The Maxwell-Lorentz system of a rigid charge

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Abstract

We prove global existence and uniqueness of classical solutions for the Maxwell-Lorentz system of a nonrotating rigid charge distribution, i.e. the relativistic dynamics of a nonrotating extended electron, which is subject to its own electromagnetic fields and an external potential. Local existence and uniqueness is achieved via the contraction mapping principle. Suitable a-priori-bounds yield global existence. We show that in case of a negative bare mass and an attracting external potential the stationary solution is unstable. We believe that this result clarifies the origin of the so-called "runaway"-solutions, which appear when the limit to a point charge is taken, formally described by the so called Lorentz-Dirac equation for the radiating electron.

1 Introduction

In this paper, we study the following system of differential equations:

$$
\begin{aligned}
\dot{q}(t) &= \frac{\alpha p(t)}{\sqrt{1+p(t)^2}}, \\
\dot{p}(t) &= -\kappa q(t) + \int \rho\big(x'-q(t)\big)\left[E(x',t) + \frac{\alpha p(t)}{\sqrt{1+p(t)^2}} \times B(x',t)\right] dx', \\
\dot{E}(t,x) &= \mathrm{curl} B(t,x) - 4\pi \frac{\alpha p(t)}{\sqrt{1+p(t)^2}} \rho\big(x-q(t)\big), \\
\dot{B}(t,x) &= -\mathrm{curl} E(t,x), \\
\mathrm{div} E(t,x) &= 4\pi\rho\big(x-q(t)\big), \\
\mathrm{div} B(t,x) &= 0.
\end{aligned}
\tag{1}
$$

This system describes the motion of a charge distribution ρ, which is subject to the electric field E and the magnetic field B. The charge distribution is rigidly carried along with a mass point, whose space coordinate q is located in its center. The motion of q is governed by the Lorentz force law and therefore by the coupling of ρ to the electromagnetic fields. On the other hand, the dynamics of the fields is determined by Maxwell's equations, where ρ and the corresponding current $\dot{q}\rho$ appear as sources. Therefore the system describes the selfinteraction of a charge distribution with its own radiation field. Additionaly, the charge distribution is subject to an external potential, which is assumed to be quadratic* with curvature κ. In our units the absolute value of the charge center's mass m as well as the velocity of light c are equal to one. However, we allow the sign $\alpha = sgn(m)$ to be negative, i.e. α may take the values $\alpha = \pm 1$. We will refer to (1) as the Maxwell-Lorentz system of a rigid charge distribution. We do not consider the generalization of the Maxwell-Lorentz system to a spinning extended

*See however remark 2 below for more general potentials.

charge [1]. With rotational degrees of freedom included in the equations of motion our analysis becomes more complicated [2].

There is a huge literature on the subject of a charge distribution interacting with the electromagnetic fields generated by itself, especially in connection with the limit to a point charge. However, despite of its physical relevance, no mathematically rigorous results in connection with the Maxwell-Lorentz system existed. This situation however changed rather recently, and now many results are available. We mention the early work [3] on the linearized problem, and more recent [4], where the long time behavior of solutions of the Maxwell-Lorentz equations is considered, [5] for soliton like solutions, [6] for conservation laws and [2], a very recent publication on the motion of a rotating extended charge, which contains also an exhaustive list of references on the subject.

We prove global existence and uniqueness of classical solutions for the above Maxwell-Lorentz system. In addition to our existence result we prove that the stationary solution of the Maxwell-Lorentz system is unstable if $\alpha = -1$ and $\kappa > 0$, i.e. if the charge center's mass is negative and the external potential is attracting. This result is interesting for the following reason. Physicists have always been tempted to derive from the Maxwell-Lorentz system an equation of motion for a point charge. There are various reasons that make a point charge theory preferable. One reason is that an equation of motion for a point charge would be more fundamental. It would not depend on the arbitrariness of the charge distribution's size and shape. Moreover, a point charge theory would be "'naturally"' relativistic invariant, and thus it could serve as a fundamental model for a charged elementary particle like e.g. an electron. But the electromagnetic fields generated by a point charge are singular at the location of the charge. Therefore the Lorentz force is no more well-defined: In case of a point charge the fields have to be evaluated at the charge's location, i.e. just where they have their singularity. Consequently, there is no Maxwell-Lorentz electrodynamics for a point charge.

Dirac now invented a limiting procedure for (1), in which a sequence ρ_n of charge distributions is considered, each having the fixed total charge Q and a radius r_n such that $r_n \to 0$ as $n \to \infty$. This leads formally to a point charge theory, and to overcome the above problem Dirac proposed that in the limiting procedure the particle's bare mass m should depend on n, too [7]. (In fact, his proposal was the first example of a so called mass renormalization, which later became an essential tool in quantum field theory.) He showed that if $m_n \to -\infty$ as $n \to \infty$ in such a way, that the sum of m_n and the particle's electromagnetic mass (i.e. the field energy of the electromagnetic fields it generates) remains finite, a mathematically well-defined equation for a point charge emerges in the limit $n \to \infty$. This equation however, now called the Lorentz-Dirac equation, is of third order in the time derivative and allows pathological solutions, the famous "runaway"-solutions, where the charge is accelerated to the speed of light without (or even with an attracting) external force [8, 9]. Since Dirac's work there has been much speculation on the physical origin of these solutions. The discussion has often been obscured by the employment of the ad hoc renormalization procedure from which the Lorentz-Dirac equation emerges. Our result now sheds light on the origin of the "runaway"-solutions as they are due to the particle's bare mass being negative. We prove that "runaway"-like behavior, i.e. instability of the stationary solution, already occurs without taking the limit of the mass renormalization, namely as soon as m in the Maxwell-Lorentz equations is negative, while in the case of positive mass and confining potential the motion of the charge is always bounded. It was in fact one motivation for our existence theory to have a mathematical basis for achieving such a result. Recently there has been renewed interest in the physical relevance of the Lorentz-Dirac equation, whose stable solutions (living on a center manifold) can be shown to be governed by an effective second order equation [10].

The paper is organized as follows. In section 2 we fix our notation and reformulate the Maxwell-Lorentz equations as an evolution equation in a Hilbert space. In section 3 we prove existence and uniqueness of local solutions via the contraction mapping principle. By means of appropriate a-priori-bounds we establish global existence in section 4. In the final section 5 we consider the case that the charge center's bare mass m is negative and the external potential is attracting, i.e. $\alpha = -1$ and $\kappa > 0$. We prove that in this case the stationary solution of the above system is unstable (in the sense of Lyapunov).

2 Notational and technical preliminaries

Throughout this paper we assume the charge distribution ρ to be smooth and of compact support*, i.e. $\rho \in C_0^\infty(\mathbb{R}^3)$. For $q \in \mathbb{R}^3$ we write $\rho_q = \rho(\cdot - q)$.

