A CHARACTERIZATION OF QUANTUM GROUPS

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Abstract. We classify pointed Hopf algebras with finite Gelfand-Kirillov dimension, which are domains, whose groups of group-like elements are abelian, and whose infinitesimal braidings are positive.

Introduction

Since the appearance of quantum groups [KR, Sk, Dr, Ji], there were many attempts to define them intrinsically. Important descriptions of the so-called "nilpotent" parts were given by Ringel [Ri], Lusztig [L2] and Rosso [Ro]. However, the question of finding an abstract characterization of the quantized enveloping algebras remained open.

In the main Theorem 5.2 of this paper, we classify all Hopf algebras over an algebraically closed field of characteristic 0 which are

- pointed, that is all their simple comodules are one-dimensional, and have an abelian group of group-like elements,
- domains of finite Gelfand-Kirillov dimension, and
- have positive infinitesimal braiding (see Section 1).

The first two conditions are natural. The positivity condition should be related to the existence of a real involution.

In Theorem 4.2, we describe these Hopf algebras by generators and relations. They are natural generalizations of quantized enveloping algebras with positive parameter. To prove our main Theorem, we combine the lifting method for pointed Hopf algebras [AS1, AS2, AS4] with a characterization obtained by Rosso of the "nilpotent part" of a quantized enveloping algebra in terms of finiteness of the Gelfand-Kirillov dimension [Ro].

Among the main differences between the new Hopf algebras and multiparametric quantized enveloping algebras, let us mention that we have one parameter of deformation for each connected component of the Dynkin diagram (this is explained as follows: in the "classical limit", one may have different scalar multiples of the Sklyanin brackets in the different connected components) and linking relations, see (4.12), generalizing the classical relations $E_iF_j - F_jE_i = \delta_{ij}(K_i - K_i^{-1})$.

Note that we are not assuming that the Hopf algebras have an *a-priori* assigned Dynkin diagram as in [W, KW]; it comes from our hypothesis, and here is where we rely on Rosso's result [Ro].

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The article is organized as follows. Section 1 contains preliminary material. In Section 2, we collect some well-known facts about quantum groups, and give details of some proofs when they are not easily available in the literature. Section 3 contains a technical result on the coradical filtration of certain Hopf algebras, generalizing an idea of Takeuchi. In Section 4, we construct a new family of pointed Hopf algebras with generic braiding and establish the main basic properties of them. The approach is similar to [AS4] but instead of dimension arguments, we use the technical results on the coradical filtration obtained in Section 3; these results should be useful also for other classes of Hopf algebras. In Section 5, we prove our main Theorem. A key point is Lemma 5.1, which implies that a wide class of Hopf algebras with finite Gelfand-Kirillov dimension is generated by group-like and skew-primitive elements.

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1. Preliminaries

Notation. Let $\Bbbk$ be an algebraically closed field of characteristic 0. Our references are [M], [Sw] for Hopf algebras; [KL], for growth of algebras and Gelfand-Kirillov dimension; and [AS5] for pointed Hopf algebras. We use standard notation for Hopf algebras: Δ, $\mathcal{S}$, ϵ, denote respectively the comultiplication, the antipode, the counit; we use a short version of Sweedler's notation: $\Delta(x) = x_{(1)} \otimes x_{(2)}$, x in a coalgebra C.

Let H be a Hopf algebra with bijective antipode. We denote by $G(H)$ be the group of group-like elements of H; and by $\mathcal{P}_{g,h}(H)$ the space of g, h skew-primitive elements x of H, that is with $\Delta(x) = g \otimes x + x \otimes h$, where $g, h \in G(H)$; then $\mathcal{P}(H) := \mathcal{P}_{1,1}(H)$. The braided category of Yetter-Drinfeld modules over H is denoted by ${}^H_H\mathcal{YD}$, *cf.* the conventions of [AS5].

The adjoint representation ad of a Hopf algebra A on itself is given by $\operatorname{ad} x(y) = x_{(1)} y \mathcal{S}(x_{(2)})$. If R is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ then there is a braided adjoint representation ad_c of R on itself given by $\operatorname{ad}_c x(y) = \mu(\mu \otimes \mathcal{S})(\mathrm{id} \otimes c)(\Delta \otimes \mathrm{id})(x \otimes y)$, where μ is the multiplication and $c \in \operatorname{End}(R \otimes R)$ is the braiding. If $x \in \mathcal{P}(R)$ then the braided adjoint representation of x is $\operatorname{ad}_c x(y) = \mu(\mathrm{id} - c)(x \otimes y) =: [x, y]_c$. The element $[x, y]_c$ defined by the second equality for any x and y, regardless of whether x is primitive, will be called a braided commutator. When $A = R\#H$, then for all $b, d \in R$, $\operatorname{ad}_{(b\#1)}(d\#1) = (\operatorname{ad}_c b(d))\#1$.

If Γ is an abelian group, we denote by $\widehat{\Gamma}$ the group of characters of Γ. If V is a $\Bbbk\Gamma$-module (resp., $\Bbbk\Gamma$-comodule), then we denote $V^\chi := \{v \in V : h.v = \chi(h)v, \forall h \in \Gamma\}$, $\chi \in \widehat{\Gamma}$; resp., $V_g := \{v \in V : \delta(v) = g \otimes v\}$, $g \in G$. A Yetter-Drinfeld module over Γ is $\Bbbk\Gamma$-module V which is also a $\Bbbk\Gamma$-comodule, and such that each homogeneous component V_g, $g \in \Gamma$, is a $\Bbbk\Gamma$-submodule. Thus, a vector space V provided with a direct sum decomposition $V = \oplus_{g\in G, \chi\in\widehat{\Gamma}} V_g^\chi$ is a Yetter-Drinfeld module over $H = \Bbbk\Gamma$.

Braided vector spaces. A braided vector space (V, c) is a finite-dimensional vector space provided with an isomorphism $c : V \otimes V \to V \otimes V$ which is a solution of the braid equation, that is $(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$. Examples of braided vector spaces are Yetter-Drinfeld modules: if $V \in {}^H_H\mathcal{YD}$, then $c : V \otimes V \to V \otimes V$, $c(v \otimes w) = v_{(-1)}.w \otimes v_{(0)}$, is a solution of the braid equation.

Definition 1.1. Let (V, c) be a finite-dimensional braided vector space. We shall say that the braiding $c : V \otimes V \to V \otimes V$ is *diagonal* if there exists a basis $x_1, \dots, x_\theta$ of V and non-zero scalars q_{ij} such that $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$, $1 \le i, j \le \theta$. The matrix (q_{ij}) is called the matrix of the braiding.

Furthermore, we shall say that a diagonal braiding with matrix (q_{ij}) is *indecomposable* if for all $i \neq j$, there exists a sequence $i = i_1, i_2, \dots, i_t = j$ of elements of $\{1, \dots, \theta\}$ such that $q_{i_s, i_{s+1}} q_{i_{s+1}, i_s} \neq 1$, $1 \le s \le t - 1$. Otherwise, we say that the matrix is decomposable.

We attach a graph to a diagonal braiding in the following way. The vertices of the graph are the elements of $\{1, \dots, \theta\}$, and there is an edge between i and j if they are different and $q_{ij} q_{ji} \neq 1$. Thus, "indecomposable" means that the corresponding graph is connected. The components of the matrix are the principal submatrices corresponding to the connected components of the graph. If i and j are vertices in the same connected component, then we write $i \sim j$. We shall denote by $\mathcal{X}$ the set of connected components of the matrix (q_{ij}). If $I \in \mathcal{X}$, then V_I denotes the subspace of V spanned by x_i, $i \in I$.

We shall say that a braiding c is *generic* if it is diagonal with matrix (q_{ij}) where q_{ii} is not a root of 1, for any i.

Let $\Bbbk = \mathbb{C}$. We shall say that a braiding c is *positive* if it is generic with matrix (q_{ij}) where q_{ii} is a positive real number, for all i.

We shall say that a diagonal braiding c with matrix (q_{ij}) is of *Cartan type* if $q_{ii} \neq 1$ for all i, and there are integers a_{ij} with $a_{ii} = 2$, $1 \le i \le \theta$, and $0 \le -a_{ij} < \operatorname{ord} q_{ii}$ (which could be infinite), $1 \le i \neq j \le \theta$, such that $q_{ij} q_{ji} = q_{ii}^{a_{ij}}$ for all i and j. Since clearly $a_{ij} = 0$ implies that $a_{ji} = 0$ for all $i \neq j$, (a_{ij}) is a generalized Cartan matrix. This generalizes the definition in [AS2, p. 4]. In this case, the braiding is indecomposable if and only if the corresponding Cartan matrix is indecomposable. We shall also denote by $\mathcal{X}$ the set of connected components of the Dynkin diagram corresponding to the matrix (a_{ij}); clearly, this agrees with the previous convention.

Let (V, c) be a braided vector space of Cartan type with generalized Cartan matrix (a_{ij}). We say that (V, c) is of *DJ-type* (or Drinfeld-Jimbo type) if there exist positive integers $d_1, \dots, d_\theta$ such that

(1.1)

For all i, j, $d_i a_{ij} = d_j a_{ji}$ (thus (a_{ij}) is symmetrizable).

(1.2)

For all $I \in \mathcal{X}$, there exists $q_I \in \Bbbk$, which is not a root of unity, such that $q_{ij} = q_I^{d_i a_{ij}}$ for all $i, j \in I$.

In particular, $q_{ij} = 1$ if $i \not\sim j$.

Nichols algebras. Let $V \in {}^H_H\mathcal{YD}$. A braided graded Hopf algebra $R = \oplus_{n\geq 0} R(n)$ in ${}^H_H\mathcal{YD}$ is called a *Nichols algebra* of V if $\Bbbk \simeq R(0)$ and $V \simeq R(1)$ in ${}^H_H\mathcal{YD}$, and

(1) $P(R) = R(1)$,
(2) R is generated as an algebra by $R(1)$.

The Nichols algebra of V exists and is unique up to isomorphisms; it will be denoted by $\mathfrak{B}(V)$. It depends, as an algebra and coalgebra, only on the underlying braided vector space (V, c). The underlying algebra is called a *quantum symmetric algebra* in [Ro]. We shall identify V with the subspace of homogeneous elements of degree one in $\mathfrak{B}(V)$. See [AS5] for more details and some historical references.

Given a braided vector space of any of the types in Definition 1.1, we will say that its Nichols algebra is of the same type.

Lemma 1.2. [AS2, Lemma 4.2]. *Let V be a finite-dimensional Yetter-Drinfeld module over an ablian group. Let $\mathcal{X} = \{I_1, \dots, I_N\}$ be a numeration of the set of connected components. Then $\mathfrak{B}(V) \simeq \mathfrak{B}(V_{I_1}) \underline{\otimes} \dots \underline{\otimes} \mathfrak{B}(V_{I_N})$ as braided Hopf algebras with the braided tensor product algebra structure $\underline{\otimes}$.* $\square$

Lifting method for pointed Hopf algebras. Recall that a Hopf algebra A is pointed if any irreducible A-comodule is one-dimensional. That is, if the coradical A_0 equals the group algebra $\Bbbk G(A)$.

