a, Corollary 3.7] one can then show that $\mathcal{R}(V)$ is connected. Since this result is not needed in the sequel we do not include a proof here.

The graph of racks defined in this way is not a geometric model for the handlebody group, as the stabilizer of a weak isotopy class of a rack contains the group generated by Dehn twists about the support system, and thus is in particular infinite. For the same reason, the graph of racks is locally infinite. Also recall that even up to the action of the group of Dehn twists about the support system of R, there may be infinitely many different racks which can be obtained from R by a single split (as demonstrated by the example in Section 6).

To define a geometric model for the handlebody group using racks, we therefore have to overcome two difficulties. On the one hand, we need to record twist parameters at the support curves so that the stabilizer of a rack with a set of such twist parameters becomes finite. On the other hand, the edges have to be more restrictive than splits so that the graph becomes locally finite.

For the purposes of this article, these problems will be addressed by considering a more restrictive notion of equivalence of racks.

Definition 7.1. i) Let R be a large rack. The union of the support system and the system of ropes of R defines a cell decomposition of the surface ∂V which we call the *cell decomposition induced by* R.

ii) Let R and R' be racks. We say that R and R' are *rigidly isotopic* if the cell decompositions induced by R and R' are isotopic as cell decompositions of the surface ∂V.

In particular, if φ is a simple Dehn twist about the boundary of a support curve of a rack R, then R and $\varphi^n(R)$ are not rigidly isotopic for $n \geq 2$. This observation and the fact that the stabilizer of a reduced disk system in the mapping class group is contained in the handlebody group imply the following.

Corollary 7.2. *The handlebody group acts on the set of rigid isotopy classes of racks with finite quotient and finite stabilizers.*

This corollary shows that the set of rigid isotopy classes of racks can be used as the set of vertices of a $\mathrm{Map}(V)$-graph which is a geometric model for $\mathrm{Map}(V)$.

To define a suitable set of edges for such a graph we note the following lemma.

Lemma 7.3. *i) There is a number $K_1 > 0$ with the following property. Let R, R' be two racks sharing the same support system. Then there is a sequence*

$$R = R_1, \dots, R_N = R'$$

of racks, such that the number of intersections between the cell decompositions induced by R_i and R_{i+1} is less than K_1 for all $i = 1, \dots, N-1$.

ii) There is a number $K_2 > 0$ with the following property. Let R be a rack and let α be a returning rope of R. Then there is a rack R' which is obtained from R by a split along α such that the number of intersections between the cell decompositions induced by R and R' is less than K_2.

Proof. Part *i)* of the lemma follows immediately from the fact that for every reduced disk system Σ of V, the stabilizer of $\partial\Sigma$ in the mapping class group of ∂V is contained in the handlebody group and acts with finite quotient on the set of all rigid isotopy classes of racks with a common support system.

To prove part *ii)*, let Σ' be the reduced disk system obtained from the support system of R by the disk exchange along the returning rope α. Every component of $\partial\Sigma'$ is homotopic to a union of uniformly few edges of the cell decomposition induced by R. Therefore, the number of intersections between Σ' and the cell decomposition induced by R can be uniformly bounded. Now the claim follows as in part *i)* since the stabilizer of $\partial\Sigma'$ in the mapping class group of ∂V is contained in the handlebody group. □

Definition 7.4. The *graph of rigid racks* $\mathcal{RR}(V)$ is the graph whose vertex set is the set of rigid isotopy classes of large racks. Two such vertices are joined by an edge if the intersection number between the cell decompositions induced by the large racks corresponding to the edges is at most K. Here K is the maximum of the constants K_1 and K_2 of Lemma 7.3.

Remark 7.5. Part *ii)* of Lemma 7.3 can be interpreted as the fact that twisting data about the support system of a rack R determines a finite number of splits which are adapted to these twist parameters. Furthermore, each of these possible splits carries a coarsely unique set of twist parameters induced by the original rack R.

Lemma 7.3 implies that $\mathcal{RR}(V)$ is connected. Since the handlebody group acts on $\mathcal{RR}(V)$ properly discontinuously and cocompactly, it is a geometric model of the handlebody group by the Svarć-Milnor-Lemma.