By L^2 we denote the Hilbert space $L^2(\mathbb{R}^3)$ of real-valued, measurable, square-integrable functions on $\mathbb{R}^3$. Then $(L^2)^3$ is the real Hilbert space of 3-tuples of functions from L^2. We write $\langle f, g\rangle$ and $x \cdot y$ for the canonical scalar products of two functions $f, g \in (L^2)^3$ and two vectors $x, y \in \mathbb{R}^3$, respectively. Both the norms in L^2 and $(L^2)^3$ will be denoted by $\|\cdot\|$, whereas both the norms in $\mathbb{R}$ and $\mathbb{R}^3$ will be denoted by $|\cdot|$. We write x_j $(j = 1, 2, 3)$ for the j-th component of a vector $x \in \mathbb{R}^3$ with respect to the canonical basis vectors of $\mathbb{R}^3$.

We treat the Maxwell-Lorentz equations as an evolution equation in the following Hilbert space $\mathcal{H}$:

$$\mathcal{H} = \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus (L^2)^3 \oplus (L^2)^3.$$

$\mathcal{H}$ is equipped with the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and the corresponding norm $\|\cdot\|_{\mathcal{H}}$, where $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is defined by

$$\langle \varphi, \varphi' \rangle_{\mathcal{H}} = q \cdot q' + p \cdot p' + \langle E, E' \rangle + \langle B, B' \rangle$$

for $\varphi, \varphi' \in \mathcal{H}$, $\varphi = (q, p, E, B)$, $\varphi' = (q', p', E', B')$. An element $\varphi = (q, p, E, B)$ of $\mathcal{H}$, interpreted as a physical state vector, consists of the charge center's location q and momentum p as well as the electromagnetic fields E and B.

Having introduced the underlying state space $\mathcal{H}$, we rewrite the first four lines in (1) in the compact form

$$\dot{\varphi}(t) = A\varphi(t) + J(\varphi(t)). \tag{2}$$

Here, $\varphi(\cdot) : t \mapsto \varphi(t) = (q(t), p(t), E(t), B(t))$ is a $\mathcal{H}$-valued map on $\mathbb{R}$ that represents the evolution of the physical state in time. The linear operator A and the nonlinear operator J are formally given by

$$A\varphi = (0, 0, \mathrm{curl} B, -\mathrm{curl} E) \tag{3}$$

and

$$J(\varphi) = \left(\frac{\alpha p}{\sqrt{1+p^2}}, -\kappa q + \int \rho_q(x) \left[E(x) + \frac{\alpha p}{\sqrt{1+p^2}} \times B(x) \right] dx, \frac{-4\pi\alpha p}{\sqrt{1+p^2}} \rho_q , 0 \right) \tag{4}$$

for $\varphi = (q, p, E, B)$.

We first analyze the operator A. Let

$$W^{\mathrm{curl}} = \left\{ E \in (L^2)^3 \,\middle|\, \mathrm{curl} E \in (L^2)^3 \right\},$$

where the partial derivatives of the curl are meant in the sense of distributions. We define the domain of A by

$$\mathcal{D}(A) = \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus W^{\mathrm{curl}} \oplus W^{\mathrm{curl}}$$

and for each $n \in \mathbb{N}$, $n \geq 2$, the domain of the n-th power A^n of A by

$$\mathcal{D}(A^n) = \left\{ \varphi \in \mathcal{D}(A) \,\middle|\, A^j \varphi \in \mathcal{D}(A), \quad j = 1, \ldots, n-1 \right\}.$$

For our purposes it will be essential that A generates a strongly continuous group of unitary operators on $\mathcal{H}$. This is established in the following lemma.

Lemma 1 *There is a strongly continuous family $\{U_t\}_{t \in \mathbb{R}}$ of unitary operators $U_t : \mathcal{H} \to \mathcal{H}$ such that*

$$\lim_{h \to 0} \frac{U_h \varphi - \varphi}{h} = A\varphi, \quad U_t \varphi \in \mathcal{D}(A) \quad \text{and} \quad AU_t \varphi = U_t A \varphi$$

for each $\varphi \in \mathcal{D}(A)$ and $t \in \mathbb{R}$.

*See however remark 2 below for more general ρ.

Proof. A standard functional analytical argument, using Fourier transform and self-adjointness, see e.g. [11]. □

We now come to analyze the operator J in (2), formally given by (4). The following lemma states that we may choose $\mathcal{D}(A)$ as domain of J and provides us with estimates on J.

Lemma 2 *The nonlinear operator J, cf. (4), has the following properties:*

(i) *J maps $\mathcal{D}(A)$ into $\bigcap_{n=0}^{\infty} \mathcal{D}(A^n)$.*

(ii) *For each $n \in \mathbb{N}$, $n \geq 0$, it holds*

$$\begin{aligned} A^{2n+1} J(\varphi) &= \left(0, 0, 0, (-1)^n \frac{4\pi\alpha}{\sqrt{1+p^2}} curl^{2n+1}(p\,\rho_q) \right), \\ A^{2n+2} J(\varphi) &= \left(0, 0, (-1)^n \frac{4\pi\alpha}{\sqrt{1+p^2}} curl^{2n+2}(p\,\rho_q), 0 \right). \end{aligned} \tag{5}$$

(iii) *There are constants $K_n > 0$ such that*

$$\begin{aligned} \left\| A^n J(\varphi) \right\|_{\mathcal{H}} &\leq K_n, \\ \left\| A^n \left(J(\varphi) - J(\widetilde{\varphi}) \right) \right\|_{\mathcal{H}} &\leq K_n \left(1 + \|\widetilde{\varphi}\|_{\mathcal{H}} \right) \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}} \end{aligned}$$

for all $\varphi, \widetilde{\varphi} \in \mathcal{D}(A)$, $n \in \mathbb{N}$, $n \geq 0$.

Proof. A straightforward calculation. For (iii) only standard inequalities are needed. Note that the second inequality immediately follows from the first one. □

3 Local existence and uniqueness

We shall use now the group $\{U_t\}_{t \in \mathbb{R}}$, cf. lemma 1, to prove via the contraction mapping principle the existence and uniqueness of local solutions of the integral equation

$$\varphi(t) = U_t \varphi_0 + \int_0^t U_{t-s} J\big(\varphi(s)\big)\, ds \tag{6}$$

and to show that a unique solution of (6) is a unique solution of (2) with initial value $\varphi(0) = \varphi_0$. Doing this, we specify the set of initial values φ_0 as well as the exact notion of uniqueness. Our existence and uniqueness result is more or less standard from the viewpoint of semigroup theory. Therefore we keep the proofs short and refer to [12], section X.13, where a very similar technique is presented in connection with nonlinear wave equations.

