Contemporary Mathematics

Pointed indecomposable Hopf algebras over Coxeter groups

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1. Introduction

Every pointed Hopf algebra can be decomposed as a crossed product of its link indecomposable component containing 1 (a Hopf subalgebra) and a group algebra [**M2**]. This means that the structure theory of pointed Hopf algebras can be reduced to the link-indecomposable case. However, the structure of such link-indecomposable Hopf algebras A, in particular in the case when A is finite-dimensional and the group of its group-like elements $G(A)$ is non-abelian, is not known. S. Westreich has shown that if A is a pointed, link-indecomposable, and quasitriangular Hopf algebra over a field of characteristic zero, then the group $G(A)$ of its group-like elements is abelian [**W**]. Moreover it seems that all finite-dimensional, link-indecomposable Hopf algebras which have appeared in the literature so far do have an abelian group of group-likes.

In this paper we present a general construction which associates to any Coxeter group W a pointed link-indecomposable Hopf algebra with group-like elements W (over an arbitrary base field $\Bbbk$). If W is one of the symmetric or dihedral groups $D_3(=S_3)$, D_4, S_4 or S_5, the Hopf algebra is finite-dimensional (see Section 6). We do not know whether our construction gives a finite-dimensional Hopf algebra whenever the Coxeter group is finite.

In Section 4 we show that link-indecomposability is an infinitesimal property in the following sense. Let A be a pointed Hopf algebra with coradical $\Bbbk G$, $G = G(A)$ the group-like elements of A. As in [**AS1**] we consider the associated graded Hopf algebra $\mathrm{gr}(A)$, where the grading is defined by the coradical filtration. The coinvariant elements R of $\mathrm{gr}(A)$ with respect to the projection onto the degree zero part $\mathrm{gr}(A)(0) = \Bbbk G$ form a Hopf algebra in the braided category of Yetter-Drinfeld modules over $\Bbbk G$ (see [**M1**] for the notions of braided categories and Yetter-Drinfeld modules). The primitive elements $V = P(R)$ of R are a Yetter-Drinfeld submodule. Then A is link-indecomposable if and only if the elements $g \in G$ with non-zero homogeneous component V_g generate the group G (Lemma 4.1). In such case the Yetter-Drinfeld module V is called link-indecomposable. Now the subalgebra of R generated by V is a Yetter-Drinfeld Hopf algebra which only depends on V.

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Following [**AS2**] we call it the Nichols algebra $\mathfrak{B}(V)$, since Nichols was the first who considered Hopf algebras of this type in [**N**].

By a theorem of Radford [**R**] the smash-product of any Yetter-Drinfeld Hopf algebra with the group-algebra gives a usual Hopf algebra. Thus the main case to study are Nichols algebras of link-indecomposable Yetter-Drinfeld modules V over the group-algebra $\Bbbk G$ of a group G. By Proposition 4.2, there exists a finite-dimensional link-indecomposable Hopf algebra A with $G(A) = G$ if and only if there is a link-indecomposable Yetter-Drinfeld module V over G such that $\mathfrak{B}(V)$ is finite-dimensional and G is finite.

The algebra $\mathfrak{B}(V)$ is defined as a certain universal quotient of the tensor algebra $T(V)$. It is one of the main difficulties that the relations are not known explicitly. And if they are known, it can be very hard to decide whether they define a finite-dimensional quotient algebra.

In section 2 we review the general theory of Nichols algebras [**N**], [**Wo**], also called quantum symmetric algebras [**Ro**], or bitensor algebras [**Sb**], in a way suitable for this paper. In Section 5 we construct a class of Yetter-Drinfeld modules over Coxeter groups and study the associated Nichols algebra $\mathfrak{B}(V)$ (Theorems 5.7, 5.8). The relations of $\mathfrak{B}(V)$ reflect the relations between the generators of the Coxeter group. In section 3 we note an abstract splitting result for braided Hopf algebras which shows in many cases that $\mathfrak{B}(V)$ is free over $\mathfrak{B}(V')$ for a subspace $V' \subset V$ which is a Yetter-Drinfeld module over a subgroup, for example a parabolic subgroup of the Coxeter group. As a consequence, the Hilbert series of $\mathfrak{B}(V')$ divides the Hilbert series of $\mathfrak{B}(V)$.

In the last section we consider some examples of the general construction. We show the finiteness results already mentioned, and we note that the quadratic algebras $\mathcal{E}_n$ introduced in [**FK**] to describe the cohomolgy of the flag variety are in fact braided Hopf algebras over the symmetric groups. Hence the conjectured Poincaré-duality about the dimensions of the homogeneous components of $\mathcal{E}_n$ in [**FK**] follows from a general fact about braided Hopf algebras in [**N**]. As a corollary of the freeness result in section 3, we conclude that $\mathcal{E}_{n+1}$ is free over $\mathcal{E}_n$, in particular that the Hilbert series of $\mathcal{E}_n$ divides the Hilbert series of $\mathcal{E}_{n+1}$.

After submission of this paper, the second author learned from C. Procesi that freeness of $\mathcal{E}_{n+1}$ over $\mathcal{E}_n$ was also shown in the preprint [**FP**] by different methods using explicitly the defining relations. The second author wishes to thank Procesi for stimulating discussions on the subject of this paper.

Both authors would like to thank D. Fischman and S. Westreich whose sketch of a finite-dimensional, link-indecomposable, pointed Hopf algebra H with group S_3 of group-like elements several years ago motivated the work presented in this paper.

2. Nichols algebras

The ground-field will be denoted by $\Bbbk$, and $\otimes = \otimes_{\Bbbk}$. Let G be a group. Recall that a Yetter-Drinfeld module V over $\Bbbk G$ is a G-graded vector space $V = \bigoplus_{g\in G} V_g$, which is a G-module such that $g \cdot V_h \subset V_{ghg^{-1}}$ for all $g, h \in G$. The category ${}^G_G\mathcal{YD}$ of $\Bbbk G$-Yetter-Drinfeld modules is braided: For $V, W \in {}^G_G\mathcal{YD}$ the braiding $c : V \otimes W \to W \otimes V$ is defined by $c(v \otimes w) = (g \cdot w) \otimes v$, $v \in V_g, w \in W$. The G-gradation and G-action on $V \otimes W$ are defined in the usual way by $(V \otimes W)_g = \bigoplus_{ab=g} V_a \otimes W_b$, and $g \cdot (v \otimes w) = g \cdot v \otimes g \cdot w, g \in G, v \in V, w \in W$.

Let V be a Yetter-Drinfeld module over G, and let $T(V) = \bigoplus_{n\geq 0} T(V)(n)$ denote the tensor algebra of the vector space V, that is $T(V)(n) = V^{\otimes n}$ and the multiplication μ is defined as tensor product. We consider $T(V)$ as an algebra in ${}^G_G\mathcal{YD}$ in the obvious way: $g \cdot (v_1 \otimes \cdots \otimes v_n) := (g \cdot v_1) \otimes \cdots \otimes (g \cdot v_n)$, $g \in G$, $v_1, \ldots, v_n \in V$; if $v_i \in V_{g_i}, 1 \leq i \leq n$, then $v_1 \otimes \cdots \otimes v_n \in V_g$, where $g = g_1 \ldots g_n$. There is a unique coalgebra structure on $T(V)$, such that $v \in P(T(V))$ for all $v \in V$ and $T(V)$ is a bialgebra in ${}^G_G\mathcal{YD}$, that is $(\mu \otimes \mu)(1 \otimes c \otimes 1)(\Delta \otimes \Delta) = \Delta\mu$, where c is the braiding of the Yetter-Drinfeld module $T(V)$ as defined above. For example we have $\Delta(vw) = 1 \otimes vw + v \otimes w + (g \cdot w) \otimes v + vw \otimes 1$ for $v \in V_g, w \in V$. We note that the canonical filtration of $T(V)$ is a coalgebra filtration. Therefore the coradical of the coalgebra $T(V)$ equals $\Bbbk$, and $T(V)$ and all its coalgebra quotients are pointed irreducible coalgebras [**M1**, 5.3.4, 5.3.5]. In particular this implies that the identity of $T(V)$ is convolution invertible by [**M1**, 5.2.10]. Hence $T(V)$ and all its bialgebra quotients in ${}^G_G\mathcal{YD}$ are Hopf algebras in ${}^G_G\mathcal{YD}$. Note that $T(V)$ is a $G \times \mathbb{N}$-graded vector space as each $T(V)(n)$ has a basis consisting of G-homogenous elements.

We consider the set $\widetilde{\mathcal{S}}$ of all ideals and coideals I of $T(V)$ which are generated as ideals by $\mathbb{N}$-homogeneous elements of degree ≥ 2. Let $\mathcal{S}$ be the subset of all $I \in \widetilde{\mathcal{S}}$ which are Yetter-Drinfeld submodules of $T(V)$, that is stable under the action of G and generated by $G \times \mathbb{N}$-homogeneous elements. Both sets are closed under addition and contain the zero ideal. Hence the sums $I(V) = \sum_{I \in \mathcal{S}} I$ and $\widetilde{I}(V) = \sum_{J \in \widetilde{\mathcal{S}}} J$ are the largest elements in $\mathcal{S}$ and $\widetilde{\mathcal{S}}$.

For all $I \in \mathcal{S}$, $T(V)/I$ is a graded Hopf algebra in ${}^G_G\mathcal{YD}$ with components $\Bbbk$ and V in degree 0 and 1. In particular, $V \subset P(T(V)/I)$.

Lemma 2.1. *(1) Let $I \in \mathcal{S}$. Then $I = I(V)$ if and only if $V = P(T(V)/I)$.*
(2) $I(V) = \widetilde{I}(V)$.

Proof. (1) We first show the equality $V = P(T(V)/I(V))$. Let $n \geq 2$ and consider the inverse image X in $T(V)(n)$ of all primitive elements of $T(V)/I(V)$ in degree n. Then X is a Yetter-Drinfeld submodule of $T(V)(n)$, and for all $x \in X$, $\Delta(x) \in x \otimes 1 + 1 \otimes x + T(V) \otimes I(V) + I(V) \otimes T(V)$. Hence the ideal generated by $I(V)$ and X is in $\mathcal{S}$, and $X \subset I(V)$ by the maximality of $I(V)$. Hence the image of X in $T(V)/I(V)$ is zero. This proves our claim since the primitive elements form a graded submodule.

Now assume that $V = P(T(V)/I)$. Then the surjective map $T(V)/I \to T(V)/I(V)$ of irreducible coalgebras is injective by [**M1**, 5.3.3] since by assumption it is injective on the primitive elements. Hence $I = I(V)$.

