QUASI-MORPHISMS ON SURFACE DIFFEOMORPHISM GROUPS

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Dedicated to Mladen Bestvina on the occasion of his 60th *birthday.*

Abstract. We show that the identity component of the group of diffeomorphisms of a closed oriented surface of positive genus admits many unbounded quasi-morphisms. As a corollary, we also deduce that this group is not uniformly perfect and its fragmentation norm is unbounded, answering a question of Burago–Ivanov–Polterovich.

To do this, we introduce an analogue of the curve graph from the theory of mapping class groups. We show that it is hyperbolic and that the natural group action by isometries satisfies the criterion of Bestvina–Fujiwara.

1. Introduction

The general problem of constructing quasi-morphisms has played a prominent role in geometric group theory, symplectic geometry and dynamics since Gromov's introduction of bounded cohomology in the early 80s [Gro82]. In the context of mapping class groups there are now many constructions of quasi-morphisms, starting with the work of Endo–Kotschick [EK01], Bestvina–Fujiwara [BF02], and Hamenstädt [Ham08] to name only a few. Using the natural projection of the full diffeomorphism group to the mapping class group this then yields many non-trivial quasi-morphisms on the *full* diffeomorphism group $\mathrm{Diff}(S_g)$ of any surface S_g of genus $g \geq 2$.

In view of this it remains to study whether the identity component $\mathrm{Diff}_0(S_g)$ admits any quasi-morphisms. If one restricts to the subgroup of area-preserving diffeomorphisms, or more precisely the subgroup of Hamiltonian diffeomorphisms, there are constructions of quasi-morphisms due to Ruelle [Rue85] and Gambaudo–Ghys [GG04] (see also [BM17]). This motivates the following question going back to Burago–Ivanov–Polterovich [BIP08]

Question 1.1. *Does the group* $\mathrm{Diff}_0(S_g)$ *admit any quasi-morphisms that are unbounded?*

To put this question in context recall that for any compact manifold the identity component of the diffeomorphism group $\mathrm{Diff}_0(M)$ is perfect by classical results of Mather and Thurston [Mat71, Mat74, Thu74] and therefore does not admit any non-trivial homomorphisms to an abelian group (cf. also [Man16]). In fact for many manifolds these groups are uniformly perfect, meaning that any element can be written as a product of commutators of uniformly bounded length. If M has odd dimension then $\mathrm{Diff}_0(M)$ is uniformly perfect. This is due to Burago–Ivanov–Polterovich [BIP08] in the 3-dimensional case and their argument was extended by Tsuboi [Tsu08] to hold in any odd dimension. If M^{2n} has even dimension and M admits a handle-decomposition without handles of index n then $\mathrm{Diff}_0(M)$ is again uniformly perfect and one can in fact write every element as the product of at most 4 commutators [Tsu08]. The existence of an unbounded quasi-morphism shows that a group is not uniformly perfect.

The simplest manifolds that are not covered by these general results are the closed surfaces of genus $g \geq 1$. Our main result shows that these groups indeed show drastically different behaviour:

Theorem 1.2. *For $g \geq 1$ the space $\widetilde{\mathrm{QH}}(\mathrm{Diff}_0(S_g))$ of unbounded quasi-morphisms on $\mathrm{Diff}_0(S_g)$ is infinite dimensional.*

Here $\widetilde{\mathrm{QH}}(G)$ denotes the space of quasi-morphisms modulo the set of bounded functions on G (see Section 2.2 for definitions).

The existence of an unbounded quasi-morphism on $\mathrm{Diff}_0(S_g)$ also has the following consequence, which answers an open problem of Burago–Ivanov–Polterovich [BIP08]

Corollary 1.3. *For $g \geq 1$ the group $\mathrm{Diff}_0(S_g)$ is not uniformly perfect and has unbounded fragmentation norm.*

Note that by the main result of [BIP08] the fragmentation norm on $\mathrm{Diff}_0(S_0)$ is uniformly bounded and so the assumptions in the corollary are optimal.

Outline of Proof: Our proof of Theorem 1.2 relies on the Bestvina–Fujiwara construction [BF02] of quasi-morphisms from group actions on hyperbolic graphs.

The hyperbolic graph in question will be a variant of the curve graph which works in this setting. Curve graphs were defined by Harvey [Har81], and have quickly become one of the central tools to study mapping class groups starting with the foundational work of Masur and Minsky [MM99, MM00]. Recently, following a strategy suggested by Calegari, variants of curve graphs have also been defined and used to construct quasi-morphisms on so-called *big mapping class groups*, e.g. the mapping class group of the plane minus a Cantor set (see [Bav16]).

Intuitively, curve graphs encode intersection patterns of *isotopy classes* of curves (or similar objects) on surfaces. Hence the group $\mathrm{Diff}_0(S_g)$ will act trivially on all of them. In this paper we therefore begin the study of a (much larger) curve graph which encodes intersection patterns between actual curves and thus admits an interesting action of $\mathrm{Diff}_0(S_g)$.

A new curve graph. In this article we define and study the graph $\mathcal{C}^\dagger(S)$ whose vertices correspond precisely to the simple closed curves on S (not their isotopy classes). An edge connects two vertices precisely when the corresponding curves are disjoint.

In order to study $\mathcal{C}^\dagger(S)$ we relate its geometry to the geometry of (usual) curve graphs whose geometry is relatively well understood. For technical reasons (see Section 3) it is easier to work with the (quasi-isometrically embedded) subgraph $\mathcal{NC}^\dagger(S) \subset \mathcal{C}^\dagger(S)$ whose vertices are non-separating curves.

The key tool enabling us to understand the geometry of $\mathcal{NC}^\dagger(S)$ is Lemma 3.4 in Section 3 which shows that the distance between two vertices in $\mathcal{NC}^\dagger(S)$ can be computed using the distance in the (usual) non-separating curve graph $\mathcal{NC}(S-P)$ of the punctured surface $S-P$, provided the puncture set P is chosen correctly.

Since all non-separating curve graphs $\mathcal{NC}(S-P)$ are hyperbolic with a constant independent of the choice of P by a result of Rasmussen [Ras17], this allows us to prove hyperbolicity of $\mathcal{NC}^\dagger(S)$ and therefore $\mathcal{C}^\dagger(S)$.

The reason we use the complex of non-separating curves is that any non-separating curve on $S-P$ (for any set of punctures) is still essential on S, which allows us to relate different $\mathcal{NC}(S-P)$ to each other and to $\mathcal{NC}^\dagger(S)$. We emphasise that this is impossible if S is a sphere (as there are no essential curves on an unpunctured sphere), and therefore our strategy does not show anything when S is a sphere. Alternatively we could consider the graph consisting of all curves—including the inessential curves—but this would have bounded diameter and therefore would not be useful for constructing quasi-morphisms.

Building quasi-morphisms. By construction we now have an action of $\mathrm{Diff}_0(S_g)$ on the hyperbolic graph $\mathcal{C}^\dagger(S)$. In order to produce quasi-morphisms using [BF02] we also need to construct elements that act *hyperbolically* (i.e. with positive asymptotic translation length) and that are *independent* (i.e. there is a bound on how far their axes fellow travel even after applying any diffeomorphism to either axis). We refer the reader to Section 6 for details on these notions.

We remark that in most applications of the Bestvina–Fujiwara construction [BF02] the independence of elements is guaranteed by showing that the action in question satisfies *WPD*. We emphasise that this is not the case here—the stabiliser of any finite collection of points in $\mathcal{C}^\dagger(S)$ contains a copy of the diffeomorphism group of a disk. In fact more is true. There is no action of $\mathrm{Diff}_0(S)$ satisfying *WPD*, for if this were the case, then $\mathrm{Diff}_0(S)$ would admit a non-elementary acylindrical action on a hyperbolic space [Osi16] and therefore have uncountably many normal subgroups [DGO17]. But $\mathrm{Diff}_0(S)$ is known to be simple (since it is perfect, and [Eps70] shows that the commutator subgroup is simple).

Constructing independent hyperbolic elements. Our hyperbolic elements will be constructed using *point-pushing pseudo-Anosov* mapping classes. These are isotopically-trivial diffeomorphisms of S which fix a set of points P but are pseudo-Anosov as mapping classes of $S - P$ (compare Section 2.6 for definitions). Using the connection of $\mathcal{NC}^\dagger(S)$ to $\mathcal{NC}(S-P)$ described above we show that any such map acts hyperbolically on $\mathcal{NC}^\dagger(S)$ (Lemma 5.2).

Even though we would expect that most hyperbolic elements one can obtain this way are independent, verifying independence is the most technical part of the paper. To do so we use more subtle geometric tools of curve graphs, namely *subsurface projections* in the sense of Masur and Minsky [MM00]. In fact for our argument we only need projections to annuli, which yield a notion to quantify how much a curve α twists about a curve β. See Sections 2.5 and 4 for details.

In this introduction we will only indicate the main idea for independence. We refer the interested reader to Section 6 where a more detailed overview of the actual strategy is given.

In very rough terms the idea is the following: suppose that φ_1 is (a smoothing of) a point-pushing pseudo-Anosov on $S-p$. On any of its quasi-axes in the curve graph of $S-p$, the maximal possible twisting between any two points about any curve b is bounded. Now to find an independent φ_2, we will choose a suitable point-pushing pseudo-Anosov which exhibits much larger twisting along its quasi-axis. Carefully controlling how twists behave in our setting (see Sections 4 and 6) will then allow us to show that these quasi-axes cannot be made to fellow-travel in $\mathcal{C}^\dagger(S)$.

Automatic continuity. In general a quasi-morphism on a topological group need not be continuous as one can always add a discontinuous bounded function to any given quasi-morphism. However for *homogeneous* quasi-morphisms on $\mathrm{Diff}_0(S_g)$ automatic continuity does indeed hold. This fact is due to Kotschick however a proof unfortunately did not appear in the published version of [Kot08] (cf. also [EPP12]) and therefore we give a proof in Section 8.

Theorem 1.4 (Kotschick)**.** *Any homogeneous quasi-morphism on* $\mathrm{Diff}_0(S_g)$ *is continuous with respect to the* C^0*-topology.*

In particular it follows that any homogeneous quasi-morphism on $\mathrm{Diff}_0(S_g)$ extends (uniquely) to its C^0-closure in $\mathrm{Homeo}_0(S)$. By classical approximation results this closure is known to be all of $\mathrm{Homeo}_0(S)$ and we deduce a topological version of our main result along with many conjugacy-invariant norms.

Theorem 1.5. *For* $g \geq 1$ *the space* $\widetilde{\mathrm{QH}}(\mathrm{Homeo}_0(S_g))$ *of unbounded quasi-morphisms on* $\mathrm{Homeo}_0(S_g)$ *is infinite dimensional.*

A closer examination of the continuity yields an equicontinuity property for homogeneous quasi-morphisms of bounded defect. Thus an application of Bavard Duality implies that the stable commutator length function is continuous as well.

Theorem 1.6 (Continuity of scl)**.** *The stable commutator length function on the group* $\mathrm{Diff}_0(S_g)$ *is continuous with respect to the* C^0*-topology.*

In fact all of these results hold for arbitrary closed manifolds but in higher dimensions it is still unknown whether there exist any non-trivial quasi-morphisms. For example in view of Tsuboi's results there can be no non-trivial homogeneous quasi-morphisms in odd dimensions.

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2. Background

2.1. **Graphs, hyperbolic geometry, hyperbolic elements, and axes.**

The main two references for this section are Bestvina–Fujiwara [BF02, Section 1] and Bridson–Haefliger [BH99, Chapter III.H.1]. All the graphs in this paper are viewed as metric spaces where the length of each edge is equal to 1.

A *path* in a graph Γ is a sequence of vertices $(v_i)_i$ such that v_i is adjacent to v_{i+1}. If the sequence is $v_0, \dots, v_n$ then we say that the path *connects* v_0 and v_n, or alternatively, the path is from v_0 to v_n. The *length* of the path $v_0, \dots, v_n$ is defined to be n.

Convention. Unless stated otherwise, we assume that all graphs in this article are connected, and all actions on graphs are by simplicial isometries.

Given two vertices a and b we define $d(a,b)$ to be the minimal possible length of a path connecting a and b. A *geodesic* is a path $(v_i)_i$ such that $|i-j| = d(v_i, v_j)$. A *quasi-geodesic* is a sequence $(v_i)_i$ such that there exist K and L with

$$\frac{1}{K}|i-j| - L \le d(v_i, v_j) \le K|i-j| + L.$$

More specifically we say that $(v_i)_i$ is a (K,L)-quasi-geodesic, or less specifically, a C-quasi-geodesic when it is a (C,C)-quasi-geodesic.

Let $k > 0$. A *k-local-geodesic* is a sequence of vertices $(v_i)_i$ such that whenever $|i-j| \le k$ then $d(v_i, v_j) = |i-j|$. A k-local-geodesic is necessarily a path.

With the exception of Section 3 we use the following definition throughout.

Definition 2.1 (Slim triangles)**.** We say that a graph Γ is *hyperbolic* if there exists δ such that for any three geodesics g_1, g_2, and g_3 that form a triangle in Γ, we have that each vertex of g_1 is within δ of a vertex of g_2 or g_3. More specifically we say that Γ is *δ-hyperbolic* or Γ has δ-slim triangles.

In Section 3 we find it more convenient to use the equivalent notion of the four-point condition. Since it is used only in Section 3 we define it there. We refer the reader to [ABC$^+$91] for various other definitions of hyperbolicity, and proofs of their equivalence.

The following standard lemma is crucial in the study of hyperbolic spaces.

Lemma 2.2. *In a δ-hyperbolic geodesic metric space a geodesic and a C-quasi-geodesic with the same endpoints have Hausdorff distance at most $R = R(C, \delta)$ regardless of their length.*

Proof. We refer to [BH99, p401 Theorem 1.7]. □

The following fact in a δ-hyperbolic geodesic metric space is well known. If a C-quasi-geodesic and a C'-quasi-geodesic have Hausdorff distance at most D and are sufficiently long (in terms of C, C', δ, and D) then they admit long subsegments that have Hausdorff distance at most $B = B(C, C', \delta)$. We emphasize the important fact that B does *not* depend on D. We make this precise in the following lemma.