Proposition 1 (Local existence and uniqueness) *For each $\varphi_0 \in \mathcal{D}(A^n)$, $n \geq 1$, there is a $T > 0$ and a map $\varphi(\cdot) : [0, T) \to \mathcal{D}(A^n)$ with the following properties:*

(i) *$\varphi(\cdot)$ is a n-times strongly continuously differentiable solution of (2) with initial value $\varphi(0) = \varphi_0$ and $d^j/dt^j \varphi(t) \in \mathcal{D}(A^{n-j})$ for all $t \in [0, T)$ and $j = 0, \dots, n$.*

(ii) *If $\widetilde{\varphi}(\cdot) : [0, \widetilde{T}) \to \mathcal{D}(A)$ is a strongly continuously differentiable solution of (2) with initial value $\widetilde{\varphi}(0) = \varphi_0$, then $\widetilde{\varphi}(t) = \varphi(t)$ for $t \in [0, \widetilde{T}) \cap [0, T)$.*

Proof. (i) For given $n \in \mathbb{N}, n \geq 1$, and $T > 0$ we define

$$X_{T,n} := \Big\{ \varphi(\cdot) : [0,T) \to \mathcal{D}(A^n) \;\Big|\; t \mapsto A^j \varphi(t) \text{ strongly continuous in } [0,T),\; j = 0,\dots,n,$$
$$\|\varphi(\cdot)\|_X := \sum_{j=0}^{n} \sup_{t \in [0,T)} \left\| A^j \varphi(t) \right\|_{\mathcal{H}} < \infty \Big\}.$$

As far as we do not refer explicitly to the dependence on n or T, we write X for $X_{T,n}$. Since A is a closed operator, X equipped with the norm $\|\cdot\|_X$ is a Banach space. The map $U_{(\cdot)}\varphi_0 : t \mapsto U_t \varphi_0$ is an element of X. Define

$$M_{T,n,\varphi_0} := \left\{ \varphi(\cdot) \in X_{T,n} \;\middle|\; \varphi(0) = \varphi_0,\; \left\| \varphi(\cdot) - U_{(\cdot)}\varphi_0 \right\|_{X_{T,n}} \leq 1 \right\},$$

which is a closed subset of $X_{T,n}$. We shall show that the map $S_{\varphi_0} : \varphi(\cdot) \mapsto \big(S_{\varphi_0}\varphi\big)(\cdot)$, where

$$\big(S_{\varphi_0}\varphi\big)(t) = U_t \varphi_0 + \int_0^t U_{t-s} J\big(\varphi(s)\big)\, ds, \quad t \in [0,T), \tag{7}$$

is a contracting self-mapping on M_{T,n,φ_0} provided T is chosen sufficiently small.

Before doing so, we observe that for each $\varphi(\cdot) \in M_{T,n,\varphi_0}$ and for all $t \in [0,T)$ we have $\|\varphi(t)\|_{\mathcal{H}} \leq \|U_t \varphi_0\|_{\mathcal{H}} + 1 = \|\varphi_0\|_{\mathcal{H}} + 1$. Hence it follows from lemma 2 that we simply have

$$\left\| A^j J\big(\varphi(t)\big) \right\|_{\mathcal{H}} \leq C_{n,\varphi_0}, \qquad \left\| A^j \Big(J\big(\varphi(t)\big) - J\big(\widetilde{\varphi}(t)\big) \Big) \right\|_{\mathcal{H}} \leq C_{n,\varphi_0} \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}} \tag{8}$$

for all $\varphi(\cdot), \widetilde{\varphi}(\cdot) \in M_{T,n,\varphi_0}$, $t \in [0,T)$ and $j = 0,\dots,n$, where

$$C_{n,\varphi_0} := \big(2 + \|\varphi_0\|_{\mathcal{H}}\big) \max_{j=0,\dots,n} K_j. \tag{9}$$

For the sake of brevity, we will often write M for M_{T,n,φ_0}, C for C_{n,φ_0} and S for S_{φ_0}.

We first assert that the map S in (7) yields for each $\varphi(\cdot) \in M$ a $\mathcal{D}(A^n)$-valued function $t \mapsto \big(S\varphi\big)(t)$ in $[0,T)$, i.e. we show that the integral in (7) is $\mathcal{D}(A^n)$-valued. Since by (8)

$$\begin{aligned}
&\left\| A^j U_{t-(s+h)} J\big(\varphi(s+h)\big) - A^j U_{t-s} J\big(\varphi(s)\big) \right\|_{\mathcal{H}} \\
&\qquad = \left\| A^j J\big(\varphi(s+h)\big) - A^j J\big(\varphi(s)\big) \right\|_{\mathcal{H}} + \left\| U_{-h} A^j J\big(\varphi(s)\big) - A^j J\big(\varphi(s)\big) \right\|_{\mathcal{H}} \\
&\qquad \leq C \|\varphi(s+h) - \varphi(s)\|_{\mathcal{H}} + \left\| U_{-h} A^j J\big(\varphi(s)\big) - A^j J\big(\varphi(s)\big) \right\|_{\mathcal{H}},
\end{aligned} \tag{10}$$

the mappings $s \mapsto A^j U_{t-s} J\big(\varphi(s)\big)$ are continuous in $[0,t)$ for all $t \in [0,T)$ and $j = 0,\dots,n$. Therefore the integrals

$$\sigma^{(j)}(t) := \int_0^t A^j U_{t-s} J\big(\varphi(s)\big)\, ds, \quad \sigma(t) := \sigma^{(0)}(t), \tag{11}$$

are well-defined as $\mathcal{H}$-valued Riemann integrals. Define

$$\sigma_N^{(j)}(t) := \frac{1}{N} \sum_{k=1}^{N} U_{t-\frac{k}{N}t} A^j J\left(\varphi\Big(\frac{k}{N}t\Big)\right), \quad \sigma_N(t) := \sigma_N^{(0)}(t),$$

as Riemann sum of the integrals in (11). We have $\sigma_N(t) \in \mathcal{D}(A^n)$ and

$$\lim_{N\to\infty} \sigma_N(t) = \sigma(t), \quad \lim_{N\to\infty} A^j \sigma_N(t) = \sigma^{(j)}(t)$$

with respect to the norm $\|\cdot\|_{\mathcal{H}}$ for all $t \in [0,T)$ and $j = 0,\dots,n$. Since A is closed, this implies $\sigma(t) \in \mathcal{D}(A^n)$ and $A^j \sigma(t) = \sigma^{(j)}(t)$, i.e.

$$A^j \int_0^t U_{t-s} J\big(\varphi(s)\big)\, ds = \int_0^t U_{t-s} A^j J\big(\varphi(s)\big)\, ds. \tag{12}$$

Thus, $t \mapsto (S\varphi)(t) = U_t\varphi_0 + \sigma(t)$ is a $\mathcal{D}(A^n)$-valued map in $[0,T)$.