(2) As in (1) we see that $T(V)/I(V) \to T(V)/\widetilde{I}(V)$ is bijective since $V = P(T(V)/I(V))$ by (1). $\square$

As in [**AS1**], we call $\mathfrak{B}(V) := T(V)/I(V)$ the *Nichols algebra* of $V \in {}^G_G\mathcal{YD}$. More generally, if U is a braided subspace of $V \in {}^G_G\mathcal{YD}$, that is a subspace such that $c(U \otimes U) \subset U \otimes U$, where c is the braiding of V, we define $\mathfrak{B}(U) := T(U)/\widetilde{I}(U)$.

Remark 2.2. 1) By Lemma 2.1, $\mathfrak{B}(V)$ as an algebra and coalgebra only depends on the braiding of V. The relations in each degree of $I(V)$ can be described by the action of the braid group (see for example [**Sb**, 2.9, 2.12]). However this does not mean that the relations are known explicitly.

2) Let $R = \bigoplus_{n\geq 0} R(n)$ be a graded Hopf algebra in ${}^G_G\mathcal{YD}$ with $R(0) = \mathbb{k}1$ and $R(1) = P(R) = V$. Then by Lemma 2.1, $\mathfrak{B}(V)$ is the subalgebra $\mathbb{k}[V]$ of R generated by V, since $\mathbb{k}[V]$ is of the form $T(V)/I$ with primitive elements V.

3) Let U be a braided subspace of $V \in {}^G_G\mathcal{YD}$. Then the canonical map $\mathfrak{B}(U) \to \mathfrak{B}(V)$ is injective since it is injective on the primitive elements by Lemma 2.1.

4) Let $R = \bigoplus_{n\geq 0} R(n)$ be a finite-dimensional graded Hopf algebra in ${}^G_G\mathcal{YD}$ with $R(0) = \mathbb{k}1$ which is generated as an algebra by $R(1)$, and let $x_1, \dots, x_\theta$ be a basis of $R(1)$. Then there is a non-zero monomial $\Lambda = x_{i_1}\cdots x_{i_N}$, $1 \leq i_1, \dots, i_N \leq \theta$, of maximal length N. Hence $x_i\Lambda = 0 = \Lambda x_i$ for all $1 \leq i \leq \theta$, and Λ is a left and right integral of R. Since R is a Frobenius algebra [**FMS**], the space of left (or right) integrals is one-dimensional. Hence $R = \bigoplus_{n=0}^N R(n)$, and $R(N) = \mathbb{k}\Lambda$. Since R is a graded coalgebra, the linear function $\lambda : R \to \mathbb{k}$ with $\lambda(\Lambda) = 1$ and $\lambda(R(n)) = 0$ for all $0 \leq n \leq N-1$ is a two-sided integral of R^*. Therefore for all $0 \leq i \leq N$,

$$R(i) \otimes R(N-i) \to \mathbb{k}, \quad x \otimes y \mapsto \lambda(xy),$$

is a non-degenerate pairing (cf. [**N**, 1.5]). In particular, $\dim R(i) = \dim R(N-i)$.

The crucial property $V = P(B(V))$ can be characterized as follows. For a graded coalgebra $R = \bigoplus_{n\geq 0} R(n)$ with projections $\pi_n : R \to R(n), n \geq 0$, we denote by

$$\Delta_{i,j} : R(i+j) \xrightarrow{\Delta} R \otimes R \xrightarrow{\pi_i \otimes \pi_j} R(i) \otimes R(j), i, j \geq 0,$$

the (i,j)-th component of the comultiplication Δ. More generally,

$$\Delta_{i_1,\dots,i_k} : R(i_1 + \cdots + i_k) \to R(i_1) \otimes \cdots \otimes R(i_k), i_1, \dots, i_k \geq 0,$$

denotes the $(i_1, \dots, i_k)$-th component of the $(k-1)$-fold iteration of Δ.

Lemma 2.3. *Let $R = \bigoplus_{n\geq 0} R(n)$ be a graded coalgebra with group-like element 1 and $R(0) = \mathbb{k}1$. Then the following are equivalent:*

(1) $P(R) = R(1)$.

(2) For all $n \geq 2$, $\Delta_{1,\dots,1} : R(n) \to R(1)^{\otimes n}$ is injective.

(3) For all $i, j \geq 0$, $\Delta_{i,j} : R(i+j) \to R(i) \otimes R(j)$ is injective.

(4) For all $n \geq 2$, $\Delta_{n-1,1} : R(n) \to R(n-1) \otimes R(1)$ is injective.

Proof. Let $x \in R(n), n \geq 2$, and $\Delta(x) = \sum_{i=0}^n y_i$ where $y_i = \Delta_{i,n-i}(x) \in R(i) \otimes R(n-i)$ for all $0 \leq i \leq n$. For all $m \geq 1$, let Φ_m be the composition $\Phi_m : R \xrightarrow{\Delta^{m-1}} R^{\otimes m} \xrightarrow{\pi_1^{\otimes m}} R(1)^{\otimes m}$. Hence $\Delta_{1,\dots,1}$ is the restriction of Φ_m to $R(m)$. Then for all $1 \leq i \leq n-1$,

$$\Phi_n(x) = (\Phi_i \otimes \Phi_{n-i})(\Delta(x)) = (\Phi_i \otimes \Phi_{n-i})(y_i).$$

This formula shows that (1) $\Rightarrow$ (2) by induction on n (as in [**AS2**, Lemma 5.5]), and also (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (4) is trivial, and (4) $\Rightarrow$ (1) is clear since by (4), $R(n) \cap P(R) = 0$ for all $n \geq 2$. $\square$

All the previous results hold more generally for Yetter-Drinfeld modules over Hopf algebras with bijective antipode instead of group algebras. We now recall an important tool introduced by Nichols [**N**, 3.3] to deal with $\mathfrak{B}(V)$ over group algebras without knowing the relations explicitly. Let $V \in {}^G_G\mathcal{YD}$ be of finite dimension θ. We choose a basis $x_i \in V_{g_i}$ with $g_i \in G, 1 \leq i \leq \theta$, of G-homogeneous elements. Let $I \in \mathcal{S}$ and $R = T(V)/I$. Then R is a graded Hopf algebra in ${}^G_G\mathcal{YD}$ with $R(0) = \mathbb{k}1$ and $R(1) = V$. For all $1 \leq i \leq \theta$ let $\sigma_i : R \to R$ be the algebra automorphism given by the action of g_i.

Proposition 2.4. *Let $R, x_i, \sigma_i, 1 \leq i \leq \theta$ as above.*

1) For all $1 \leq i \leq \theta$, there exists a uniquely determined (id, σ_i)-derivation $D_i : R \to R$ with $D_i(x_j) = \delta_{i,j}$ (Kronecker δ) for all j.

2) $I = I(V)$, that is $R = \mathfrak{B}(V)$, if and only if $\bigcap_{i=1}^{\theta} ker(D_i) = \mathbb{k}1$.

Proof. 1) For all $1 \leq i \leq \theta$, let $D_i(1) = 0$, and define for all $x \in R(n), n \geq 1$, elements $D_i(x) \in R(n-1)$ by

$$\Delta_{n-1,1}(x) = \sum_{i=1}^{\theta} D_i(x) \otimes x_i.$$

Then for all $n, m \geq 1, x \in R(n), y \in R(m)$,

$$\begin{aligned}\Delta(xy) = \Delta(x)\Delta(y) &= (x \otimes 1 + D_i(x) \otimes x_i + \dots)(y \otimes 1 + D_i(y) \otimes x_i + \dots) \\ &= xy \otimes 1 + xD_i(y) \otimes x_i + D_i(x)(g_i \cdot y) \otimes x_i + \dots\end{aligned}$$

by the definition of the braided multiplication in $R \otimes R$. Hence $D_i(xy) = xD_i(y) + D_i(x)\sigma_i(y)$. Since the elements x_j are primitive, $D_i(x_j) = \delta_{ij}$ for all i, j, and D_i is uniquely determined by this equation since R is generated by $x_1, \dots, x_\theta$.

2) By Lemma 2.1, $I = I(V)$ if and only if $P(R) = R(1)$. Hence the claim follows from Lemma 2.3, since $\bigcap_{i=1}^{\theta} \ker(D_i) = \mathbb{k}1$ if and only if $\Delta_{n-1,1}$ is injective for all $n \geq 2$. $\square$

For later use we note

Lemma 2.5. *Let $V, x_i, D_i, 1 \leq i \leq \theta$, be as in Proposition 2.4.*

1) For all $1 \leq i \leq \theta$, let $\mathfrak{B}(V)^{(i)}$ be the subalgebra of $\mathfrak{B}(V)$ generated by all $x_j, 1 \leq i \leq \theta, j \neq i$. Assume $a, b \in \mathfrak{B}(V)^{(i)}$ with $a + bx_i = 0$. Then $a = 0$ and $b = 0$.

2) Let $1 \leq i_1, \dots, i_n \leq \theta$, $n \geq 1$, and $i_k \neq i_l$ for all $k \neq l$. Then $x_{i_1}x_{i_2}\dots x_{i_n} \neq 0$ in $\mathfrak{B}(V)$.

Proof. To prove 1) note that $D_i(\mathfrak{B}(V)^{(i)}) = 0$. Hence $b = D_i(a + bx_i) = 0$. 2) follows from 1) by induction on n. $\square$

3. Non-categorical splitting of Yetter-Drinfeld modules

In this section we use a braided version of the fundamental theorem on Hopf modules to prove freeness of certain extensions of Yetter-Drinfeld Hopf algebras.

Let L be a Hopf algebra with bijective antipode, H a Hopf algebra in ${}^L_L\mathcal{YD}$ (see [**M1**, 10.6.10] for the general definition) and V a right H-module and a right H-comodule with structure maps $\mu_V : V \otimes H \to V$ and $\rho_V : V \to V \otimes H$. Then V is a right H-*Hopf module*, if the braided Hopf module axiom holds, that is if ρ_V is a right H-linear map where the H-module structure of $V \otimes H$ is $(\mu_V \otimes \mu)(id \otimes c \otimes id)(id \otimes id \otimes \Delta)$. Here, μ and Δ denote the multiplication and comultiplication of H. Note that we do not assume that V is a Yetter-Drinfeld module.

LEMMA 3.1. *Let L be a Hopf algebra with bijective antipode, H a Hopf algebra in ${}^L_L\mathcal{YD}$, and V a right H-Hopf module. Define $V^{coH} = \{v \in V \mid \rho_V(v) = v \otimes 1\}$. Then*

$$V^{coH} \otimes H \to V, \quad v \otimes h \mapsto vh,$$

is bijective.