Lemma 2.3. *Let Γ be a δ-hyperbolic graph and let $D > 0$. Suppose that $(x_0, \ldots, x_n)$ is a C-quasi-geodesic and $(y_0, \ldots, y_m)$ is a C'-quasi-geodesic such that each x_i is within D of some y_j. Then there exist $A = A(C, C', \delta, D)$ and $B = B(C, C', \delta)$ such that whenever*

$$i, n - i > A,$$

then there exists j such that $d(x_i, y_j) \leq B$.

Proof. Without loss of generality we take C large enough so that the aforementioned quasi-geodesics are C-quasi-geodesics. We set

$$A = C(R + 2\delta + D + C),$$

and

$$B = 2R + 2\delta,$$

where $R = R(C, \delta)$ is as in Lemma 2.2.

There exist y_a such that $d(x_0, y_a) \leq D$ and y_b such that $d(x_n, y_b) \leq D$. Find geodesics g_1 and g_3 connecting x_0 and y_a, and y_b and x_n respectively. Find geodesics g_2 connecting y_a and y_b, and g_4 connecting x_n and x_0. Then g_1, g_2, g_3, g_4 form a geodesic square.

Now suppose that $i, n - i \geq A$. Then by Lemma 2.2 we have that x_i is within R of some vertex v of g_4. Because g_1, g_2, g_3, g_4 is a geodesic square, we have that g_4 is contained within the 2δ-neighborhood of $g_1 \cup g_2 \cup g_3$ (use the δ-slim triangle condition twice with an extra geodesic from x_0 to y_b). If v is within 2δ of g_1 or g_3 then x_i is within $R + 2\delta + D$ of x_0 or x_n respectively.

Suppose that $d(x_i, x_0) \le R + 2\delta + D$ (the case with x_n in place of x_0 is similar). This implies that

$$\frac{1}{C} i - C \le d(x_i, x_0) \le R + 2\delta + D,$$

and so $i \le A$, contradicting our earlier assumption. We conclude that v is within 2δ of some vertex of g_2 and therefore by Lemma 2.2 there exists some y_j such that $d(x_i, y_j) \le 2R + 2\delta$. □

Now let a group G act on Γ (by simplicial isometries). For $g \in G$ we define

$$|g| \coloneqq \lim_{k\to\infty} \frac{1}{k} d(x, g^k x),$$

to be the *asymptotic translation length* of g. The limit exists because $d(x, g^k x)$ is a non-negative and subadditive function with respect to $k > 0$. We say that g is a *hyperbolic element* if $|g| > 0$. If g is a hyperbolic element then any orbit of g is a C-*quasi-axis*, i.e. a g-invariant C-quasi-geodesic, for some C depending on the orbit.

To establish our main theorem we use the following notion from [BF02].

Definition 2.4 ([BF02])**.** Let $g_1, g_2 \in G$ be two hyperbolic elements with a C-quasi-axis A_1 and a C'-quasi-axis A_2 respectively. We write $g_1 \sim g_2$ if for any arbitrarily long subsegment J in A_1 there is an element $h \in G$ such that hJ is within the B-neighborhood of A_2, where $B = B(C, C', \delta)$ is as in Lemma 2.3. In light of Lemma 2.3 this definition is independent of the choice of quasi-axes for g_1 and g_2. We also have that $\sim$ is an equivalence relation.

2.2. **Quasi-morphisms, homogenization, and norms.** A map $\varphi\colon G \to \mathbb{R}$ is called a *quasi-morphism* (of defect $D(\varphi)$) if

$$\sup_{g,g' \in G} |\varphi(gg') - \varphi(g) - \varphi(g')| = D(\varphi) < \infty.$$

Furthermore it is *homogeneous* if $\varphi(g^k) = k\varphi(g)$ for all integers $k \in \mathbb{Z}$ and $g \in G$. We denote by $\widetilde{\mathrm{QH}}(G)$ the space of unbounded quasi-morphisms modulo the subspace of bounded functions (which are also quasi-morphisms), which can naturally be identified with the space of homogeneous quasi-morphisms via the process of *homogenization*.

Lemma 2.5 (Homogenization)**.** *Suppose $\varphi\colon G \to \mathbb{R}$ is a quasi-morphism. Then the homogenization $\widetilde{\varphi}(g) = \lim_{n\to\infty} \frac{\varphi(g^n)}{n}$ is a homogeneous quasi-morphism of defect $D(\widetilde{\varphi}) \le 4D(\varphi)$. Moreover $|\widetilde{\varphi}(g) - \varphi(g)| \le D(\varphi)$ for all $g \in G$ and the homogenization is non-trivial if and only if φ is unbounded.*

Proof. For a proof see [Cal09, Lemma 2.21]. □

A key property of non-trivial homogeneous quasi-morphisms is their conjugacy invariance, which in particular means that their absolute values define conjugacy-invariant quasi-norms in the sense of [BIP08].

Lemma 2.6 (Conjugation invariance)**.** *Suppose $\varphi\colon G \to \mathbb{R}$ is a homogeneous quasi-morphism. Then $\varphi(g) = \varphi(hgh^{-1})$ for all $g, h \in G$.*

Proof. Using homogeneity as well as the quasi-morphism property twice we have:

$$\begin{aligned} n|\varphi(g) - \varphi(hgh^{-1})| &= |\varphi(g^n) + \varphi(hg^{-n}h^{-1})| \\ &\leq |\varphi(g^n) + \varphi(h) + \varphi(g^{-n}) + \varphi(h^{-1})| + 2D(\varphi) \\ &= |\varphi(g^n) + \varphi(h) - \varphi(g^n) - \varphi(h)| + 2D(\varphi) = 2D(\varphi). \end{aligned}$$

Letting $n \to \infty$ we conclude that the left hand side must vanish proving the lemma. □

Corollary 2.7. *A uniformly perfect group does not admit an unbounded quasi-morphism.*

We now recall how to bulid *conjugacy-invariant norms* out of homogeneous quasi-morphisms as in [BIP08]. Suppose $\varphi\colon G \to \mathbb{R}$ is a non-trivial homogeneous quasi-morphism of defect $D(\varphi)$. Define $\nu_\varphi(g) = |\varphi(g)| + D(\varphi)$ for any non-trivial $g \in G$ and set $\nu(e) = |\varphi(g)| = 0$.

Lemma 2.8 (Conjugation-invariant norms)**.** *For any non-trivial homogeneous quasi-morphism the function ν_φ is an unbounded conjugation-invariant norm.*

Proof. Since conjugation preserves non-trivial elements, the conjugation invariance follows from Lemma 2.6. Assume that g, h and their product are non-trivial. Then we have

$$|\varphi(gh)| - |\varphi(g)| - |\varphi(h)| \leq |\varphi(gh) - (\varphi(g) + \varphi(h))| \leq D(\varphi)$$

and hence

$$\nu_\varphi(gh) - (\nu_\varphi(g) + \nu_\varphi(h)) = |\varphi(gh)| - |\varphi(g)| - |\varphi(h)| - D(\varphi) \leq 0.$$

Or in other words $\nu_\varphi(gh) \leq \nu_\varphi(g) + \nu_\varphi(h)$. The other case is more easily verified and we deduce that ν_φ is a conjugacy-invariant norm that is unbounded if φ is non-trivial. □

Example 2.9 (Fragmentation Norm)**.** *For a closed manifold M of dimension n, it is well known that any diffeomorphism $f \in \mathrm{Diff}_0(M)$ can be written as a product of diffeomorphisms supported on balls (see eg.* [Man16]*). Such a factorisation is called a* fragmentation*. We define the fragmentation norm*

$$\|f\|_{Frag} = \min\{N \mid f = h_1 \cdots h_N\ ,\ \mathrm{supp}(h_i) \subset U_i \cong B^n\}.$$

As shown by [BIP08] this norm is universal in the sense that any conjugacy-invariant norm on $\mathrm{Diff}_0(M)$ satisfies

$$\nu \leq C_\nu \|f\|_{Frag}.$$

In particular the existence of an unbounded norm is equivalent to the unboundedness of the fragmentation norm. Another important pseudo-norm is the stable commutator length.

Example 2.10 (Stable Commutator Length)**.** *Let G be a perfect group (for example* $\mathrm{Diff}_0(M)$ *where M is a closed manifold). Then the* commutator length *is defined to be*

$$\mathrm{cl}(g) = \min\{N \mid g = [f_1, h_1] \cdots [f_N, h_N]\},$$

where $[f, h]$ denotes the commutator of two elements in G. It is natural to consider the stable commutator length

$$\mathrm{scl}(g) = \lim_{n\to\infty} \frac{\mathrm{cl}(g^n)}{n}.$$

The commutator length is a conjugacy-invariant pseudo-norm and hence so is $\mathrm{scl}(g)$*. By adding some positive number to the value of* $\mathrm{scl}(g)$ *for any non-trivial $g \neq e$, this can easily be made into a conjugate invariant norm.*

Whilst the fragmentation norm detects unboundedness for diffeomorphism groups, it is an open question whether the same is true of stable commutator lengths.

2.3. **Actions on hyperbolic spaces and counting quasi-morphisms.** Consider an isometric action of a group G on a δ-hyperbolic graph considered with the path metric (X, d_X). Fujiwara [Fuj98] described certain "counting quasi-morphisms". These generalise the counting quasi-morphisms of Brooks [Bro81] for free groups, whereby one counts non-overlapping copies of some word. While Fujiwara assumes in [Fuj98] that the group action is properly discontinuous, this is not required for the results we use (compare also [BF02] for a discussion of this point). We follow the notation of [BF02] in our description below.

Definition 2.11 (Counting paths)**.** Let $w \in X$ be any oriented path of length $|w|$ and let $0 < W < |w|$. For any two vertices $x, y \in X$ we set

$$c_{w,W}(x, y) = d_X(x, y) - \inf_{\alpha}(|\alpha| - W|\alpha|_w),$$

where $|\alpha|_w$ denotes the maximal number of non-overlapping copies of translates of w under the action and the infimum is taken over all oriented paths from x to y.

In the case of a tree the definition above reduces to counting oriented subpaths of the unique geodesic from x to y. Now choose a base point $x_0 \in X$ and define $h_{w,W} \colon G \to \mathbb{R}$ by setting

$$h_{w,W}(g) = c_{w,W}(x_0, gx_0) - c_{w^{-1},W}(x_0, gx_0).$$

Note that the definition above ensures that the map is anti-symmetric with respect to taking inverses. Furthermore, the hyperbolicity implies that the resulting map is a quasi-morphism.

Proposition 2.12 ([Fuj98, Proposition 3.10])**.** *The map $h_{w,W} \colon G \to \mathbb{R}$ is a quasi-morphism whose defect is bounded by some D depending only on w, W, and δ.*

Note that there is no claim that the quasi-morphism described in Proposition 2.12 above is unbounded. In order to achieve this one needs the existence of elements satisfying the condition in Definition 2.4. Following [Fuj98] one fixes W sufficiently large and considers only w such that $|w| > W$, in which case we simply write h_w for $h_{w,W}$.

Following Bestvina–Fujiwara [BF02] for every $f \in G$ choose a geodesic γ_f from x_0 to $f(x_0)$. We write f^a for the path obtained by concatenation of $\gamma_f, f(\gamma_f), f^2(\gamma_f), \ldots, f^{a-1}(\gamma_f)$.

With this notation, we now have

Theorem 2.13 ([BF02, Proposition 5]). *Let G act on a δ-hyperbolic graph X by isometries. Suppose that a hyperbolic element satisfies $f \not\sim f^{-1}$. Then there is $a > 0$ such that h_{f^a} is unbounded and grows linearly on $\langle f \rangle$. In particular its homogenization is non-trivial on $\langle f \rangle$.*

In order to find elements satisfying this property Bestvina–Fujiwara instead show that it is sufficient to find *any* two hyperbolic elements such that $g_1 \not\sim g_2$.

Theorem 2.14 ([BF02, Theorem 1 and Proposition 2]). *Suppose that $g_1, g_2 \in G$ act hyperbolically on X and that $g_1 \not\sim g_2$. Then there are hyperbolic elements satisfying $f \not\sim f^{-1}$.*

Moreover in this case the space of unbounded and homogeneous quasimorphisms is infinite dimensional.

2.4. **Curve graphs.** In this section we collect some basic results on (usual) curve graphs. Throughout we denote by $\mathcal{C}(S)$ the *curve graph* of the (finite-type) surface S, and by $\mathcal{NC}(S)$ the *non-separating curve graph* of the surface S. The curve graph is the 1-skeleton of the *curve complex* introduced by W. J. Harvey [Har81].

The vertex set of $\mathcal{C}(S)$ (respectively $\mathcal{NC}(S)$) is the set of isotopy classes of (non-separating) simple closed curves not homotopic to a point i.e. *essential*, and not homotopic into a puncture i.e. *non-peripheral*. Edges connect two distinct vertices precisely when they admit disjoint representatives.

As we frequently need to use both actual curves and isotopy classes we adopt the following notational convention.

Convention. We use Greek letters for actual simple closed curves on S and Latin letters for isotopy classes. Furthermore all curves are smooth.

For two curves a and b we may define the *geometric intersection number* $i(a, b)$ of a and b to be the minimal possible value of $|\alpha \cap \beta|$ where α and β are transverse, and, are representatives of the isotopy classes a and b respectively. Therefore $i(a, b) = 0$ if and only if a and b are adjacent vertices.

When α is an essential and non-peripheral simple closed curve on S and $P \subset S$ is a set of points disjoint from α we denote by $[\alpha]_{S-P}$ the isotopy class defined by α on $S - P$.

For a pair of transverse curves α and β disjoint from a finite subset $P \subset S$, we say that α and β are in *minimal position* in $S - P$ if $|\alpha \cap \beta|$ is minimal among the representatives of $[\alpha]_{S-P}$ and $[\beta]_{S-P}$. This is equivalent to the well-known topological condition known as the bigon criterion. A *bigon* of α and β in $S-P$ is a complementary region of $\alpha \cup \beta$ in $S-P$ that is homeomorphic to a disk and bounds exactly one subarc of α and one subarc of β.