As a next step we define a distinguished class of paths in $\mathcal{RR}(V)$. These paths are sufficiently well-behaved to obtain a coarse geometric control for the handlebody group. The length estimates for these paths use markings and Corollary A.4 which relates word norms of mapping class group elements to intersection numbers of cell decompositions. The necessary definitions and statements are given in the Appendix.

In order to simplify the notation for the rest of the paper, we usually do not specify constants or additive and multiplicative errors in formulas, but rather state that a quantity x is "coarsely bounded" by some other quantity y (or "uniformly bounded"). By this we mean that there are constants C_1, C_2 which only depend on the genus (and the number of marked points) of V, such that x is bounded by $C_1 \cdot y + C_2$ (or C_1).

Lemma 7.6. *There is a number $k > 0$ satisfying the following. Let P be a pants decomposition of ∂V all of whose components bound disks* *in V. Let R be a large rack with support system $\Sigma(R)$. Then there is a large rack R' with the following properties.*

i) The support system $\Sigma(R')$ of R' agrees with the one of R.
ii) Each component of P which intersects the support system of R essentially is carried by R'.
iii) Each component of $P \setminus \partial\Sigma(R')$ intersects the cell decomposition induced by R' in at most k points.
iv) The distance between R and R' in $\mathcal{RR}(V)$ is coarsely bounded by $i(P, \partial\Sigma(R))$.

Proof. Denote the cell decomposition induced by R by C. Let S' be the surface obtained from ∂V by cutting at $\partial\Sigma(R)$. The intersection of P with S' is a union of simple closed curves and arcs connecting the boundary components of S'. We call these arcs the *arcs induced by P*. Let $\hat{R}$ be the rack whose support support system agrees with the one of R and whose ropes are given by the arcs induced by P. If $\hat{R}$ is not a large rack, then we can add ropes to $\hat{R}$ which intersect P in uniformly few points, and which intersect ropes of R in at most $i(P, C)$ points. Call the result R'.

From the construction of the rack R', properties *i)* to *iii)* are immediate. Property *iv)* follows by applying Corollary A.4 to the cell decomposition C and the cell decomposition induced by R' on the subsurface S'. □

Definition 7.7. If P and R' satisfy the conclusions *ii)* and *iii)* of Lemma 7.6 above, we say that P *is effectively carried by R'*.

The following lemma is the main step towards the upper distortion bound for the handlebody group and contains the construction of the distinguished paths in the handlebody group.

Lemma 7.8. *Let P be a pants decomposition all of whose components bound disks in V. Suppose P is effectively carried by a rack R with support system $\Sigma(R)$. If at least one component of P intersects $\partial\Sigma(R)$ essentially, there is a rack R' with the following properties.*

i) The support system $\Sigma(R')$ is obtained from $\Sigma(R)$ by a disk exchange move in the direction of a component of P.
ii) P is effectively carried by R'.
iii) The distance of R and R' in $\mathcal{RR}(V)$ is coarsely bounded by $i(P, \partial\Sigma(R))$.

Proof. Since the intersection of P with $\partial\Sigma(R)$ is nonempty, the rack R has a returning rope α corresponding to an arc induced by P.

Let Σ' be the reduced disk system obtained from $\Sigma(R)$ by a disk exchange along the returning leaf α. Each component of $\partial\Sigma'$ intersects the cell decomposition induced by R in uniformly few points. Define a rack $\hat{R}$ with support system Σ' by choosing the arcs induced by P relative to Σ' as ropes. By construction, each rope of $\hat{R}$ is obtained as a concatenation of ropes of R (as in the definition of the split of a rack). Furthermore, each rope of $\hat{R}$ intersects $\Sigma(R)$ in at most as many points as P does. Therefore, the intersection number between a rope of $\hat{R}$ and the cell decomposition induced by R can be coarsely bounded by $i(P, \partial\Sigma(R))$. We can extend $\hat{R}$ in any way to a large rack R' such that every rope of R' has the same property. Both R' and R intersect $\partial\Sigma'$ in uniformly few points. The mapping class group of $\partial V \setminus \partial\Sigma'$ is contained in the handlebody group and undistorted in the mapping class group. Hence Corollary A.4 applied in the subsurface $\partial V \setminus \partial\Sigma'$ implies that the distance between R and R' in $\mathcal{RR}(V)$ is coarsely bounded by $i(P, \partial\Sigma(R))$. Now we can apply Lemma 7.6 to R' to obtain a rack with the desired properties. ☐

The following theorem is an easy consequence of Lemma 7.8.