PROOF. This is shown as for usual Hopf algebras (see [**M1**, 1.9.4]). □

THEOREM 3.2. *Let L' and L be Hopf algebras with bijective antipode, and $\alpha : L' \to L$ a Hopf algebra map. Let R be a bialgebra in ${}^L_L\mathcal{YD}$, R' a Hopf algebra in ${}^{L'}_{L'}\mathcal{YD}$, and $i : R' \to R$, $\phi : R \to R'$ algebra and coalgebra maps such that $\phi i = id$. Assume that i is an L'-linear map, and that i, ϕ are L-colinear, where R is a left L'-module and R' a left L-comodule by restriction via α. Define $K = R^{co\phi} = \{r \in R \mid (id \otimes \phi)(\Delta(r)) = r \otimes 1\}$. Then the multiplication map*

$$K \otimes R' \to R, x \otimes y \mapsto xi(y),$$

is bijective, and K is a subalgebra of R with $\Delta(K) \subset R \otimes K$.

PROOF. We consider R as a right R'-module by restriction via i, and as a right R'-comodule with structure map $\bar{\Delta} : R \xrightarrow{\Delta} R \otimes R \xrightarrow{\mathrm{id} \otimes \phi} R \otimes R'$. Here we used that i is an algebra map and ϕ is a coalgebra map. Then one checks that R is a right R'-Hopf module using that i is a coalgebra map and L'-linear, ϕ is an algebra map and L-colinear, and $\phi i = id$. Hence the multiplication map $K \otimes R' \to R$ is bijective by 3.1. Moreover it follows from the definition of K that $\Delta(K) \subset R \otimes K$. To see that K is a subalgebra of R, one checks that $(id \otimes i)\bar{\Delta} : R \to R \otimes R$ is an algebra map, where the algebra structure on $R \otimes R$ is given by the braiding of R, using that i, ϕ are L-colinear algebra maps. □

Theorem 3.2 is true in a more general version which only uses the braidings $c : R \otimes R \to R \otimes R$ and $c' : R' \otimes R' \to R' \otimes R'$ of R and R'. It suffices to assume that i, ϕ are algebra and coalgebra maps with $\phi i = id$, which satisfy the following identities: $(i \otimes id)c'(\phi \otimes id) = (id \otimes \phi)c(id \otimes i)$, and $c(i\phi \otimes id) = (id \otimes i\phi)c$.

If $X = \bigoplus_{n=0}^{\infty} X_n$ is a graded vector space with finite-dimensional components X_n, $n \geq 0$, the *Hilbert series* of X is the formal power series $P_X(t) = \sum_{i=0}^{\infty} \dim(X_n)t^n$.

Corollary 3.3. *Let L be a Hopf algebra with bijective antipode, $V \in {}_L^L\mathcal{YD}$, and $L' \subset L$ a Hopf subalgebra with bijective antipode, and $V' \subset V$ a subspace such that $L'V' \subset V'$, $\delta(V') \subset L' \otimes V'$, where $\delta : V \to L \otimes V$ is the coaction of V. Then V is a left L'-module by restriction and V' is a left L-comodule via δ. Let $\varphi : V \to V'$ be an L'-linear and L-colinear map with $\varphi \mid V' = id$.*

Then V' is a Yetter-Drinfeld module over L', $\mathfrak{B}(V')$ is a graded subalgebra of $\mathfrak{B}(V)$, and $\mathfrak{B}(V)$ is a free right $\mathfrak{B}(V')$-module. If V is finite-dimensional, then $P_{\mathfrak{B}(V')}$ divides $P_{\mathfrak{B}(V)}$.

Proof. The tensor algebras $T(V)$ and $T(V')$ are Hopf algebras in ${}_L^L\mathcal{YD}$ and ${}_{L'}^{L'}\mathcal{YD}$ as in the first section where the elements of V and V' are primitive. The natural map $\phi : T(V) \to T(V')$ given by $\varphi : V \to V'$ is an algebra and a coalgebra map with $\phi i = id$, where $i : T(V') \to T(V)$ denotes the inclusion map. Moreover, ϕ is L-colinear and L'-linear, where $T(V')$ is an L-comodule and $T(V)$ is an L'-module via the inclusion map $\alpha : L' \to L$. Therefore ϕ induces an L-colinear algebra and coalgebra map $\mathfrak{B}(V) \to \mathfrak{B}(V')$ which will again be denoted by ϕ. Since $\mathfrak{B}(V)$ and $\mathfrak{B}(V')$ are graded algebras and coalgebras and ϕ is a graded algebra map (of degree 0), $K = \mathfrak{B}(V)^{co\phi}$ is a graded subalgebra of $\mathfrak{B}(V)$. Hence we conclude from Theorem 3.2 with $R = \mathfrak{B}(V)$ and $R' = \mathfrak{B}(V')$ that $K \otimes \mathfrak{B}(V') \cong \mathfrak{B}(V)$. In particular, $\mathfrak{B}(V)$ is free over $\mathfrak{B}(V')$, and if V is finite-dimensional, $P_{\mathfrak{B}(V')} P_K = P_{\mathfrak{B}(V)}$. □

When $\mathfrak{B}(V')$ is finite-dimensional, Corollary 3.3 also follows from [**G1**, Theorem 3.11] with a completely different proof. But the argument in Theorem 3.2 is quite general and does not depend on Nichols algebras.

An example of the situation of Corollary 3.3 will be given in Section 5. The next remark shows that the existence of the splitting map φ in Corollary 3.3 is a rather mild assumption.

Remark 3.4. Let G be a group, $G' \subset G$ a finite subgroup such that the characteristic of $\Bbbk$ does not divide the order of G'. Let $V \in {}_G^G\mathcal{YD}$ and $V' \subset V$ a subspace such that $G'V' \subset V'$, and $\delta(V') \subset \Bbbk G' \otimes V'$, where $\delta : V \to \Bbbk G \otimes V$ is the coaction of V. Then there is always a $\Bbbk G'$-linear and $\Bbbk G$-colinear map $\varphi : V \to V'$ with $\varphi \mid V' = id$. Hence we are in the situation of Corollary 3.3 with $L' = \Bbbk G'$ and $L = \Bbbk G$.

Proof. Since $\Bbbk G$ is cosemisimple, there is a $\Bbbk G$-colinear map $f : V \to V'$ with $f \mid V' = id$. Define $\varphi : V \to V'$ by $\varphi(v) = \frac{1}{\mathrm{ord}(G')} \sum_{g \in G'} g f(g^{-1} v)$ for all $v \in V$. Then φ is a G'-linear map with $\varphi \mid V' = id$, and φ is $\Bbbk G$-colinear, since for all $h \in G$ and $v \in V_h$, by the Yetter-Drinfeld condition $g^{-1} v \in V_{g^{-1}hg}$ and $g f(g^{-1} v) \in V'_h$ for all $g \in G'$. □

4. Link-indecompsable pointed Hopf algebras

Let A be a pointed Hopf algebra with group-like elements $G = G(A)$. For $g, h \in G$, $P_{g,h}(A) = \{x \in A \mid \Delta(x) = g \otimes x + x \otimes 1\}$ will denote the (g,h)-*primitive elements* of A. $P_{g,h}(A)$ is called *non-trivial* if $\Bbbk(g-h) \subsetneq P_{g,h}(A)$.

We recall the definition of the *quiver* Γ_A *of* A in [**M2**]. The vertices of Γ_A are the elements of G; for $g,h \in G$, there exists an arrow from h to g if $P_{g,h}(A)$ is non-trivial.

Note that $P_{g,h}(A)$ is non-trivial if and only if $P_{gh^{-1},1}(A)$ or $P_{h^{-1}g,1}(A)$ is non-trivial. Then $g,h \in G$ are *connected* (or $g \sim h$) if they are in the same connected component of Γ_A as an undirected graph. The Hopf algebra A is called *link-indecomposable* if any two elements in G are connected.

We call $\mathrm{Supp}(A) = \{g \in G \mid P_{g,1}(A) \text{ is non-trivial}\}$ the *support* of A. If $V = \bigoplus_{g\in G} V_g$ is a left $\Bbbk G$-comodule with homogeneous components V_g, then we define $\mathrm{Supp}(V) = \{g \in G \mid V_g \neq 0\}$.

Let $A_0 = \Bbbk G \subset A_1 \subset \dots$ be the coradical filtration of A and $\mathrm{gr}(A) = A_0 \oplus A_1/A_0 \oplus A_2/A_1 \oplus \dots$ the associated graded Hopf algebra. Then

$$\mathrm{gr}(A) \cong R\#\Bbbk G,$$

where $R = (\mathrm{gr}(A)^{\mathrm{co}A_0}$ is the *diagram* of A ; R is a graded Hopf algebra in ${}^G_G\mathcal{YD}$ and $P(R)$ is in ${}^G_G\mathcal{YD}$ (see [**AS1**]).

Lemma 4.1. *Let A be a pointed Hopf algebra with $G = G(A)$ and diagram R. Define $N = \{g \in G \mid g \sim 1\}$, and $V = P(R)$. Then*

1) N is a normal subgroup of G and the connected components of Γ_A are the cosets of N.

2) N is generated by Supp(A).

3) Supp(A) = Supp(V).

Proof. 1) is noted in [**M2**, 3.2]. By definition there exists an arrow between $g,h \in G$ if and only if gh^{-1} or $hg^{-1} \in \mathrm{Supp}(A)$. Suppose $g \sim 1$. Then there are $g_1,\dots,g_n \in G, n \geq 1, g_1 = g, g_n = 1$ with $g_i g_{i+1}^{-1} \in \mathrm{Supp}(A)$ or $(\mathrm{Supp}(A))^{-1}$ for all i, and $g = g_1g_2^{-1}g_2g_3^{-1}\dots g_{n-1}g_n^{-1}$ is generated by Supp(A). Conversely, if $u_1,\dots,u_{n-1} \in \mathrm{Supp}(A)$ or $(\mathrm{Supp}(A))^{-1}$, $n \geq 1$, and $g = u_1\dots u_{n-1}$, define $g_1 = g$, and g_{i+1} by $u_i = g_ig_{i+1}^{-1}, 1 \leq i \leq n-1$. Then g is connected to $g_n = 1$. This shows 2). For 3), note that $P(R)\#\Bbbk G \cong A_1/A_0$ by [**AS1**, Lemma 2.4]. For all $g,h \in G$ choose a decomposition of vector spaces $P_{g,h}(A) = \Bbbk(g-h) \oplus P_{g,h}(A)'$. Then by the Theorem of Taft and Wilson [**M1**, 5.4.1], $\bigoplus_{g,h\in G} P_{g,h}(A)' \cong A_1/A_0$. Thus for all $g \in G$ we obtain an isomorphism $P(R)_g\#1 \cong P_{g,1}(A)/\Bbbk(1-g)$. □

As suggested by the previous Lemma, we say a Yetter-Drinfeld module V over G is *link-indecomposable* if G is generated by Supp(V).