Let A be a pointed Hopf algebra let $A_0 = \Bbbk G(A) \subseteq AA_1 \subseteq \dots$ be the coradical filtration and let $\operatorname{gr} A = \oplus_{n\geq 0} \operatorname{gr} A(n)$ be the associated graded coalgebra, which is a graded Hopf algebra [M]. The graded projection $\pi : \operatorname{gr} A \to \operatorname{gr} A(0) \simeq \Bbbk G(A)$ is a Hopf algebra map and a retraction of the inclusion. Let $R = \{a \in A : (\mathrm{id} \otimes \pi)\Delta(a) = a \otimes 1\}$ be the algebra of coinvariants of π; R is a braided Hopf algebra in the category ${}^{G(A)}_{G(A)}\mathcal{YD}$ of Yetter-Drinfeld modules over $\Bbbk G(A)$ and $\operatorname{gr} A$ can be reconstructed from R and $\Bbbk G(A)$ as a bosonization: $\operatorname{gr} A \simeq R\#\Bbbk G(A)$. Moreover, $R = \oplus_{n\geq 0} R(n)$, where $R(n) = \operatorname{gr} A(n) \cap R$ is a graded braided Hopf algebra. We then have several invariants of our initial pointed Hopf algebra A: The graded braided Hopf algebra R; it is called the *diagram* of A. The braided vector space (V, c), where $V := R(1) = P(R)$ and $c : V \otimes V \to V \otimes V$ is the braiding in ${}^{G(A)}_{G(A)}\mathcal{YD}$. It will be called the *infinitesimal braiding* of A. The dimension of $V = P(R)$, called the *rank* of A, or of R. The subalgebra R' of R generated by $R(1)$, which is the Nichols algebra of V: $R' \simeq \mathfrak{B}(V)$. See [AS5] for more details.

2. Nichols algebras of Cartan type

Nichols algebras of diagonal type. In this section, (V, c) denotes a finite-dimensional braided vector space; we assume that the braiding c is diagonal with matrix (q_{ij}), with respect to a basis $x_1, \dots x_\theta$.

Let Γ be the free abelian group of rank θ with basis $g_1, \dots, g_\theta$. We define characters $\chi_1, \dots, \chi_\theta$ of Γ by

$$\chi_i(g_j) = q_{ji}, \qquad 1 \leq i, j \leq \theta.$$

We consider V as a Yetter-Drinfeld module over $\Bbbk\Gamma$ by defining $x_i \in V_{g_i}^{\chi_i}$, for all i.

We begin with some relations that hold in any Nichols algebra of diagonal type.

Lemma 2.1. *Let R be a braided Hopf algebra in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$, such that $V \hookrightarrow P(R)$.*

(a). If q_{ii} is a root of 1 of order $N > 1$ for some $i \in \{1, \dots, \theta\}$, then $x_i^N \in P(R)$.

(b). Let $i \neq j \in \{1, \dots, \theta\}$ such that $q_{ij}q_{ji} = q_{ii}^{1-r}$, where r is an integer such that $0 \leq r-1 < \operatorname{ord} q_{ii}$ (which could be infinite). Then $(\operatorname{ad} x_i)^r(x_j)$ is primitive in R.

Assume for the rest of the Lemma that $R = \mathfrak{B}(V)$.

(c). In the situation of (a), resp. (b), $x_i^N = 0$, resp. $\operatorname{ad} x_i^r(x_j) = 0$.

(d). If $R = \mathfrak{B}(V)$ is an integral domain, then $q_{ii} = 1$ or it is not a root of 1, for all i.

(e). If $i \neq j$, then $\operatorname{ad}_c(x_i)^r(x_j) = 0$ if and only if $(r)!_{q_{ii}} \prod_{0 \leq k \leq r-1} \left(1 - q_{ii}^k q_{ij} q_{ji}\right) = 0$.

Proof. (a) and (b) are consequences of the quantum binomial formula, see *e. g.* [AS2, Appendix] for (b). Then (c) and (d) follow; the second statement in (c) is also a consequence of (e). Part (e) is from [Ro, Lemma 14]; it can also be shown using skew-derivations as in [AS5, Section 2.2]. □

We now recall a variation of a well-known result of Reshetikhin on twisting [Re]. Let $(\widehat{V}, \widehat{c})$ be another braided vector space of the same dimension as V, such that the braiding $\widehat{c}$ is diagonal with matrix $(\widehat{q}_{ij})$ with respect to a basis $\widehat{x}_1, \dots \widehat{x}_\theta$. We define characters $\widehat{\chi}_1, \dots, \widehat{\chi}_\theta$ of Γ by

$$\widehat{\chi}_i(g_j) = \widehat{q}_{ji}, \qquad 1 \leq i, j \leq \theta.$$

We consider $\widehat{V}$ as a Yetter-Drinfeld module over Γ by defining $\widehat{x}_i \in \widehat{V}_{g_i}^{\widehat{\chi}_i}$, for all i.

Proposition 2.2. *Assume that for all i, j, $q_{ii} = \widehat{q}_{ii}$ and*

$$q_{ij}q_{ji} = \widehat{q}_{ij}\widehat{q}_{ji}. \tag{2.1}$$

Then there exists an $\mathbb{N}$-graded isomorphism of $\Bbbk\Gamma$-comodules $\psi : \mathfrak{B}(V) \to \mathfrak{B}(\widehat{V})$ such that

$$\psi(x_i) = \widehat{x}_i, \qquad 1 \leq i \leq \theta. \tag{2.2}$$

Let $\sigma : \Gamma \times \Gamma \to \Bbbk^\times$ be the unique bilinear form such that $\sigma(g_i, g_j) = \widehat{q}_{ij} q_{ij}^{-1}$, if $i \leq j$, and is equal to 1 otherwise; σ is a group 2-cocycle and we have, for all $g, h \in \Gamma$,

$$\psi(xy) = \sigma(g,h)\psi(x)\psi(y), \qquad x \in \mathfrak{B}(V)_g, \quad y \in \mathfrak{B}(V)_h; \tag{2.3}$$

$$\psi([x,y]_c) = \sigma(g,h)[\psi(x), \psi(y)]_c, \qquad x \in V_g^\chi, \quad y \in \mathfrak{B}(V)_h^\eta. \tag{2.4}$$

Proof. See [AS5, Prop. 3.9 and Remark 3.10]. □

Remark 2.3. In the situation of the proposition, we say that $\mathfrak{B}(V)$ and $\mathfrak{B}(\widehat{V})$ are *twist-equivalent*; note that $\mathfrak{B}(V)$ is twist-equivalent to a $\mathfrak{B}(\widehat{V})$ with $\widehat{q}_{ij} = \widehat{q}_{ji}$ for all i and j, since all the $q_{ij}q_{ji}$'s have square roots in $\Bbbk$.

Lemma 2.4. *Assume that the braiding with matrix (q_{ij}) is generic and of Cartan type with generalized Cartan matrix (a_{ij}). Then (a_{ij}) is symmetrizable.*

Proof. By [K, Ex. 2.1], it is enough to show that $a_{i_1i_2}a_{i_2i_3}\dots a_{i_{t-1}i_t}a_{i_ti_1} = a_{i_2i_1}a_{i_3i_2}\dots a_{i_ti_{t-1}}a_{i_1i_t}$ for all $i_1, i_2, \dots, i_t$. But

$$q_{i_1i_1}^{a_{i_1i_2}a_{i_2i_3}\dots a_{i_{t-1}i_t}a_{i_ti_1}} = q_{i_2i_2}^{a_{i_2i_1}a_{i_2i_3}\dots a_{i_{t-1}i_t}a_{i_ti_1}} = \dots = q_{i_1i_1}^{a_{i_2i_1}a_{i_3i_2}\dots a_{i_ti_{t-1}}a_{i_1i_t}},$$

by substituting $q_{i_1i_1}^{a_{i_1i_2}} = q_{i_2i_2}^{a_{i_2i_1}}$, then $q_{i_2i_2}^{a_{i_2i_1}} = q_{i_3i_3}^{a_{i_3i_2}}$ and so on. The claim follows because $q_{i_1i_1}$ is not a root of one. □

The following result is due to Rosso, who sketched an argument in [Ro, Th. 2.1]. We include a proof for completeness.

Lemma 2.5. [Ro]. *Let $\Bbbk = \mathbb{C}$. Assume that the braiding with matrix (q_{ij}) is positive and of Cartan type with generalized Cartan matrix (a_{ij}). Then (a_{ij}) is symmetrizable, with symmetrizing diagonal matrix (d_i); and there is a collection of positive numbers $(q_I)_{I\in\mathcal{X}}$ such that (q_{ij}) is twist-equivalent to $(\widehat{q}_{ij})$, where*

$$\widehat{q}_{ij} = q_I^{d_ia_{ij}} \text{ for all } i, j \in I.$$

That is, the braiding associated to $(\widehat{q}_{ij})$ is of DJ-type.

Proof. We can assume that the braiding is indecomposable; write $I = \{1, \dots, \theta\}$. By Remark 2.3, we can assume that $q_{ij} = q_{ji}$, for all $i, j \in I$. Given $j \in I$, there exists a sequence $i_1 = 1, i_2, \dots, i_t = j$ of elements in I, such that $a_{i_\ell i_{\ell+1}} \neq 0$ for all ℓ, $1 \le \ell < t$. Then

$$q_{11}^{a_{i_1i_2}a_{i_2i_3}\dots a_{i_{t-1}i_t}} = q_{jj}^{a_{i_2i_1}a_{i_3i_2}\dots a_{i_ti_{t-1}}},$$

as in the proof of the previous Lemma. Since $\alpha_j := a_{i_1i_2}a_{i_2i_3}\dots a_{i_{t-1}i_t}$ and $\beta_j := a_{i_2i_1}a_{i_3i_2}\dots a_{i_ti_{t-1}}$ are integers of the same sign, we can find $b \in \mathbb{N}$ and a family $(d_i)_{i\in I}$ of positive integers such that $\dfrac{\alpha_j}{\beta_j} = \dfrac{d_j}{b}$. Let q_I be the unique positive number such that $q_I^b = q_{11}$. Then $q_{jj}^{\beta_j} = q_{11}^{\alpha_j} = q_{11}^{\beta_j\frac{d_j}{b}} = q_I^{d_j\beta_j}$. Thus $q_{jj} = q_I^{d_j}$; and for all $i, j \in I$, $q_{ij}q_{ji} = q_I^{d_ia_{ij}} = q_I^{d_ja_{ji}}$, and the claim follows. □

Remark 2.6. The diagonal braiding with matrix

$$\begin{pmatrix} q & q^{-1} \\ q^{-1} & -q \end{pmatrix},$$

where q is not a root of one, is generic of Cartan type but not of DJ-type.