Theorem 7.9. *Let $g \geq 2$ be arbitrary. Then the handlebody group* $\mathrm{Map}(V_g)$ *is at most exponentially distorted in the mapping class group.*

Together with the results from Sections 3 and 4 this theorem implies the main theorem from the introduction.

Proof of Theorem 7.9. There is a number $K > 0$ such that for every large rack R there is a pants decomposition P_R whose geometric intersection number with the cell decomposition $C(R)$ induced by R is bounded by K. This is due to the fact that the handlebody group acts cocompactly on the graph of rigid racks.

Let R_0 be a rack, and P_0 such a pants decomposition. Let f be an arbitrary element of the handlebody group. Put $P = f(P_0)$. By Proposition A.3 the geometric intersection number between P and P_0 is coarsely bounded exponentially in the word norm of f in the mapping class group. Denote this bound by N.

As a first step, apply Lemma 7.6 to R_0 and P to construct a rack R_1 which effectively carries P and whose distance to R_0 is coarsely bounded by N. Next, use Lemma 7.8 to construct a rack R_2 whose distance to R_1 is again coarsely bounded by N, and such that the number of intersections between P and $\Sigma(R_2)$ is strictly less than the number of intersections between P and $\Sigma(R_1)$. Inductively repeating this procedure we find a sequence $R_1, \ldots, R_K$ of racks of length K coarsely bounded by N^2, and such that P is disjoint from $\Sigma(R_K)$. In particular, there is a handlebody group element g which maps P_0 to P and whose word norm in the handlebody group is also coarsely bounded by N^2. The difference $f^{-1} \circ g$ fixes the pants decomposition P_0 and hence is a Dehn multitwist about P_0. As the group of Dehn multitwists about P_0 is contained in the handlebody group, and undistorted in the mapping class group, the word norm of $f^{-1} \circ g$ in the handlebody group is also coarsely bounded by N^2. This shows the theorem. □

Appendix A. Markings and intersection numbers

In this Appendix we recall some facts about markings and intersection numbers which are used several times in this work.

Our terminology deviates slightly from the one used in [MM00], so we also recall the necessary definitions.

Definition A.1. A *marking* μ of a surface S is a pants decomposition P of S together with a clean transversal for each curve in P. Here, a *clean transversal* to a pants curve $\gamma \in P$ is a curve c which is disjoint from all curves $\gamma' \in P \setminus \gamma$ and which intersects γ in the minimal number of points.

Two clean transversals to a curve α in a pants decomposition P differ by a Dehn twist about α (after possibly applying a half-twist about α). In this way, the set of clean transversals can be thought of as a twist normalization about the pants decomposition curves.

Note that the object we denote by "marking" is called "complete clean marking" in the terminology of [MM00]. The more general notion of marking used in [MM00] does not play any role in the present work.

Let S be a oriented surface of finite type and negative Euler characteristic (possibly with punctures and boundary components). *Subsurface projections to annuli* in S are defined in the following way (compare [MM00]). Recall that the arc complex of a closed annulus A is the graph whose vertex set is the set of arcs connecting the two boundary components of A up to isotopy fixing ∂A pointwise. Two such vertices are connected by an edge of length one, if the corresponding arcs can be realized with disjoint interior.

Let α be an essential simple closed curve on S. By S_α we denote the annular cover corresponding to α. Explicitly, S_α is a covering surface of S corresponding to the (conjugacy class of the) cyclic subgroup of $\pi_1(S)$ generated by α. Since S has negative Euler characteristic, it carries a hyperbolic metric which lifts to a hyperbolic metric on the annulus S_α. In particular, S_α has a natural boundary compactifying it to a closed annulus.

Let β be a simple closed curve or essential arc on S intersecting α. Consider the set of lifts $\widetilde{\beta}$ of β to S_α which connect the two boundary components of S_α. Every element of this set defines a vertex in the arc complex of the annulus S_α. We call the set of all these vertices the *subsurface projection of β to α*. The subsurface projection of β to α has diameter at most one as all lifts of β to S_α are disjoint.