Lemma 2.15 (Bigon Criterion)**.** *For transverse simple closed curves α and β we have that α and β are in minimal position in $S-P$ if and only if there are no bigons of α and β in $S-P$.*

Proof. We refer to [FM12, Proposition 1.7]. □

For any finite set $P \subset S$ there is a well-defined forgetful map

$$\mathcal{NC}(S-P) \to \mathcal{NC}(S),$$

which is 1-Lipschitz. We can rephrase the Lipschitz property of the forgetful map by saying that for any non-separating α and β which are disjoint from P, we have

$$d_{\mathcal{NC}(S-P)}([\alpha]_{S-P}, [\beta]_{S-P}) \geq d_{\mathcal{NC}(S)}([\alpha]_S, [\beta]_S).$$

Remark 2.16. *These forgetful maps are the reason why we work with non-separating curve graphs and the graphs $\mathcal{C}^s(S-P)$ whose vertices are all isotopy classes of curves which are still essential as curves on S, where one has a similar forgetful map $\mathcal{C}^s(S-P) \to \mathcal{C}^s(S)$ which is 1-Lipschitz and surjective. When S is closed we have $\mathcal{C}(S) = \mathcal{C}^s(S)$.*

In general the natural inclusion of the non-separating curve graph $\mathcal{NC}(S)$ into $\mathcal{C}(S)$ is not a quasi-isometric embedding (in fact it is arbitrarily distorted, compare [MS13]). However if S is a surface of genus at least two with at most one marked point then one can easily arrange curves to have the same distance in both graphs in the sense of the following lemma, which is well known.

Lemma 2.17. *Let S be a closed surface of genus at least two with at most one puncture. Then given any $a, b \in \mathcal{NC}(S-p)$ we have that*

$$d_{\mathcal{C}(S)}(a, b) = d_{\mathcal{NC}(S)}(a, b).$$

Proof. Because there is at most one puncture we observe that if there is an essential simple closed curve γ that is disjoint from $\alpha, \beta \subset S$ such that $\alpha \cap \beta \neq \emptyset$ then there is a non-separating γ' which is disjoint from α and β. Using this repeatedly by induction we may convert a geodesic in $\mathcal{C}(S)$ to one in $\mathcal{NC}(S)$. □

2.5. **Subsurface Projections.** We use the notion of *subsurface projection to annuli* defined by Masur and Minsky in [MM00]. We briefly recall the necessary notions here and we refer the reader to [MM00, Section 2.4] for details.

Let $P \subset S$ be finite. Let Y be an isotopy class of a closed annulus in $S-P$ (isotopies rel P) with essential and non-peripheral core curve β in $S-P$.

We simply call this an *annulus* in $S-P$. Then Y is determined by the conjugacy class of the subgroup $\pi_1(Y)$ in $\pi_1(S-P)$. We may endow $S-P$ with a complete finite area hyperbolic metric, then we see that the metric pulls back to the universal cover $\widetilde{S-P}$ and is isometric to the hyperbolic plane $\mathbb{H}^2$. Similarly the cover $(S-P)_Y$ of $S-P$ corresponding to $\pi_1(Y)$ inherits a similar metric, which can be compactified to a closed annulus $\overline{(S-P)_Y}$ in much the same way that $\mathbb{H}^2$ compactifies to a closed disk.

Given an essential non-peripheral curve $v=[\alpha]_{S-P}$ in $S-P$ we may consider the preimage $\widetilde{\alpha} \subset (S-P)_Y$ and the closure $\overline{\alpha} \subset \overline{(S-P)_Y}$. Assuming $[\beta]_{S-P} \neq v$ we have that $\overline{\alpha}$ consists of an infinite number of closed intervals, though only finitely many connect both boundary components of $\overline{(S-P)_Y}$. Note that any isotopy of α rel P lifts to an isotopy of $\overline{\alpha}$ rel $\partial\overline{(S-P)_Y}$. This motivates the following definition. We define the graph $\mathcal{C}(Y)$ in the following way. The vertices are the ambient isotopy classes (rel the boundary) of properly embedded arcs that connect both boundary components of $\overline{(S-P)_Y}$. Edges connect two distinct vertices precisely when they can be realised disjointly in the interior of $\overline{(S-P)_Y}$. For $a,b \in \mathcal{C}(Y)$ we define $|a.b|$ to be the smallest possible $|\alpha \cap \beta|$ in the interior of the annulus between transverse (in the interior) representatives α and β of a and b respectively.

Lemma 2.18 ([MM00, Section 2.4])**.** *Distinct vertices $a,b \in \mathcal{C}(Y)$ satisfy*

$$d(a,b) = |a.b| + 1.$$

We may define $\kappa_Y(v) \subset \mathcal{C}(Y)$ by taking the finitely many arcs of $\overline{\alpha}$ that connect both boundary components of $\partial\overline{(S-P)_Y}$. This is always a complete subgraph if non-empty. Notice that $\kappa_Y(v)$ is non-empty if and only if α cannot be isotoped to be disjoint from β, or equivalently, $[\alpha]_{S-P}$ and $[\beta]_{S-P}$ are not adjacent vertices.

If v and w are curves that are not adjacent to $[\beta]_{S-P}$ then we define

$$d_\beta(v,w) := \mathrm{diam}_{\mathcal{C}(Y)}(\kappa_Y(v) \cup \kappa_Y(w)).$$

A crucial tool is the following Lipschitz property for twists, which is [MM00, Lemma 2.3]. Whenever $[\beta]=b$ then we define d_b to be equal to d_β.

Lemma 2.19 (Lipschitz projection)**.** *Let S be a surface of finite type. Suppose that*

$$a = v_0, \ldots, v_k = a',$$

is a path in $\mathcal{C}(S)$ such that $k>0$ and each v_i is not adjacent to b. Then

$$d_b(a,a') \leq k.$$

Proof. Each v_i has non-empty $\kappa_Y(v_i)$. It is straightforward to see that for adjacent v_i and v_{i+1} we have that $d_b(v_i,v_{i+1}) \leq 1$ and so the result follows by induction on k. $\square$

We use d_β to measure the amount of twisting between two curves around β. We need a couple of lemmas about the effect on d_β under Dehn twists.

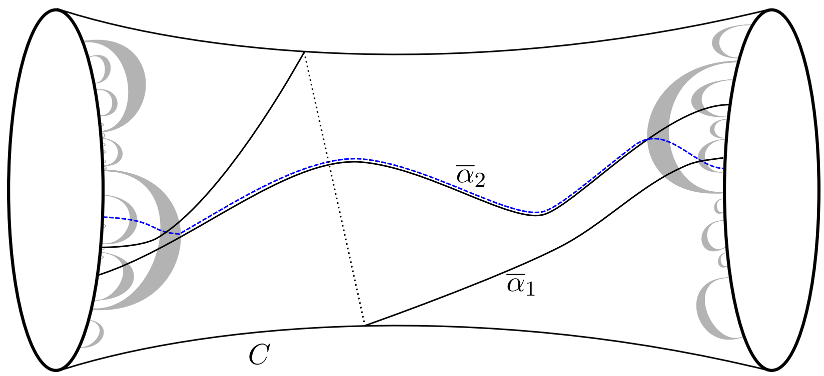

FIGURE 1. The proof of Lemma 2.21. The shaded regions make up $\widetilde{N}$. In dotted blue is $\Phi\overline{\alpha}_2$.

A *Dehn twist* of S about β is a homeomorphism $S \to S$, well defined up to isotopy, which is constructed by taking a closed regular neighborhood N of β, parametrising it as $S^1 \times [0,1]$, and then performing a homeomorphism supported on N using the map $(x,t) \mapsto (xe^{2\pi i t}, t)$. Whenever β is isotopic to β' then the corresponding Dehn twists are isotopic, therefore we simply write T_b for a Dehn twist about $b = [\beta]$. Our surfaces are orientable so the usual convention is that positive Dehn twists T_b^{+1} turn left and negative Dehn twists T_b^{-1} turn right. See [FM12, Chapter 3] for more details.

Lemma 2.20. *Suppose a intersects b essentially as isotopy classes of curves on $S - p$. Then for $n \neq 0$ we have that*

$$d_b(a, T_b^n a) = 2 + |n|.$$

Proof. We refer to [MM00, Equation (2.6)]. □

Lemma 2.21. *Let b_- and b_+ be distinct isotopy classes of disjoint curves on $S - p$. Then*

$$d_{b_-}(a, T_{b_+}^n a) \leq 3,$$

for any $n \in \mathbb{Z}$ and any isotopy class of curve a intersecting b_- essentially.

Proof. The lemma is straightforward if a is disjoint from b_+ so we now suppose otherwise.

Let $Z \to S - p$ be the annular cover corresponding to the isotopy class b_-, and let $\overline{Z}$ be its closure as discussed in the beginning of Section 2.5.

Let α, β_-, and β_+ represent a, b_-, and b_+ respectively such that each pair of these curves are in minimal position. Let $\overline{\alpha}$ be the closure of a lift of α to Z which connects the two boundary components of $\overline{Z}$.

We can choose a representative φ of $T_{b_+}^n$ which is supported on a small open neighborhood N of β_+. We write $\widetilde{N}$ for the preimage of N in Z. Now $Z - \widetilde{N}$ has infinitely many connected components only one of which is not simply connected namely the one containing the homeomorphic lift of β_-

(recall that β_- and β_+ do not intersect). We write C for this component of $Z - \widetilde{N}$. We have that $Z - C$ is an infinite disjoint union of open disks, each of them incident to the boundary of $\overline{Z}$.

Because β_+ and α are in minimal position we have that $\overline{\alpha}$ is a concatenation

$$\overline{\alpha} = \gamma_1 * \gamma * \gamma_2$$

where $\gamma \subset C$ and $\gamma_i \subset D_i$ where D_i are connected components of $Z - C$.

We can choose a lift $\Phi\colon Z \to Z$ of φ which is the identity on C (since φ is the identity on β_-). Therefore we have that Φ preserves each connected component of $Z - C$. From this we can see that $\Phi(\overline{\alpha})$ is a concatenation

$$\Phi(\overline{\alpha}) = \gamma_1' * \gamma * \gamma_2'$$

where $\gamma_i' \subset D_i$. Up to isotopy with fixed endpoints two arcs in a disk intersect in at most one point. This shows that

$$i(\overline{\alpha}, \Phi(\overline{\alpha})) \leq 2,$$

so by Lemma 2.18 we are done. The slightly more general case of considering $\overline{\alpha}_1$ and $\Phi\overline{\alpha}_2$, where $\overline{\alpha}_1$ and $\overline{\alpha}_2$ are the closures of two different lifts of α, is similar. □

2.6. **Pseudo-Anosovs and point-pushing maps.** Finally we recall the Nielsen–Thurston Classification of mapping classes and the dynamics of pseudo-Anosov mapping homeomorphisms. We refer the reader to [FM12] for a proof and more background on pseudo-Anosov maps.

Theorem 2.22 (Nielsen–Thurston Classification)**.** *Let Σ be a surface of finite type and let $f \in \mathrm{Mcg}(\Sigma)$ be a mapping class. Then (at least) one of the following occurs:*

- *(Finite Order): f^n is isotopic to the identity for some $n > 0$.*
- *(Reducible): The map fixes a non-peripheral multi-curve C up to isotopy.*
- *(Pseudo-Anosov): The map f is isotopic to a homeomorphism φ^{Th} that preserves a pair of (singular) measured foliations $\mathcal{F}_u, \mathcal{F}_s$ that are exponentially contracted resp. expanded under iterates of φ^{Th}.*

The representative of a pseudo-Anosov mapping class guaranteed by the theorem is called the *Thurston (or dynamical) representative* and is a smooth diffeomorphism except at finitely many points. The foliations $\mathcal{F}_u, \mathcal{F}_s$ are called the *unstable* resp. *stable foliations* of f. Moreover, the (singular) foliations $\mathcal{F}_u, \mathcal{F}_s$ have singularities that are of "prong-type" (cf. Figure 2) with 1-prongs allowed only at punctures; see e.g. [FM12, 13.2.3] for more details.

The foliations also define a singular flat metric on the surface S, in which the stable foliation becomes horizontal and the unstable becomes vertical. In such a singular flat metric, in general a geodesic consists of a concatenation of straight segments meeting with angle $\geq \pi$ on both sides. Typically there

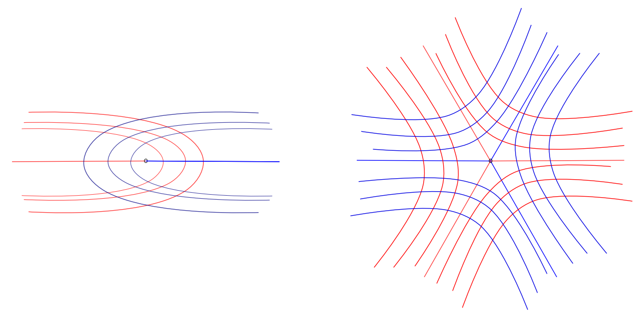

FIGURE 2. A one-pronged singularity is shown on the left and a three-pronged singularity on the right. The red/blue lines indicate the stable/unstable foliations respectively.

is a single geodesic in a given homotopy class which passes through singular points. However there can be geodesics which do not pass through singular points at all. Such curves are called *cylinder curves* as they come in parallel families foliating (metric, flat) cylinders. We use the fact that such cylinder curves exist on our singular flat metrics:

Theorem 2.23. *For any singular flat metric on S with finitely many singularities, each with an integer multiple of π, there is a cylinder curve.*

Proof. This was originally proved in [Mas86] for translation structures (i.e. all monodromies are translations). See also [Vor05] for an effective version. Given any singular flat surface X in our setting, one can consider a cover $Y \to X$, branched over the singularities, so that Y is a translation structure. Hence, there is a cylinder curve on Y, which is in particular disjoint from all branch points of the cover. Its image in X does not intersect the singular set either, and has constant slope as $Y \to X$ is a local isometry away from the singularities. Hence, this yields the desired cylinder curve. $\square$

We also want to emphasise at this point that an affine automorphism of a singular flat structure (e.g. a Thurston representative of a pseudo-Anosov map) preserves locally straight lines and parallelelism, and therefore maps a cylinder curve to another cylinder curve.