We now state a very elegant description of Nichols algebras used by Lusztig in a fundamental way [L2].

Proposition 2.7. *Let (V, c) be as above and assume that $q_{ij} = q_{ji}$ for all i, j. Let $B_1, \dots, B_\theta$ be non-zero elements in $\Bbbk$. There is a unique bilinear form $(\,|\,) : T(V) \times T(V) \to \Bbbk$ such that* $(1|1) = 1$

and

(2.5) $(x_j|x_j) = \delta_{ij} B_i$, *for all* i, j;

(2.6) $(x|yy') = (x_{(1)}|y)(x_{(2)}|y')$, *for all* $x, y, y' \in T(V)$;

(2.7) $(xx'|y) = (x|y_{(1)})(x'|y_{(2)})$, *for all* $x, x', y \in T(V)$.

This form is symmetric and also satisfies

$$(x|y) = 0, \qquad \textit{for all } x \in T(V)_g, y \in T(V)_h, g \neq h \in \Gamma. \tag{2.8}$$

The homogeneous components of $T(V)$ *with respect to its usual* $\mathbb{N}$*-grading are also orthogonal with respect to* $(\,|\,)$.

The quotient $T(V)/I(V)$, *where* $I(V) = \{x \in T(V) : (x|y) = 0 \forall y \in T(V)\}$ *is the radical of the form, is canonically isomorphic to the Nichols algebra of* V. *Thus,* $(\,|\,)$ *induces a non-degenerate bilinear form on* $\mathfrak{B}(V)$, *which will be again be denoted by* $(\,|\,)$.

Proof. The existence and uniqueness of the form, and the claims about symmetry and orthogonality, are proved exactly as in [L2, 1.2.3]. It follows from the properties of the form that $I(V)$ is a Hopf ideal. We now check that $T(V)/I(V)$ is the Nichols algebra of V; it is enough to verify that the primitive elements of $T(V)/I(V)$ are in V. Let x be a primitive element in $T(V)/I(V)$, homogeneous of degree $n \geq 2$. Then $(x|yy') = 0$ for all y, y' homogeneous of degrees $m, m' \geq 1$ with $m + m' = n$; thus $x = 0$. $\square$

Nichols algebras arising from quantum groups. Let $(a_{ij})_{1\leq i,j\leq\theta}$ be a generalized symmetrizable Cartan matrix [K]; let $(d_1, \dots, d_\theta)$ be positive integers such that $d_i a_{ij} = d_j a_{ji}$. Let $q \in \Bbbk$, $q \neq 0, 1$, and not a root of 1. We assume that the braided vector space (V, c) is given by the matrix $q_{ij} = q^{d_i a_{ij}}$.

We now want to derive some precise information about the algebra $\mathfrak{B}(V)$ mainly from [L2]. We need to consider vector spaces over the field of rational functions $\mathbb{Q}(v)$.

Let (W, d) denote a finite-dimensional braided vector space over $\mathbb{Q}(v)$; we assume that the braiding d is diagonal with matrix $(v^{d_i a_{ij}})$, with respect to a basis $y_1, \dots y_\theta$. We define characters $\eta_1, \dots, \eta_\theta$ of Γ by

$$\eta_i(g_j) = v^{d_i a_{ij}}, \qquad 1 \leq i, j \leq \theta.$$

We consider W as a Yetter-Drinfeld module over $\mathbb{Q}(v)\Gamma$ by defining $y_i \in W_{g_i}^{\eta_i}$, for all i.

Let $v_i := v^{d_i}$, $1 \leq i \leq \theta$. We now take $B_i := \left(1 - v_i^{-2}\right)^{-1} \in \mathbb{Q}(v)$ as in [L2, 1.2.3]. By proposition 2.7, $\mathfrak{B}(W)$ is Lusztig's braided Hopf algebra $\mathbf{f}$ (with Cartan datum given by $i.j = d_i a_{ij}$, $1 \leq i, j \leq \theta$.

Let $\mathcal{A} := \mathbb{Q}[v, v^{-1}]$; let $[n]_i := \dfrac{v_i^n - v_i^{-n}}{v_i - v_i^{-1}}$, $[r]_i! = [1]_i[2]_i \dots [r]_i$. Let $\mathfrak{B}(W)_{\mathcal{A}}$ be the $\mathcal{A}$-subalgebra of $\mathfrak{B}(W)$ generated by all

$$y_i^{(r)} := \frac{y_i^r}{[r]_i!}, \qquad 1 \leq i \leq \theta, \quad r \geq 0.$$

The canonical bilinear form $(\,|\,) : \mathfrak{B}(W) \times \mathfrak{B}(W) \to \mathbb{Q}(v)$ does not restrict to an $\mathcal{A}$-bilinear form $\mathfrak{B}(W)_{\mathcal{A}} \times \mathfrak{B}(W)_{\mathcal{A}} \to \mathcal{A}$, since by [L2, 1.4.4],

$$(y_i^{(r)}|y_i^{(r)}) = v_i^{\frac{r(r+1)}{2}} \frac{\left(v_i - v_i^{-1}\right)^{-r}}{[r]_i!}.$$

Following an idea of Müller [Mu], we define $\widetilde{\mathfrak{B}}(W)_{\mathcal{A}}$ as the $\mathcal{A}$-subalgebra of $\mathfrak{B}(W)$ generated by all

$$\widetilde{y}_i := \left(1 - v_i^{-2}\right) y_i = B_i^{-1} y_i, \qquad 1 \leq i \leq \theta.$$

Then $(\,|\,)$ restricts to an $\mathcal{A}$-bilinear form $(\,|\,) : \mathfrak{B}(W)_{\mathcal{A}} \times \widetilde{\mathfrak{B}}(W)_{\mathcal{A}} \to \mathcal{A}$, by the argument in [Mu, Lemma 2.2 (a)].

Note that $\mathfrak{B}(W)_{\mathcal{A}}$ and $\widetilde{\mathfrak{B}}(W)_{\mathcal{A}}$ inherit the Hopf algebra structure from $\mathfrak{B}(W)$, since $\mathcal{A}$ is a principal ideal domain with quotient field $\mathbb{Q}(v)$.

Let $\mathcal{W}$ be the Weyl group of the generalized Cartan matrix (a_{ij}) [L2, 2.1.1]; then $\mathcal{W}$ is finite if and only if (a_{ij}) is of finite type. Let $w \in \mathcal{W}$ be an element with reduced expression $w = s_{i_1} s_{i_2} \dots s_{i_P}$, $P \in \mathbb{N}$. For any $c = (c_1, \dots, c_P) \in \mathbb{N}^P$, let

$$L(c) := y_{i_1}^{(c_1)} T_{i_1}\left(y_{i_2}^{(c_2)}\right) \dots T_{i_1} T_{i_2} \dots T_{i_{P-1}}\left(y_{i_P}^{(c_P)}\right),$$

where T_{i_ℓ} are the $\mathbb{Q}(v)$-algebra automorphisms of $\mathbf{U}$ named $T'_{i_\ell,-1}$ in [L2, 37.1.3]. Note that $L(c) = L(\mathbf{h}, c, 0, 1)$, with $\mathbf{h} := (i_1, \dots, i_P)$, in [L2, 38.2.3]. By [L2, 41.1.3],

$$T_{i_1} T_{i_2} \dots T_{i_{\ell-1}}\left(y_{i_\ell}^{(r)}\right) \in \mathfrak{B}(W)_{\mathcal{A}}, \qquad 1 \leq \ell < P, \quad r \geq 0,$$

where we have identified $\mathfrak{B}(W)$ with $\mathbf{U}^+$ by [L2, 3.2.6], that is we identify $E_i := \theta_i^+$ in Lusztig's notation with y_i. We denote

$$z_\ell := T_{i_1} T_{i_2} \dots T_{i_{\ell-1}}\left(y_{i_\ell}\right), \qquad 1 \leq \ell \leq P.$$

Then $z_\ell \in \mathfrak{B}(W)_{\mathcal{A}}$ and

$$z_\ell^r = [r]_{i_\ell}! \, T_{i_1} T_{i_2} \dots T_{i_{\ell-1}}\left(y_{i_\ell}^{(r)}\right), \qquad 1 \leq \ell \leq P, \quad r \geq 0.$$

Theorem 2.8. *(Lusztig). For all $c = (c_1, \dots, c_P), c' \in \mathbb{N}^P$*

$$(L(c)|L(c')) = \delta_{c,c'} \prod_{1 \leq s \leq P} \prod_{1 \leq t \leq c_s} \left(1 - v_i^{-2t}\right)^{-1}. \tag{2.9}$$

Proof. This follows from [L2, 38.2.3 and 1.4.4]. □

We regard $\Bbbk$ as an $\mathcal{A}$-algebra via the algebra map $\varphi : \mathcal{A} \to \Bbbk$ given by $\varphi(v) = q$. We define

$$\mathfrak{B}(W)_\Bbbk := \mathfrak{B}(W)_{\mathcal{A}} \otimes_{\mathcal{A}} \Bbbk, \qquad \widetilde{\mathfrak{B}}(W)_\Bbbk := \widetilde{\mathfrak{B}}(W)_{\mathcal{A}} \otimes_{\mathcal{A}} \Bbbk.$$

Then $\mathfrak{B}(W)_\Bbbk$, $\widetilde{\mathfrak{B}}(W)_\Bbbk$ are graded braided Hopf algebras in ${}_{\Bbbk\Gamma}^{\Bbbk\Gamma}\mathcal{YD}$ and by tensoring with $\Bbbk$ over $\mathcal{A}$, we get a bilinear form

$$(\,|\,) : \mathfrak{B}(W)_\Bbbk \times \widetilde{\mathfrak{B}}(W)_\Bbbk \to \Bbbk.$$

Define $\pi : T(V) \to \mathfrak{B}(W)_{\Bbbk}$ and $\widetilde{\pi} : T(V) \to \widetilde{\mathfrak{B}}(W)_{\Bbbk}$ by $\pi(x_i) := y_i \otimes 1$, $\widetilde{\pi}(x_i) := \widetilde{y}_i \otimes 1$, $1 \le i \le \theta$. Since π and $\widetilde{\pi}$ induce isomorphisms of braided vector spaces of (V, c) with $\mathfrak{B}(W)_{\Bbbk}(1)$ and $\widetilde{\mathfrak{B}}(W)_{\Bbbk}(1)$, the composition

$$(\,|\,) : T(V) \times T(V) \xrightarrow{\pi\times\widetilde{\pi}} \mathfrak{B}(W)_{\Bbbk} \times \widetilde{\mathfrak{B}}(W)_{\Bbbk} \xrightarrow{(\,|\,)} \Bbbk$$

is the canonical bilinear form of $T(V)$ as in Proposition 2.7 with scalars $B_i = 1$, $1 \le i \le \theta$. These arguments allow to adapt many results of Lusztig to the case when q is not a root of 1.