Definition A.2. The *marking graph* of S is the graph whose vertex set is the set of isotopy classes of markings. Two such markings μ and μ' are joined by an edge of length one if they differ by an *elementary move*. An elementary move from μ to μ' is one of the following two operations.

i) μ' has the same underlying pants decomposition as μ. The transversals of μ' are obtained from the ones of μ by applying one primitive Dehn twist about one of the pants curves.
ii) Replace a pants curve α by its corresponding clean transversal β in μ. Then modify α to a clean transversal of β ("cleaning the marking" in the terminology of [MM00]).

The cleaning operation is described in detail in [MM00, Lemma 2.4] (also compare the discussion on page 21 of [MM00]).

Since the details are not relevant for the current work, we do not review them here. The marking graph is a connected, locally finite graph on which the mapping class group of S acts with finite point stabilizers and finite quotient (compare [MM00]). Therefore, it is quasi-isometric to the mapping class group.

The following proposition is well-known to experts and relates distances in the marking graph to intersection numbers. Since we did not find a proof in the literature, we include one here for completeness.

Proposition A.3. *Let μ_1, μ_2 be markings of a surface S. If μ_1 and μ_2 are of distance k in the marking graph, then the total number of intersections between μ_1 and μ_2 is bounded exponentially in k. Conversely, the total intersection number between μ_1 and μ_2 is a coarse upper bound for the distance between μ_1 and μ_2 in the marking graph of S.*

Proof. We begin with the lower bound for the distance in the marking graph. Let μ_1 and μ_2 be two markings. For a number $\epsilon > 0$, we say a marked Riemann surface X belongs to the ϵ-thick part of Teichmüller space if the length of each simple closed geodesic on X is at least ϵ. We will simply speak of the thick part, if the corresponding ϵ is understood from the context. There are points X_i the ϵ-thick part of Teichmüller space for S such that each curve in μ_i is shorter than some universal constant C on X_i. Here, ϵ is a universal constant depending only on the genus of the surface S. Explicitly, let P_i be the underlying pants decomposition of the marking μ_i. The pants decomposition P_i

defines Fenchel-Nielsen coordinates for the Teichmüller space of S. This implies that there is a marked Riemann surface X_i' such that each curve in P_i has hyperbolic length 1 on X_i'. On a hyperbolic pair of pants all of whose boundary components have lengths equal one the distance between any two boundary components is uniformly bounded. This implies that on X_i' there are clean transversals to P_i whose hyperbolic length is also uniformly bounded. By changing the marking on X_i' by Dehn twists about P_i we obtain the desired surfaces X_i.

If the distance between μ_1 and μ_2 in the marking graph is bounded by k, then the Teichmüller distance between X_1 and X_2 is also coarsely bounded by k since the mapping class group acts properly and cocompactly on the thick part of Teichmüller space. Thus the total hyperbolic length of μ_2 on X_1 is bounded by $e^{2k} \cdot C$ by Wolpert's lemma ([W79, Lemma 3.1]). But each curve in μ_1 has a collar of definite width on X_1 since its length is bounded by C, and therefore the total number of intersections of μ_1 and μ_2 is also coarsely bounded by e^{2k}.

Next we show the upper bound for the distance in the marking graph. In the proof we will use singular Euclidean structures as in [B06] and the relation between the mapping class group of a surface and the corresponding Teichmüller space.

Let P_1 and P_2 be the underlying pants decompositions of the markings μ_1, μ_2. We may assume that $P_1 \cup P_2$ fills the surface, i.e. that all components of $S \setminus (P_1 \cup P_2)$ are simply connected. Namely, if $P_1 \cup P_2$ does not fill, then P_1 and P_2 share a common curve α. We can then change the transversal to α in μ_1 such that the diameter of the subsurface projection to α of the transversals to α in μ_1 and μ_2 is at most one. The number of steps necessary for this modification is bounded by the intersection number between the two transversals. We can then pass to the subsurface obtained by cutting S along the common curve α and discarding the corresponding transversal. Repeat this procedure until $P_1 \cup P_2$ fills.

Furthermore, we can assume that the twist about a pants curve $\delta \in P_1$ defined by μ_1 coarsely agrees with the one defined by P_2. By this we mean the following. Since P_1 and P_2 fill the surface, there is at least one curve of P_2 which intersects δ. Denote by c_δ the transversal to δ in μ_1. The diameter of the subsurface projection of P_2 and c_δ to δ is bounded from above by the intersection number between μ_1 and μ_2. Hence, after modifying the transversal to δ in μ_1 by at most $i(\mu_1, \mu_2)$ Dehn twists about δ, the diameter of the projection is at most 3. Similarly, we modify μ_2 such that the twist about the pants curves in P_2 given by μ_2 agrees with the one defined by P_1.