Proposition 4.2. *Let G be a finite group. Then the following are equivalent:*

(1) There exists a finite-dimensional pointed and link-indecomposable Hopf algebra A such that $G \cong G(A)$.

(2) There exists a link-indecomposable Yetter-Drinfeld module V over G such that $\mathfrak{B}(V)$ is finite-dimensional.

Proof. Assume (1) and let R be the diagram of A. Then $V = P(R)$ is link-indecomposable by Lemma 4.1. By 2.2 2), $\Bbbk[V] \cong \mathfrak{B}(V)$ is finite-dimensional.

Assume (2). Define $R = \mathfrak{B}(V)$, and let $A = R\#\Bbbk G$ be the Radford biproduct. It is well-known and not difficult to see that $P_{g,1}(A) = P(R)_g\#1 + \Bbbk(g-1)$ (if $x \in P_{g,1}(A)$, write $(\varepsilon_R \otimes id)(x) = \alpha(g-1), \alpha \in \Bbbk$, and check that $x - \alpha(g-1) \in P(A)_g^{\mathrm{co}\Bbbk G}$). Hence $\mathrm{Supp}(A) = \mathrm{Supp}(P(R)) = \mathrm{Supp}(V)$ generates G, and A is link-indecomposable by 4.1. $\square$

5. Link-indecomposable pointed Hopf algebras over Coxeter groups

Let W be a group and $T \subset W$ a subset such that for all $g \in W, t \in T, gtg^{-1} \in T$. Let $\chi : W \times T \to \Bbbk \setminus \{0\}$ be a function such that for all $g, h \in W$ and $t \in T$,

$$\chi(1,t) = 1, \tag{5.1}$$

$$\chi(gh,t) = \chi(g, hth^{-1})\chi(h,t). \tag{5.2}$$

We can then define a Yetter-Drinfeld module $V = V(W,T,\chi)$ over W with $\Bbbk$-basis $x_t, t \in T$, and action and coaction of W given by

$$gx_t = \chi(g,t)x_{gtg^{-1}}, \tag{5.3}$$

$$\delta(x_t) = t \otimes x_t \tag{5.4}$$

for all $g \in W, t \in T$.

Conversely, if the function χ defines a Yetter-Drinfeld module on the vector space V by (5.3), (5.4), then χ satisfies (5.1), (5.2).

Note that the braiding c of $V(W,T,\chi)$ is determined by

$$c(x_s \otimes x_t) = \chi(s,t)x_{sts_{-1}} \otimes x_t \text{ for all } s,t \in T,$$

hence by the values of χ on $T \times T$.

Our main example comes from the theory of Coxeter groups ([**B**, Chapitre IV]). Let S be a subset of the group W of elements of order 2. For all $s, s' \in S$ let $m(s,s')$ be the order of ss'. (W,S) is called a *Coxeter system* and W a *Coxeter group* if W is generated by S with defining relations $(ss')^{m(s,s')} = 1$ for all $s, s' \in S$ such that $m(s,s')$ is finite.

Let (W,S) be a Coxeter system. For any $g \in W$ there is a sequence $(s_1, \dots, s_q)$ of elements in S with $g = s_1 \cdot \dots \cdot s_q$. If q is minimal among all such representations, then $q = l(g)$ is called the *length* of g, and $(s_1, \dots, s_q)$ is a *reduced representation* of g.

Definition 5.1. Let (W,S) be a Coxeter system, and $T = \{gsg^{-1} \mid g \in W, s \in S\}$. Define $\chi : W \times T \to \Bbbk \setminus \{0\}$ by

$$\chi(g,t) = (-1)^{l(g)} \text{ for all } g \in W, t \in T. \tag{5.5}$$

Let $V = V(W,T,\chi) \in {}^W_W\mathcal{YD}$. The associated Hopf algebras will be denoted by $R(W,S) = \mathfrak{B}(V)$ and $A(W,S) := \mathfrak{B}(V)\#\Bbbk W$.

By Proposition 4.2, $A(W,S)$ is a pointed link-indecomposable Hopf algebra with group-like elements W.

Remark 5.2. We assume W, T, χ as in the beginning of this section, and define $V = V(W,T,\chi)$.

1) By definition, T is a union of conjugacy classes of W. Let $t_1 \in T$ and T_1 the conjugacy class containing t_1. Let $s_i, 1 \leq i \leq \theta$, be representatives of the right residue classes of the centralizer $C_W(t_1)$, $W = \bigcup_{i=1}^{\theta} s_i C_W(t_1)$, and define $t_i := s_i t_1 s_i^{-1}, 1 \leq i \leq \theta$. Then $W/C_W(t_1) \to T_1, \bar{s_i} \mapsto t_i, 1 \leq i \leq \theta$, is bijective. Let $\widehat{\chi} : C_W(t_1) \to \mathbb{k} \setminus \{0\}$ be the group homomorphism defined by $\widehat{\chi}(g) = \chi(g, t_1)$ for all $g \in C_W(t_1)$. We now change the basis to

$$x_i := \chi(s_i, t_1) x_{t_i}, 1 \leq i \leq \theta. \tag{5.6}$$

For all $g \in W, 1 \leq i \leq \theta$, define $1 \leq j \leq \theta$ by $t_j = g t_i g^{-1}$. Then $s_j^{-1} g s_i \in C_W(t_1)$, and

$$\chi(g, t_i) = \widehat{\chi}(s_j^{-1} g s_i) \chi(s_j, t_1) \chi(s_i, t_1)^{-1}, \tag{5.7}$$

$$g x_i = \eta(g, t_i) x_j \text{ where } \eta(g, t_i) = \widehat{\chi}(s_j^{-1} g s_i). \tag{5.8}$$

Thus $\eta : W \times T_1 \to \mathbb{k} \setminus \{0\}$ satisfies (5.1), (5.2), and $\bigoplus_{1 \leq i \leq \theta} \mathbb{k} x_{t_i} = V(W, T_1, \eta)$ with respect to the new basis $y_{t_i} = x_i, 1 \leq i \leq \theta$.

Proof. From $t_j = g t_i g^{-1}$ and the definition of t_i, t_j it follows that $s_j^{-1} g s_i \in C_W(t_1)$. Hence by (5.2), $\chi(s_j(s_j^{-1} g s_i), t_1) = \chi(s_j, t_1) \chi(s_j^{-1} g s_i, t_1)$. On the other hand by (5.2), $\chi(g s_i, t_1) = \chi(g, t_i) \chi(s_i, t_1)$, and (5.7) and (5.8) follow. □

2) Conversely, any character $\widehat{\chi} : C_W(t_1) \to \mathbb{k} \setminus \{0\}$ defines a Yetter-Drinfeld module $\bigoplus_{t \in T_1} \mathbb{k} x_t$ over W by (5.8) and (5.6), and by (5.7) a function $\chi : W \times T_1 \to \mathbb{k} \setminus \{0\}$ satisfying (5.1) and (5.2) where $\chi(s_i, t_1), 1 \leq i \leq \theta$, can be arbitrary non-zero scalars. This construction is a special case of the well-known description of the simple Yetter-Drinfeld modules over a finite group (cf. [**CR**], [**AG**]).

3) Assume that $\mathfrak{B}(V)$ is finite-dimensional. Then for all $t \in T$ there is a natural number $n \geq 2$ such that $1 + \chi(t, t) + \cdots + (\chi(t, t))^{n-1} = 0$, and if $t^2 = 1$ and $\mathrm{char}(\mathbb{k}) = 0$, then $\chi(t, t) = -1$. Note that in 1), $\chi(t_i, t_i) = \widehat{\chi}(t_i) = \eta(t_i, t_i)$ for all $1 \leq i \leq \theta$.

Proof. Let $D_t, t \in T$ be the skew-derivations corresponding to the basis $x_t, t \in T$. Let $t \in T$ and $q = \chi(t, t)$. It follows easily by induction on n that $D_t(x_t^n) = (1 + q + \cdots + q^{n-1}) x_t^{n-1}$ for all $n \geq 1$. Since $x_t^n \in \mathfrak{B}(V)(n)$, x_t^n must be zero for large n, hence $1 + q + \cdots + q^{n-1} = 0$, if $x_t^n = 0$ and $x_t^{n-1} \neq 0$. In particular, if $t^2 = 1$, then $(\chi(t, t))^2 = 1$ by (5.1), (5.2), and $\chi(t, t) = -1$ if $\mathrm{char}(\mathbb{k}) = 0$. □

We describe the possible functions χ up to base change as in Remark 5.2, 1) for the symmetric and the dihedral groups assuming $\chi(t, t) = -1$ for all $t \in T$ as in Remark 5.2, 3).

Example 5.3. Let $W = S_n$ be the symmetric group of permutations of $n \geq 2$ elements, and $S = \{(12), (23), \ldots, (n-1, n)\}$. Then (W, S) is a Coxeter system, and T is the set of all transpositions in S_n. Up to base change as described in 5.2 1), there are exactly two functions χ (which coincide if $\mathrm{char}(\mathbb{k}) = 2$) satisfying

(5.1), (5.2) and $\chi(t,t) = -1$ for all $t \in T$. Either χ is defined as in (5.5) or for all $g \in S_n, 1 \le i < j \le n$,

$$\chi(g,(ij)) = \begin{cases} 1 & \text{, if } g(i) < g(j), \\ -1 & \text{, if } g(i) > g(j). \end{cases} \tag{5.9}$$

Proof. Let $t_1 = (12)$. The centralizer of t_1 is $\{g \in S_n \mid \{g(1), g(2)\} = \{1,2\}\} = \langle (34),(45),\dots,(n-1,n)\rangle \cup \langle (34),(45),\dots,(n-1,n)\rangle(12)$. Hence the characters $\widehat{\chi} : C_{S_n}(t_1) \to \Bbbk \setminus \{0\}$ are $\chi_{\varepsilon_1,\varepsilon_2}$, with $\varepsilon_1, \varepsilon_2 \in \{1,-1\}$, and $\chi_{\varepsilon_1,\varepsilon_2}((ij)) = \varepsilon_1$ if $3 \le i < j \le n$, and $\chi_{\varepsilon_1,\varepsilon_2}((12)) = \varepsilon_2$. Moreover $\varepsilon_2 = -1$ since we assumed $\eta(t_1,t_1) = -1$. The characters $\chi_{-1,-1}$ and $\chi_{1,-1}$ define the functions χ in (5.5) and (5.9). □

Example 5.4. Let $W = D_m = \langle t, t'\rangle, m \ge 1$, be the dihedral group of order $2m$, where $t, t' \in T$ are generators of order 2 and tt' is of order m. Let $S = \{t, t'\}$. Then (W, S) is a Coxeter sytem. Define $t_i = (tt')^{i-1}t$ for all $i \in \mathbb{Z}$. Then $T = \{t_i \mid 1 \le i \le m\}$, and for all i, j, $t_i t_j t_i = t_{2i-j}$, and $t_i = t_j$ if and only if $i \equiv j \bmod m$. If m is odd, then T is one conjugacy class. For $m = 2l$ even, T is the union of the two conjugacy classes $T_0 = \{t_{2k} \mid 1 \le k \le l\}$ and $T_1 = \{t_{2k-1} \mid 1 \le k \le l\}$, and $z = (tt')^l$ is a central element of order 2.