Theorem 2.9. [Ro, Theorem 15]; [L2, Section 37]. *Let (V, c) be a braided vector space of DJ-type. Then $\mathfrak{B}(V) \simeq \Bbbk\langle x_1, \dots, x_\theta | \operatorname{ad}_c(x_i)^{1-a_{ij}} = 0, 1 \le i \ne j \le \theta\rangle$.* □

The following Theorem is part of the folklore of quantum groups.

Theorem 2.10. *Let (V, c) be a braided vector space of DJ-type, with generalized Cartan matrix (a_{ij}).*

(i). If the Gelfand-Kirillov dimension of $\mathfrak{B}(V)$ is finite, then (a_{ij}) is a finite Cartan matrix.

(ii). If (a_{ij}) is a finite Cartan matrix, then the Gelfand-Kirillov dimension of $\mathfrak{B}(V)$ is finite and equal to the number of positive roots.

Proof. (i). We can assume that the braiding is connected. Let $w \in \mathcal{W}$ be an element with reduced expression $w = s_{i_1} s_{i_2} \dots s_{i_P}$, $P \in \mathbb{N}$. We keep the notation above. For all ℓ, $1 \le \ell \le P$, we choose an element $t_\ell \in T(V)$ with $\pi(t_\ell) = z_\ell \otimes 1$, and set $b_\ell :=$ canonical image of t_ℓ in $\mathfrak{B}(V)$. We claim that the ordered monomials $b_1^{c_1} b_2^{c_2} \dots b_P^{c_P}$, $c_1, \dots, c_P \ge 0$, are linearly independent.

To prove the claim, we choose $a_c \in \mathcal{A}$ such that $a_c L(c) \in \widetilde{\mathfrak{B}}(W)_{\mathcal{A}}$ for all $c = (c_1, \dots, c_P) \in \mathbb{N}^P$. If $\widetilde{L}(c) := a_c L(c) \prod_{1\le s\le P} \prod_{1\le t\le c_s} \left(1 - v_i^{-2t}\right) \in \widetilde{\mathfrak{B}}(W)_{\mathcal{A}}$, then

$$(L(c)|\widetilde{L}(c')) = \delta_{c,c'} a_c, \tag{2.10}$$

for all $c, c' \in \mathbb{N}^P$, by (2.9). It follows that $\pi(t_1^{c_1} t_2^{c_2} \dots t_P^{c_P}) = \alpha_c L(c) \otimes 1 \in \widetilde{\mathfrak{B}}(W)_{\Bbbk}$, where $\alpha_c = \varphi\left(\prod_{1\le s\le P} ([c_s]_{i_s}!)^{-1}\right)$ is a non-zero scalar in $\Bbbk$. Choose elements $\widetilde{\ell}(c) \in T(V)$ with $\widetilde{\pi}(\widetilde{\ell}(c)) = \widetilde{L}(c) \otimes 1$, $c \in \mathbb{N}^P$. Then for all $c, c' \in \mathbb{N}^P$, we conclude that $(t_1^{c_1} t_2^{c_2} \dots t_P^{c_P} | \widetilde{\ell}(c')) = 0$, if $c \ne c'$, and $(t_1^{c_1} t_2^{c_2} \dots t_P^{c_P} | \widetilde{\ell}(c)) \ne 0$. Since the form $(\ \ |\ \ )$ factorizes over $\mathfrak{B}(V) \times \mathfrak{B}(V)$, the elements $b_1^{c_1} b_2^{c_2} \dots b_P^{c_P}$, $c_1, \dots, c_P \ge 0$, are linearly independent.

The following statement is well-known:

Let A be a k-algebra, where k is a field. Let $a_1, \dots, a_P \in A$ such that the ordered monomials $a_1^{c_1} a_2^{c_2} \dots a_P^{c_P}$, $c_1, \dots, c_P \ge 0$, are linearly independent. Then the Gelfand-Kirillov dimension of A is $\ge P$.

We conclude that the Gelfand-Kirillov dimension of $\mathfrak{B}(V)$ is $\ge P$. If the Cartan matrix (a_{ij}) is not finite, then there are elements in the Weyl group of arbitrary lenght, and (i) follows.

(ii). Now the Weyl group is finite, and we take the longest element w_0. Let us consider the natural map $\varphi : \Bbbk\langle x_1, \dots, x_\theta | \operatorname{ad}_c(x_i)^{1-a_{ij}}, 1 \le i \ne j \le \theta\rangle \to \mathfrak{B}(V)$. Since the elements $b_1^{c_1} b_2^{c_2} \dots b_P^{c_P}$, $c_1, \dots, c_P \ge 0$, are the image of the PBW-basis of $\Bbbk\langle x_1, \dots, x_\theta | \operatorname{ad}_c(x_i)^{1-a_{ij}}, 1 \le i \ne j \le \theta\rangle$, see *e. g.* [DCK, Proposition 1.7], the claim follows. □

Rosso's characterization of Nichols algebras. We recall some important results of Rosso.

Theorem 2.11. [Ro, Lemma 19] *Let (V, c) be a braided vector space of diagonal type. If $\mathfrak{B}(V)$ has finite Gelfand-Kirillov dimension, then for all $i \neq j$, there exists $r > 0$ such that* $\operatorname{ad}_c(x_i)^r(x_j) = 0$. □

Corollary 2.12. *Let (V, c) be a braided vector space of diagonal type with indecomposable matrix. Assume that $\mathfrak{B}(V)$ has finite Gelfand-Kirillov dimension.*

(a). If there exists i such that $q_{ii} = 1$, then $\theta = 1$.

(b). If the braiding is generic then it is of Cartan type.

Proof. This follows from Theorem 2.11 and Lemma 2.1 (e). □

Theorem 2.13. [Ro, Theorem 21] *Let (V, c) be a finite-dimensional braided vector space with positive braiding. Then the following are equivalent:*

(a). $\mathfrak{B}(V)$ has finite Gelfand-Kirillov dimension.

(b). (V, c) is twist-equivalent to a braiding of DJ-type with finite Cartan matrix.

Proof. We can assume that the matrix (q_{ij}) is indecomposable. (b) $\implies$ (a) follows from Theorem 2.10 (ii). (a) $\implies$ (b). (V, c) is of Cartan type by Corollary 2.12 (b) with Cartan matrix (a_{ij}). We know that (a_{ij}) is symmetrizable by Lemma 2.4 and that (V, c) is twist-equivalent to a braiding of DJ-type by Lemma 2.5. By Theorem 2.10 (i), the Cartan matrix (a_{ij}) is finite. □

Remark 2.14. See [AS2] for the analogous problem of characterizing finite-dimensional braided vector spaces with diagonal braiding such that $\mathfrak{B}(V)$ has finite dimension.

3. Coradically graded coalgebras

In this Section we prove a general criterion to determine the coradical filtration of certain Hopf algebras. We generalize a method of Takeuchi [T], who computed the coradical filtration of $U_q(g)$ in this way; see also [Mu]. We first extend the definition of coradically graded coalgebras [CM].

Let $T \geq 1$ be a natural number. If $\mathbf{i} = (i_1, \dots, i_T) \in \mathbb{N}^T$, then we set $|\mathbf{i}| = i_1 + \cdots + i_T$.

Definition 3.1. An $\mathbb{N}^T$*-graded coalgebra* is a coalgebra C provided with an $\mathbb{N}^T$-grading $C = \oplus_{\mathbf{i} \in \mathbb{N}^T} C(\mathbf{i})$ such that $\Delta C(\mathbf{i}) \subset \oplus_{\mathbf{j}} C(\mathbf{j}) \otimes C(\mathbf{i} - \mathbf{j})$. An $\mathbb{N}^T$-graded coalgebra C is *coradically graded* if the n-th term of the coradical filtration is

$$C_n = \oplus_{\mathbf{i} \in \mathbb{N}^T, |\mathbf{i}| \leq n} C(\mathbf{i}), \quad \forall n \in \mathbb{N}.$$

We denote by $\pi_{\mathbf{i}} : C \to C(\mathbf{i})$ the projection associated to the grading.

An $\mathbb{N}^T$-graded coalgebra C is *strictly coradically graded* if $C(0) = C_0$, the coradical of C, and $\Delta_{\mathbf{i},\mathbf{j}} : C(\mathbf{i} + \mathbf{j}) \to C(\mathbf{i}) \otimes C(\mathbf{j})$, $\Delta_{\mathbf{i},\mathbf{j}} = (\pi_{\mathbf{i}} \otimes \pi_{\mathbf{j}}) \circ \Delta$, is injective, for all $\mathbf{i}, \mathbf{j} \in \mathbb{N}^{\mathbf{T}}$.

Lemma 3.2. *(a). Let C be a strictly coradically $\mathbb{N}^T$-graded coalgebra and let D be a strictly coradically $\mathbb{N}^S$-graded coalgebra. Then $C \otimes D$ is strictly coradically $\mathbb{N}^{T+S}$-graded with respect to the tensor product grading.*

(b). If C is strictly coradically $\mathbb{N}^T$-graded, then it is coradically graded.

(c). If C is coradically $\mathbb{N}$-graded, then it is strictly coradically graded.

Proof. (a) follows from the definition. We prove (b) by induction on n, the case $n = 0$ being part of the hypothesis. Assume $n > 0$. If $c \in \oplus_{|\mathbf{i}|\le n} C(\mathbf{i})$, then $\Delta(c) \in C(\mathbf{0}) \otimes C(\mathbf{n}) + \oplus_{|\mathbf{j}|<n} C(\mathbf{n}-\mathbf{j}) \otimes C(\mathbf{j}) \subset C_0 \otimes C + C \otimes C_{n-1}$; hence $c \in C_n$ (the argument does not need the second hypothesis). Conversely, let $c \in C_n \cap C(\mathbf{i})$ with $|\mathbf{i}| > n$. By the recursive hypothesis, $\Delta(c) \in C(\mathbf{0}) \otimes C + \oplus_{|\mathbf{j}|<n} C(\mathbf{i}-\mathbf{j}) \otimes C(\mathbf{j})$. If $\mathbf{e} \in N^T$ has $|\mathbf{e}| = 1$, then $\Delta_{\mathbf{e},\mathbf{i}-\mathbf{e}}(c) = 0$, thus $c = 0$.

(c). See for example [MiS, Lemma 2.3]. □

We now consider the following situation. Let U be a Hopf algebra, H a Hopf subalgebra and $N_1, \dots, N_T$ subalgebras of U, such that the multiplication induces a linear isomorphism

$$\mu : N_1 \otimes \cdots \otimes N_T \otimes H \to U.$$

We assume that the following condition holds:

For all l, $1 \le l \le T$, $N_l H$ is a Hopf subalgebra of U. Furthermore, the projection $\pi_l : N_l H \to H$, defined by $\pi_l(nh) = \varepsilon(n)h$, $n \in N_l$, $h \in H$, is a Hopf algebra map, and $N_l = (N_l H)^{\mathrm{co}\,\pi_l}$.