Next, we assert the continuity of the mappings $t \mapsto A^j(S\varphi)(t)$ in $[0,T)$ for $j = 0,\ldots,n$, which holds if the mappings $t \mapsto A^j\sigma(t)$ are continuous. Using (8) and (12), we have

$$\begin{aligned} &\left\|A^j\sigma(t+h) - A^j\sigma(t)\right\|_{\mathcal{H}} \\ &\quad \le \int_0^t \left\|U_h A^j J\big(\varphi(s)\big) - A^j J\big(\varphi(s)\big)\right\|_{\mathcal{H}} ds + hC. \end{aligned}$$

Here, the integrand converges to zero as $h \to 0$ for each $s \in [0,T)$ and is bounded uniformly in h according to

$$\left\|U_h A^j J\big(\varphi(s)\big) - A^j J\big(\varphi(s)\big)\right\|_{\mathcal{H}} \le 2\left\|A^j J\big(\varphi(s)\big)\right\|_{\mathcal{H}} \le 2C.$$

Therefore, by the dominated convergence theorem, the integral converges to zero as $h \to 0$, and so does $\left\|A^j\sigma(t+h) - A^j\sigma(t)\right\|_{\mathcal{H}}$. This implies the continuity of $t \mapsto A^j\sigma(t)$ in $[0,T)$ for each $j \in \mathbb{N}$ with $0 \le j \le n$.

Finally we observe by use of (8) that

$$\left\|(S\varphi)(\cdot) - U_{(\cdot)}\varphi_0\right\|_X = \sum_{j=0}^n \sup_{t\in[0,T)} \left\|\int_0^t U_{t-s}A^j J\big(\varphi(s)\big)\right\|_{\mathcal{H}} \le TC(n+1) \tag{13}$$

an similarly

$$\left\|(S\varphi)(\cdot) - (S\widetilde{\varphi})(\cdot)\right\|_X \le TC(n+1)\,\|\varphi(\cdot) - \widetilde{\varphi}(\cdot)\|_X \tag{14}$$

for all $\varphi(\cdot), \widetilde{\varphi}(\cdot) \in M$. Hence, if $T \le 1/C(n+1)$, (13) completes showing that S is a self-mapping on M and (14) asserts S to be a contraction.

By the Banach fixed point theorem, we thus obtain that for each $\varphi_0 \in \mathcal{D}(A^n)$, $n \ge 1$, there is a $T = T_{n,\varphi_0} > 0$, given by

$$T_{n,\varphi_0} = \frac{1}{2C_{n,\varphi_0}(n+1)} \tag{15}$$

with C_{n,φ_0} cf. (9), such that the map S_{φ_0} has exactly one fixed point $\varphi(\cdot)$ in M_{T,n,φ_0}.

For the proof of the differentiability of $\varphi(\cdot)$ in $[0,T)$ we need the following two lemmas.

Lemma 3 *For each $j = 0,\ldots,n-1$ the map $t \mapsto A^j\varphi(t)$ is strongly continuously differentiable in $[0,T)$ and*

$$\frac{d}{dt}A^j\varphi(t) = A^{j+1}\varphi(t) + A^j J\big(\varphi(t)\big). \tag{16}$$

Proof. We consider

$$\begin{aligned} \frac{A^j\varphi(t+h) - A^j\varphi(t)}{h} &= \frac{(U_h - I)}{h}A^j U_t\varphi_0 + \frac{1}{h}\int_t^{t+h} U_{t+h-s}A^j J\big(\varphi(s)\big)\,ds \\ &\quad + \int_0^t \frac{(U_h - I)}{h}A^j U_{t-s}J\big(\varphi(s)\big)\,ds. \end{aligned} \tag{17}$$

Since $\varphi_0 \in \mathcal{D}(A^n)$ the first term converges to $A^{j+1}U_t\varphi_0$ and since the integrand of the second term is continous the second term converges to $A^j J\big(\varphi(t)\big)$ as h tends to zero. The integrand of the third term converges to $A^{j+1}U_{t-s}J\big(\varphi(s)\big)$ for each $s \in [0,t)$ and according to

$$\left\|\frac{(U_h - I)}{h}A^j U_{t-s}J\big(\varphi(s)\big)\right\|_{\mathcal{H}} = \left\|\frac{1}{h}\int_0^h U_{h'}A^{j+1}J\big(\varphi(s)\big)\,dh'\right\|_{\mathcal{H}} \le C$$

it is bounded uniformly in h. Thus, by the dominated convergence theorem, the third term converges to

$$\lim_{h\to 0}\int_0^t \frac{(U_h - I)}{h}A^j U_{t-s}J\big(\varphi(s)\big)\,ds = \int_0^t A^{j+1}U_{t-s}J\big(\varphi(s)\big)\,ds = A^{j+1}\int_0^t U_{t-s}J\big(\varphi(s)\big)\,ds.$$

To sum up, $t \mapsto A^j\varphi(t)$ is differentiable in $[0,T)$ and, taking the terms together, $d/dtA^j\varphi(t)$ satisfies (16). Note that the right-hand side of this equation is continuous because of $\varphi(\cdot) \in M$, $j \leq n-1$ and lemma 2, so that $t \mapsto A^j\varphi(t)$ is even strongly continuously differentiable. $\square$

Lemma 4 *Assume that the map $\varphi(\cdot) : [0,T) \to \mathcal{D}(A^n)$ is j times strongly continuously differentiable for some $j \in \mathbb{N}$, where $d^k/dt^k\varphi(t) \in \mathcal{D}(A)$ for all $t \in [0,T)$ and $k \leq j$. Then $t \mapsto A^l J\big(\varphi(t)\big)$ is j times strongly continuously differentiable,*

$$\frac{d^k}{dt^k} J\big(\varphi(t)\big) \in \bigcap_{l=0}^{\infty} \mathcal{D}(A^l) \quad \textit{and} \quad A^l \frac{d^k}{dt^k} J\big(\varphi(t)\big) = \frac{d^k}{dt^k} A^l J\big(\varphi(t)\big) \tag{18}$$

for all $t \in [0,T)$ and $k, l \in \mathbb{N}$ with $k \leq j$.

Proof. Let $\varphi(\cdot) = \big(q(\cdot), p(\cdot), E(\cdot), B(\cdot)\big)$ be as supposed in the lemma. Then the component functions $q(\cdot)$, $p(\cdot)$, $E(\cdot)$ and $B(\cdot)$ of $\varphi(\cdot)$ are j times continuously differentiable in $[0,T)$ and $E^{(k)}(t), B^{(k)}(t) \in \left(L^2\right)^3$ for all $t \in [0,T)$ and $k \leq j$. The lemma is an immediate consequence of $\rho \in C_0^\infty(\mathbb{R}^3)$ and the definition of A and J, cf. (4) and (5). The integrand in (4), as a $\left(L^2\right)^3$-valued function of t in $[0,T)$, is j times strongly continuously differentiable, and all derivatives with respect to t are again $\left(L^2\right)^3$-valued. Therefore also the integral, as a function of t in $[0,T)$, is j times continuously differentiable, integration over $\mathbb{R}^3$ and differentiation with respect to t being exchangeable. Moreover, for each $l \in \mathbb{N}$, the $C_0^\infty(\mathbb{R}^3)^3$-valued maps

$$t \mapsto \frac{p(t)}{\sqrt{1+p(t)^2}} \operatorname{curl}^l \Big(p(t)\rho(\,\cdot - q(t))\Big)$$

are j times continuously differentiable in $[0,T)$, the derivatives are again $C_0^\infty(\mathbb{R}^3)^3$-valued maps in $[0,T)$, and differentiation with respect to t commutes with the curl. $\square$