For a pair of measured laminations λ_1, λ_2 which jointly fill the surface and satisfy $i(\lambda_1, \lambda_2) = 1$ we denote by $q(\lambda_1, \lambda_2)$ the quadratic differential whose horizontal measured lamination is λ_1 and whose vertical measured lamination is λ_2. Now let ρ be the Teichmüller geodesic defined by P_1 and P_2; that is $\rho_t = q(e^{-t}P_1, e^t/i(P_1, P_2)P_2)$ (compare the construction in [B06] for pairs of curves). Recall that on every hyperbolic surface of genus g there is a pants decomposition such that the hyperbolic length of each pants curve is bounded by a universal constant B (the Bers constant) which depends only on the genus. By the collar lemma, a curve whose hyperbolic length is bounded by B has extremal length coarsely bounded by B. Thus the length of such a curve in any singular Euclidean metric in the same conformal class is bounded by a universal constant B'.

We set $T = \log(2B')$. Then for the singular Euclidean metric defined by ρ_{-T}, a curve whose length is smaller than B' cannot intersect P_1. Hence, P_1 is the only Bers short pants decomposition for ρ_{-T}. Similarly, P_2 is the only Bers short pants decomposition on $\rho_{\log(i(P_1,P_2))+T}$. In particular, there are two points X_1, X_2 in Teichmüller space, whose Teichmüller distance is bounded by $2T + \log(i(P_1, P_2))$ and such that P_i is Bers short on X_i.

Now for any k which is sufficiently large, by work of Rafi we have the following estimate for the Teichmüller distance $d_{\mathcal{T}}(X_1, X_2)$ (compare [R07, Equation (19)]).

$$d_{\mathcal{T}}(X_1, X_2) \succ \sum_{Y} [d_Y(\mu_1', \mu_2')]_k + \sum_{\alpha \notin \Gamma} \log [d_\alpha(\mu_1', \mu_2']_k .$$

Here, μ_1' and μ_2' are shortest markings on X_1 and X_2, respectively, and $[x]_k$ is a cutoff function which is 0 if $x \leq k$ and x otherwise. The expression $a \succ b$ means that a is coarsely bounded by b. The first sum is taken over all subsurfaces $Y \subset S$, while the indexing set Γ of the second sum is the set of (isotopy classes of) simple closed curves which are short on either X_1 or X_2. Note that in our case Γ agrees with the union of the pants curves in P_1 and P_2. In both cases d_Y (or d_α) denotes the diameter of the set of subsurface projections of μ_1' and μ_2' to Y (or α).

In our case, since P_1 and P_2 fill, we can replace the subsurface projections of μ_i' by those of P_i, except maybe in the cases where the subsurface is bounded by curves contained in Γ. Hence we get

$$d_{\mathcal{T}}(X_1, X_2) \succ \sum_{\partial Y \not\subset \Gamma} [d_Y(P_1, P_2)]_k + \sum_{\alpha \notin \Gamma} \log [d_\alpha(P_1, P_2)]_k .$$

Now, since $d_{\mathcal{T}}(X_1, X_2) \prec \log(i(P_1, P_2))$ we have

$$i(P_1, P_2) \succ \sum_{\partial Y \not\subset \Gamma} [d_Y(P_1, P_2)]_k + \sum_{\alpha \notin \Gamma} [d_\alpha(P_1, P_2)]_k \,.$$

Since the number of subsurfaces whose boundary is completely contained in Γ is uniformly bounded, and the total intersection of μ_1 and μ_2 bounds each of these projections, we get

$$i(\mu_1, \mu_2) \succ \sum_{Y} [d_Y(\mu_1, \mu_2)]_k + \sum_{\alpha} [d_\alpha(\mu_1, \mu_2)]_k \,.$$

where now the sums are taken over all subsurfaces and all curves respectively. By [MM00, Theorem 6.12], the right hand side of this inequality is coarsely equal to the distance of μ_1 and μ_2 in the marking graph. This shows the first claim. ☐

In the proof of the upper bound on distortion of the handlebody group the following corollary is used in an essential way.