Let χ be a function satisfying (5.1), (5.2) and $\chi(t,t) = -1$ for all $t \in T$.

a) If m is odd, then the centralizer of t_1 is $C_W(t_1) = \{1, t_1\}$. We choose representatives s_i, $1 \le i \le m$, of the residue classes of $C_W(t_1)$ by $s_i = t_k$ with $2k - 1 \equiv i \bmod m$, hence $s_i t_1 s_i^{-1} = t_i$. In (5.8) we then have $\eta(t_k, t_i) = -1$ for all k, i, that is we are in case (5.5).

Proof. By (5.8) with $g = t_k$, $\eta(t_k, t_i) = \chi(s_j^{-1} t_k s_i, t_1) = -1$, since $\chi(t_1, t_1) = -1$ by assumption, and $s_j^{-1} t_k s_i = t_1$ (since $s_j^{-1} t_k s_i \neq 1$ as a product of three reflections). □

b) If $m = 2l$ is even, we pick t_0 in the conjugacy class T_0. Then $C_W(t_0) = \{1, t_0, z, t_0 z\}$. We choose representatives s_i, $1 \le i \le l$, of the residue classes of $C_W(t_0)$ by $s_i = t_i$, $1 \le i \le l$. Then $s_i t_0 s_i^{-1} = t_{2i}$,$1 \le i \le l$, and for clarity we denote the elements of T_0 by $\tau_i = t_{2i}$, $1 \le i \le l$. Let $g = t_k$, $1 \le k \le m$, and $1 \le i \le l$. Define $1 \le j \le l$ by $\tau_j = g\tau_i g^{-1}$, that is $t_{2j} = t_{2(k-i)}$, and $j \equiv k - i \bmod l$. In (5.6) we write x_{2i} instead of x_i and extend this definition by $x_{2i} = \chi(t_{\phi(i)}, t_0) x_{2i}$, where $\phi(i)$ is the integer with $1 \le \phi(i) \le l$ and $\phi(i) \equiv i \bmod l$. We then have $t_k x_{2i} = \eta(t_k, t_{2i}) x_{2(k-i)}$. Define $\varepsilon_0 \in \{1, -1\}$ by $\chi(z, t_0) = -\varepsilon_0$. Then for all $1 \le k \le m, 1 \le i \le l$,

$$\eta(t_k, t_{2i}) = \begin{cases} -1 & \text{, if } 1 \le k - i \le l, \\ \varepsilon_0 & \text{otherwise.} \end{cases} \tag{5.10}$$

PROOF. By (5.8),

$$\eta(t_k,\tau_i) = \widehat{\chi_0}((tt')^{j-k+i}t'), \text{ since } s_j^{-1}t_k s_i = t_j t_k t_i = (tt')^{j-k+i}t'.$$

The homomorphism

$$\widehat{\chi_0} : \{1, t_0, z, t_0 z\} \to \mathbb{k} \setminus \{0\}$$

is given by $\widehat{\chi_0}(t_0) = \chi(t_0,t_0) = -1$ and $\widehat{\chi_0}(z) = \chi(z,t_0) = -\varepsilon_0$. If $1 \le k-i \le l$, then $j = k-i$, and $(tt')^{j-k+i}t' = t' = t_0$, and $\eta(t_k,\tau_i) = \widehat{\chi_0}(t_0) = -1$. If $k-i = j \pm l$, then $(tt')^{j-k+i}t' = zt_0$, and $\eta(t_k,\tau_i) = \widehat{\chi_0}(zt_0) = \varepsilon_0$. □

c) If $m = 2l$, we pick t_1 in the conjugacy class T_1. Then $C_W(t_1) = \{1, t_1, z, t_1 z\}$. We choose representatives $s_i, 1 \le i \le l$, by $s_i = t_i$. Then $s_i t_1 s_i^{-1} = t_{2i-1}$, and for clarity we set $\tau_i = t_{2i-1}, 1 \le i \le l$. Let $g = t_k, 1 \le k \le m$, and $1 \le i \le l$. Define $1 \le j \le l$ by $\tau_j = g\tau_i g^{-1}$, that is $t_{2j-1} = t_{2(k-i+1)-1}$, or $j \equiv k-i+1 \bmod l$. In (5.6) we write x_{2i-1} instead of x_i. As in b) we define $x_{2i-1} = \chi(t_{\phi(i)}, t_1)x_{t_{2i-1}}$ for all integers i. We then have $t_k x_{2i-1} = \eta(t_k, t_{2i-1})x_{2(k-i+1)-1}$. Define $\varepsilon_1 \in \{1,-1\}$ by $\chi(z,t_1) = -\varepsilon_1$. Then for all $1 \le k \le m, 1 \le i \le l$,

$$\eta(t_k, t_{2i-1}) = \begin{cases} -1 & , \text{ if } 1 \le 1+k-i \le l, \\ \varepsilon_1 & \text{otherwise.} \end{cases} \tag{5.11}$$

PROOF. By (5.8) with $g = t_k$, $\eta(t_k,\tau_i) = \widehat{\chi_1}((tt')^{j-k+i}t')$ as before. The homomorphism

$$\widehat{\chi_1} : \{1, t_1, z, t_1 z\} \to \mathbb{k} \setminus \{0\}$$

is given by $\widehat{\chi_1}(t_1) = -1, \widehat{\chi_1}(z) = -\varepsilon_1$. If $1 \le k-i+1 \le l$, then $j = k-i+1$, and $(tt')^{j-k+i}t' = t = t_1$, hence $\eta(t_k,\tau_i) = \widehat{\chi_1}(t_1) = -1$. If $k-i+1 = j \pm l$, then $(tt')^{j-k+i}t' = z(tt')t' = zt_1$, hence $\eta(t_k,\tau_i) = \varepsilon_1$. □

d) Conversely, if $m = 2l$ is even and $\varepsilon_0, \varepsilon_1$ are arbitrary elements in $\{1,-1\}$, then (5.10), (5.11) define a function η satisfying (5.1), (5.2) and $\eta(t,t) = -1$ for all $t \in T$.

Note that the case $\varepsilon_0 = -1 = \varepsilon_1$ is the example (5.5) for $W = D_m$.

LEMMA 5.5. *Assume W, T and χ as in the beginning of this section and let $V = V(W,T,\chi)$. Then the antipode $S : \mathfrak{B}(V) \to \mathfrak{B}(V)$ can be explicitly computed as follows. Let $s_1, \dots, s_q \in T$ and define $t_1, \dots, t_q \in T$ by $t_i = (s_1 \cdots s_{i-1})s_i(s_1 \cdots s_{i-1})^{-1}$, $1 \le i \le q$. Then*

$$S(x_{s_1}x_{s_2}\cdots x_{s_q}) = \varepsilon(s_1, s_2, \dots, s_q)x_{t_q}\cdots x_{t_2}x_{t_1},$$

where $\varepsilon(s_1,\dots,s_q) = (-1)^q \prod_{i=2}^q \chi(s_1 s_2 \cdots s_{i-1}, s_i)$.

PROOF. For all $x, y \in \mathfrak{B}(V)$, $S(xy) = S(x_{-1}\cdot y)S(x_0)$. Hence

$$\begin{aligned} S(x_{s_1}\cdots x_{s_{q-1}}x_{s_q}) &= S((s_1\cdots s_{q-1})\cdot x_{s_q})S(x_{s_1}\cdots x_{s_{q-1}}) \\ &= -\chi(s_1\cdots s_{q-1}, s_q)x_{t_q}S(x_{s_1}\cdots x_{s_{q-1}}), \end{aligned}$$

and the claim follows by induction on q. $\square$

We now describe some of the relations of $\mathfrak{B}(V)$. In general we do not know whether these are defining relations. We begin with the dihedral group of order 4. In the terminology of [**AS2**], the Hopf algebras are then quantum linear spaces or of type A_2. For completeness we give a proof in the context of this paper.

LEMMA 5.6. *Let $W = D_2$ be generated by $t = t_1, t' = t_2$ as in Example 5.4 with $T = \{t_1, t_2\}$. Let $\varepsilon_1, \varepsilon_0 \in \{1, -1\}$ and define $\chi : W \times T \to \Bbbk \setminus \{0\}$ by* (5.10) *and* (5.11). *Let $V = V(D_2, T, \eta)$, and $x = x_t, y = x_{t'}$. Thus the action is given by $t \cdot x = -x, t' \cdot y = -y$ and $t' \cdot x = \varepsilon_1 x, t \cdot y = \varepsilon_0 y$.*

1) Assume $\varepsilon_1 = \varepsilon_0$ and let $\varepsilon = \varepsilon_1 = \varepsilon_0$. Then

$$x^2 = 0, y^2 = 0, \tag{5.12}$$

$$xy = \varepsilon yx, \tag{5.13}$$

are defining relations of $\mathfrak{B}(V)$, and $1, x, y, xy$ is a basis of $\mathfrak{B}(V)$.

2) Assume $\varepsilon_1 = -\varepsilon_0$, and char$(\Bbbk) \neq 2$. Then

$$x^2 = 0, y^2 = 0, \tag{5.14}$$

$$xyxy + yxyx = 0, \tag{5.15}$$

are defining relations of $\mathfrak{B}(V)$, and $1, x, y, xy, yx, xyx, yxy, xyxy$ is a basis of $\mathfrak{B}(V)$.

PROOF. Let D_1, D_2 be the skew-derivations in Proposition 2.4 with respect to the basis x, y of V. By Proposition 2.4 it suffices to show that the identities hold after applying D_i for all i.

The relations (5.12) and (5.14) follow from $\eta(t,t) = \eta(t',t') = -1$, since $D_1(x^2) = x + \eta(t,t)x = 0$, $D_2(x^2) = 0$, and by the same argument $D_i(y) = 0, i = 1, 2$.

We compute $D_1(xy) = \eta(t_1, t_2)y = \varepsilon_0 y, D_2(xy) = x, D_1(yx) = y$ and $D_2(yx) = \eta(t_2, t_1)x = \varepsilon_1 x$. Then $D_1((xy)^2) = -yxy, D_2((xy)^2) = xyx$, and $D_1((yx)^2) = yxy, D_2((yx)^2) = -xyx$, since $x^2 = 0, y^2 = 0$. This implies (5.13) and (5.15).