Proof of Proposition 1 (continued). It is now straightforward to show that the fixed point $\varphi(\cdot)$ of S_{φ_0} in M_{T,n,φ_0} has the following property: $\varphi(\cdot)$ is n times strongly continuously differentiable, $d^j/dt^j\varphi(t) \in \mathcal{D}(A^{n-j})$ and

$$\frac{d^j}{dt^j}\varphi(t) = A^j\varphi(t) + \sum_{\nu=0}^{j-1} A^{j-1-\nu} \frac{d^\nu}{dt^\nu} J\big(\varphi(t)\big)$$

for all $t \in [0,T)$ and $j = 0, \ldots, n$. We prove this statement by induction with respect to j. The case $j = 0$ is obvious. Now choose some $j \in \mathbb{N}$ with $j \leq n-1$ and assume that the statement is valid vor all $k \in \mathbb{N}$ with $k \leq j$. Then $\varphi(\cdot)$ is j times continuously differentiable and $d^k/dt^k\varphi(t) \in \mathcal{D}(A^{n-k})$ for all $t \in [0,T)$ and $k \leq j$. Therefore, by lemma 4, the maps

$$t \mapsto A^{j-1-\nu} \frac{d^\nu}{dt^\nu} J\big(\varphi(t)\big) = \frac{d^\nu}{dt^\nu} A^{j-1-\nu} J\big(\varphi(t)\big)$$

are continuously differentiable for each $\nu \leq j-1$ and

$$\frac{d}{dt} A^{j-1-\nu} \frac{d^\nu}{dt^\nu} J\big(\varphi(t)\big) = A^{j-1-\nu} \frac{d^{\nu+1}}{dt^{\nu+1}} J\big(\varphi(t)\big) \in \mathcal{D}(A^{n-(j+1)})$$

for all $t \in [0,T)$. By lemma 3, also $t \mapsto A^j\varphi(t)$ is continuously differentiable and (16) holds, where the right-hand side is in $\mathcal{D}(A^{n-(j+1)})$ for each $t \in [0,T)$. We thus have obtained that $d^j/dt^j\varphi(\cdot)$ is continuously differentiable in $[0,T)$ and

$$\begin{aligned}
\frac{d^{j+1}}{dt^{j+1}}\varphi(t) &= A^{j+1}\varphi(t) + A^j J\big(\varphi(t)\big) + \sum_{\nu=0}^{j-1} A^{j-1-\nu} \frac{d^{\nu+1}}{dt^{\nu+1}} J\big(\varphi(t)\big) \\
&= A^{j+1}\varphi(t) + \sum_{\nu=0}^{j} A^{j-\nu} \frac{d^\nu}{dt^\nu} J\big(\varphi(t)\big) \in \mathcal{D}(A^{n-(j+1)}),
\end{aligned}$$

i.e. the above statement holds for $j+1$. This yields part (i) of the proposition.

(ii) Let $\varphi(\cdot)$ be the unique fixed point of S_{φ_0} in M_{T,n,φ_0}, where $T = T_{n,\varphi_0}$. Then for each T_0 with $0 < T_0 \leq T$ the restriction $\varphi(\cdot)|_{[0,T_0)}$ of $\varphi(\cdot)$ to $[0, T_0)$ is unique fixed point of S_{φ_0} in $M_{T_0,1,\varphi_0}$, since by (15) and (9) T_{n,φ_0} depends monotonously decreasing on n.

Now let $\widetilde{\varphi}(\cdot) : [0, \widetilde{T}) \to \mathcal{D}(A)$ be a strongly continuously differentiable solution of (2) with initial value $\widetilde{\varphi}(0) = \varphi_0$. By the differential equation (2), $t \mapsto A\widetilde{\varphi}(t)$ is strongly continous in $[0, \widetilde{T})$. Therefore $\widetilde{\varphi}(\cdot)|_{[0,T_0)} \in M_{T_0,1,\varphi_0}$ for some T_0 with $0 < T_0 \leq \min(T, \widetilde{T})$. Since $\widetilde{\varphi}(\cdot)$ satisfies (2), $\widetilde{\varphi}(\cdot)|_{[0,T_0)}$ is a fixed point of S_{φ_0} in $M_{T_0,1,\varphi_0}$. This implies $\widetilde{\varphi}(t) = \varphi(t)$ for all t with $0 \leq t < T_0$. Let T_1 be the supremum of such T_0 and suppose $T_1 < \min(T, \widetilde{T})$. By the definition of $M_{T,1,\varphi_0}$ we must have $\widetilde{\varphi}(T_1) = \varphi(T_1)$. We can thus repeat the above argument, with initial value $\varphi(T_1)$ instead of φ_0, to show that $\widetilde{\varphi}(t) = \varphi(t)$ in a small interval of $T_1 \leq t < T_2$. This contradicts the maximality of T_1 and hence $\widetilde{\varphi}(t) = \varphi(t)$ holds for all t with $0 \leq t < \min(T, \widetilde{T})$, i.e. for $t \in [0, T) \cap [0, \widetilde{T})$. □

4 Regularity and global existence

So far we did not take into account the last two lines in (1). From now on however we will always impose on the initial value $\varphi_0 = \big(q_0, p_0, E_0, B_0\big)$ the constraint

$$\mathrm{div} E_0 = 4\pi\rho(\cdot - q_0), \qquad \mathrm{div} B_0 = 0, \tag{19}$$

where the partial derivatives of the divergence are meant in the sense of distributions. The following proposition states that this constraint implies for a solution of (2) with initial value φ_0 the validity of the last two lines in (1) on the whole interval of existence as well as certain regularity properties.

Proposition 2 *Suppose $\varphi_0 = \big(q_0, p_0, E_0, B_0\big) \in \mathcal{D}(A^n)$, $n \geq 1$. Suppose further that q_0, E_0 and B_0 satisfy (19). Then the solution $\varphi(\cdot) = \big(q(\cdot), p(\cdot), E(\cdot), B(\cdot)\big) : [0, T) \to \mathcal{D}(A^n)$ of (2) with initial value $\varphi(0) = \varphi_0$, cf. proposition 1, has the following properties:*

(i) *For all $t \in [0, T)$, the functions $q(\cdot)$, $E(\cdot)$ and $B(\cdot)$ satisfy*

$$\mathrm{div} E(t) = 4\pi\rho\big(\cdot - q(t)\big), \qquad \mathrm{div} B(t) = 0, \tag{20}$$

where the partial derivatives of the divergence are meant in the sense of distributions.