Then N_l is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ and $N_l H \simeq N_l \# H$. Let Δ_l be the comultiplication of the braided Hopf algebra N_l and let $j_l : N_l H \otimes N_l \to N_l H \otimes N_l$ be the map given by $j_l(rh \otimes s) = rs_{(-1)}h \otimes s_{(0)}$; we know that

$$\Delta(s) = j_l \Delta_l(s), \text{ for all } s \in N_l. \tag{3.1}$$

Lemma 3.3. *The map $j : N_1 \otimes N_1 \otimes N_2 \otimes N_2 \otimes \cdots \otimes N_T \otimes N_T \otimes H \otimes H \to U \otimes U$ given by*

$$j(u_1 \otimes v_1 \otimes u_2 \otimes v_2 \otimes \cdots \otimes u_T \otimes v_T \otimes h \otimes k) = j_1(u_1 \otimes v_1) j_2(u_2 \otimes v_2) \dots j_T(u_T \otimes v_T)(h \otimes k) \tag{3.2}$$

(product in $U \otimes U$), is a linear isomorphism, and the following diagram commutes:

$$\begin{array}{ccc} N_1 \otimes \cdots \otimes N_T \otimes H & \xrightarrow{\mu} & U \\ \Big\downarrow{\scriptstyle \Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_H} & & \Big\downarrow{\scriptstyle \Delta} \\ N_1 \otimes N_1 \otimes N_2 \otimes N_2 \otimes \cdots \otimes N_T \otimes N_T \otimes H \otimes H & \xrightarrow{j} & U \otimes U. \end{array} \tag{3.3}$$

Proof. The commutativity of (3.3) follows from (3.1) since Δ is multiplicative. We prove now that j is a linear isomorphism. Let $z = u_1 \otimes v_1 \otimes u_2 \otimes v_2 \otimes \cdots \otimes u_T \otimes v_T \otimes h \otimes k \in N_1 \otimes N_1 \otimes N_2 \otimes N_2 \otimes \cdots \otimes N_T \otimes N_T \otimes H \otimes H$. Then

$$\begin{aligned} j(z) &= u_1 (v_1)_{(-1)} u_2 (v_2)_{(-1)} \dots u_T (v_T)_{(-1)} h \otimes (v_1)_{(0)} (v_2)_{(0)} \dots (v_T)_{(0)} k \\ &= u_1 \left(\mathrm{ad}\,(v_1)_{(-T)} u_2\right) \left(\mathrm{ad}\,((v_1)_{(-T+1)} (v_2)_{(-T+1)}) u_3\right) \dots \left(\mathrm{ad}\,((v_1)_{(-2)} (v_2)_{(-2)} \dots (v_{T-1})_{(-2)}) u_T\right) \times \\ &\quad \times (v_1)_{(-1)} (v_2)_{(-1)} \dots (v_T)_{(-1)} h \otimes (v_1)_{(0)} (v_2)_{(0)} \dots (v_T))_{(0)} k \\ &= (\mu \otimes \mu) \big(u_1 \otimes (v_1)_{(-2)} \cdot \big(u_2 \otimes (v_2)_{(-2)} \cdot \big(\cdots \otimes (v_{T-1})_{(-2)} \cdot u_T\big)\big) \otimes (v_1)_{(-1)} (v_2)_{(-1)} \dots (v_T)_{(-1)} h \\ &\quad \otimes (v_1)_{(0)} \otimes (v_2)_{(0)} \otimes \cdots \otimes (v_T)_{(0)} \otimes k\big). \end{aligned}$$

Here, in the first equality we applied the rule $(v_i)_{(-1)} u_j = (\mathrm{ad}\,(v_i)_{(-2)} u_j)(v_i)_{(-1)}$ for $i < j$ systematically; in the second equality, we use $\cdot$ for the tensor product of copies of the adjoint representation. We now apply successively several isomorphisms to this expression. First, if M is any left H-comodule and N is any left H-module, the map $\phi_{N,M} : N\otimes M \to N\otimes M$, $\phi_M(n\otimes m) = m_{(-1)}\cdot n\otimes m_{(0)}$ is an isomorphism with inverse $\phi_M^{-1}(n\otimes m) = \mathcal{S}^{-1}(m_{(-1)})\cdot n\otimes m_{(0)}$. We apply $\operatorname{id}_{N_1\otimes\cdots\otimes N_T}\otimes\phi^{-1}_{H,N_1\otimes\cdots\otimes N_T}\otimes\operatorname{id}_H$ to $(\mu\otimes\mu)^{-1}j(z)$ and get

$$u_1\otimes (v_1)_{(-1)}\cdot\big(u_2\otimes (v_2)_{(-1)}\cdot\big(\cdots\otimes (v_{T-1})_{(-1)}\cdot u_T\big)\big)\otimes h\otimes (v_1)_{(0)}\otimes (v_2)_{(0)}\otimes\cdots\otimes (v_T)_{(0)}\otimes k;$$

to this expression, we apply $\operatorname{id}_{N_1}\otimes\phi^{-1}_{N_2\otimes\cdots\otimes N_T\otimes H,N_1}\otimes\operatorname{id}_{N_2\otimes\cdots\otimes N_T\otimes H}$ and get

$$u_1\otimes u_2\otimes (v_2)_{(-1)}\cdot\big(\cdots\otimes (v_{T-1})_{(-1)}\cdot u_T\big)\otimes h\otimes v_1\otimes (v_2)_{(0)}\otimes\cdots\otimes (v_T)_{(0)}\otimes k;$$

iterating this procedure, we obtain z. This shows that j is bijective. □

Theorem 3.4. *We keep the notations above. We assume that $N_l = \oplus_{i\in\mathbb{N}}N_l(i)$ is a coradically graded Hopf algebra in ${}^H_H\mathcal{YD}$, for all l, $1\le l\le T$, and that H is cosemisimple. Let $U(\mathbf{i}) = \mu\left(N_1(i_1)\otimes\cdots\otimes N_T(i_T)\otimes H\right)$, for all $\mathbf{i} = (i_1,\dots,i_T)\in\mathbb{N}^T$. Then $U = \oplus_{\mathbf{i}\in\mathbb{N}^T}U(\mathbf{i})$ is a coradically $\mathbb{N}^T$-graded coalgebra.*

Proof. The tensor product coalgebra $N_1\otimes\cdots\otimes N_T\otimes H$ is a strictly coradically $\mathbb{N}^T$-graded coalgebra by Lemma 3.2 (a) and (c) since each N_l is coradically graded and H is cosemisimple. Since each $N_l(i)$ is a Yetter-Drinfeld submodule of N_l, the map j in Lemma 3.1 is homogeneous. Hence, it follows from Lemma 3.3 that U is strictly coradically $\mathbb{N}^T$-graded coalgebra. Then U is coradically $\mathbb{N}^T$-graded by Lemma 3.2 (b). □

4. A family of pointed Hopf algebras

In this Section, we fix

- a free abelian group Γ of finite rank s,
- a finite Cartan matrix $(a_{ij})\in\mathbb{Z}^{\theta\times\theta}$ [K]; we denote by $(d_1,\dots,d_\theta)$ a diagonal matrix of positive integers such that $d_ia_{ij} = d_ja_{ji}$, which is minimal with this property;
- a family $(q_I)_{I\in\mathcal{X}}$ of elements in $\Bbbk$ which are not roots of 1;
- elements $g_1,\dots,g_\theta\in\Gamma$, characters $\chi_1,\dots,\chi_\theta\in\widehat{\Gamma}$ such that $\langle\chi_i,g_i\rangle = q_I^{d_i}$ for all i, and

$$\langle\chi_j,g_i\rangle\langle\chi_i,g_j\rangle = q_I^{d_ia_{ij}},\qquad \text{for all } 1\le i,j\le\theta,\quad i\in I. \tag{4.1}$$

Definition 4.1. [AS4]. We say that two vertices i and j *are linkable* (or that i *is linkable to* j) if

(4.2) $i\not\sim j$,

(4.3) $g_ig_j\neq 1$ and

(4.4) $\chi_i\chi_j = \varepsilon$.

Here, ε denotes the trivial representation of Γ. One can easily see, *cf.* [AS4], that:

(4.5) If i is linkable to j, then $\chi_i(g_j) = \chi_j(g_i)^{-1} = \chi_i(g_i) = \chi_j(g_j)^{-1}$.

(4.6) If i and k, resp. j and ℓ, are linkable, then $a_{ij} = a_{k\ell}$, $a_{ji} = a_{\ell k}$.

(4.7) A vertex i can not be linkable to two different vertices j and h.

A *linking datum* for Γ, (a_{ij}), $(q_I)_{I\in\mathcal{X}}$, $g_1,\dots,g_\theta$ and $\chi_1,\dots,\chi_\theta$ is a collection $(\lambda_{ij})_{1\le i<j\le\theta, i\nsim j}$ of elements in $\{0,1\}$ such that λ_{ij} is arbitrary if i and j are linkable but 0 otherwise. Given a linking datum, we say that two vertices i and j *are linked* if $\lambda_{ij}\neq 0$.

The collection $\mathcal{D}=\mathcal{D}((a_{ij}),(q_I),(g_i),(\chi_i),(\lambda_{ij}))$, where (λ_{ij}) is a linking datum, will be called a *generic datum of finite Cartan type* for Γ. If $\Bbbk=\mathbb{C}$, a generic datum of finite Cartan type will be called *positive* if $q_I>0$, for all $I\in\mathcal{X}$.

Let $\mathcal{D}'$ be a generic datum of finite Cartan type over a free abelian group Γ' of finite rank, formed by $(a'_{ij})\in\mathbb{Z}^{\theta'\times\theta'}$, $(q'_I)_{I\in\mathcal{X}'}$, $g'_1,\dots,g'_\theta$, $\chi'_1,\dots,\chi'_\theta$ and a linking datum $(\lambda'_{ij})_{1\le i<j\le\theta,i\nsim j}$. The data $\mathcal{D}$ and $\mathcal{D}'$ are called *isomorphic* if $\theta=\theta'$, and if there exist a group isomorphism $\varphi:\Gamma\to\Gamma'$, a permutation $\sigma\in\mathbb{S}_\theta$, and elements $0\neq\alpha_i\in\Bbbk$, for all $1\le i\le\theta$ such that

- $\varphi(g_i)=g'_{\sigma(i)}$, for all $1\le i\le\theta$;
- $\chi_i=\chi'_{\sigma(i)}\varphi$, for all $1\le i,j\le\theta$;
- $\lambda_{ij}=\begin{cases}\alpha_i\alpha_j\lambda'_{\sigma(i)\sigma(j)}, & \text{if } \sigma(i)<\sigma(j)\\ -\alpha_i\alpha_j\chi_j(g_i)\lambda'_{\sigma(j)\sigma(i)}, & \text{if } \sigma(i)>\sigma(j)\end{cases}$, for all $1\le i<j\le\theta, i\nsim j$.