Recall that if S is a surface of genus $g \geq 2$, and $P \subset S$ is a non-empty finite set of points, then there is a *Birman exact sequence*

$$1 \to \mathcal{P} \to \mathrm{PMcg}(S - P) \to \mathrm{Mcg}(S) \to 1$$

where the kernel $\mathcal{P}$ consists of *point-pushing maps*. We refer the reader to [FM12, Section 4.2] for details. If $P = \{p\}$ is a singleton then the kernel $\mathcal{P}$ can be identified with the fundamental group $\pi_1(S, p)$:

$$1 \to \pi_1(S, p) \to \mathrm{Mcg}(S - p) \to \mathrm{Mcg}(S) \to 1$$

The following theorem shows that there is a large supply of point-pushing pseudo-Anosov mapping classes in the point-pushing subgroup. A (not simple) closed curve γ is *filling* if there is no non-trivial homotopy class α such that, after homotopies of both curves, α and γ are disjoint.

Theorem 2.24 (Kra [Kra81]). *The point-pushing map $\gamma \in \pi_1(S, p)$ defines a pseudo-Anosov mapping class in $\mathrm{Mcg}(S - p)$ if and only if γ is a filling curve on S.*

3. A hyperbolic graph

Throughout this section we let S be a closed surface of genus $g \geq 2$ unless otherwise stated.

Definition 3.1.
1. Let $\mathcal{NC}^\dagger(S)$ be the graph whose vertices correspond to non-separating simple closed curves in S. Two such vertices are joined by an edge precisely when the corresponding curves are disjoint.
2. Let $\mathcal{C}^\dagger(S)$ be the graph whose vertices correspond to essential simple closed curves in S. Two such vertices are joined by an edge precisely when the corresponding curves are disjoint.

We denote by $d^\dagger$ the distance in $\mathcal{NC}^\dagger(S)$ and by $d^\dagger_{\mathcal{C}}$ the distance in $\mathcal{C}^\dagger(S)$.

The natural action of $\mathrm{Diff}_0(S)$ on the set of curves induces an isometric action on $\mathcal{C}^\dagger(S)$. We wish to show that $\mathcal{C}^\dagger(S)$ is hyperbolic. In order to do this we first show that $\mathcal{NC}^\dagger(S)$ is hyperbolic and utilise the fact that the two graphs are quasi-isometric. Though this theorem can be proved in several ways, for brevity and convenience we use the four-point condition. See [ABC+91] for various definitions of hyperbolicity and proofs of their equivalence.

Definition 3.2. For points x, y, w of a metric space (X, d) the Gromov product is defined to be

$$\langle x, y\rangle_w := \frac{1}{2}(d(w, x) + d(w, y) - d(x, y)).$$

We say that X is δ-hyperbolic if for all $w, x, y, z \in X$ we have that

$$\langle x, z\rangle_w \geq \min\{\langle x, y\rangle_w, \langle y, z\rangle_w\} - \delta.$$

We require the following result due to Alexander Rasmussen.

Theorem 3.3 ([Ras17]). *There is a number $\delta > 0$ such that $\mathcal{NC}(\Sigma)$ is δ-hyperbolic whenever Σ is a finite-type surface with positive genus.*

We wish to "approximate" $\mathcal{NC}^\dagger(S)$ with (usual) non-separating curve graphs of finite-type surfaces. For $\alpha \in \mathcal{NC}^\dagger(S)$ disjoint from a finite subset $P \subset S$ we remind the reader that we write $[\alpha]_{S-P}$ for the isotopy class of α in $S - P$. We also use this notation for maps $S \to S$ later. The following Lemma 3.4 is key. Recall the notion of minimal position from Section 2.4.

Lemma 3.4. *Suppose that $\alpha, \beta \in \mathcal{NC}^\dagger(S)$ are transverse, and that α and β are in minimal position in $S - P$ where $P \subset S$ is finite and disjoint from α and β. Then*

$$d_{\mathcal{NC}(S-P)}([\alpha]_{S-P}, [\beta]_{S-P}) = d^\dagger(\alpha, \beta).$$

We emphasize that α and β in this lemma need not be in minimal position when seen as curves on S, but only when seen as curves on $S-P$ (i.e. bigons between α and β in S are allowed provided they contain at least one point of P, in which case they are not bigons in $S - P$).

The proof of Lemma 3.4 is a corollary of the following two lemmas, which are stated in a broader context.

Lemma 3.5. *Suppose that $\alpha_1, \dots, \alpha_n$ are curves that are pairwise in minimal position in $S - P$. Let $\beta_1, \dots, \beta_m$ be a collection of curves that are disjoint from P. Then the β_i can be isotoped in $S - P$ such that any two elements of $\{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m\}$ are all pairwise in minimal position.*

Proof. By induction it suffices to show this for $m = 1$. After an isotopy of β_1, we may assume β_1 is transverse to each α_k.

So suppose that $\{\alpha_1, \dots, \alpha_n, \beta_1\}$ contains two curves which are not in minimal position in $S-P$. As we assume that the α_i are in pairwise minimal position by Lemma 2.15 (Bigon Criterion) there is then a bigon B bounded by subarcs of β_1 and some α_j in $S - P$. We may assume that the bigon is innermost among all bigons between β_1 and any α_k. Pushing β_1 past this bigon (as in the proof of the bigon criterion, compare [FM12]) decreases $\sum i(\beta_1, \alpha_j)$ by exactly two. Hence this process terminates after finitely many steps, producing a curve isotopic to β_1 in $S-P$ which is in minimal position with respect to each α_i. Since at each stage the curves α_i are fixed the lemma follows. □

Lemma 3.6. *Let $\alpha, \beta \in \mathcal{C}^\dagger(S)$ and $P \subset S$ be a finite set. Then we may find a geodesic $\alpha = \nu_0, \dots, \nu_k = \beta$ such that $\nu_i \cap P = \emptyset$ for all $0 < i < k$.*

Proof. Pick any geodesic $\alpha = \nu'_0, \dots, \nu'_k = \beta$ and set $\nu_0 = \nu'_0$ and $\nu_k = \nu'_k$. Then inductively find a perturbation ν_i of each ν'_i (for $0 < i < k$) such that ν_i is disjoint from P, ν_{i-1} and ν'_{i+1}. □

Proof of Lemma 3.4. We first prove

$$d_{\mathcal{NC}(S-P)}([\alpha]_{S-P}, [\beta]_{S-P}) \geq d^\dagger(\alpha, \beta).$$

Indeed, by Lemma 3.5 any geodesic between $[\alpha]_{S-P}$ and $[\beta]_{S-P}$ is realised by vertices $\alpha = \alpha_0, \dots, \alpha_k = \beta$ in $S - P$ pairwise in minimal position. In particular α_i is disjoint from α_{i+1}, and each α_i is non-separating, so we are done.

We now prove

$$d_{\mathcal{NC}(S-P)}([\alpha]_{S-P}, [\beta]_{S-P}) \leq d^\dagger(\alpha, \beta).$$

Indeed, by Lemma 3.6 we can find a geodesic $\alpha = \nu_0, \dots, \nu_k = \beta$ disjoint from P then simply consider the path $[\nu_i]_{S-P}$ in $\mathcal{NC}(S - P)$. □

Theorem 3.7. *The graph $\mathcal{NC}^\dagger(S)$ is hyperbolic.*

Proof. In fact we prove that $\mathcal{NC}^\dagger(S)$ is $(\delta+2)$-hyperbolic, where δ satisfies Theorem 3.3. More precisely we show for arbitrary vertices $\mu, \alpha, \beta, \gamma \in \mathcal{NC}^\dagger(S)$ we have that

$$\langle \alpha, \gamma \rangle_\mu \geq \min\{\langle \alpha, \beta \rangle_\mu, \langle \beta, \gamma \rangle_\mu\} - \delta - 2. \tag{1}$$

In order to prove this we relate the vertices $\mu, \alpha, \beta, \gamma$ with vertices of some $\mathcal{NC}(S-P)$ for some finite $P \subset S$. The first obstacle is to remove the pathology of pairs of vertices of $\mathcal{NC}^\dagger(S)$ that are not transverse.

To do so, we find vertices α', β', γ' with properties (i)–(iii) below. One should think of the primed curves as perturbations of the unprimed curves.

We set $\mu' = \mu$.

(i) We have that $d^\dagger(\alpha, \alpha'), d^\dagger(\beta, \beta'), d^\dagger(\gamma, \gamma') \leq 1$,
(ii) the vertices $\mu', \alpha', \beta', \gamma'$ are transverse, and
(iii) $d^\dagger(\kappa', \lambda') \leq d^\dagger(\kappa, \lambda)$ for any $\kappa, \lambda \in \{\mu, \alpha, \beta, \gamma\}$.

Construction of α', β', γ'**:** We now explain how to ensure the above three items. First, for each pair of $\mu, \alpha, \beta, \gamma$, fix one geodesic between that pair, and let F be a finite set of vertices of $\mathcal{NC}^\dagger(S)$ that contains the vertices of these geodesics. We now find perturbations α', β', γ' of α, β, γ respectively, such that if $\eta \in F$ and η is disjoint from α (or β or γ), then η is also disjoint from α' (or β' or γ')—this ensures item (iii) above. This is easy to ensure because F is only a finite set and we can perturb α in such a way that it remains disjoint from the finite set of such curves $\eta \in F$, and similarly for β and γ. Finally to ensure item (i) we first find an α'' disjoint from and isotopic to α and then find a small enough perturbation α' of α'', such that item (ii) holds (and similarly do this for β and γ).

We now choose a finite subset $P \subset S$ large enough such that any bigon between a pair of $\mu', \alpha', \beta', \gamma'$ contains a point of P. There are only finitely many such bigons so P exists. By Lemma 2.15 (Bigon Criterion) this ensures that $\mu', \alpha', \beta', \gamma'$ are pairwise in minimal position in $S - P$.

By Lemma 3.4 we have that

$$d^\dagger(\kappa', \lambda') = d_{\mathcal{NC}(S-P)}([\kappa']_{S-P}, [\lambda']_{S-P}), \tag{2}$$

moreover by construction we see that

$$d^\dagger(\mu, \kappa) - 1 \leq d^\dagger(\mu, \kappa'), \text{ and} \tag{3}$$

$$d^\dagger(\kappa, \lambda) - 2 \leq d^\dagger(\kappa', \lambda') \leq d^\dagger(\kappa, \lambda) \tag{4}$$

whenever $\kappa, \lambda \in \{\alpha, \beta, \gamma\}$. By (2), (3), and (4) above, we may approximate the four-point condition for $\mu, \alpha, \beta, \gamma \in \mathcal{NC}^\dagger(S)$ using the four-point condition for $\mu', \alpha', \beta', \gamma'$ and $\mathcal{NC}(S-P)$ via Lemma 3.4, and so the theorem is proved. ◻

We now deduce the hyperbolicity of $\mathcal{C}^\dagger(S)$. The following is the analogue Lemma 2.17.

Lemma 3.8. *Let S be a closed surface of genus at least two. Then given any $\alpha, \beta \ \in \mathcal{NC}^\dagger(S)$ we have that*

$$d^\dagger_{\mathcal{C}}(\alpha,\beta) = d^\dagger(\alpha,\beta).$$

Proof. Arguing exactly as in Lemma 2.17 we can replace any geodesic between α,β by one of the same length that consists only of sequence of edges between non-separating curves except at the end points. $\square$

In view of Lemma 2.17 we no longer distinguish between the distances in $\mathcal{C}^\dagger(S)$ and $\mathcal{NC}^\dagger(S)$ and denote henceforth both by $d^\dagger$.

Corollary 3.9. *Let S be a closed surface of genus at least two. Then the inclusion $\mathcal{NC}^\dagger(S) \to \mathcal{C}^\dagger(S)$ is a quasi-isometry and in particular $\mathcal{C}^\dagger(S)$ is hyperbolic.*

Proof. By Lemma 3.8 the inclusion $\mathcal{NC}^\dagger(S) \to \mathcal{C}^\dagger(S)$ is isometric. Since any curve has distance at most one from a non-separating one we deduce that the inclusion map is a quasi-isometry (with constant 1). $\square$

Another important fact is the $\mathcal{C}^\dagger(S)$ version of Lemma 3.4 whose proof is very similar if not the same:

Lemma 3.10. *Suppose that $\alpha,\beta \in \mathcal{C}^\dagger(S)$ are transverse, and that α and β are in minimal position in $S-P$ where $P \subset S$ is finite and disjoint from α and β. Then*

$$d_{\mathcal{C}^s(S-P)}([\alpha]_{S-P},[\beta]_{S-P}) = d^\dagger(\alpha,\beta).$$

4. Twists

We wish to construct two independent hyperbolic elements of $\mathrm{Diff}_0(S)$ for the action on $\mathcal{C}^\dagger(S)$. To show that they are indeed independent we use the notion of subsurface projection, see Section 2.5. To keep our situation simple we only consider projections to annuli to prove the main result though we are sure that other subsurface projections are useful.

We require the following lemma which is similar to one of Masur and Schleimer, compare the bottom claim of p. 19 and its proof in [MS13, Section 10]. The idea is that if two curves $\alpha,\gamma \in \mathcal{C}^\dagger(S-P)$ have large projection distance to an annulus with core curve β then this has consequences for the topology of $\alpha \cup \beta$ in $S-P$, provided that this pair is in minimal position in $S-P$ (see Section 2.4 for the definition of minimal position). Informally speaking, γ twists about β with respect to α.

Lemma 4.1. *Let $P \subset S$ be a finite subset and let $\alpha,\beta,\gamma \in \mathcal{C}^\dagger(S)$ be pairwise in minimal position in $S-P$. Suppose that*

$$d_\beta([\alpha]_{S-P},[\gamma]_{S-P}) \geq K \geq 4,$$

then there exist a closed annulus $\mathcal{Y} \subset S-P$ containing β, with $\partial\mathcal{Y}$ in minimal position with α and γ in $S-P$, and two subarcs $\varepsilon_1 \subset \alpha$ and $\varepsilon_2 \subset \gamma$ such that $|\varepsilon_1 \cap \varepsilon_2| \geq K-3$ and $\varepsilon_1,\varepsilon_2 \subset \mathcal{Y}$.