It is clear from the relations that the above elements generate the vector space $\mathfrak{B}(V)$. Hence linear independency of these elements (of the same degree in the variables x, y) implies that the relations are defining relations. This is clear in case 1) since $D_1(xy) \neq 0$, hence $xy \neq 0$ in $\mathfrak{B}(V)$, and follows easily in case 2) by using the values of the derivations D_i which we have just computed. For example, suppose $\alpha xy + \beta yx = 0$ in case 2), where $\alpha, \beta \in \Bbbk$. Then $0 = D_1(\alpha xy + \beta yx) = (\alpha\varepsilon_0 + \beta)y$, and $0 = D_2(\alpha xy + \beta yx) = (\alpha + \beta\varepsilon_1)x$, hence $\alpha = \beta = 0$ since char$(\Bbbk \neq 2$. $\square$

THEOREM 5.7. *Let W be a group, $T \subset W$ a subset which is stable under conjugation with elements in W, $\chi : W \times T \to \Bbbk \setminus \{0\}$ a function satisfying* (5.1), (5.2) *and $V = V(W, T, \chi) \in {}^W_W\mathcal{YD}$. Let $t, t' \in T$ be elements of order 2 and assume that tt' is of order $m \geq 1$.*

Then the subgroup $D \subset W$ generated by t, t' is the dihedral group of order $2m$. For all $i \in \mathbb{Z}$, define $t_i = (tt')^{i-1}t$ as in Example 5.4. Then $t_i = t_j$ if and only if $i \equiv j \bmod m$, and the sequence $t_1, \dots, t_m$ is defined inductively by

$$t_0 = t', t_1 = t \text{ and } t_i t_{i-1} t_i = t_{i+1}, 1 \leq i \leq m-1.$$

Assume $\chi(t_i, t_i) = -1$ for all i.

If $m = 2l+1$ is odd, define $x_i = \chi(t_k, t_1) x_{t_i}$ for all $i \in \mathbb{Z}$ where $k \in \mathbb{Z}$ with $i \equiv 2k-1 \bmod m$. If $m = 2l$ is even, define $x_{2i-1} = \chi(t_{\phi(i)}, t_1) x_{t_{2i-1}}$ and $x_{2i} = \chi(t_{\phi(i)}, t_1) x_{t_{2i}}$ for all $i \in \mathbb{Z}$, where $\phi(i)$ is the natural number with $1 \leq \phi(i) \leq l$ and $\phi(i) \equiv i \bmod l$.

Then $x_a = x_b$ if $a, b \in \mathbb{Z}, a \equiv b \bmod m$, and for all $i, j \in \mathbb{Z}$,

$$t_i x_j = \eta(t_i, t_j) x_{2i-j}$$

where the function η is defined by (5.5) if m is odd, and by (5.10), (5.11) if $m = 2l$ is even, and where $\varepsilon_0, \varepsilon_1 \in \{1, -1\}$ are given by $\chi((tt')^l, t_i) = -\varepsilon_i, 0 \leq i \leq 1$.

The following relations are satisfied in $\mathfrak{B}(V)$:

$$x_i^2 = 0 \text{ for all } i. \tag{5.16}$$

If m is even, assume $\varepsilon_0 = \varepsilon_1$, and define $\varepsilon = \varepsilon_0 = \varepsilon_1$. Then

$$x_1 x_2 + x_2 x_3 + \cdots + x_{m-1} x_m = \varepsilon x_m x_1, \tag{5.17}$$

$$x_1 \cdots x_m = \varepsilon x_m \cdots x_1, \tag{5.18}$$

$$(x_m x_1)^l = \varepsilon (x_1 x_m)^l. \tag{5.19}$$

If $m = 2l+1$ is odd, then

$$x_1 x_2 + x_2 x_3 + \cdots + x_{m-1} x_m + x_m x_1 = 0, \tag{5.20}$$

$$x_1 x_2 \cdots x_m = x_m x_{m-1} \cdots x_1, \tag{5.21}$$

$$(x_m x_1)^l x_m = (x_1 x_m)^l x_1. \tag{5.22}$$

Assume $m = 2l$ and l are even and $\varepsilon_0 = \varepsilon_1 = -1$.

Let $x_\pm = \sum_{i=1}^{l} (\pm 1)^{i-1} x_{2i-1}$, and $y_\pm = \sum_{i=1}^{l} (\pm 1)^{i-1} x_{2i}$.

If $x = x_+, y = y_-$ or $x = x_-, y = y_+$, then

$$x^2 = 0, y^2 = 0, \tag{5.23}$$

$$xyxy + yxyx = 0, \tag{5.24}$$

and if char$(\mathbb{k}) \neq 2$, then the subalgebra generated by x, y has dimension 8.

Proof. Let $U \subset V$ be the braided subspace with basis $x_1, \dots, x_m$. Then U is a Yetter-Drinfeld module over D and $U = V(D, \{t_1, \dots, t_m\}, \eta)$ with respect to the basis $y_{t_i} = x_i, 1 \leq i \leq m$, as shown in Example 5.4. Since the obvious natural map $\mathfrak{B}(U) \to \mathfrak{B}(V)$ is an algebra map, we can assume that $U = V$.

For all $1 \leq i \leq m$, let D_i be the skew-derivation defined in Proposition 2.4 with respect to $x_i \in V_{t_i}$. As in the previous Lemma it suffices to show that the identities hold after applying D_i for all i.

The relations (5.16) follow from $\eta(t_i,t_i) = \chi(t_i,t_i) = -1$ as in the proof of (5.12).

To show (5.17) and (5.20), let $a = x_1x_2 + x_2x_3 + \cdots + x_{m-1}x_m - \varepsilon x_m x_1$, where $\varepsilon = -1$ when m is odd. Then

$$\begin{aligned}
D_1(a) &= t_1 \cdot x_2 - \varepsilon x_m = (\eta(t_1,t_2) - \varepsilon)x_0,\\
D_2(a) &= x_1 + t_2 \cdot x_3 = (\eta(t_2,t_3) + 1)x_1,\\
&\dots\\
D_{m-1}(a) &= x_{m-2} + t_{m-1} \cdot x_m = (\eta(t_{m-1},t_m) + 1)x_{m-2},\\
D_m(a) &= x_{m-1} - \varepsilon t_m \cdot x_1 = (-\varepsilon\eta(t_m,t_1 + 1)x_{m-1}.
\end{aligned}$$

Assume m is even. From (5.10) and (5.11) we see that $\eta(t_i,t_{i+1}) = -1$ for all $2 \le i \le m-1$, and $\eta(t_1,t_2) = \varepsilon_0, \eta(t_m,t_1) = \varepsilon_1$. Thus $D_i(a) = 0$ for all i since $\varepsilon_0 = \varepsilon_1$.

If m is odd, then $D_i(a) = 0$ for all i, since $-1 = \varepsilon = \eta(t_k,t_i)$ for all k, i.

To prove (5.18) and (5.21), we first note that

$$D_1(x_1(x_2\cdots x_m)) = t_1 \cdot (x_2 \cdots x_m) = \eta(t_1,t_2)\dots\eta(t_1,t_m)x_mx_{m-1}\cdots x_2.$$

On the other hand, $D_1((x_m\cdots x_2)x_1) = x_m \cdots x_2$.

Assume $m = 2l$. By (5.10) and (5.11) we compute $\eta(t_1,t_2)\dots\eta(t_1,t_m) = \varepsilon_0^l\varepsilon_1^{l-1} = \varepsilon$. Hence $D_1(x_1\cdots x_m) = D_1(\varepsilon x_m\cdots x_1)$. Similarly we obtain the same equality for D_m since $\eta(t_m,t_{m-1})\dots\eta(t_m,t_1) = \varepsilon_0^{l-1}\varepsilon_1^l = \varepsilon$.

If m is odd, then $\eta(t_k,t_i) = -1$ for all k, i, and $D_i(x_1\cdots x_m) = D_i(x_m\cdots x_1)$ for $i = 1$ and $i = m$.

Finally, if $2 \le i \le m-1$, then

$$\begin{aligned}
D_i((x_1\cdots x_{i-1}x_i)(x_{i+1}\cdots x_m)) &=\\
D_i((x_1\cdots x_{i-1})x_i)\; t_i \cdot (x_{i+1}\cdots x_m) &=\\
(x_1\cdots x_{i-1})\; t_i \cdot (x_{i+1}\cdots x_m) &= 0,
\end{aligned}$$

since $t_i \cdot (x_{i+1}\cdots x_m)$ begins with the element x_{i-1}, and $x_{i-1}^2 = 0$ by (5.16).

Similarly we have $D_i(x_m\cdots x_1) = (x_m\cdots x_{i+1})\; t_i \cdot (x_{i-1}\cdots x_m) = 0$, since $x_{i+1}^2 = 0$.

To show (5.19), consider the sequence $(s_1,\dots,s_{2l}) = (t,t',t,t',\dots,t,t')$. Then $s_1 = t = t_1, s_1s_2s_1^{-1} = tt't = t_2, s_1s_2s_3(s_1s_2)^{-1} = tt'tt't = t_3,\dots$. Thus $t_1,\dots,t_{2l}$ coincide with the t_i's corresponding to $s_1,\dots,s_{2l}$ in Lemma 5.5. Hence by Lemma 5.5,

$$S((x_1x_m)^l) = \varepsilon(t,t',\dots,t,t')x_mx_{m-1}\cdots x_1. \tag{5.25}$$

Changing the roles of t, t' we obtain in the same way

$$S((x_mx_1)^l) = \varepsilon(t',t,\dots,t',t)x_1x_2\cdots x_m, \tag{5.26}$$

since $(t't)^{m-i}t' = (tt')^i t' = (tt')^{i-1}t = t_i$ for all i. By the definition in Lemma 5.5

$$\varepsilon(t,t',t,t',\dots,t,t') = \prod_{i=2}^{2l} \eta(s_1\cdots s_{i-1},s_i) = \eta(t,t')\eta(tt',t)\cdots\eta(tt'\cdots tt't,t').$$

From (5.10), (5.11),

$$\eta((tt')^{i-1}t,t') = \eta(t_i,t_{2l}) = \varepsilon_0 \text{ for all } 1\le i\le l,$$

and

$$\eta((t't)^i,t) = \eta(t_{i+1}t,t) = \eta(t_{i+1},t)\eta(t,t) = 1 \text{ for all } 1\le i\le l-1.$$

Hence we have

$$\varepsilon(t,t',t,t',\dots,t,t') = \varepsilon_0^l, \text{ and similarly } \varepsilon(t',t,t',t,\dots,t',t) = \varepsilon_1^l.$$

Therefore we obtain from (5.25), (5.26) and (2) the equality

$$S((x_1x_m)^l) = S(\varepsilon(x_mx_1)^l),$$

and the claim follows from the bijectivity of the antipode S.