(ii) *If $n \geq 2\lambda + 2$ for some $\lambda \in \mathbb{N}$, then, for every $t \in [0, T)$, each of $E(t)$ and $B(t)$ is equal almost everywhere in $\mathbb{R}^3$ to a function in $C^{2\lambda}(\mathbb{R}^3)^3$. If $\varphi_0 \in \bigcap_{n=0}^{\infty} \mathcal{D}(A^n)$, then for every $t \in [0, T)$ each of $E(t)$ and $B(t)$ is equal almost everywhere in $\mathbb{R}^3$ to a function in $C^\infty(\mathbb{R}^3)^3$.*

Proof. (i) Since $t \mapsto B(t)$ is continuously differentiable in $[0, T)$, we may write

$$\mathrm{div} B(t) = \mathrm{div}\left(B_0 + \int_0^t \dot{B}(s) ds\right) = -\mathrm{div} \int_0^t \mathrm{curl} E(s) ds$$

for each $t \in [0, T)$. Now choose some arbitrary test function ϕ. We have

$$-\int \left(\mathrm{div} \int_0^t \mathrm{curl} E(x, s) ds\right) \phi(x) dx = \int_0^t \left(\int \mathrm{curl} E(x, s) \cdot \mathrm{grad}\phi(x) dx\right) ds = 0, \tag{21}$$

using Fubini's theorem, which may be applied since the integral

$$\int_0^t \left(\int \big|\mathrm{curl} E(x, s) \cdot \mathrm{grad}\phi(x)\big| dx\right) ds \leq ||\mathrm{grad}\phi|| \int_0^t ||\mathrm{curl} E(s)||\, ds$$

exists. We further have

$$
\begin{aligned}
\operatorname{div} E(t) &= \operatorname{div}\left(E_0 + \int_0^t \dot{E}(s)ds\right) \\
&= 4\pi\rho(\cdot - q_0) + \operatorname{div}\int_0^t \operatorname{curl} B(s)ds - 4\pi \operatorname{div}\int_0^t \frac{p(s)}{\sqrt{1+p(s)^2}}\rho\big(\,\cdot - q(s)\big)\,ds.
\end{aligned}
$$

Here, the second term vanishes by an argument completely analogous to (21). In the third term we are allowed to commute the divergence with the integration, since $p(\cdot)$ and $q(\cdot)$ are continuous and $\rho \in C_0^\infty(\mathbb{R}^3)$. This yields

$$
\begin{aligned}
\operatorname{div} E(t) &= 4\pi\rho(\cdot - q_0) - 4\pi\int_0^t \left(\frac{p(s)}{\sqrt{1+p(s)^2}} \cdot \operatorname{grad}\rho\big(\,\cdot - q(s)\big)\right) ds \\
&= 4\pi\rho(\cdot - q_0) - 4\pi\int_0^t \left(\frac{d}{ds}\rho\big(\,\cdot - q(s)\big)\right) ds = 4\pi\rho\big(\,\cdot - q(t)\big).
\end{aligned}
$$

(ii) Assume $n \geq 2\lambda + 2$ for some $\lambda \in \mathbb{N}$. Fix some arbitrary $t \in [0,T)$. Since $\varphi(t) \in \mathcal{D}(A^n)$, we have $\operatorname{curl}^{2l} E(t), \operatorname{curl}^{2l} B(t) \in (L^2)^3$ for all $l \in \mathbb{N}$ with $0 \leq l \leq \lambda + 1$, where the partial derivatives of the curl are meant in the sense of distributions. For $l \geq 1$ we have

$$
\begin{aligned}
\operatorname{curl}^{2l} E(t) &= \left(\operatorname{curl}^2\right)^{l-1} \operatorname{curl}^2 E(t) = \left(\operatorname{graddiv} - \Delta\right)^{l-1} \operatorname{curl}^2 E(t) \\
&= \sum_{\nu=0}^{l-1} \binom{l-1}{\nu} (-\Delta)^{l-1-\nu} (\operatorname{graddiv})^{\nu} \operatorname{curl}^2 E(t) \qquad (22) \\
&= (-\Delta)^{l-1} \operatorname{curl}^2 E(t) = (-1)^{l-1}\Delta^{l-1} \operatorname{graddiv} E(t) + (-1)^l \Delta^l E(t)
\end{aligned}
$$

in the distributional sense, where Δ denotes the Laplacian. The same calculation as (22) also holds for $B(t)$. By $\operatorname{div} E(t) = 4\pi\rho\big(\,\cdot - q(t)\big)$ and $\operatorname{div} B(t) = 0$ according to (i) we thus obtain

$$
\begin{aligned}
\operatorname{curl}^{2l} E(t) &= 4\pi(-1)^{l-1}\Delta^{l-1} \operatorname{grad}\rho\big(\,\cdot - q(t)\big) + (-1)^l \Delta^l E(t), \\
\operatorname{curl}^{2l} B(t) &= (-1)^l \Delta^l B(t).
\end{aligned}
$$

Thus, by $\rho \in C_0^\infty(\mathbb{R}^3)$, we conclude $\Delta^l E(t), \Delta^l B(t) \in (L^2)^3$ for $0 \leq l \leq \lambda + 1$. For the Fourier transforms $\widehat{\Delta^l E(t)}$ of $\Delta^l E(t)$ and $\widehat{E(t)}$ of $E(t)$ in the sense of tempered distributions we have $\widehat{\Delta^l E(t)} = (-1)^l |k|^{2l} \widehat{E(t)}$, and by $\Delta^l E(t) \in (L^2)^3$, the Plancherel theorem and $|k|^{2l} \geq k_j^{2l}$ this yields $k_j^{2l}\widehat{E(t)} \in (L^2)^3$ for $0 \leq l \leq \lambda + 1$ and $j = 1,2,3$. By Schwartz' inequality, $k_j^{2l} k_j^{2l'} \big|\widehat{E(t)}\big|^2 = \big|k_j^{l+l'}\widehat{E(t)}\big|^2 \in L^1(\mathbb{R}^3)$ and therefore $k_j^{l+l'}\widehat{E(t)} \in (L^2)^3$ for $0 \leq l, l' \leq \lambda + 1$, i.e. $k_j^l \widehat{E(t)} \in (L^2)^3$ for all $l \in \mathbb{N}$ with $0 \leq l \leq 2\lambda + 2$ and $j = 1,2,3$. Applying Fourier transform and the Plancherel theorem again, this implies $\partial^l/\partial x_j^l E(t) \in (L^2)^3$ for $0 \leq l \leq 2\lambda + 2$ and $j = 1,2,3$, with the partial derivatives $\partial^l/\partial x_j^l$ in the sense of distributions. The same holds for $B(t)$. By this property of $E(t)$ and $B(t)$, Sobolev's lemma [13] assures that each of $E(t)$ and $B(t)$ are equal almost everywhere in $\mathbb{R}^3$ to a function in $C^{2\lambda}(\mathbb{R}^3)^3$. □

We now come to formulate and prove our central theorem, the existence and uniqueness of global classical solutions of the Maxwell-Lorentz system. It remains to show that the local solutions of (2), cf. propositions 1 and 2, may be continued to the whole interval $[0,\infty)$, i.e. that they exist globally. The natural idea would be to use conservation of energy*. We shall not do that, also because we have included the case of negative mass, for which the energy is no more bounded below.