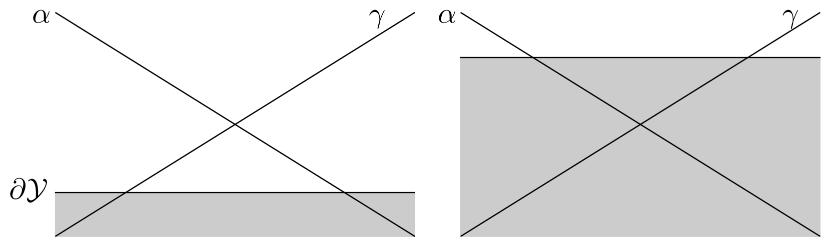

FIGURE 3. Pushing $\partial\mathcal{Y}$ past a triangle.

Proof. We follow the proof found on p. 19 and p. 20 of [MS13, Section 10]. We start with a closed regular neighborhood $\mathcal{Y}$ of β in $S-P$ and we can assume that its boundary components are in minimal position with α and γ in $S-P$. Now, as on [MS13, p. 19], we would like to ensure that any embedded triangle in $S-P$ formed by segments of $\partial\mathcal{Y}$, α and γ is contained in $\mathcal{Y}$. Masur and Schleimer achieve this by modifying α and γ by an isotopy, which we are not allowed to do because α and γ are fixed simple closed curves. Instead we do the following. For any such triangle found outside $\mathcal{Y}$ we may push $\partial\mathcal{Y}$ across the triangle in $S-P$, see Figure 3. We observe that $\partial\mathcal{Y}$ continues to be in minimal position with both α and γ in $S-P$. This process terminates after finitely many steps, because a new intersection point of $\alpha\cap\gamma$ is contained in $\mathcal{Y}$ each time this is performed. Therefore there exists a closed annulus $\mathcal{Y}$ containing β, such that $\partial\mathcal{Y}$ is in minimal position with α and γ in $S-P$, and such that any embedded triangle in $S-P$ formed by segments of $\partial\mathcal{Y}$, α and γ is contained in $\mathcal{Y}$.

We now show that this is the required $\mathcal{Y}$. Write $a=[\alpha]_{S-P}$ and $c=[\gamma]_{S-P}$. Let Y be the isotopy class in $S-P$ of $\mathcal{Y}$. By Lemma 2.18 there exist arcs δ^* of $\kappa_Y(a)$ and ε^* of $\kappa_Y(c)$ that intersect at least $K-1$ times in the interior of the annulus $\overline{Y}$, as described in Section 2.5. Now following p. 20 and Figure 10.3 of [MS13], the arcs δ^* and ε^* intersect at least $K-3$ times in the homeomorphic lift $\mathcal{Y}'$ of $\mathcal{Y}$ in $\overline{Y}$. Taking $\delta^*\cap\mathcal{Y}'$ and going back downstairs to $S-P$, this is the required subarc $\varepsilon_1\subset\alpha$, and ε_2 is similarly defined in terms of ε^* and γ. $\square$

We now abuse notation by writing $S-p=S-\{p\}$ and $S-p-q=S-\{p,q\}$ where $p\neq q\in S$. With little effort the following two lemmas can be generalised to the case of more marked points but we have chosen for now to keep our statements and proofs simple.

Lemma 4.2 (Puncturing keeps twists)**.** *Suppose that γ, γ', and β are essential curves that are pairwise in minimal position in $S-p$ and*

$$d_\beta([\gamma]_{S-p},[\gamma']_{S-p})=K\geq 7,$$

then for any $q\in S$ disjoint from γ, γ', and β, there exists $\beta'\subset S-p-q$ such that

(i) β and β' are disjoint, and $[\beta]_{S-p}=[\beta']_{S-p}$,

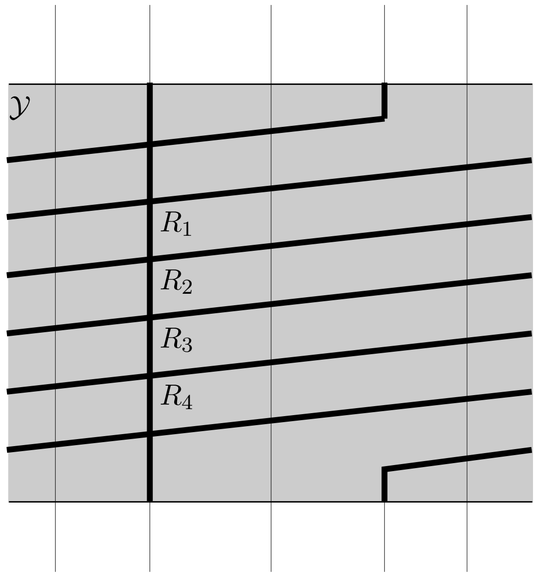

FIGURE 4. The rectangles R_i contained in the annulus $\mathcal{Y}$. These are components of $S - p - \varepsilon_1 - \varepsilon_2$ homeomorphic to disks. The subarcs ε_1 and ε_2 are in bold.

(ii) the simple closed curves γ, γ', β, and β' are pairwise in minimal position in $S - p$ and $S - p - q$, and
(iii) we have that

$$d_{\beta'}([\gamma]_{S-p-q}, [\gamma']_{S-p-q}) \geq \frac{K}{2}.$$

Proof. By Lemma 4.1 there is a closed annulus $\mathcal{Y} \subset S - p$ containing β and subarcs $\varepsilon_1 \subset \gamma$ and $\varepsilon_2 \subset \gamma'$ such that $|\varepsilon_1 \cap \varepsilon_2| \geq K - 3$. Now γ and γ' are in minimal position in $S - p$ so they must share no bigons in $\mathcal{Y}$, which implies that $S - p - \varepsilon_1 - \varepsilon_2$ contains at least $K - 5 \geq 2$ components with 4 sides i.e. at least $K - 5$ distinct rectangles $R_1, \ldots, R_m$ such that R_i is adjacent to R_{i+1}, and each R_i is inside $\mathcal{Y}$.

Now q is contained in one of the components of $S - \gamma - \gamma'$. Without loss of generality q does not belong to the rectangles $R_1, \ldots, R_J$ where $J \geq \frac{K}{2} - 3$. We pick β' to be the component of $\partial\mathcal{Y}$ which is closest to R_1, which is in minimal position with γ and γ' in $S - p$ by Lemma 4.1, and in $S - p - q$ automatically, which proves (ii). It is clear that β and β' are disjoint, and isotopic in $S - p$, which proves (i).

Now we prove (iii). On $S - p$ we have that β' is isotopic into the closed annulus $\mathcal{Y}'$ obtained by taking a closed regular neighborhood of the union of $R_1, \ldots, R_J$. We set $\varepsilon'_i = \varepsilon_i \cap \mathcal{Y}'$. Then $|\varepsilon'_1 \cap \varepsilon'_2| = J + 2$. Because minimal position holds in $S - p - q$ we may now consider the covering of $S - p - q$ corresponding to $\mathcal{Y}'$. We observe that $\mathcal{Y}'$ has a homeomorphic lift, containing lifts of ε_1' and ε_2' that intersect at least $J + 2 \geq \frac{K}{2} - 1$ times, which by Lemma 2.18 completes the proof. $\square$

Lemma 4.3 (Forgetting keeps twists)**.** *Suppose that α and α' are in minimal position in $S - q$ and there exists a curve β' in $S - p - q$ that is essential in $S - q$ such that*

$$d_{\beta'}([\alpha]_{S-p-q}, [\alpha']_{S-p-q}) \geq L \geq 4,$$

then

$$d_{\beta'}([\alpha]_{S-q}, [\alpha']_{S-q}) \geq L - 2.$$

Proof. By Lemma 4.1 there is a closed annulus $\mathcal{Y}'$ in $S - p - q$ containing subarcs $\varepsilon_1 \subset \alpha$ and $\varepsilon_2 \subset \alpha'$ such that $|\varepsilon_1 \cap \varepsilon_2| \geq L - 3$. It is clear that $\mathcal{Y}'$ is not null-homotopic in $S - q$ because it contains β'. Now α and α' are in minimal position in $S - q$ so they must share no bigons in $\mathcal{Y}'$.

We may consider the covering of $S - q$ corresponding to $\mathcal{Y}'$. We observe that $\mathcal{Y}'$ has a homeomorphic lift, containing lifts of ε_1 and ε_2 that intersect at least $L - 3$ times, which by Lemma 2.18 completes the proof. $\square$

5. Constructing isotopically-trivial diffeomorphisms that act hyperbolically

5.1. **Constructing hyperbolic elements.**

Let S be a hyperbolic closed orientable surface and $P \subset S$ be finite. Recall the definition of asymptotic translation length $|g|$ from Section 2.1. For $f \in \mathrm{Mcg}(S - P)$ we define $|f|$ to be the asymptotic translation length of the action of f on $\mathcal{C}^s(S - P)$, and for $\varphi \in \mathrm{Diff}(S)$ we define $|\varphi|$ similarly via its action on $\mathcal{C}^\dagger(S)$.

We now construct hyperbolic elements of $\mathrm{Diff}(S)$ on $\mathcal{C}^\dagger(S)$. We require the following landmark theorem of Masur and Minsky [MM99].

Theorem 5.1. *Depending only on the topology of $S - P$ there exists $c > 0$ such that for any pseudo-Anosov $f \in \mathrm{Mcg}(S - P)$ we have $|f| \geq c > 0$.*

Proof. By [MM99, Proposition 4.6], depending only on the topology of $S-P$, there exists $c > 0$ such that

$$d_{\mathcal{C}(S-P)}(f^n v, v) \geq c|n|,$$

for any $v \in \mathcal{C}(S - P)$ and $n \in \mathbb{Z}$. The same result follows immediately for the case of $\mathcal{C}^s(S - P)$ in place of $\mathcal{C}(S - P)$ and therefore $|f| \geq c > 0$. $\square$

Similar to our convention and notation with curves, we write $[\varphi]_{S-P}$ for the isotopy class of $\varphi \in \mathrm{Diff}(S)$ rel P. Whenever we write this, we also assert that $\varphi(P) = P$. We are now ready to state a general construction of hyperbolic elements of $\mathrm{Diff}(S)$ on $\mathcal{C}^\dagger(S)$.

Lemma 5.2. *Let $P \subset S$, $f \in \mathrm{Mcg}(S - P)$, and $\varphi \in \mathrm{Diff}(S)$ be such that $\varphi(P) = P$ and $f = [\varphi]_{S-P}$. Then for any $\alpha \in \mathcal{C}^\dagger(S)$ with $\alpha \subset S - P$ and any $i \in \mathbb{Z}$ we have that*

$$d_{\mathcal{C}^s(S-P)}([\alpha]_{S-P}, f^i[\alpha]_{S-P}) \leq d^\dagger(\alpha, \varphi^i\alpha). \tag{5}$$

Furthermore $|f| \le |\varphi|$. *In particular if* f *is pseudo-Anosov then* φ *is a hyperbolic element.*

Proof. We observe that $[\varphi^i \alpha]_{S-P} = f^i[\alpha]_{S-P}$. Given any $i \in \mathbb{Z}$ by Lemma 3.6 there exists a geodesic in $\mathcal{C}^\dagger(S)$ connecting α and $\varphi^i\alpha$ with each vertex disjoint from P. Consider the sequence of isotopy classes of these curves on $S - P$. This sequence is a path in $\mathcal{C}^s(S-P)$ of the same length, and this proves the first inequality.

Now we show that $|f| \le |\varphi|$. Given arbitrary $i \in \mathbb{Z}$ and $\alpha \in \mathcal{C}^\dagger(S)$ we claim that

$$|f| \le \frac{1}{i} d^\dagger(\alpha, \varphi^i\alpha).$$

If $\alpha \cap P = \emptyset$ then this is immediate by Equation 5. So now we assume otherwise i.e. $\alpha \cap P \neq \emptyset$. By Lemma 3.6 there exists a geodesic $\nu_0, \ldots, \nu_k$ between α and $\varphi^i\alpha$ such that whenever $0 < i < k$ then $\nu_i \cap P = \emptyset$. We now pick a sufficiently small perturbation α' of α about $\alpha \cap P$ in a neighborhood disjoint from ν_1 and $\varphi^{-i}\nu_{k-1}$, which is possible because the latter two curves are closed subsets disjoint from P. Hence the ν_i also connect α' and $\varphi^i\alpha'$. Therefore

$$d^\dagger(\alpha', \varphi^i\alpha') \le k = d^\dagger(\alpha, \varphi^i\alpha).$$

Since $\alpha' \subset S - P$ we obtain $i|f| \le d^\dagger(\alpha', \varphi^i\alpha') \le k$ as required.

Finally if f is pseudo-Anosov then we have $0 < |f|$ by Theorem 5.1 and therefore $0 < |\varphi|$ because $|f| \le |\varphi|$. $\square$

5.2. **Isotopically-trivial diffeomorphisms acting hyperbolically.** In this section, we construct hyperbolic elements of $\mathrm{Diff}_0(S)$ on $\mathcal{C}^\dagger(S)$. One robust way of finding these is via pseudo-Anosov maps of punctured surfaces in the following way.

Take any point-pushing pseudo-Anosov $f \in \mathrm{Mcg}(S-P)$. We may find a homeomorphism $\varphi \in \mathrm{Homeo}_0(S)$ such that $\varphi(P) = P$ and $f = [\varphi]_{S-P}$. A particularly useful choice for us is to pick φ a Thurston representative of f, see Section 2.6.

We may pick $\varphi' \in \mathrm{Diff}_0(S)$ such that $\varphi'(P) = P$ and φ' is a perturbation of φ in some (small) neighborhood of P (i.e. $\varphi = \varphi'$ outside this small neighborhood). By Alexander's trick we have that $[\varphi]_{S-P} = [\varphi']_{S-P}$. Lemma 5.2 shows that $\varphi' \in \mathrm{Diff}_0(S)$ acts hyperbolically on $\mathcal{C}^\dagger(S)$.

