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Functional Analysis II
Assignment 13

Problem 49. Prove that a densely defined operator T on a Hilbert space $\mathcal{H}$ satisfying $\sigma(T) \subsetneq \mathbb{C}$ is necessarily closed.

Problem 50. Let P, Q be densely defined linear operators on a Hilbert space $\mathcal{H}$ such that $\mathcal{D}(PQ) \cap \mathcal{D}(QP)$ is dense in $\mathcal{H}$, and

$$[P, Q] := PQ - QP = i\mathbb{I}\,.$$

Show that at least one of the operators P and Q has to be unbounded.

Problem 51 (Momentum operator on $[0, 2\pi]$). Consider the operators A_0 and A in $L^2([0, 2\pi])$ given by

$$\begin{aligned} A_0 f &= -if'\,, \quad \mathcal{D}(A_0) = \{f \in C^1([0, 2\pi]) \,|\, f(0) = f(2\pi) = 0\}, \\ Af &= -if'\,, \quad \mathcal{D}(A) = \{f \in C^1([0, 2\pi]) \,|\, f(0) = f(2\pi)\}. \end{aligned}$$

(*i*) Prove that A_0 and A are symmetric, and that $A_0 \subset A$.

(*ii*) Find A_0^*.

(*iii*) Find $\overline{A_0}$.

(*iv*) Find A^* and prove that A is essentially self-adjoint.

(*v*) Prove that A_0 has no eigenvalues.

(*vi*) Prove that A admits an orthonormal basis of eigenvectors.

(*vii*) Find all self-adjoint extensions of A_0.