In this case the triple $(\varphi,\sigma,(\alpha_i))$ will be called an *isomorphism* from $\mathcal{D}$ to $\mathcal{D}'$.

Note that then $a_{ij}=a'_{\sigma(i)\sigma(j)}$, for all $1\le i,j\le\theta$. Thus σ is an isomorphism of the corresponding Dynkin diagrams. Indeed, for all i,j, $\chi_j(g_i)=\chi'_{\sigma(i)}(g'_{\sigma(i)})$, hence $\chi_j(g_i)\chi_i(g_j)=\chi_i(g_i)^{a_{ij}}=\chi_i(g_i)^{a'_{\sigma(i)\sigma(j)}}$, and $a_{ij}=a'_{\sigma(i)\sigma(j)}$, since $\chi_i(g_i)$ is not a root of 1.

Let $\operatorname{Isom}(\mathcal{D},\mathcal{D}')$ be the set of all isomorphisms from $\mathcal{D}$ to $\mathcal{D}'$.

To state the next theorem, we follow the conventions of [AS4, Section 4]. We assume that the Cartan matrix is a matrix of blocks corresponding to the connected components; that is, for each $I\in\mathcal{X}$, there exist c_I,d_I such that $I=\{j: c_I\le j\le d_I\}$.

Let Φ_I, resp. Φ_I^+, be the root system, resp. the subset of positive roots, corresponding to the Cartan matrix $(a_{ij})_{i,j\in I}$; then $\Phi=\bigcup_{I\in\mathcal{X}}\Phi_I$, resp. $\Phi^+=\bigcup_{I\in\mathcal{X}}\Phi_I^+$ is the root system, resp. the subset of positive roots, corresponding to the Cartan matrix $(a_{ij})_{1\le i,j\le\theta}$. Let $\alpha_1,\dots,\alpha_\theta$ be the set of simple roots.

Let $\mathcal{W}_I$ be the Weyl group corresponding to the Cartan matrix $(a_{ij})_{i,j\in I}$; we identify it with a subgroup of the Weyl group $\mathcal{W}$ corresponding to the Cartan matrix (a_{ij}). We fix a reduced decomposition of the longest element $\omega_{0,I}$ of $\mathcal{W}_I$ in terms of simple reflections. Then we obtain a reduced decomposition of the longest element $\omega_0=s_{i_1}\dots s_{i_P}$ of $\mathcal{W}$ from the expression of ω_0 as product of the $\omega_{0,I}$'s in some fixed order of the components, say the order arising from the order of the vertices. Therefore $\beta_j:=s_{i_1}\dots s_{i_{j-1}}(\alpha_{i_j})$ is a numeration of Φ^+.

We fix a finite-dimensional Yetter-Drinfeld module V over Γ with a basis $x_1,\dots,x_\theta$ with $x_i\in V_{g_i}^{\chi_i}$, $1\le i\le\theta$. Note that

$$V_{g_i}^{\chi_i}\not\simeq V_{g_j}^{\chi_j} \text{ in } {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}, \text{ for all } 1\le i,j\le\theta, i\neq j. \tag{4.8}$$

For suppose that $V_{g_i}^{\chi_i} \cong V_{g_j}^{\chi_j}$, and $i \neq j$. Then $g_i = g_j$, $\chi_i = \chi_j$ and

$$\chi_i(g_i)^2 = \chi_i(g_j)\chi_j(g_i) = \chi_i(g_i)^{a_{ij}}.$$

Thus $a_{ij} = 2$, since $\chi_i(g_i)$ is not a root of one; but this is a contradiction.

Lusztig defined root vectors X_α, $\alpha \in \Phi^+$ [L2], in the case of braidings of DJ-type; these are the elements $b_1, \dots, b_P$ in the proof of Theorem 2.10; they can be expressed as iterated braided commutators. As in [AS4], this definition can be extended to generic braidings of finite Cartan type, first in the tensor algebra $T(V)$, and then in suitable quotients.

We fix a $\mathbb{Z}$-basis $Y_h, 1 \leq h \leq s$ of Γ.

Theorem 4.2. *Let $\mathcal{D} = \mathcal{D}((a_{ij}), (q_I), (g_i), (\chi_i), (\lambda_{ij}))$ be a generic datum of finite Cartan type for the free abelian group Γ of finite rank. Let $U(\mathcal{D})$ be the algebra presented by generators $a_1, \dots, a_\theta$, $y_1^{\pm 1}, \dots, y_s^{\pm 1}$ and relations*

$$y_m^{\pm 1} y_h^{\pm 1} = y_h^{\pm 1} y_m^{\pm 1}, \quad y_m^{\pm 1} y_m^{\mp 1} = 1, \qquad 1 \leq m, h \leq s, \tag{4.9}$$

$$y_h a_j = \chi_j(Y_h) a_j y_h, \qquad 1 \leq h \leq s,\ 1 \leq j \leq \theta, \tag{4.10}$$

$$(\operatorname{ad} a_i)^{1-a_{ij}} a_j = 0, \qquad 1 \leq i \neq j \leq \theta, \quad i \sim j, \tag{4.11}$$

$$a_i a_j - \chi_j(g_i) a_j a_i = \lambda_{ij}(1 - g_i g_j), \qquad 1 \leq i < j \leq \theta, \quad i \not\sim j; \tag{4.12}$$

then $U(\mathcal{D})$ is a pointed Hopf algebra with structure determined by

$$\Delta y_h = y_h \otimes y_h, \qquad \Delta a_i = a_i \otimes 1 + g_i \otimes a_i, \qquad 1 \leq h \leq s,\ 1 \leq i \leq \theta. \tag{4.13}$$

Furthermore, $U(\mathcal{D})$ has a PBW-basis given by monomials in the root vectors $b_1 := a_{\beta_1}, \dots, b_P := a_{\beta_P}$. The coradical filtration of $U(\mathcal{D})$ is given by

$$U(\mathcal{D})_N = \text{ span of } a_{i_1} a_{i_2} \dots a_{i_s} y : \quad s \leq \mathbb{N}, \quad y \in G(H). \tag{4.14}$$

There is an isomorphism of graded Hopf algebras $\psi : \mathfrak{B}(V) \# \Bbbk\Gamma \to \operatorname{gr} U(\mathcal{D})$, given by $x_i \# 1 \mapsto \overline{a_i}$, $1 \leq i \leq \theta$, $1 \# Y_h \mapsto y_h$, $1 \leq h \leq s$.

$U(\mathcal{D})$ has finite Gelfand-Kirillov dimension and is a domain.

In (4.12) and (4.13), the elements g_i, $1 \leq i \leq \theta$, must be read as the word in the generators $y_h, 1 \leq h \leq s$ given by $y_1^{t_{i,1}} \cdots y_s^{t_{i,s}}$, if $g_i = Y_1^{t_{i,1}} \cdots Y_s^{t_{i,s}}$, where $t_{i,1}, \cdots, t_{i,s}$ are natural numbers.

The relations (4.11) are the quantum Serre relations. By (4.10) and (4.13), $(\operatorname{ad} a_i)(a_j) = a_i a_j - \chi_j(g_i) a_j a_i = (\operatorname{ad}_c a_i)(a_j)$. Here, ad_c is the braided adjoint representation in the tensor algebra of V. Hence the left hand side of (4.11) should be more formally written as

$$(\operatorname{ad}_c a_i)^{1-aij}(a_j) = \sum_{l=0}^{1-a_{ij}} (-1)^l \binom{1-a_{ij}}{l}_{q_{ii}} q_{ii}^{\frac{l(l-1)}{2}} q_{ij}^l a_i^{1-a_{ij}-l} a_j a_i^l,$$

where $q_{ij} := \chi_j(g_i)$ for all i, j.

Proof. Step I. It is not difficult to see that Δ is well-defined by (4.13), using Lemma 2.1 (b) for the quantum Serre relations.

Step II. As in [AS4, Th. 4.5], we deduce from Theorem 2.10, via Proposition 2.2, that $\mathfrak{B}(V) \simeq \Bbbk\langle x_1, \dots, x_\theta | \operatorname{ad}_c(x_i)^{1-a_{ij}} = 0, 1 \leq i \neq j \leq \theta\rangle$.

Step III. We now prove the statement about the PBW-basis. We argue exactly as in the proof of [AS4, Th. 5.17]; we proceed by induction on the number of connected components. If the Dynkin diagram is connected, the claim follows from Step II as in [AS4, Lemma 5.18]. For the inductive step, one repeats the argument in [AS4, Lemma 5.19]. We assume there exists $\widehat{\theta} < \theta$ such that $J = \{1, \dots, \widehat{\theta}\} \in \mathcal{X}$. Let $\Upsilon := < Z_1 > \oplus \cdots \oplus < Z_{\widehat{\theta}} >$, be a free abelian group of rank $\widehat{\theta}$. Let η_j be the unique character of Υ such that $\eta_j(Z_i) = \chi_j(g_i)$, $1 \leq i, j \leq \widehat{\theta}$.

Let $\mathcal{D}_1$ be the generic datum over Γ given by $(a_{ij})_{\widehat{\theta}<i,j\leq\theta}$, $(g_i)_{\widehat{\theta}<i\leq\theta}$, $(\chi_j)_{\widehat{\theta}<j\leq\theta}$, $(\lambda_{ij})_{\widehat{\theta}<i<j\leq\theta, i\not\sim j}$. Let $\mathcal{B} := U(\mathcal{D}_1)$, with generators $b_{\widehat{\theta}+1}, \dots, b_\theta$ (instead of the a_i's) and $y_1, \dots, y_s$.

Let $\mathcal{D}_2$ be the generic datum over Υ given by $(a_{ij})_{1\leq i,j\leq\widehat{\theta}}$, $(Z_i)_{1\leq i\leq\widehat{\theta}}$, $(\eta_j)_{1\leq j\leq\widehat{\theta}}$, with empty linking datum. Let $\mathcal{U} := U(\mathcal{D}_2)$ with generators $u_1, \dots, u_{\widehat{\theta}}$ (instead of the a_i's) and $z_1, \dots, z_{\widehat{\theta}}$.