*See remark 3 below.

Theorem 1 *Let $\varphi_0 = (q_0, p_0, E_0, B_0) \in \mathcal{D}(A^n)$, $n \geq 1$. Suppose further that q_0, E_0 and B_0 satisfy* (19)*. Then the following is true:*

(i) *(Global existence) There is a function $\varphi(\cdot) = (q(\cdot), p(\cdot), E(\cdot), B(\cdot)) : \mathbb{R} \to \mathcal{D}(A^n)$ with the following properties: $\varphi(\cdot)$ is a n times strongly continuously differentiable solution of* (2) *with initial value $\varphi(0) = \varphi_0$ and $\varphi^{(j)}(t) \in \mathcal{D}(A^{n-j})$ for all $t \in \mathbb{R}$ and $j = 0, \ldots, n$. The functions $q(\cdot)$, $E(\cdot)$ and $B(\cdot)$ satisfy* (20) *for all $t \in \mathbb{R}$. Thus, $\varphi(\cdot)$ is a global solution of the Maxwell-Lorentz system, with curl and divergence in the distributional sense.*

(ii) *(Uniqueness) Suppose $\Lambda \subset \mathbb{R}$ is an interval and $T_0 \in \Lambda$. If $\widetilde{\varphi}(\cdot) : \Lambda \to \mathcal{D}(A)$ is a continuously differentiable solution of (2) with initial value $\widetilde{\varphi}(T_0) = \varphi(T_0)$, then $\widetilde{\varphi}(t) = \varphi(t)$ for all $t \in \Lambda$.*

(iii) *(Regularity) If $n \geq 2\lambda + 2$ for some $\lambda \in \mathbb{N}$, then for every $t \in \mathbb{R}$ each of $E(t)$ and $B(t)$ is equal almost everywhere in $\mathbb{R}^3$ to a function in $C^{2\lambda}(\mathbb{R}^3)^3$. If $\varphi_0 \in \bigcap_{n=0}^{\infty} \mathcal{D}(A^n)$, then for every $t \in \mathbb{R}$ each of $E(t)$ and $B(t)$ is equal almost everywhere in $\mathbb{R}^3$ to a function in $C^\infty(\mathbb{R}^3)^3$.*

Proof. (i) According to proposition 1 there is a $T > 0$ and a function $\varphi(\cdot) : [0, T) \to \mathcal{D}(A^n)$ such that $\varphi(\cdot)$ is a n times strongly continuously differentiable solution of (2) in $[0, T)$ with initial value $\varphi(0) = \varphi_0$. Let $\overline{T}$ be the sup of all $T > 0$ with this property and suppose $\overline{T} < \infty$. In order to derive a contradiction, observe

$$\begin{aligned} \|\varphi(t)\|_{\mathcal{H}} &= \left\| U_t \varphi_0 + \int_0^t U_{t-s} J(\varphi(s)) ds \right\|_{\mathcal{H}} \leq \|\varphi_0\|_{\mathcal{H}} + \int_0^t \|J(\varphi(s))\|_{\mathcal{H}} \, ds \\ &\leq \|\varphi_0\|_{\mathcal{H}} + K_1 \int_0^t \|\varphi(s)\|_{\mathcal{H}} \, ds \end{aligned}$$

for all $t \in [0, \overline{T})$, using lemma 2. Gronwall's lemma yields

$$\|\varphi(t)\|_{\mathcal{H}} \leq \|\varphi_0\|_{\mathcal{H}} \, e^{K_1 t} \leq \|\varphi_0\|_{\mathcal{H}} \, e^{K_1 \overline{T}} < \infty, \tag{23}$$

i.e. the map $t \mapsto \|\varphi(t)\|_{\mathcal{H}}$ is bounded in $[0, \overline{T})$. Since by (15) and (9) the length of the interval over which one can use the contraction mapping principle only depends on this bound, the solution can be continued beyond $\overline{T}$. This violates the maximality of $\overline{T}$ and hence $\overline{T} = \infty$. Note that nowhere in our proof we make use of the sign of t. Our existence result and the properties of the solutions thus are valid for $(-\infty, 0]$ as well as for $[0, \infty)$, i.e. for whole $\mathbb{R}$.

(ii) For $\Lambda = [0, \widetilde{T})$ and $T_0 = 0$ what has to be shown follows immediately from part (ii) of proposition 1, whose proof also includes the case $\widetilde{T} = \infty$. For general intervals Λ the proof may be traced back easily to this special case by translation or inversion of the interval.

(iii) This statement is the content of part (ii) of proposition 2, whose proof includes the case that $\varphi(\cdot)$ is a global solution. □

Remarks. 1. Part (iii) of the theorem states that if $\varphi_0 \in \mathcal{D}(A^n)$ for all $n \geq 0$, then for each $t \in \mathbb{R}$ the equivalence class of functions in $(L^2)^3$ that differ from $E(t)$ only on a null set of $\mathbb{R}^3$ has a representative in $C^\infty(\mathbb{R}^3)^3$. The same holds for $B(t)$. If we denote by $E(t)$ and $B(t)$ this representative and no more the whole equivalence class, we have $E(t), B(t) \in C^\infty(\mathbb{R}^3)^3$. Furthermore, it follows from (i) that for each interval $\Lambda \subset \mathbb{R}$ the maps $t \mapsto E(t)$ and $t \mapsto B(t)$ as well as their $(L^2)^3$-derivatives $t \mapsto \partial^l/\partial t^l E(t)$ and $t \mapsto \partial^l/\partial t^l B(t)$, $l \geq 0$, are elements of $L^2(\Lambda, (L^2)^3)$, this space being isomorphic to $L^2(\Lambda \times \mathbb{R}^3)^3$. Since $C_0^\infty \subset L^2$, the derivative in the time direction with respect to the $(L^2)^3$-norm is the distributional derivative. Therefore we can use Sobolev's lemma [13] to conclude from $\partial^l/\partial t^l E$, $\partial^l/\partial t^l B \in L^2(\Lambda \times \mathbb{R}^3)^3$, $l \geq 0$, that E and B are not only strongly continuously differentiable with respect to the $(L^2)^3$-norm, but even pointwise up to all orders. Thus $E, B \in C^\infty(\mathbb{R}\times\mathbb{R}^3)^3$, and $\varphi(\cdot)$ is a classical solution of the Maxwell-Lorentz system (lesser smoothness of the initial conditions would yield in a similar way classical solution which are less smooth).

2. We wrote the Maxwell-Lorentz system (1) with a linear force term acting in addition to the Lorentz force, i.e. we have a quadratic external potential. We note that this is only for notational simplicity and that the consideration of more general potentials would require only minor adjustments. For example, as long as a more general potential is bounded by a quadratic function, the estimates in lemma 2 remain unaltered and global existence is still valid. In this respect we also remark that our results apply in a straightforward manner to the Maxwell-Lorentz system with physical constants mass m and velocity of light c written out. Our assumption $\rho \in C_0^\infty(\mathbb{R}^3)$ is again for notational simplicity. In case of more general ρ, the regularity properties of the solution would reflect the degree of differentiability of ρ.