Corollary A.4. *Let $N > 0$ be given. Let C be a cell decomposition of the surface S with at most N cells. Let $f \in \mathrm{Map}(S)$ be arbitrary. The intersection number between C and $f(C)$ is coarsely bounded by an exponential of the word norm of f. Here, the constants depend on the genus of S and the number N.*

Similarly, let C and C' are cell decomposition with at most N cells and which intersect in K points. Then there is a mapping class g whose word norm is bounded coarsely in K, and such that $g(C)$ and C' intersect in uniformly few points.

Proof. Note that up to the action of the mapping class group there are only finitely many cell decompositions C of S with at most N cells. Hence, there is a constant $K > 0$ such that for any such cell decomposition C there is a marking μ_C whose intersection number with C is bounded by K.

By the preceding Proposition A.3 the number of intersections between μ_C and $f(\mu_C)$ is coarsely bounded exponentially in the word norm of f. Since the intersection number between $f(\mu_C)$ and $f(C)$ is uniformly bounded, the corollary follows.

Similarly, if C and C' intersect in K points, then the intersection number between μ_C and $\mu_{C'}$ can be coarsely bounded by K. Hence, Proposition A.3 implies the second claim of the corollary. ☐

References

[A02] E. Alibegović, *Translation Lengths in* $\mathrm{Out}(F_n)$, Geom. Dedicata 92 (2002), 87–93.

[Bi74] J. S. Birman, *Braids, links and mapping class groups*, Ann. Math. Studies 82, Princeton University Press, Princeton, N.J. 1974.

[B06] B. Bowditch, *Intersection numbers and the hyperbolicity of the curve complex*, J. Reine Angew. Math. 598 (2006), 105–129.

[BFP07] N. Broaddus, B. Farb, A. Putman, *Irreducible Sp-representations and subgroup distortion in the mapping class group*, arXiv:0707.2262, to appear in Comm. Math. Helv.

[FLM01] B. Farb, A. Lubotzky, Y. Minsky, *Rank one phenomena for mapping class groups*, Duke Math. J. 106 (2001), 581–597.

[FM11] B. Farb, D. Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, Princeton University Press, Princeton N.J. 2011.

[H09a] U. Hamenstädt, *Geometry of the mapping class group I: Boundary amenability*, Invent. Math. 175 (2009), 545–609.

[H09b] U. Hamenstädt, *Geometry of the mapping class group II: A biautomatic structure*, arXiv:0912.0137.

[Ha95] A. Hatcher, *Homological stability for automorphism groups of free groups*, Comm. Math. Helv. 70 (1995), 39–62.

[M86] H. Masur, *Measured foliations and handlebodies*, Erg. Th. & Dynam. Sys. 6 (1986), 99–116.

[MM99] H. Masur, Y. Minsky, *Geometry of the complex of curves I: Hyperbolicity*, Invent. Math. 138 (1999), 103–149.

[MM00] H. Masur, Y. Minsky, *Geometry of the complex of curves II: Hierarchical structure*, Geom. Funct. Anal. 10 (2000), 902–974.

[Mc85] D. McCullough, *Twist groups of compact 3–manifolds*, Topology 24 (1985), no. 4, 461–474.

[R07] K. Rafi, *A combinatorial model for the Teichmüller metric*, Geom. Funct. Anal. 17 (2007), no. 3, 936–959.

[St99] J. Stallings, *Whitehead graphs on handlebodies*, Geometric group theory down under (Canberra, 1996), 317–330, de Gruyter, Berlin, 1999.

[S77] S. Suzuki, *On homeomorphisms of a 3-dimensional handlebody*, Canad. J. Math. 29 (1977), no. 1, 111-124.

[Wa98] B. Wajnryb, *Mapping class group of a handlebody*, Fund. Math. 158 (1998), no. 3, 195–228.

[W79] S. Wolpert, *The length spectra as moduli for compact Riemann surfaces*, Ann. of Math. (2) 109 (1979), no. 2, 323–351.

Ursula Hamenstädt and Sebastian Hensel, Mathematisches Institut der Universität Bonn, Endenicher Allee 60, D-53115 Bonn, Germany

E-mail address: ursula@math.uni-bonn.de

E-mail address: loplop@math.uni-bonn.de