Then the analogue of [AS4, Lemma 5.19] holds replacing the dual of $\mathcal{B}$ by the Hopf dual. Notice that the argument in *loc. cit.* works since algebra maps and skew-derivations are in the Hopf dual. The proof is actually easier because there are less relations to check. Finally, we form the Hopf algebra $(\mathcal{U} \otimes \mathcal{B})_\sigma$ as in *loc. cit.*, and consider the quotient $\widetilde{\mathcal{A}}$ of $(\mathcal{U} \otimes \mathcal{B})_\sigma$ by the central Hopf subalgebra $\Bbbk[(z_i \otimes g_i^{-1}) : 1 \leq i \leq \widehat{\theta}]$. The same argument as in *loc. cit.* shows that $\widetilde{\mathcal{A}} \simeq U(\mathcal{D})$ as Hopf algebras. On the other hand, the monomials $b_1^{c_1} b_2^{c_2} \dots b_P^{c_P} y$, $c_j \in \mathbb{N}$, $1 \leq j \leq P$, $y \in \Gamma \times \Upsilon$ form a basis of $(\mathcal{U} \otimes \mathcal{B})_\sigma$. Since $\Gamma \times \Upsilon$ splits as the product of Γ and the group generated by $(z_i \otimes g_i^{-1})$, $1 \leq i \leq \widehat{\theta}$, we conclude that the images of the monomials $b_1^{c_1} b_2^{c_2} \dots b_P^{c_P} y$, $c_j \in \mathbb{N}$, $1 \leq j \leq P$, $y \in \Gamma$ form a basis of $\widetilde{\mathcal{A}}$.

Step IV. The claim about the GK-dimension follows from the previous step. The claim about the coradical filtration also follows from the previous step, together with Theorem 3.4. Indeed, Theorem 3.4 applies since Nichols algebras are coradically graded by definition.

Step V. The existence of ψ follows from the definition of Nichols algebras, since by the statement about the coradical filtration we have a monomorphism of Yetter-Drinfeld modules $V \to U(\mathcal{D})_1 / U(\mathcal{D})_0$. Since the restriction of ψ to the first term of the coradical filtration $(\mathfrak{B}(V)\#\Bbbk\Gamma)_1$ is injective, ψ is injective [M, Th. 5.3.1]. It is surjective by the PBW-basis claim. Therefore ψ is bijective.

Step VI. We finally prove that $U(\mathcal{D})$ is a domain. Here we follow [DCK, Corollary 1.8]. We introduce an $\mathbb{N}^{P+1}$-filtration on $U(\mathcal{D})$ by the degree defined by

$$\deg(b_1^{c_1} b_2^{c_2} \dots b_P^{c_P} y) = (c_1, c_2, \dots, c_P, \sum_{1\leq j\leq P} c_j \ \operatorname{ht}\ \beta_j), \quad , c_j \in \mathbb{N}, \quad 1 \leq j \leq P, \quad y \in \Gamma;$$

here $\operatorname{ht}\ \beta$ is the height of the root β as in [DCK]. We claim that this is an algebra filtration. If $\alpha = e_1\alpha_1 + \cdots + e_\theta\alpha_\theta$, we set $g_\alpha = g_1^{e_1} \dots g_\theta^{e_\theta}$, $\chi_\alpha = \chi_1^{e_1} \dots \chi_\theta^{e_\theta}$. Recall that $b_k = a_{\beta_k}$, $1 \leq k \leq P$. To prove the claim one has to verify that for all $k > l$

$$b_k b_l - \chi_{\beta_l}(g_{\beta_k}) b_l b_k = \sum_{\mathbf{c}\in\mathbb{N}^P} \rho_{\mathbf{c}}\, b_1^{c_1} b_2^{c_2} \dots b_P^{c_P}, \tag{4.15}$$

where $\rho_{\mathbf{c}} \in \Bbbk$, and $\rho_{\mathbf{c}} = 0$ unless $\deg(b_1^{c_1} b_2^{c_2} \dots b_P^{c_P}) < \deg(b_l b_k)$.

If β_l and β_k have support in the same connected component, then (4.15) follows from the formula of Levendorskii and Soibelman [DCK, Lemma 1.7] using Lemma 2.2. Note that here we are using the special order of the set of positive roots, in which the roots with support on the first component are smaller than the roots in the second, and so on.

If β_l and β_k have support in different connected components, then (4.15) follows from the linking relations since the monomials in any root vector are homogeneous.

As in [DCK, Corollary 1.8], it is not difficult to verify that the associated graded ring is a domain, which implies that $U(\mathcal{D})$ is a domain. □

To describe the isomorphisms between Hopf algebras $U(\mathcal{D})$ and $U(\mathcal{D}')$, we first formulate a Lemma which is needed in this general form in the proof of the main theorem 5.2.

Lemma 4.3. *Let Γ be a finitely generated abelian group, A a pointed Hopf algebra with coradical $A_0 = \Bbbk\Gamma$, $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ with $\Bbbk$-basis $x_i \in V_{g_i}^{\chi_i}, g_i \in \Gamma, \chi_i \in \widehat{\Gamma}, 1 \le i \le \theta$.*

Assume that $grA \cong \mathfrak{B}(V)\#\Bbbk\Gamma$ as graded Hopf algebras, and

$$\chi_i \neq \varepsilon, \text{ for all } 1 \le i \le \theta. \tag{4.16}$$

Then the first term of the coradical filtration of A is

$$A_1 = A_0 \oplus \bigoplus_{g\in\Gamma, 1\le i\le\theta} \mathcal{P}_{gg_i,g}(A)^{\chi_i}.$$

If $(g_i, \chi_i) \neq (g_j, \chi_j)$ for all $i \neq j$, then the vector spaces $\mathcal{P}_{gg_i,g}(A)^{\chi_i}$ are one-dimensional for all $g \in \Gamma, 1 \le i \le \theta$.

Proof. By assumption, $A_1/A_0 \cong V\#\Bbbk\Gamma$, and the elements $x_i\#g \in \mathcal{P}_{gg_i,g}(\mathfrak{B}(V)\#\Bbbk\Gamma)^{\chi_i}$, $1 \le i \le \theta$, $g \in \Gamma$, $x \in \mathcal{P}_{gg_i,g}(A)^{\chi_i}$ form a $\Bbbk$-basis of $V\#\Bbbk\Gamma$. Hence

$$A_1/A_0 \cong \bigoplus_{1\le i\le\theta} (A_1/A_0)^{\chi_i}.$$

Let $h_1, \cdots, h_s$ be generators of Γ. Fix $1 \le i \le \theta$. Then for any $a \in A_1$ with $\bar{a} \in (A_1/A_0)^{\chi_i}$, and $1 \le l \le s$, the adjoint action of h_l on a is given by $h_l \cdot a = \chi_i(h_l)a + v_l$, for some $v_l \in A_0$. Since Γ is abelian, the vector space spanned by $a, v_1, \cdots, v_s$ is Γ-stable containing a. This shows that A_1 is locally finite under the adjoint action of Γ. We claim that A_1 is completely reducible as Γ-module. Indeed, let U be any locally finite Γ-module, let $\chi \in \widehat{\Gamma}$, and let $U^{(\chi)} = \{u \in U : \exists s > 0$ such that $(g - \chi(g))^s(u) = 0 \forall g \in \Gamma\}$. Then $U = \oplus_{\chi\in\widehat{\Gamma}} U^{(\chi)}$ (see for instance [D, Th. 1.3.19]). Now, $A_0 \subset (A_1)^{\varepsilon} \subset (A_1)^{(\varepsilon)}$, but by (4.16), they are all three equal. The claim follows.

Hence $A_1 = A_0 \oplus \bigoplus_{1\leq i\leq\theta}(A_1)^{\chi_i}$. By the theorem of Taft and Wilson [M, Theorem 5.4.1], $A_1 = A_0 + (\bigoplus_{g,h\in\Gamma}\mathcal{P}_{g,h}(A))$. Hence

$$A_1 = A_0 \oplus \bigoplus_{g,h\in\Gamma,\varepsilon\neq\chi\in\widehat{\Gamma}} \mathcal{P}_{g,h}(A)^{\chi},$$

and the lemma follows. □

Note that (4.16) holds for generic braidings, since in this case no $\chi_i(g_i)$ is a root of 1.

If A, B are Hopf algebras, we denote the set of all Hopf algebra isomorphisms from A to B by $\mathrm{Isom}(A, B)$.

Theorem 4.4. *Let $\mathcal{D}$ and $\mathcal{D}'$ be generic data of finite Cartan type for Γ. Then the Hopf algebras $U(\mathcal{D})$ and $U(\mathcal{D}')$ are isomorphic if and only if $\mathcal{D}$ is isomorphic to $\mathcal{D}'$.*

More precisely, let $a_1, \cdots, a_\theta$ resp. $a'_1, \cdots, a'_\theta$ be the simple root vectors in $U(\mathcal{D})$ resp. $U(\mathcal{D}')$ of Theorem 4.2, and let $g_1, \cdots, g_\theta$ resp. $g'_1, \cdots, g'_\theta$ be the group-like elements in $\mathcal{D}$ resp. $\mathcal{D}'$. Then the map

$$Isom(U(\mathcal{D}), U(\mathcal{D}')) \to Isom(\mathcal{D}, \mathcal{D}'),$$

given by $\phi \mapsto (\varphi, \sigma, (\alpha_i))$, where $\varphi(g) = \phi(g), \varphi(g_i) = g'_{\sigma(i)}, \phi(a_i) = \alpha_i a'_{\sigma(i)}$, for all $g \in \Gamma, 1 \leq i \leq \theta$, is bijective.

Proof. Let V resp. V' be the Yetter-Drinfeld module of the infinitesimal braiding of $A := U(\mathcal{D})$ resp. of $A' := U(\mathcal{D})$. Let $\phi : A \to A'$ be an isomorphism of Hopf algebras. Then ϕ induces isomorphisms $A_0 \to A'_0$ and $A_1 \to A'_1$. Hence ϕ defines an isomorphism of groups $\varphi : \Gamma \to \Gamma$, and for all $g, h \in \Gamma, \chi \in \widehat{\Gamma}$, a linear isomorphism

$$\mathcal{P}_{g,h}(A)^{\chi} \cong \mathcal{P}_{\varphi(g),\varphi(h)}(A')^{\chi\varphi^{-1}}.$$

By Theorem 4.2, the assumptions of Lemma 4.3 are satisfied for A, V and A', V'. Then it follows from Lemma 4.3 and (4.8) that there is a uniquely determined permutation $\sigma \in \mathbb{S}_\theta$ such that ϕ induces an isomorphism

$$\mathcal{P}_{g_i,1}(A)^{\chi_i} \cong \mathcal{P}_{g'_{\sigma(i)},1}(A')^{\chi'_{\sigma(i)}}, \text{ with } \varphi(g_i) = g'_{\sigma(i)}, \chi_i\varphi^{-1} = \chi'_{\sigma(i)}, \text{ for all } 1 \leq i \leq \theta.$$

Moreover, since for all i, $\mathcal{P}_{g_i,1}(A)^{\chi_i}$ and $\mathcal{P}_{g'_{\sigma(i)},1}(A)^{\chi'_{\sigma(i)}}$ are one-dimensional with basis a_i and a'_i, there are non-zero scalars $\alpha_i \in \Bbbk$ with $\phi(a_i) = \alpha_i a'_{\sigma(i)}$, for all $1 \leq i \leq \theta$.