To prove (5.22), we proceed as in the previous case. Let $(s_1,\dots,s_{2l+1})$ be the sequence $(t,t',t,t',\dots,t,t',t)$. Then $t_1,\dots,t_{2l+1}$ coincide with the t_i's corresponding to $s_1,\dots,s_{2l+1}$ in Lemma 5.5. By Lemma 5.5,

$$S((x_1x_m)^lx_1) = \varepsilon(t,t',\dots,t)x_mx_{m-1}\cdots x_1,$$

where $\varepsilon(t,t',\dots,t) = -\eta(t,t')\eta(tt',t)\cdots\eta(tt'\cdots tt',t) = (-1)^{1+l}$. By changing t,t' we obtain

$$S((x_mx_1)^lx_m) = (-1)^{1+l}x_1x_2\cdots x_m,$$

since $(t't)^{i-1}t' = t_{m-i+1}$, $1\le i\le 2l+1$. Therefore the claim follows from (5.21).

The last part of the Theorem follows from the fact that the elements x,y span a braided subspace of type A_2. Write $x=\sum_{i=1}^l \alpha^{i-1}x_{2i-1}$ and $y=\sum_{i=1}^l \beta^{i-1}x_{2i}$, where $\alpha,\beta\in\{\pm1\}$ and $\alpha\beta=-1$. We compute the braiding c with respect to x,y:

$$c(x\otimes y) = \sum_{i,j}\alpha^{i-1}\beta^{j-1}c(x_{2i-1}\otimes x_{2j}),$$

where the indices i,j are in $\mathbb{Z}/(l)$ (since l is even). Since

$$c(x_{2i-1}\otimes x_{2j}) = -x_{2(2i-j-1)}\otimes x_{2i-1}$$

, we substitute $k=2i-j-1$ and obtain

$$c(x\otimes y) = -\sum_{i,k}\alpha^{i-1}\beta^k x_{2k}\otimes x_{2i-1} = b_{12}(y\otimes x) \text{ ,where } b_{12}=-\beta.$$

In the same way one shows $c(y\otimes x) = b_{21}(x\otimes y), c(x\otimes x) = b_{11}(x\otimes x), c(y\otimes y) = b_{22}(y\otimes y)$, where $b_{21}=-\alpha, b_{11}=b_{22}=-1$. Hence the braiding on x,y is the braiding of case 2) in Lemma 5.6. Since the relations only depend on the braiding (see Remark 2.2, 1)), the claim follows from Lemma 5.6. □

In the next theorem we assume that for all elements t, t' such that the product tt' has even order $2l$, we have $\varepsilon_0 = \varepsilon_1$ in the sense of Theorem 5.7. Note that this condition is satisfied in the example (5.5) where $\varepsilon_0 = \varepsilon_1 = -1$, and also for the symmetric group in (5.9) where $\varepsilon_0 = \varepsilon_1 = 1$ if $l = 1$.

THEOREM 5.8. *Let (W, S) be a Coxeter system, T the set of all W-conjugates of elements in S, $\chi : W \times T \to \Bbbk \setminus \{0\}$ a function satisfying (5.1), (5.2), $V = V(W, T, \chi)$, and $R = \mathfrak{B}(V)$. Assume for all $t, t' \in T$ such that the product tt' has even order $2l$ that $\chi((tt')^l, t) = \chi((tt')^l, t')$.*
For all $t \in T$ let $R^{(t)}$ be the subalgebra of R generated by all $x_{t'}, t' \in T, t' \neq t$.
Then the generators $x_t, t \in T$ of R satisfy the relations (5.16) – (5.22), and

1) For all $t \in T$ and $x \in R$ there are uniquely determined elements $a, b \in R^{(t)}$ such that $x = a + bx_t$. If x is a W-homogeneous element in R_g, $g \in W$, then $a \in R_g$ and $b \in R_{gt}$.

2) For all $g \in W$, the W-homogeneous components R_1 and R_g are isomorphic as vector spaces.

3) For all $g \in W$, choose a reduced representation $g = s_1 \cdots s_q$, $s_1, \cdots, s_q \in S$, of g, and define

$$x_g = x_{s_1} \cdots x_{s_q}.$$

Then x_g is a non-zero element in R_g.

4) For all $g = t_1 \cdots t_p$, where $t_1, \cdots, t_p$ are elements of S,

$$x_{t_1} \cdots x_{t_p} = \begin{cases} \pm x_g, & \text{if } (t_1, \cdots, t_p) \text{ is a reduced representation of } g \\ 0, & \text{otherwise.} \end{cases} \tag{5.27}$$

PROOF. 1) To prove the existence of the representation we have to show for any $s_1, \ldots, s_q \in T$ and $x = x_{s_1}, \cdots x_{s_q}$ the existence of W-homogeneous elements $a, b \in R^{(t)}$ with $x = a + bx_t$. This follows easily by induction on the length q from (5.17), (5.20), since these relations are W-homogeneous since all their monomials have the same degree $t_{i-1}t_i = t_i t_{i+1}$. The uniqueness of the representation is clear from Lemma 2.5 1).

2) Fix an element $t \in T$. By 1) there is well-defined linear isomorphism

$$\phi_t : R \to R, \text{ with } \phi_t(a + bx_t) = b + ax_t \text{ for all } a, b \in R^{(t)}.$$

For any $g \in W$, the restriction of ϕ_t defines an isomorphism $R_g \cong R_{gt}$, and 2) follows, since T generates the group W.

3) Define the sequence $(t_1, \ldots, t_q)$ of elements in T with respect to $(s_1, \cdots, s_q)$ as in Lemma 5.5. Since $(s_1, \ldots, s_q)$ is a reduced representation of g, the elements t_i are pairwise distinct by [**B**, 1.4, Lemma 2]. Hence $S(x_g) \neq 0$ by Lemma 5.5 and Lemma 2.5 2).

4) We consider $M = \mathfrak{B}(V)/\{1, -1\}$ as a multiplicative monoid, and denote by $\pi : \mathfrak{B}(V) \to M$ the quotient map. We claim that there exists a map $\Phi : W \to M$ satisfying $\Phi(g) = \pi(x_{s_1} \cdots x_{s_q})$ whenever $(s_1, \ldots, s_q)$ is a reduced representation of

g. By [**B**, 1.6, Proposition 5] it is sufficient to show that $a(s,s') = a(s',s)$ for all $s, s' \in S$, $s \neq s'$, where

$$a(s,s') = \begin{cases} \pi(x_s x_{s'})^l, & \text{if } \mathrm{ord}(ss') = 2l, \\ \pi((x_s x_{s'})^l x_s), & \text{if } \mathrm{ord}(ss') = 2l+1. \end{cases} \tag{5.28}$$

This follows directly from (5.19), (5.22).

Finally we have to show that $x_{t_1} \cdots x_{t_p} = 0$ if the length of g is $< p$. We may assume that $(t_2, \cdots, t_p)$ is reduced. Then by the exchange property in [**B**, 1.4, Lemma 3] there exists $2 \leq j \leq p$ such that $t_1 t_2 \cdots t_{j-1} = t_2 \cdots t_{j-1} t_j$. Hence $(x_{t_1} \cdots x_{t_{j-1}})(x_{t_j} \cdots x_{t_p}) = \pm(x_{t_2} \cdots x_{t_j})(x_{t_j} \cdots x_{t_p}) = 0$ by what we have already shown, since $(x_{t_j})^2 = 0$ by (5.16). □

The following Corollary is an immediate consequence of Theorem 5.8.

Corollary 5.9. *Assume the situation of Theorem 5.8. Then the subalgebra of R generated by all $x_s, s \in S$ has the $\Bbbk$-basis $x_g, g \in W$.*

If R is finite-dimensional, then W is finite and $dim(R) = ord(W) dim(R_1)$. Here, R_1 is the W-homogeneous component of R of the unit element in W.

Finally we note that the results of Section 3 give some information about the Hopf algebras considered in this Section, in particular about the parabolic subalgebras of the examples in Theorem (5.8).

Proposition 5.10. *Let W be a finite group, $T \subset W$ a subset closed under conjugation, and $\chi : W \times T \to \Bbbk \setminus \{0\}$ a function satisfying* (5.1), (5.2). *Let $W' \subset W$ be a subgroup, $T' \subset T$ a subset which is stable under conjugation with elements in W', and let χ' be the restriction of χ. Define $V = V(W,T,\chi)$ and $V' = V(W',T',\chi')$. Then*

(1) $\mathfrak{B}(V)$ is a free right $\mathfrak{B}(V')$-module, and the Hilbert series $P_{\mathfrak{B}(V')}$ divides $P_{\mathfrak{B}(V)}$.

(2) As in Theorem (5.8) *assume for all $t, t' \in T$ such that the product tt' has even order $2l$ that $\chi((tt')^l, t) = \chi((tt')^l, t')$. Let $\widetilde{\mathfrak{B}}(V)$ resp. $\widetilde{\mathfrak{B}}(V')$ be the tensor algebra $T(V)$ resp. $T(V')$ modulo all the quadratic relations* (5.16) *and* (5.17), (5.20). *Then $\widetilde{\mathfrak{B}}(V)$ is a free right $\widetilde{\mathfrak{B}}(V')$-module, and $P_{\widetilde{\mathfrak{B}}(V')}$ divides $P_{\widetilde{\mathfrak{B}}(V)}$.*

Proof. Define $\varphi : V \to V'$ for all $t \in T$ by $\varphi(x_t) = x_t$, if $t \in T'$, and $\varphi(x_t) = 0$, otherwise. Then φ is a $\Bbbk W$-colinear map, and it is W'-linear, since for all $g \in W'$, $gtg^{-1} \in T'$ if and only if $t \in T'$. Hence (1) follows from Corollary 3.3.

To prove (2) let $\phi : T(V) \to T(V')$ be the map induced by φ. Since the quadratic relations define in fact primitive elements in the tensor algebras, $\widetilde{\mathfrak{B}}(V)$ and $\widetilde{\mathfrak{B}}(V')$ are braided Hopf algebras. The map ϕ induces a map on the quotients since the quadratic relations are preserved. This is clear for (5.16), and is also true for (5.17), (5.20), since all the $t_i, 1 \leq i \leq m$, are in T', if for one i both t_i and t_{i+1} are in T'. The claim now follows from Theorem 3.2 in the same way as Corollary 3.3. □

6. Examples

We consider some special cases of Theorem 5.8.