3. We mention that the total energy of the Maxwell-Lorentz system is conserved. It can be used as an a-priori-bound on the norm of local solutions if $\alpha = 1$ and $\kappa \geq 0$. However, we proved global existence by other means for general α and κ.

Lemma 5 (conservation of energy) *Let $\varphi(\cdot) = \big(q(\cdot), p(\cdot), E(\cdot), B(\cdot)\big) : [0,T) \to \mathcal{D}(A^n)$, $n \geq 1$, be a solution of (2) according to proposition 1. Then the quantity H,*

$$H = \alpha\sqrt{1+p(t)^2} + \frac{1}{2}\kappa q(t)^2 + \frac{1}{8\pi}\int \left(E(t,x)^2 + B(t,x)^2\right) dx, \tag{24}$$

is independent of t. We call H the total energy of the Maxwell-Lorentz system.

Proof. A straightforward calculation. □

5 Negative bare mass and "runaway"-solutions

Throughout this section we consider the situation that the charge distribution's bare mass m is negative and that the external potential is attracting, i.e. $\alpha = \mathrm{sgn}(m) = -1$ and $\kappa > 0$. In addition, we assume the charge distribution ρ to be spehrically symmetric. Our investigation is concerned with the stationary solution of the Maxwell-Lorentz system (1). By this we mean the following time-independent state vector ψ_s:

$$\psi_s = (0, 0, E_C, 0),$$

where

$$E_C(x) = \int \rho(x')\frac{x-x'}{|x-x'|^3}dx'$$

is the electrostatic Coulomb field of the charge distribution ρ. It is easy to check, making use of the symmetry of ρ, that ψ_s is indeed a solution of (1). The following theorem states that in an arbitrarily small neighbourhood of ψ_s there are initial data $\varphi(0)$ such that the corresponding solution $t \mapsto \varphi(t)$ of (1) has "runaway"-character, i.e. the charge center runs away infinitely far from the origin and its velocity tends to the speed of light as $t \to \infty$.

Theorem 2 *Let $\alpha = -1$, $\kappa > 0$ and let ρ be sperically symmetric. Then the stationary solution ψ_s of the Maxwell-Lorentz system (1) is unstable in the sense of Lyapunov: For each $\epsilon > 0$ there is a solution $t \mapsto \varphi(t) = \big(q(t), p(t), E(t), B(t)\big)$ of (1) with $||\psi_s - \varphi(0)||_{\mathcal{H}} < \epsilon$ and*

$$\lim_{t\to\infty}|q(t)| = \infty,\ \lim_{t\to\infty}|p(t)| = \infty,\ i.e.\ \lim_{t\to\infty}||\psi_s - \varphi(t)||_{\mathcal{H}} = \infty.$$

Proof. Let u be some unit vector, $u \in \mathbb{R}^3$ and $|u| = 1$. We choose the following initial data:

$$\begin{aligned} q_0 &= x_0 u, \quad x_0 > 0, \\ \dot{q}_0 &= \frac{-p_0}{\sqrt{1+p_0^2}} = v_0 u, \quad v_0 > 0, \end{aligned}$$

$$
\begin{aligned}
E_0(x) &= \int \rho(x')\frac{x-q(0)-x'}{|x-q(0)-x'|^3}dx' = E_C\big(x-q(0)\big),\\
B_0(x) &= 0.
\end{aligned}
$$

These initial data satisfy (19). We may assume that

$$
\varphi_0 = \big(q_0, p_0, E_0, B_0\big) = \left(q_0, \frac{-\dot q_0}{\sqrt{1-\dot q_0^2}}, E_C\big(\,\cdot - q_0\big), 0\right)
$$

satisfies $\|\psi_s - \varphi_0\|_{\mathcal{H}} < \epsilon$. For this we only have to choose x_0 and v_0 sufficiently small. By theorem 1, there is a global solution $t \mapsto \varphi(t) = \big(q(t), p(t), E(t), B(t)\big)$ of the Maxwell-Lorentz system (1) with $\varphi(0) = \varphi_0$. Due to the symmetry of the initial data, i.e. $q_0, p_0 || u$ and E_0, B_0 being rotational symmetric around the axis through u, it is assured that the motion of q is one-dimensional: We may write

$$
q(t) = x(t)u \quad (t \in \mathrm{I\!R}), \tag{25}
$$

with a real-valued function $t \mapsto x(t)$ satisfying $x(0) = x_0$ and $\dot x(0) = v_0$. Making use now of energy conservation, cf. lemma 5, we have

$$
\begin{aligned}
H &= \frac{-1}{\sqrt{1-\dot x(t)^2}} + \frac{1}{2}\kappa x(t)^2 + \frac{1}{8\pi}\int \big(E(t,x)^2 + B(t,x)^2\big)\,dx \\
&= \frac{-1}{\sqrt{1-v_0^2}} + \frac{1}{2}\kappa x_0^2 + \frac{1}{8\pi}\int E_C^2 dx
\end{aligned} \tag{26}
$$

for all $t \in \mathrm{I\!R}$. We decompose the electric field into a longitudinal and a transversal part: $E = E_l + E_t$, where $\mathrm{div} E_t = 0$ and $\mathrm{rot} E_l = 0$. Note that $E_l(x,t) = E_C\big(x - q(t)\big)$ and $\int E_C \cdot E_t dx = 0$ since $\mathrm{div} E(x,t) = \mathrm{div} E_C\big(x-q(t)\big) = 4\pi\rho\big(x-q(t)\big)$. Therefore

$$
\int E^2 dx = \int \big(E_C^2 + 2E_C\cdot E_t + E_t^2\big)dx \geq \int E_C^2 dx.
$$

From (26) we thus obtain

$$
\frac{1}{\sqrt{1-\dot x(t)^2}} \geq \frac{1}{2}\kappa\big(x(t)^2 - x_0^2\big) + \frac{1}{\sqrt{1-v_0^2}}
$$

for all $t \in \mathrm{I\!R}$. Since $x_0, v_0 > 0$, this implies $\dot x(t) > v_0$ for all $t > 0$ and therefore

$$
\lim_{t\to\infty} |x(t)| = \infty, \;\; \lim_{t\to\infty} |\dot x(t)| = 1.
$$

Because of (25) this is what we had to show. $\square$

Remark. As we mentioned in the introduction, this result sheds light on the genesis of the "runaway"-solutions of the Lorentz-Dirac equation. We wish to remark that the Lorentz-Dirac equation exhibits "runaway"-solutions also without external potential. We employed the potential merely as a tool to have a quick argument for instability. Clearly, also without external potential, as soon as the particle accelerates, it will continue to accelerate, thereby loosing energy to the fields. A rigorous argument for this would need a more detailed analysis of the fields.

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