Then the elements $\phi(a_i), 1 \leq i \leq \theta$, satisfy the Serre relations (4.11), and they satisfy (4.12) if and only if the triple $(\varphi, \sigma, (\alpha_i))$ is an isomorphism of generic data. Thus the map $\mathrm{Isom}(U(\mathcal{D}), U(\mathcal{D}')) \to \mathrm{Isom}(\mathcal{D}, \mathcal{D}')$ in the theorem is well-defined and injective. Surjectivity of this map follows from the description of the Hopf algebras $U(\mathcal{D})$ and $U(\mathcal{D}')$ in Theorem 4.2. □

The main reason why the proof of the preceding theorem works is the knowledge of the coradical filtration. The same ideas allow to determine all Hopf subalgebras of $U(\mathcal{D})$.

5. Pointed Hopf algebras with generic braidings

We are going to show that the class of Hopf algebras described in the previous section has an intrinsic description.

The following key Lemma implies that pointed Hopf algebras belonging to a natural class are generated by group-like and skew primitive elements.

Lemma 5.1. *(a). Let $S = \oplus_{n\in\mathbb{N}} S(n)$ be a graded braided Hopf algebra such that $S(0) = \mathbb{C}.1$, $V := S(1)$ is finite-dimensional and generates S as an algebra. Assume that S has finite Gelfand-Kirillov dimension and that V has positive braiding. Then S is a Nichols algebra.*

(b). Let R be as S in (a), except that we assume $P(R) = R(1)$ instead of generation in degree 1. Then R is a Nichols algebra.

Proof. (a). $\mathfrak{B}(V)$ has finite Gelfand-Kirillov dimension since it is a quotient of S. Assume first that the matrix is indecomposable. We can then apply Theorem 2.13; let (a_{ij}), $(d_1, \dots, d_\theta)$ and q such that $q_{ij}q_{ji} = q^{d_i a_{ij}}$ for all $i \neq j$ and $q_{ii} = q^{d_{ii}}$.

Let $i \neq j$. We claim that $z_2 = \operatorname{ad}_c(x_i)^{1-a_{ij}}(x_j) = 0$ in S. Indeed, let $z_1 = x_i$, suppose that $z_2 \neq 0$ and consider the two-dimensional subspace W of S generated by the primitive elements z_1 and z_2.

We claim that $\mathfrak{B}(W)$ has finite Gelfand-Kirillov dimension. For, let T be the subalgebra of S generated by W; then the graded Hopf algebra $\operatorname{gr}(T\#\mathbb{C}\Gamma)$ has finite Gelfand-Kirillov dimension, and contains $\mathfrak{B}(W)$.

Also, the braiding of W is given by the matrix:

$$\begin{pmatrix} q_{ii} & q_{ii}^{1-a_{ij}} q_{ij} \\ q_{ii}^{1-a_{ij}} q_{ji} & q_{ii}^{(1-a_{ij})^2} q_{ij}^{1-a_{ij}} q_{ji}^{1-a_{ij}} q_{jj} \end{pmatrix} = \begin{pmatrix} q^{d_i} & q^{d_i(1-a_{ij})} q_{ij} \\ q^{d_i(1-a_{ij})} q_{ji} & q^{d_i - d_i a_{ij} + d_j} \end{pmatrix}$$

By Theorem 2.11 and Lemma 2.1, there exists $k \geq 0$ such that $1 = q^{d_i k + 2d_i(1-a_{ij})} q_{ij} q_{ji}$, hence $0 = d_i k + 2d_i(1 - a_{ij}) + d_i a_{ij} = d_i(k + 2 - a_{ij})$, a contradiction. This shows that $z_2 = 0$.

Therefore, we have an epimorphism of braided graded Hopf algebras $\mathfrak{B}(V) \to S$, by Step II of Theorem 4.2, which is the identity in degree 1. Hence $\mathfrak{B}(V) \simeq S$.

Assume now that the matrix is decomposable. Let i, j belong to different components; in particular $q_{ij}q_{ji} = 1$. We claim that $x_i x_j = q_{ij} x_j x_i$. If not, let $z_1 := x_i$ and $z_2 := x_i x_j - q_{ij} x_j x_i$ is primitive by Lemma 2.1 (b). Let $z_1 = x_i$ and consider as before the subspace W of S generated by z_1 and z_2. As before, $\mathfrak{B}(W)$ has finite Gelfand-Kirillov dimension. Now the braiding of W is given by the matrix:

$$\begin{pmatrix} \widetilde{q}_{11} & \widetilde{q}_{12} \\ \widetilde{q}_{21} & \widetilde{q}_{22} \end{pmatrix} := \begin{pmatrix} q_{ii} & q_{ii} q_{ij} \\ q_{ii} q_{ji} & q_{ii} q_{jj} \end{pmatrix}.$$

By Theorem 2.11 again, there exists $k \geq 0$ such that $1 = q_{ii}^{k+2} q_{ij} q_{ji} = q_{ii}^{k+2}$, a contradiction. This concludes the proof of (a).

Finally, (a) and (b) are equivalent by [AS2, Lemma 5.5] and the definition of finite Gelfand-Kirillov dimension. □

Theorem 5.2. *Let A be a pointed Hopf algebra with abelian coradical and diagram R. Assume that $R(1)$ is finite-dimensional with positive braiding. Then the following are equivalent:*

(a). A is a domain with finite Gelfand-Kirillov dimension.

(b). The group $\Gamma := G(A)$ is free abelian of finite rank, and there exist a positive datum $\mathcal{D}$ for Γ such that $A \simeq U(\mathcal{D})$ as Hopf algebras.

Proof. (b) $\implies$ (a): this is Theorem 4.2.

(a) $\implies$ (b). By [KL, 6.5], $\operatorname{gr} A$ has finite GK-dimension; hence both R and $\Bbbk\Gamma$ also have. It is clear then that Γ should be a free abelian group of finite rank, say s. Consider the diagram R of A. From Theorem 2.13 and Lemma 5.1, we conclude the existence of the finite Cartan matrix (a_{ij}), the family $(q_I)_{I\in\mathcal{X}}$, and the elements $g_1,\dots,g_\theta \in \Gamma$, $\chi_1,\dots,\chi_\theta \in \widehat{\Gamma}$ satisfying (4.1), and such that no q_I is a root of 1 (in fact $q_I > 0$ and not 1 for all I), $R = \mathfrak{B}(V)$ where $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ has a basis $x_i \in V_{g_i}^{\chi_i}, 1 \le i \le \theta$.

Since $\operatorname{gr} A \cong \mathfrak{B}(V)\#\Bbbk\Gamma$ as graded Hopf algebras, and $\chi_i(g_i) \neq 1$ for all $1 \le i \le \theta$, it follows from Lemma 4.3 that the first term of the coradical filtration of A is

$$A_1 = A_0 \oplus \bigoplus_{g\in\Gamma, 1\le i\le\theta} \mathcal{P}_{gg_i,g}(A)^{\chi_i}. \tag{5.1}$$

We can then choose $a_i \in \mathcal{P}_{g_i,1}(A)^{\chi_i}$ such that the class of a_i in $\operatorname{gr} A(1)$ coincides with x_i. Let $y_1,\cdots,y_s$ be free generators of $G(A)$. It is clear that relations (4.9) and (4.10) hold.

Let $i \neq j$. We claim:

(i). There exists no ℓ, $1 \le \ell \le \theta$, such that $g_i^{1-a_{ij}} g_j = g_\ell$, $\chi_i^{1-a_{ij}}\chi_j = \chi_\ell$.

(ii). If $i \sim j$, then $\chi_i^{1-a_{ij}}\chi_j \neq \varepsilon$.

We prove (i). Assume that $g_i^{1-a_{ij}} g_j = g_\ell$, $\chi_i^{1-a_{ij}}\chi_j = \chi_\ell$ for some ℓ. Then

$$q_I^{d_i a_{i\ell}} = \langle \chi_\ell, g_i\rangle\langle\chi_i, g_\ell\rangle = q_I^{2d_i(1-a_{ij})}\langle\chi_j, g_i\rangle\langle\chi_i, g_j\rangle = q_I^{d_i(2-a_{ij})};$$

we conclude that $2 = a_{ij} + a_{i\ell}$. The only possibility is $a_{ij} = 0$ and $l = i$. Then $g_j = 1$ which is impossible.

We prove (ii). Assume that $\chi_i^{1-a_{ij}}\chi_j = \varepsilon$, $i \neq j$, $i \sim j$. Evaluating at g_i, we get $1 = q_{ii}^{1-a_{ij}} q_{ij} = q_{ii}(q_{ij}q_{ji})^{-1} q_{ij} = q_{ii} q_{ji}^{-1}$, so that $q_{ii} = q_{ji}$. Evaluating at g_j, we get $1 = q_{ji}^{1-a_{ij}} q_{jj}$; hence $q_{jj} = q_{ii}^{a_{ij}-1}$. Hence $0 = d_i(1-a_{ij}) + d_j$; this is a contradiction.

If $i \neq j$, then $(\operatorname{ad} a_i)^{1-a_{ij}} a_j \in \mathcal{P}_{g_i^{1-a_{ij}} g_j,1}(H)^{\chi_i^{1-a_{ij}}\chi_j}$ by Lemma 2.1 (b). If $i \sim j$, taking into account (5.1), (i) and (ii), we see that the quantum Serre relations (4.11) hold in A.

Finally, assume that $i \not\sim j$; if $0 \neq (\operatorname{ad} a_i) a_j \in \mathcal{P}_{g_i g_j,1}(A)^{\chi_i\chi_j}$, then $\chi_i\chi_j = \varepsilon$ by (5.1) and (i). So that $a_i a_j - \chi_j(g_i) a_j a_i = \lambda_{ij}(1 - g_i g_j)$ for some $\lambda_{ij} \in \Bbbk$, where $\lambda_{ij} = 0$ when $\chi_i\chi_j \neq \varepsilon$. But we can also choose $\lambda_{ij} = 0$ when $g_i g_j = 1$. By (4.7), we can rescale a generator a_i with $\lambda_{ij} \neq 0$ to have $\lambda_{ij} = 1$. Hence, (λ_{ij}) is a linking datum for (a_{ij}), $g_1,\dots,g_\theta$ and $\chi_1,\dots,\chi_\theta$; and (4.12) holds.

We have found a positive datum $\mathcal{D}$ for Γ and constructed a homomorphism of Hopf algebras $\varphi : U(\mathcal{D}) \to A$. Now $\operatorname{gr}\varphi : \operatorname{gr} U(\mathcal{D}) \to \operatorname{gr} A$ is an isomorphism by Theorem 4.2; indeed $\operatorname{gr}\varphi$ is surjective and the restriction of $\operatorname{gr}\varphi$ to the first term of the coradical filtration is injective; thus $\operatorname{gr}\varphi$ is injective [M, Th. 5.3.1]. Hence φ is is an isomorphism. $\square$

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