EXAMPLE 6.1. Let $W = S_n, n \geq 2$, and $T = \{(ij) \mid 1 \leq i < j \leq n\}$ the set of all transpositions. Define χ by (5.5) and let $V = V(W,T,\chi)$. Then the following relations hold in $\mathfrak{B}(V)$ for all $1 \leq i < j \leq n, 1 \leq k < l \leq n$:

$$x_{(ij)}^2 = 0. \tag{6.1}$$

$$\text{If } \{i,j\} \cap \{k,l\} = \emptyset, \text{ then } \quad x_{(ij)}x_{(kl)} + x_{(kl)}x_{(ij)} = 0. \tag{6.2}$$

$$\text{If } i < j < k, \text{ then } \quad x_{(ij)}x_{(jk)} + x_{(jk)}x_{(ik)} + x_{(ik)}x_{(ij)} = 0, \tag{6.3}$$
$$x_{(jk)}x_{(ij)} + x_{(ik)}x_{(jk)} + x_{(ij)}x_{(ik)} = 0.$$

EXAMPLE 6.2. Let $W = S_n, n \geq 2$, and $T = \{(ij) \mid 1 \leq i < j \leq n\}$ the set of all transpositions. Define χ by (5.9) and let $V = V(W,T,\chi)$. Then the following relations hold in $\mathfrak{B}(V)$ for all $1 \leq i < j \leq n, 1 \leq k < l \leq n$:

$$x_{(ij)}^2 = 0. \tag{6.4}$$

$$\text{If } \{i,j\} \cap \{k,l\} = \emptyset, then \quad x_{(ij)}x_{(kl)} - x_{(kl)}x_{(ij)} = 0. \tag{6.5}$$

$$\text{If } i < j < k, then \quad x_{(ij)}x_{(jk)} - x_{(jk)}x_{(ik)} - x_{(ik)}x_{(ij)} = 0, \tag{6.6}$$
$$x_{(jk)}x_{(ij)} - x_{(ik)}x_{(jk)} - x_{(ij)}x_{(ik)} = 0.$$

The algebras generated by all $x_{(ij)}, 1 \leq i < j \leq n$, with the quadratic relations in the examples 6.1 and 6.2 are examples of the braided Hopf algebras $\widetilde{\mathfrak{B}}(V)$ in Corollary (5.10). $\widetilde{\mathfrak{B}}(V)$ in example 6.2 is the algebra $\mathcal{E}_n$ introduced in [**FK**] to describe the cohomology ring of the flag variety. We believe that indeed the quadratic relations in the examples 6.1 and 6.2 are defining relations for $\mathfrak{B}(V)$, that is $\widetilde{\mathfrak{B}}(V) = \mathfrak{B}(V)$ in these cases. Since the algebras $\mathcal{E}_n$ are braided Hopf algebras we immediately obtain the

COROLLARY 6.3. *Let $n \geq 1$.*

(1) $\mathcal{E}_{n+1}$ is a free module over $\mathcal{E}_n$, and $P_{\mathcal{E}_n}$ divides $P_{\mathcal{E}_{n+1}}$.

(2) If $\mathcal{E}_n$ is finite-dimensional with highest non-zero homogeneous component $\mathcal{E}_n(N)$, then for all $0 \leq i \leq N$, $dim\mathcal{E}_n(i) = dim\mathcal{E}_n(N-i)$.

PROOF. (1) follows from Corollary 5.10 (2) with $W = S_{n+1}$ and $W' = \{g \in W \mid g(n+1) = n+1\} = S_n$, and (2) is a special case of Remark 2.2 4). $\square$

The 'Poincaré-duality" in 6.3 was a conjecture in [**FK**], and 6.3 (1) is also shown in [**FP**] by different methods.

EXAMPLE 6.4. Let $\mathfrak{B}(V_n)$ be the Hopf algebra of Example 6.1. If $n = 3$ or 4, then (6.1), (6.2) and (6.3) are defining relations of $B(V_n)$, and $\dim\mathfrak{B}(V_3) = 12, \dim\mathfrak{B}(V_4) = 24^2$. In both cases the integral can be described in terms of the longest element in the Coxeter groups S_3, S_4.

PROOF. 1) We first consider $R = \mathfrak{B}(V_3)$. Let $t_1 = (12), t_2 = (23), t_3 = (13)$ and $x_i = x_{t_i}, 1 \leq i \leq 3.$. Then $S = \{t_1, t_2\}, T = \{t_1, t_2, t_3\}$. By Corollary 5.9 $R^{(s_3)} = \mathbb{k}[x_1, x_2]$ has dimension $\mathrm{ord} S_3 = 6$. Thus $\dim R = 2 \dim R^{(s_3)}$ by Theorem 5.8 1). On the other hand it is not difficult to check (using the diamond lemma) that the algebra with relations (6.1), (6.2) and (6.3) in this case has dimension 12. The longest element is $s_1 s_2 s_1$. Therefore $x_1 x_2 x_1 x_3$ is a non-zero monomial of maximal length, hence an integral.

2) Now let $R = \mathfrak{B}(V_4)$. We denote $s_1 = (12), s_2 = (23), s_3 = (34), s_4 = (24), s_5 = (14), s_6 = (13)$, and $x_i = x_{s_i}, 1 \leq i \leq 6$. Hence $S = \{s_1, s_2, s_3\}, T = \{s_1, \dots, s_6\}$. The relations are $x_i^2 = 0$, for all $1 \leq i \leq 6$, and

$$x_1x_3 + x_3x_1 = 0,$$
$$x_2x_5 + x_5x_2 = 0,$$
$$x_4x_6 + x_6x_4 = 0,$$
$$x_1x_2 + x_2x_6 + x_6x_1 = 0, \quad x_2x_1 + x_1x_6 + x_6x_2 = 0,$$
$$x_1x_4 + x_4x_5 + x_5x_1 = 0, \quad x_4x_1 + x_1x_5 + x_5x_4 = 0,$$
$$x_2x_3 + x_3x_4 + x_4x_2 = 0, \quad x_3x_2 + x_2x_4 + x_4x_3 = 0,$$
$$x_3x_5 + x_5x_6 + x_6x_3 = 0, \quad x_5x_3 + x_3x_6 + x_6x_5 = 0.$$

From these relations one can see that $\mathfrak{B}(V_4) = \mathbb{k}[x_1, x_2, x_3] \cdot \mathbb{k}[x_4, x_5, x_6]$. By Corollary 5.9, $\mathbb{k}[x_1, x_2, x_3]$ has dimension $\mathrm{ord} S_4 = 24$.

Let $\sigma = (1243)$ in S_4. Then $\sigma s_1 \sigma^{-1} = s_4, \sigma s_2 \sigma^{-1} = s_5, \sigma s_3 \sigma^{-1} = s_6$. Hence the action of σ defines an algebra isomorphism $\sigma\cdot : \mathbb{k}[x_1, x_2, x_3] \to \mathbb{k}[x_4, x_5, x_6]$. Hence $\dim \mathfrak{B}(V_4) \leq 24^2$. Using the method of the proof of Lemma 5.6, it is possible to show that $\dim \mathfrak{B}(V_4) = 24^2$. On the other hand it follows from the diamond lemma that the algebra with the above realations also has dimension 24^2.

The longest element is $w_0 = s_1 s_2 s_1 s_3 s_2 s_1$. Since $\sigma w_0 \sigma^{-1} = s_4 s_5 s_4 s_6 s_5 s_4$, $x_1x_2x_1x_3x_2x_1x_4x_5x_4x_6x_5x_4$ is a non-zero monomial of maximal length, hence an integral. $\square$

According to [**FK**, (2.8)] also $\mathcal{E}_5$ is finite-dimensional. However we do not know whether $\mathcal{E}_5 = \mathfrak{B}(V_5)$.

EXAMPLE 6.5. Assume $\mathrm{char}(\mathbb{k}) \neq 2$. We consider the dihedral group D_4 in Example 5.4. Then $T = \{t_1, \dots, t_4\}$. Let χ be defined by (5.5), and $V = V(D_4, T, \chi)$. Then $R = \mathfrak{B}(V)$ is generated by the elements $x_i = x_{t_i}, 1 \leq i \leq 4$ and defining relations are the quadratic relations $x_i^2 = 0, 1 \leq i \leq 4$ and

$$x_1x_3 + x_3x_1 = 0,$$
$$x_2x_4 + x_4x_2 = 0,$$
$$x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 = 0,$$
$$x_1x_4 + x_4x_3 + x_3x_2 + x_2x_1 = 0,$$

and the relations of degree 4

$$xyxy + yxyx = 0, \quad uvuv + vuvu = 0,$$

where $x = x_1 + x_3, y = x_2 - x_4, u = x_1 - x_3, v = x_2 + x_4$.

The dimension of R is 64, and the integral can be described in terms of the longest element of the Coxeter group D_4.

PROOF. These relations are part of the relations of Theorem 5.7. Hence R is an epimorphic image of the algebra $\widetilde{R}$ generated by x, y, u, v with the relations

$$xyxy + yxyx = 0, \quad uvuv + vuvu = 0,$$

and $x^2 = 0, y^2 = 0, u^2 = 0, v^2 = 0, xu+ux = 0, xv+vx = 0, yu-uy = 0, yv+vy = 0$, which follow from the quadratic relations above. Using Lemma 5.6 it is easy to see that $\dim \widetilde{R} = 64$.

On the other hand, $\dim \Bbbk[x_1, x_2] = \operatorname{ord} D_4 = 8$, by Corollary 5.9. Since $t_1t_2t_1t_2t_1 = t_3$, and $t_1t_2t_2t_2t_1 = t_4$, the action of t_1t_2 defines an isomorphism $\Bbbk[x_1, x_2] \to \Bbbk[x_3, x_4]$. Again one can show similarly to the proof of Lemma 5.6 that $\Bbbk[x_1, x_2] \cdot \Bbbk[x_3, x_4]$ has dimension 8^2. Hence $R = \Bbbk[x_1, x_2] \cdot \Bbbk[x_3, x_4]$, and we have a monomial basis in the elements x_i.

The longest element in D_4 is $t_1t_2t_1t_2$. Therefore $x_1x_2x_1x_2x_3x_4x_3x_4$ is a non-zero integral. □

In an earlier version of this paper we showed $\dim R = 64$ in the last example in a different way. In addition to the quadratic relations, we proved the following relations: If $y_1 = x_1x_2 + x_4x_1, y_2 = x_1x_4 + x_2x_1$, then

$$y_1^2 = 0, y_2^2 = 0, y_1y_2 + y_2y_1 = 0.$$

It is not difficult to see that these relations define an algebra of dimension 64. Then Graña [**G2**, 5.2.1] noted that the braiding in the basis x, y, u, v of V reduces the computation of $\mathfrak{B}(V)$ to the case of A_2 in Lemma 5.6.

REMARK 6.6. The examples we have constructed show that the symmetric groups S_3, S_4 and S_5 and the dihedral group D_4 occur as the groups of group-like elements of finite-dimensional pointed and link-indecomposable Hopf algebras. In the other examples of link-indecomposable Hopf algebras over finite Coxeter groups we do not know whether the algebras $\mathfrak{B}(V)$ are finite-dimensional.

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