However, the following issue needs to be addressed. If the perturbation to construct φ' is too large then a priori $|\varphi'|$ is much larger than $|f|$. We would like the dynamics of φ' to mimic those of f in order to use mapping class group machinery. We deal with this in the next section.

5.3. **Two constructions for the main theorem.** We describe two general constructions of hyperbolic elements of $\mathrm{Diff}_0(S)$ on $\mathcal{C}^\dagger(S)$ which we use in Section 6 to prove Theorem 6.1 and therefore Theorem 1.2.

We abuse notation by writing $S - p = S - \{p\}$.

Proposition 5.3. *Let $p \in S$ and $f_1 \in \mathrm{Mcg}(S-p)$ be a point-pushing pseudo-Anosov. Let $\varphi_1^{\mathrm{Th}} \in \mathrm{Homeo}_0(S)$ be a Thurston representative of f_1 and let α_1 be a cylinder curve of a singular flat structure associated to φ_1^{Th} (see Theorem 2.23).*

Then given any number $n_1 > 0$ there is $\varphi_1 \in \mathrm{Diff}_0(S)$ such that

i) $[\varphi_1]_{S-p} = f_1$ and $|\varphi_1| \geq |f_1|$,
ii) whenever $0 \leq i \leq n_1$ then $\varphi_1^m \alpha_1$ and $\varphi_1^{m+i}\alpha_1$ are in minimal position in $S-p$ for any $m \in \mathbb{Z}$, and
iii) $A_1^\dagger := (\varphi_1^n \alpha_1)_n$ is a C-quasi-axis for φ_1 in $\mathcal{C}^\dagger(S)$ where $C = C(f_1, \alpha_1)$.

Proof. Let $\varphi_0 = \varphi_1^{\mathrm{Th}}$ be a Thurston representative of f_1. This is a homeomorphism of S which is smooth except at a finite set Q (containing p) corresponding to the prongs of the stable/unstable foliations of φ_0.

Since α_1 is a cylinder curve, and the Thurston representative φ_0 acts as an affine map on the singular flat structure, the curves $\varphi_0^i(\alpha_1)$ are also cylinder curves, hence in particular are simple closed geodesics. By the Gauss–Bonnet theorem two geodesics in a singular flat metric cannot bound a bigon in $S-p$ (note that there is only one singular point of angle π which is p) and therefore the curves $\varphi_0^i(\alpha_1)$ are pairwise in minimal position in $S-p$ by Lemma 2.15.

Now consider the set $\Gamma = \alpha_1 \cup \varphi_0\alpha_1 \cup \ldots \cup \varphi_0^{n_1}\alpha_1$. For every $q \in Q$ choose an embedded disk D_q such that $D_q \cap \Gamma = \emptyset$. This is possible since $Q \cap \Gamma = \emptyset$ and Γ is closed. Let U be the union of the D_q and let $\gamma_1 \in \mathrm{Diff}_0(S)$ be a perturbation of φ_0 such that $\varphi_1|_{S-U} = \varphi_0|_{S-U}$. By Alexander's trick and Lemma 5.2 we have that i) holds. Observe that $\varphi_1^i\alpha_1 = \varphi_0^i\alpha_1$ for any $i \in \{0, \ldots, n_1\}$ by construction hence α_1 and $\varphi_1^i\alpha_1$ are in minimal position in $S-p$ and hence ii) holds.

Now we prove iii). By Theorem 5.1 and Lemma 5.2 we have that

$$0 < |f_1| \leq |\varphi_1|.$$

Write $a_1 = [\alpha_1]_{S-p}$. We have that α_1 and $\varphi_1\alpha_1$ are in minimal position in $S-p$ so by Lemma 3.10 we observe that

$$d^\dagger(\alpha_1, \varphi_1\alpha_1) \leq d_{\mathcal{C}^s(S-p)}(a_1, f_1 a_1),$$

and hence

$$d^\dagger(\varphi_1^n\alpha_1, \varphi_1^m\alpha_1) \leq |n-m| d_{\mathcal{C}^s(S-p)}(a_1, f_1 a_1),$$

which altogether shows that $A_1^\dagger$ is a C-quasi-axis of φ_1 where C depends on $|f_1|$ and $d_{\mathcal{C}^s(S-p)}(a_1, f_1 a_1)$ as required. □

Now we require a lemma for use in the proof of Theorem 6.1 later.

Lemma 5.4. *Given f_1, α_1, and $n_1 \in \mathbb{N}$ as in Proposition 5.3, and writing $a_1 = [\alpha_1]_{S-p}$, there exists T_1 such that*

$$d_b(a_1, f_1^n a_1) \leq T_1,$$

whenever $0 \leq n \leq n_1$ and $b \in \mathcal{C}(S-p)$.

Proof. There is an upper bound I on the geometric intersection number (see Section 2.4) between the finitely many curves to consider. By construction we have that α_1 and $\varphi_1^n\alpha_1$ are in minimal position in $S-p$ and represent a_1 and $f_1^n a_1$ respectively. However by Lemma 4.1 if $d_b(a_1, f_1^n a_1) \geq K \geq 4$ then there exist two subarcs $\varepsilon_1 \subset \alpha_1$ and $\varepsilon_2 \subset \varphi_1^n\alpha_1$ such that $|\varepsilon_1 \cap \varepsilon_2| \geq K-3$ and therefore $i(a_1, f_1^n a_1) \geq K-3$ by definition of minimal position. Therefore $T_1 = I+4$ suffices. $\square$

Remark 5.5. *In fact there is a known stronger version of Lemma 5.4 in which there exists T_1 such that the conclusion holds for all $b \in \mathcal{C}(S-p)$ and $j \in \mathbb{Z}$. This can be proved using the machinery developed by Masur and Minsky* [MM00]. *However we strive to keep this paper self contained and have found it possible to prove our main theorem without using this stronger statement.*

In the next proposition we write δ for a constant such that both $\mathcal{C}^s(S-p)$ and $\mathcal{C}^\dagger(S)$ are δ-hyperbolic. The proposition collects all information we use in the proof of the main theorem.

Proposition 5.6. *Given $8\delta < B \in \mathbb{N}$ and $T_2 > 0$ there exist a point-pushing mapping class $f_2 \in \mathrm{Mcg}(S-p)$ and $\varphi_2 \in \mathrm{Diff}_0(S)$ such that*

- *i) $[\varphi_2]_{S-p} = f_2$ and φ_2 act as hyperbolic elements on $\mathcal{C}^s(S-p)$ and $\mathcal{C}^\dagger(S)$ respectively,*
- *ii) there is a φ_2-invariant B-local-geodesic axis $A_2^\dagger$ in $\mathcal{C}^\dagger(S)$,*
- *iii) $A_2^\dagger$ is a C'-quasi-geodesic for some $C' = C'(\delta)$,*
- *iv) there exists $\gamma \in A_2^\dagger$ disjoint from p such that γ and $\varphi_2\gamma$ are in minimal position in $S-p$,*
- *v) $d^\dagger(\gamma, \varphi_2\gamma) = 2B+2$,*
- *vi) there exists an essential curve $\beta \subset S-p$ such that β is in minimal position with γ and $\varphi_2\gamma$ in $S-p$,*
- *vii) $d_\beta([\gamma]_{S-p}, [\varphi_2\gamma]_{S-p}) \geq T_2$, and*
- *viii) $d^\dagger(\beta, \gamma) = d^\dagger(\beta, \varphi_2\gamma) = B+2$.*

Proof. We start with $\alpha, \beta \in \mathcal{C}^\dagger(S)$ that are in minimal position in S so that $d^\dagger(\alpha, \beta) = B+3$, and $p \in \alpha \cap \beta$. By Lemma 3.6 there exists a geodesic $\alpha = \gamma_0, \dots, \gamma_k = \beta$ such that $\gamma_i \subset S-p$ whenever $0 < i < k$. Moreover by Lemma 3.5 we may assume that every pair γ_i and γ_j are in minimal position in S.

We take α_+ and α_- to be the boundary components of a (small enough) closed regular neighborhood of α, and similarly define β_+ and β_-. We have that $\alpha_\pm$ and $\beta_\pm$ are in minimal position in S and therefore in $S-p$ by Lemma 2.15. We may assume that $\alpha_\pm$ and $\beta_\pm$ are in minimal position with each γ_i on S and $S-p$ also.

By Lemma 3.10 we have that $d^\dagger(\alpha_\pm, \beta_\pm) = d_{\mathcal{C}^s(S-p)}(a_\pm, b_\pm)$ where $a_\pm = [\alpha_\pm]_{S-p}$ and $b_\pm = [\beta_\pm]_{S-p}$. Writing $c_i = [\gamma_i]_{S-p}$ we therefore have that

$$a_\pm, c_1, \dots, c_{k-1}, b_\pm$$

is a geodesic in $\mathcal{C}^s(S-p)$.

Set $K \geq \max(T_2+1, 2B+4)$. We define

$$f_2 \coloneqq T_{b_+}^K T_{b_-}^{-K} T_{a_+}^K T_{a_-}^{-K},$$

which is a point-pushing mapping class, as T_{b_+} and T_{b_-} are isotopic as maps on S (and similarly for a_- and a_+).

Write $\gamma = \gamma_1$. We take any $\varphi_2 \in \mathrm{Diff}_0(S)$ such that γ and $\varphi_2\gamma$ are in minimal position in $S-p$, and, $[\varphi_2]_{S-p} = f_2$. This can be achieved by taking an arbitrary representative and then applying isotopies rel p to remove all bigons.

Observe that $c_{k-1} = T_{b_+}^K T_{b_-}^{-K} c_{k-1}$ (as c_{k-1} is disjoint from b_+ and b_-) and similarly

$$f_2 c_1 = T_{b_+}^K T_{b_-}^{-K} c_1. \tag{6}$$

Therefore we may concatenate the geodesic $c_1, \ldots, c_{k-1}$ with the geodesic

$$T_{b_+}^K T_{b_-}^{-K} c_{k-1}, \ldots, T_{b_+}^K T_{b_-}^{-K} c_1,$$

to obtain a path A of length $2k-4 = 2B+2$ from c_1 to f_2c_1. Let us now take the $\langle f_2\rangle$-orbit of the path A, namely $\ldots, f_2^{-1}A, A, f_2A, \ldots$, and concatenate these in the obvious way to form an f_2-invariant path A_2.

We claim that A_2 is a B-local-geodesic. First we argue that the path A

$$c_1, \ldots, c_{k-1}, T_{b_+}^K T_{b_-}^{-K} c_{k-2}, \ldots, T_{b_+}^K T_{b_-}^{-K} c_1,$$

is a geodesic from c_1 to $T_{b_+}^K T_{b_-}^{-K} c_1$. It then follows that each subpath is a geodesic too. To do this we prove that $d_{\mathcal{C}^s(S-p)}(c_1, T_{b_+}^K T_{b_-}^{-K} c_1) = 2B+2$. This follows from the following:

Claim: Any geodesic $c_1 = v_1, \ldots, v_m = T_{b_+}^K T_{b_-}^{-K} c_1$ must admit some vertex v_i which is adjacent to b_-.

Indeed if the claim were true then $i-1 \geq k-2$ because $c_1, \ldots, c_{k-1}, b_-$ is a geodesic, and similarly, $m-i \geq k-2$ because $b_-, T_{b_+}^K T_{b_-}^{-K} c_{k-1}, \ldots, T_{b_+}^K T_{b_-}^{-K} c_1$ is a geodesic, and therefore $m \geq 2k-4 = 2B+2$ as required.

We now prove the claim that any geodesic $c_1 = v_1, \ldots, v_m = T_{b_+}^K T_{b_-}^{-K} c_1$ must admit some vertex v_i such that v_i is adjacent to b_-. So suppose it were not the case. Then by Lemma 2.19 we have that

$$d_{b_-}(c_1, T_{b_+}^K T_{b_-}^{-K} c_1) \leq m \leq 2B+2.$$

However we have by Lemma 2.20 that

$$d_{b_-}(c_1, T_{b_-}^{-K} c_1) = |K| + 2,$$

moreover by Lemma 2.21 we also have

$$d_{b_-}(T_{b_+}^K T_{b_-}^{-K} c_1, T_{b_-}^{-K} c_1) \leq 3,$$

and so

$$d_{b_-}(c_1, T_{b_+}^K T_{b_-}^{-K} c_1) \geq |K| - 1 \geq \max(T_2, 2B+3), \tag{7}$$

which is a contradiction. We conclude that there is a v_i that is adjacent to b and therefore that A is a geodesic.

To finish the claim that A_2 is a B-local-geodesic it suffices to prove that

$$T_{a_+}^{-K} T_{a_-}^{K} c_{k-1}, \ldots, T_{a_+}^{-K} T_{a_-}^{K} c_2, c_1, c_2, \ldots, c_{k-1},$$

is a geodesic also. The proof is analogous to the above. We conclude that A_2 is a B-local-geodesic. Because $B > 8\delta$ we may invoke [BH99, p405 Theorem 1.13] to show that A_2 is a C'-quasi-axis for f_2 where $C' = C'(\delta)$. It follows that $|f_2| > 0$ and by Lemma 5.2 we have that $|\varphi_2| > 0$ therefore i) holds.

Using Lemma 3.5 we may find a geodesic $A^\dagger = (\gamma_i')_i$ connecting γ and $\varphi_2\gamma$ in $\mathcal{C}^\dagger(S)$ such that $[\gamma_i']_{S-p}$ is a vertex of A for each i. Recall that $d_{\mathcal{C}^s(S-p)}(c_1, f_2 c_1) = 2B + 2$ above so by Lemma 3.10 the length of $A^\dagger$ is precisely $2B + 2$, so v) holds.

Using Lemma 3.5 we pick any $\beta \subset S - p$ representing b_+ such that β is in minimal position with γ and $\varphi_2\gamma$ in $S - p$. We have that vi) holds. Moreover by Equation 6 and Equation 7 we have that vii) holds. By Lemma 3.10 we have that viii) holds.

Set $A_2^\dagger$ to be the obvious concatenation of the $\langle\varphi_2\rangle$-orbit of $A^\dagger$, compare A_2 above. We have that iv) holds. Finally ii) follows from Lemma 3.6 and the fact that A_2 is a B-local-geodesic. By [BH99, p405 Theorem 1.13] we have that iii) holds. $\square$

Remark 5.7. *A stronger statement than Proposition 5.6 is true. If T_2 is large enough and $B \geq 0$ then $A_2^\dagger$ is a geodesic axis in Proposition 5.6. This can be shown by using the Bounded Geodesic Image Theorem of Masur and Minsky* [MM00, Theorem 3.1]. *For more details on why $A_2^\dagger$ is a geodesic axis see* [AT14, Lemma 5.1]. *In an effort to keep this paper self contained we weakened Lemma 5.6 to make it easier to prove while strong enough to prove our main theorem.*

6. Proving the main theorem

For $\varphi_1, \varphi_2 \in \mathrm{Diff}_0(S)$ hyperbolic elements acting on $\mathcal{C}^\dagger(S)$ recall the definition of $\varphi_1 \sim \varphi_2$, see Definition 2.4. We aim to show the following.

Theorem 6.1. *There are elements $\varphi_1, \varphi_2 \in \mathrm{Diff}_0(S)$ such that $\varphi_1 \not\sim \varphi_2$.*

We can then deduce our main result.

Corollary 6.2. *The space of unbounded quasi-morphisms on* $\mathrm{Diff}_0(S)$ *is infinite dimensional for any hyperbolic closed surface S.*

Proof. In view of Theorem 6.1 this is an immediate consequence of Bestvina–Fujiwara's results as in Theorem 2.13. $\square$

The proof of Theorem 6.1 occupies the remainder of this section. Before giving the details of the argument, we describe the general strategy.

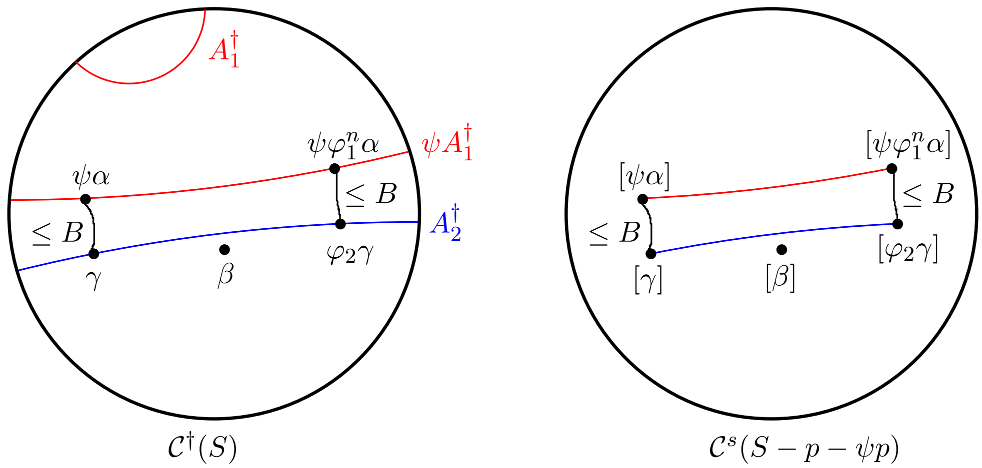

FIGURE 5. The configuration of the axes in the proof of the main theorem.

We begin by choosing a (smoothing of a) point-pushing pseudo-Anosov map φ_1 on $S-p$. On any of its quasi-axes in the curve graph of $S-p$, the maximal possible twisting between any two points about any curve b is bounded. Technically, we use Lemma 5.4, and only control twisting of a specific curve and its image, but for this summary we ignore this technicality (compare also the remark after Lemma 5.4).

Next, we construct a point-pushing pseudo-Anosov map φ_2 with various properties (which we obtain using Proposition 5.6). For this intuition, the main property is that along a quasi-axis of φ_2, twisting about some β can be ensured to be much larger than the twist bound for φ_1 above.

Now suppose that $\varphi_1 \sim \varphi_2$. Let $A_i^\dagger$ be a quasi-axis for φ_i. This would mean that we could find ψ such that a large enough part of $\psi A_1^\dagger$ is in the B-neighborhood of $A_2^\dagger$; compare Figure 5.

By construction, along $A_2^\dagger$ there are curves of large distance which have enormous twisting about some curve β. We have to arrange that the curve β also has large distance from the two curves. Then since there are nearby curves on $\psi A_1^\dagger$ and the amount of twisting can be measured by a Lipschitz map (Lemma 2.19), some curves on $\psi A_1^\dagger$ must also have very large twisting, which contradicts the bound on the maximal twisting of φ_1.

Apart from the being careful about interdependence of coarse constants, and the choice of smoothings and maps, there is one further conceptual obstacle in implementing this strategy. Namely, the curves on $A_2^\dagger$ have their twist when seen as curves on $S-p$, whereas the twist bound for $\psi A_1^\dagger$ is for curves on $S-\psi p$. This is where the lemmas from Section 4 come into play, as they let us transfer twist information from $S-p$ to $S-p-\psi p$ to $S-\psi p$.

In order to apply them, we need to choose the smoothing for φ_1 and φ_2 correctly so that the curves we are interested in stay in minimal position.

6.1. **The construction.** We now begin with the construction in earnest. Let p be any point in S. Both elements φ_1 and φ_2 that we need are representatives of pseudo-Anosov mapping classes on $S - p$. We begin with the choice of the first pseudo-Anosov.

Choice 1. *We make the following choices as in Proposition 5.3.*

i) Pick $f_1 \in \mathrm{Mcg}(S-p)$ to be a point-pushing pseudo-Anosov,
ii) $\varphi_1^{\mathrm{Th}} \in \mathrm{Homeo}_0(S)$ a Thurston representative of f_1, and
iii) α_1 cylinder curve of a singular flat structure defined by φ_1^{Th} guaranteed by Theorem 2.23.

For any choice of $n_1 > 0$ in Proposition 5.3 we have that $A_1^\dagger := (\varphi_1^n \alpha_1)_n$ is a C-quasi-axis for φ_1 in $\mathcal{C}^\dagger(S)$. Here, C is only dependent on φ_1^{Th} and α_1.

Our choice of φ_2 below has a k-local-geodesic axis where $k > 8\delta$. Therefore it has a C'-quasi-axis where $C' = C'(\delta)$. Recall the constant $B = B(C, C', \delta)$ from Section 2.1. Picking a larger B if necessary we may assume that $B > 8\delta$ and that $B \in \mathbb{N}$. The constant C' is independent from any choices we make therefore B is only dependent on the Thurston representative φ_1^{Th} and the cylinder curve α_1.

For our argument we require $\varphi_1 \in \mathrm{Diff}_0(S)$ from Proposition 5.3 to maintain minimal position in $S - p$ between many curves in $A_1^\dagger$, so we add an additional requirement on φ_1 in the following way, which only depends on B. Recall that $|\varphi_1| \geq |f_1| > 0$.

Choice 2. *Pick $n_1 > 0$ in Proposition 5.3 such that*

$$n_1|\varphi_1| \geq n_1|f_1| \geq 4B + 3.$$

Then by Proposition 5.3 we can find a $\varphi_1 \in \mathrm{Diff}_0(S)$ such that $[\varphi_1]_{S-p} = f_1$ and whenever $0 \leq i \leq n_1$ then $\varphi_1^m \alpha_1$ and $\varphi_1^{m+i}\alpha_1$ are in minimal position in $S - p$ for any $m \in \mathbb{Z}$.

We write T_1 for the twist bound constant from Lemma 5.4 applied to f_1, $a_1 = [\alpha_1]_{S-p}$, and n_1. We recall that this means

$$d_b([\alpha_1]_{S-p}, [\varphi_1^n \alpha_1]_{S-p}) = d_b(a_1, f_1^n a_1) \leq T_1, \tag{8}$$

for all $b \in \mathcal{C}(S - p)$ and $0 \leq n \leq n_1$.

We now apply Proposition 5.6 to find a suitable point-pushing pseudo-Anosov $f_2 \in \mathrm{Mcg}(S - p)$ and representative $\varphi_2 \in \mathrm{Diff}_0(S)$ that has enough twisting for our strategy to work. The twisting and the curves involved depend on T_1 and B. We summarise the necessary properties below:

Choice 3. *We earlier assumed that $B > 8\delta$. Pick $T_2 \geq 2T_1 + 4B + 6$. Using Proposition 5.6 we find point-pushing $f_2 \in \mathrm{Mcg}(S - p)$ and $\varphi_2 \in \mathrm{Diff}_0(S)$ such that*

i) $[\varphi_2]_{S-p} = f_2$ and φ_2 acts as hyperbolic elements on $\mathcal{C}^s(S-p)$ and $\mathcal{C}^\dagger(S)$,

ii) *there is a φ_2-invariant B-local-geodesic axis $A_2^\dagger$ in $\mathcal{C}^\dagger(S)$,*
iii) *$A_2^\dagger$ is a C'-quasi-geodesic for some $C' = C'(\delta)$,*
iv) *there exists $\gamma \in A_2^\dagger$ disjoint from p such that γ and $\varphi_2\gamma$ are in minimal position in $S - p$,*
v) $d^\dagger(\gamma, \varphi_2\gamma) = 2B + 2$,
vi) *there exists an essential curve $\beta \subset S - p$ such that β is in minimal position with γ and $\varphi_2\gamma$ in $S - p$,*
vii) $d_\beta([\gamma]_{S-p}, [\varphi_2\gamma]_{S-p}) \geq T_2 \geq 2T_1 + 4B + 6,$ *and*
viii) $d^\dagger(\beta, \gamma) = d^\dagger(\beta, \varphi_2\gamma) = B + 2.$

6.2. **Proof of Theorem 6.1.** We are now ready to prove Theorem 6.1 and therefore our main theorem Theorem 1.2.

Proof of Theorem 6.1. We claim that the above φ_1 and φ_2 as hyperbolic isometries of $\mathcal{C}^\dagger(S)$ satisfy $\varphi_1 \not\sim \varphi_2$. Assume the contrary i.e. suppose that we have a diffeomorphism $\psi \in \mathrm{Diff}_0(S)$ with the property that $A_2^\dagger$ has a finite subsegment of length L contained in the B-neighborhood of $\psi A_1^\dagger$ such that

$$L \geq 4B + 3,$$

then without loss of generality (by replacing ψ by $\varphi_2^j\psi$ for some $j \in \mathbb{Z}$) this subsegment contains γ and $\varphi_2\gamma$. We can do this because $d^\dagger(\gamma, \varphi_2\gamma) = 2B+2$ by Choice 3.

Therefore without loss of generality (by replacing α_1 with $\varphi_1^j\alpha_1$ for some $j \in \mathbb{Z}$) we have that $d^\dagger(\gamma, \psi\alpha_1) \leq B$ and $d^\dagger(\varphi_2\gamma, \psi\varphi_1^n\alpha_1) \leq B$ for some n.

As an aside we make the notation easier to read by writing

$$\gamma' = \varphi_2\gamma,\ \alpha = \alpha_1,\ \text{and } \alpha' = \varphi_1^n\alpha_1.$$

We claim that $|n| \leq n_1$ above. Indeed if $|n| > n_1$ then

$$d^\dagger(\psi\alpha_1, \psi\varphi_1^n\alpha_1) \geq |n||\varphi_1| > n_1|\varphi_1| \geq 4B + 3,$$

by Choice 2. Therefore $d^\dagger(\gamma, \gamma') \geq 2B + 3$, which contradicts $d^\dagger(\gamma, \gamma') = 2B + 2$ in Choice 3. Therefore we have $|n| \leq n_1$ so α and α' are in minimal position in $S - p$ by Choice 2 and so $\psi\alpha$ and $\psi\alpha'$ are in minimal position in $S - \psi p$.

By Choice 3 there exists $\beta \in \mathcal{C}^\dagger(S)$ such that $\beta \subset S - p$ and

$$d_\beta([\gamma]_{S-p}, [\gamma']_{S-p}) \geq T_2 \geq 2T_1 + 4B + 6. \tag{9}$$

Now we want ψp to be disjoint from β, γ, and γ'. This is automatic if $\psi p = p$. Otherwise we do this in the following way. First we perturb ψ about p in the domain such that ψp is moved off β, γ, and γ'. Since α and α' are disjoint from p, if the perturbation is small enough about p in the domain, then this keeps $\psi\alpha$ and $\psi\alpha'$ fixed while ensuring that ψp is disjoint from β, γ, and γ'.

We also want $\psi\alpha$ and $\psi\alpha'$ to be disjoint from p. This is automatic if $\psi p = p$. Otherwise we need to be careful. We perturb ψ about $\psi^{-1}p$ in the domain in order to move $\psi\alpha$ (and $\psi\alpha'$) off p, if necessary. But we also want to retain the bounds $d^\dagger(\gamma, \psi\alpha) \leq B$ and $d^\dagger(\gamma', \psi\alpha') \leq B$. In order to ensure this we fix geodesics in $\mathcal{C}^\dagger(S)$ connecting $\psi\alpha$ to γ and $\psi\alpha'$ to γ'. Whenever $\psi\alpha$ (or $\psi\alpha'$) intersects p then the next vertex along the geodesic cannot intersect p, hence there is a perturbation small enough that maintains this geodesic and hence the distance bound.

Now we have that γ, γ', β, $\psi\alpha$, and $\psi\alpha'$ are contained in $S-p-q$ where $q = \psi p$. Since γ, γ', and β' are pairwise in minimal position in $S-p$ they are also in minimal position in $S-p-q$. Since $\psi\alpha$ and $\psi\alpha'$ are in minimal position in $S-q$ they are also in minimal position in $S-p-q$ as well. This follows from Lemma 2.15.

We are now in a position to use the lemmas from Section 4. We have that γ, γ', and β are pairwise in minimal position on $S-p$. Therefore by Lemma 4.2 (Puncturing keeps twists) we can find $\beta' \subset S-p-q$ such that β and β' are disjoint, $[\beta]_{S-p} = [\beta']_{S-p}$, and

$$d_{\beta'}([\gamma]_{S-p-q}, [\gamma']_{S-p-q}) \geq \frac{T_2}{2} \geq T_1 + 2B + 3. \tag{10}$$

By Lemma 3.6 there exists a geodesic (ν_i) in $\mathcal{C}^\dagger(S)$ between γ and $\psi\alpha$ of length at most B such that each vertex ν_i is disjoint from p and q. For each ν_i we claim that $[\nu_i]_{S-p-q}$ and $[\beta']_{S-p-q}$ are not adjacent in $\mathcal{C}^s(S-p-q)$. Indeed if they were adjacent then the path $([\nu_i]_{S-p})$ in $\mathcal{C}^s(S-p)$ can be used to construct a path that connects $[\gamma]_{S-p}$ to $[\beta']_{S-p}$ with length at most $B+1$. Since $[\beta]_{S-p} = [\beta']_{S-p}$ we have that

$$d_{\mathcal{C}^s(S-p)}([\gamma]_{S-p}, [\beta]_{S-p}) \leq B+1,$$

which by Lemma 3.10 shows that $d^\dagger(\beta, \gamma) \leq B+1$, contradicting Choice 3. Therefore each $[\nu_i]_{S-p-q}$ is not adjacent to $[\beta']_{S-p-q}$.

We also have the analogous construction of a geodesic (ν_i') between γ' and $\psi\alpha'$. Similarly as before each $[\nu_i']_{S-p-q}$ is not adjacent to $[\beta']_{S-p-q}$.

Therefore we may apply Lemma 2.19 for the paths $([\nu_i]_{S-p-q})$ and $([\nu_i']_{S-p-q})$, which combined with Equation (10) above shows that

$$d_{\beta'}([\psi\alpha]_{S-p-q}, [\psi\alpha']_{S-p-q}) \geq T_1 + 3.$$

Finally $\psi\alpha$ and $\psi\alpha'$ are in minimal position in $S-q = S-\psi p$ therefore we may apply Lemma 4.3 (Forgetting keeps twists), which shows that

$$d_{\beta'}([\psi\alpha]_{S-\psi p}, [\psi\alpha']_{S-\psi p}) \geq T_1 + 1.$$

Taking $b = [\psi^{-1}\beta']_{S-p}$, this implies that

$$d_b([\alpha]_{S-p}, [\alpha']_{S-p}) \geq T_1 + 1.$$

Since $\alpha = \alpha_1$ and $\alpha' = \varphi_1^n \alpha_1$ this contradicts the choice of T_1, recall Equation (8). We conclude that whenever a finite subsegment of $A_2^\dagger$ is contained in the B-neighborhood of $\psi A_1^\dagger$ then its length L satisfies $L \leq 4B+2$ and therefore $\varphi_1 \not\sim \varphi_2$. $\square$

Remark 6.3. *We expect that the proof idea here would also work for any initial choice of hyperbolic element $\varphi_1 \in \mathrm{Diff}_0(S)$ on $\mathcal{C}^\dagger(S)$. Furthermore it may seem to the reader that our construction of φ_2 is restrictive. This is because we decided to prove that φ_2 has a B-local-geodesic axis $A_2^\dagger$. This is not necessary for the proof of Theorem 6.1. With more effort other choices of φ_2 should also work provided that they twist enough about suitable curves. The local-geodesic axis simply makes our argument shorter, have fewer constants, and easier to read.*

7. Genus one

We are left with proving the main theorem in the case where S is the 2-torus. The proof idea is essentially the same, but the details are different in places. In this section, we sketch the necessary changes in the proofs.

7.1. **A hyperbolic graph.** We require the following definition.

Definition 7.1. We define $\mathcal{NC}^\dagger(S)$ to have the same vertex set as in Section 3, but an edge connects two distinct vertices precisely when they intersect at most once transversely.

Rasmussen's result (Theorem 3.3) also holds for non-separating curve graphs of punctured tori, where again edges correspond to curves intersecting (at most) once. Now, the rest of the arguments in this section carry over to this setting verbatim.

Note that the Lipschitz property for twists on $S - P$ is still true, but the Lipschitz constant is 2 instead of 1.

7.2. **Twists.** Concerning the twist lemmas in Section 4, their proofs already work for any hyperbolic surface S. Hence, in particular, they are true when S is a punctured torus. However, we need versions that allow to add or remove two marked points at once. In Lemma 4.2, this changes the multiplicative lower bound from $1/2$ to $1/k$, if k is the total number of points. In Lemma 4.3, nothing changes.

7.3. **Constructing isotopically-trivial diffeomorphisms that act hyperbolically.** Here, the second main difference between genus 1 and higher genus comes to bear. Namely, on the torus with one marked point, there are no point pushing mapping classes. Thus, we need to apply the constructions of Proposition 5.3 and Proposition 5.6 on a once-punctured torus $S = T - p_0$. The proofs generalise in an obvious way (with different constants, owing to the fact that twists are now 2-Lipschitz).

7.4. **Proving the main theorem.** The argument is essentially the same, except for the fact that φ_1 and φ_2 now each have two marked points associated to them, and we need to argue that one can exchange both.

8. Automatic continuity

It was observed by Entov–Polterovich–Py [EPP12] that homogeneous quasi-morphisms of area-preserving maps of surfaces are automatically continuous in the C^0-topology due to a certain bounded fragmentation property. The fact that such a statement holds was suggested by Kotschick [Kot08] to whom the idea is attributed. Kotschick also observed that this continuity holds in the setting of diffeomorphism groups, where the corresponding fragmentation properties are well known. However, the proof of this fact did not appear in [Kot08].

Most of the results on continuity of quasi-morphisms in this section are not new, but as they have not previously appeared in print in the form we present them, we choose to include them. What does appear to be new is the fact that the stable commutator length function is continuous. Our arguments follow the lines of [EPP12], but things are significantly simpler than in the area preserving case they consider.

Bounded fragmentation. For the sake of giving a uniform account we let $\mathrm{Diff}_0^r(M)$ denote the identity component of the group of C^r-diffeomorphisms of a closed manifold M, where $\mathrm{Diff}_0^r(M)$ is just the identity component of the group of diffeomorphisms and $\mathrm{Diff}_0^\infty(M) = \mathrm{Diff}_0(M)$ is the group of smooth diffeomorphisms. The following is just the observation that the standard fragmentation procedure for diffeomorphisms on compact manifolds yields factorisations of bounded length. In the case of C^r-diffeomorphisms this is elementary (see e.g. [Man16, Lemma 2.1]) and in the case of homeomorphisms it follows from classical results of Edwards–Kirby [EK71].

Lemma 8.1 (General Case: C^r-topology)**.** *There is a neighborhood $\mathcal{U}_{Id} \subseteq \mathrm{Diff}_0^r(M)$ of the identity map with respect to the C^r-topology such that any $f \in \mathcal{U}_{Id}$ can be written as a product of C_M diffeomorphisms supported on open disks for some constant C_M depending only on the manifold.*

In the case of surfaces one can essentially model the argument of Edwards–Kirby to obtain a bounded fragmentation property with respect to the C^0-topology. This is simply due to the fact that for a closed disk of *fixed* small radius D_r and any diffeomorphism f which is C^0-close to the identity, if $D \cup fD \subseteq D_{r+\varepsilon}$ lie in the interior a larger disk can be moved to one another by diffeomorphism with support inside $D_{r+\varepsilon}$ in a C^0-small manner. This can easily be arranged using for example [EPP12, Lemma 7.1]. This then allows one to build some f_D that agree with f on the disk D and has support in $D_{r+\varepsilon}$. Also if the original diffeomorphism was the identity on some open neighborhoods N of points on ∂D, the same can be assumed of f_D, up to shrinking N slightly. Repeated application of this procedure applied to say a handle decomposition of the surface then gives a fragmentation of bounded length.

Lemma 8.2 (Surface Case: C^0-topology)**.** *There is a neighborhood $\mathcal{U}_{Id} \subseteq \mathrm{Diff}_0^r(S)$ of the identity map with respect to the C^0-topology such that any*

$f \in \mathcal{U}_{Id}$ can be written as a product of C_S diffeomorphisms supported on open disks for some constant C_S depending only on the surface.

The constant C_S in Lemma 8.2 can actually be chosen to be 3, i.e. independent of the topology of the surface. To see this pick a triangulation $\mathcal{T}$. First find F_0 agreeing with any given f on a neighborhood $N_0 = N_\varepsilon(\mathcal{T}^{(0)})$ of the 0-skeleton, which is a disjoint union of disks and thus lies in a single disk. Let $N_1 = N_{\varepsilon'}(\mathcal{T}^{(1)})$ be a neighborhood of the 1-skeleton, with $\varepsilon' \ll \varepsilon$. Choose a diffeomorphism F_0 agreeing with $F_0^{-1} \circ f$ on $N_1 \setminus N_0'$ for a slightly smaller neighborhood $N_0' \subseteq N_0$ of the 0-skeleton. Finally note that $F_2 = F_1^{-1} \circ F_0^{-1} \circ f$ has support on the complement of a neighborhood of the 1-skeleton which is a disjoint union of disks. Thus we obtain a factorisation of $f = F_0 \circ F_1 \circ F_2$ as a product of three diffeomorphisms supported on disjoint unions of disks. Since any disjoint union of disks is itself contained in a disk we obtain a fragmentation of length 3.

Boundedness near the identity implies continuity. It is well known that homogeneous quasi-morphisms vanish on diffeomorphisms supported on disks. This follows for example from the fact that the group of compactly supported diffeomorphisms of the (open) unit disk is uniformly perfect (cf. [BIP08], [Kot08]), [Tsu08]).

Lemma 8.3. *Let $\varphi\colon \mathrm{Diff}_0^r(M) \to \mathbb{R}$ be a homogeneous quasi-morphism. Then $\varphi(g) = 0$ for any element with support contained in a ball* $\mathrm{supp}(g) \subset U \cong B^n$.

We can then deduce that there is a uniform bound on some open neighborhood of the identity for *any* homogeneous quasi-morphism in terms of the defect. The following fact is often attributed to [Sht01].

Lemma 8.4. *There is a C^r-neighborhood $\mathcal{U}_{Id} \subseteq \mathrm{Diff}_0^r(M)$ so that any homogeneous quasi-morphism is bounded by some constant multiple of the defect on $\mathcal{U}_{Id}$. In the case of surfaces this also holds for a C^0-neighborhood.*

Proof. Since $f = g_1 g_2 \cdots g_k$ can be factored as a product of $k = C_M$ diffeomorphisms supported on disks we conclude that

$$\begin{aligned}|\varphi(g_1 g_2 \cdots g_k)| &= |\varphi(g_1 g_2 \cdots g_k) - (\varphi(g_1) + \cdots + \varphi(g_k))| \\ &\le (C_M - 1) D(\varphi).\end{aligned}$$

Here we use the quasi-morphism property repeatedly as well as the fact that homogeneous quasi-morphisms vanish on maps that are supported on balls. □

Theorem 8.5. *Any homogeneous quasi-morphism $\varphi\colon \mathrm{Diff}_0^r(M) \to \mathbb{R}$ is continuous with respect to the C^r-topology.*

Proof. Let $f \in \mathrm{Diff}_0(M)$. Choose a neighborhood $\mathcal{V}_n$ so that for any $g \in \mathcal{V}_n$ we have that $f^n g^{-n}$ lies in the neighborhood $\mathcal{U}_{Id}$ for some fixed n. Then using homogeneity and the quasi-morphism property we have that

$$n|\varphi(f)-\varphi(g)| = |\varphi(f^n)+\varphi(g^{-n})| \leq |\varphi(f^n g^{-n})|+D(\varphi) \leq (C_S-1)D(\varphi)+D(\varphi).$$

Then dividing and letting $n \to \infty$ the continuity follows. □

In fact for quasi-morphisms on surface diffeomorphism groups we have continuity in the C^0-sense in view of Lemma 8.2.

Theorem 8.6 (Kotschick)**.** *Any homogeneous quasi-morphism* $\varphi\colon \mathrm{Diff}_0(S) \to \mathbb{R}$ *is continuous with respect to the* C^0*-topology.*

Thus we can now deduce the existence of quasi-morphisms on the identity component of the group of homeomorphisms. This uses the fact, which is special to surfaces, that any homeomorphism can be uniformly approximated by diffeomorphisms (cf. [Mun60]).

Corollary 8.7. *The space of unbounded quasi-morphisms on* $\mathrm{Homeo}_0(S)$ *is infinite dimensional for any closed surface* S *of genus greater than one.*

Alternatively one could actually construct quasi-morphisms directly on the group $\mathrm{Homeo}_0(S)$ by considering a curve graph of *topologically embedded* curves with the obvious edge relation. The arguments used to prove Theorem 6.1 readily extend, although some care is needed in dealing with topological transversality, minimal position and so on.

A closer inspection of the proof of Theorem 8.5 actually shows that the set of homogeneous quasi-morphisms of bounded defect is *point-wise equicontinous* which in view of Bavard Duality shows that the stable commutator length function is too.

Theorem 8.8. *The stable commutator function* $\mathrm{scl}\colon \mathrm{Diff}^r_0(S) \to \mathbb{R}$ *is continuous with respect to the* C^0*-topology.*

Proof. For a group G let $\widetilde{\mathrm{QH}}(G)$ denote the group of homogeneous quasi-morphisms and let $\widetilde{\mathrm{QH}}_1(G)$ denote the subset of defect $D(\varphi) = 1$. By Bavard Duality we have

$$\mathrm{scl}(g) = \sup_{\varphi \in \widetilde{\mathrm{QH}}(G)} \frac{|\varphi(g)|}{2D(\varphi)} = \sup_{\varphi \in \widetilde{\mathrm{QH}}_1(G)} \frac{|\varphi(g)|}{2}.$$

For $G = \mathrm{Diff}^r_0(S)$ the family of real-valued functions $\widetilde{\mathrm{QH}}_1(\mathrm{Diff}^r_0(S))$ is equicontinuous at each point so it follows that the right hand side is continuous in g, whence we deduce that the stable commutator length function is continuous. □

Remark 8.9. *The result for* scl *is not specific to surface groups. However it is unclear whether or not it is vacuous in other dimensions.*

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