Bachelor's Thesis

Bohmian Mechanics
&
Position Measurements

Robin Thomas Schlenga

Ludwig-Maximilians-Universität München
Fakultät für Physik

Bachelor's Thesis

Bohmian Mechanics
&
Position Measurements

Vorgelegt von:
Robin Thomas Schlenga
Geboren am 23. Juli 1987 in Kirchheimbolanden
Matrikelnummer 8076195

Betreuer:
Prof. Dr. Peter Pickl

München, den 7. Juli 2011

Für meine Eltern, Rolf und Pia, in Liebe und Dankbarkeit.
Und für Emmi, die die Arbeit niemals lesen wird.

Contents

BOHMIAN MECHANICS & POSITION MEASUREMENTS

R. T. SCHLENGA

Abstract

The task of this work is to show that one of the research highlights 2010 of the Journal of Mathematical Physics, [KW10], erroneously claims to find an experimentum crucis with which Bohmian mechanics could be disproved. We will include a revision of Bohmian mechanics and its relevant aspects as well as a discussion of common misunderstandings of the theory, some of which occured also in the article in question. We will show that Bohmian mechanics provides a consistent account of measurements in a quantum setting and that its predictions are equivalent in statistics to those of standard quantum mechanics.

1. Introduction

Bohmian mechanics, or more exactly 'de Broglie-Bohm theory', is a realistic, non-relativistic quantum theory about moving particles. As such, it represents an attempt to formulate a quantum theory without measurement problem. Put differently, Bohmian mechanics is an attempt to find the fundamental theory behind the merely effective description of Nature in usual quantum mechanics. Sadly, still today, almost sixty years after its publication, Bohmian mechanics divides the scientific community into two lairs. On the one side are very few physicists and mathematicians working on Bohmian mechanics, who are convinced that Bohmian mechanics is the key to a deeper understanding of the quantum world. The other, much larger division of scientists reject Bohmian mechanics: Either they follow Feynman's premise, 'shut up and calculate', and do not take interest in the fundamental problems of quantum theory, or they subscribe to an unfounded point of view in which Bohmian mechanics seems to disagree with Nature. Of course, the latter attitude is plainly wrong, since Bohmian mechanics agrees with all measurements whenever ordinary quantum mechanics does. Under ordinary, or orthodox, or standard quantum meachnics (QM) we understand the study of the wave function endowed with several well-known axioms about the representation of states in a Hilbert space and the replacement of physical variables by self-adjoint operators. It is of no relevance to our work which interpretation of standard quantum mechanics is used since the main problem is the same in all of them: the (in)famous measurement problem, which Schrödinger led to an extreme exaggeration in his gedankenexperiment about macroscopic superposition of states, better known as his cat paradox.

Yet another attack on Bohmian mechanics was recently led by Kiukas and Werner in their article [KW10], even a research highlight of the Journal of Mathematical Physics. It is our goal in this work to show that and where they went wrong.

First, we will give an introduction to Bohmian mechanics, where we want to clarify the notion of position and its measurement. Then we will encounter [KW10]. In this paper position measurements are used to construct quantum states which (maximally) violate a Bell-type inequality, known as CHSH inequality (see [CHSH69]) which is experimentally easier to test than the original setting by Bell. While succeeding in this endeavour, the authors failed in a different one: They planned to use these measurements to distinguish between orthodox and Bohmian quantum mechanics – but they made glaring mistakes which demand for refutation. Since such errors are found rather often, we will very precisely show what really is the content of Bohmian Mechanics and that the POVMs (positive-operator valued measures) constructed in [KW10] are, like all POVMs used in the description of any quantum system, a prediction of Bohmian Mechanics.

The source for most misunderstandings of Bohmian Mechanics lies in the fact that many physicists have severe difficulties understanding that the 'measurement'[1] of the position operator is not, in general, a measurement of the Bohmian particle positions. It is true that in Bohmian mechanics the position measurement is also described mathematically by the expectation value of the position operator – which sometimes, but certainly not always, coincides with the value of the position variable.

Date: July 7, 2011.

[1] In fact, the usage of the word "measurement" is a bad habit in most cases – what do we mean by it? For criticism, see chapter 23, 'Against Measurement', in [Bel04]. A nice example of how to do it better is found in [DGZ04].

Another great difference between Bohmian mechanics and orthodox quantum mechanics lies within the collapse of the wave function. This nonunitary process is more or less an axiom of orthodox quantum mechanics, and thus it is taught very emphatically as an admittedly strange but seemingly fundamental law of Nature. Not only is it a pitty to have a process in the theory which is not actually described by the theory, it is also interesting to note that orthodox quantum mechanics holds this process for real – without having an observable for it. Nevertheless, the notion of a collapse diffused into the intuitive thinking about Nature. But Bohmian mechanics does not know any collapse and hence no such process occurs in the Bohmian universe. There is only the concept of an "effective" wave function, describing a system after a measurement. This function resembles the collapsed wave funtion known from orthodox QM which may lead to misunderstandings. It is, however, often critizised that the particle's dynamics would be very strange and especially incontinuous if there were a collapse – one could think of a "quantum leaping" particle. But since there is no collapse in Bohmian mechanics, all particle motions are perfectly continuous. This is a typical example for the style of criticism against Bohmian mechanics: Unneccessary and even unjustified assumptions are added to Bohmian mechanics in a rather 'handwaving' manner, and then these exact assumptions are – mathematically rigorously, at best – refuted.

In [KW10], the line of reasoning is similar, but the exact argument is not really clear. The authors state that Bohmian mechanics yields a distribution for position measurement which disagrees with usual quantum mechanics, unless the theory would be 'made' contextual. Since Bohmian mechanics *is* contextual, this is not much of a problem, obviously.

Many of the questions concerning Bohmian mechanics are answered already. A short summary of the theory, in nontechnical terms, can be found in [Tum04], a somewhat more technical one in [DGTZ09].

Acknowledgements. I wish to express my thanks to my advisor Prof. Dr. Peter Pickl for clarifying discussions, for his advice and his helpfulness. I am especially grateful to Prof. Dr. Detlef Dürr for his exposition of Bohmian mechanics and the effort he made to help his students understand the shortcomings of orthodox quantum mechanics and the beauty of Bohmian mechanics.
This work had some careful human spellcheckers: *Sebastian Lehrack*, *Max Mehlhorn* und *Peter Zeidler*. Thank you very much for your time and help!

2. Bohmian Mechanics

Hence we must inquire first what nature is: for thus we shall also see what natural science treats of.
– Aristotle: Metaphysics

In his two 1952 papers, [Boh52], David Bohm rediscovered an idea introduced by Louis de Broglie already in 1929. It was the most natural way to merge the wave-like distribution of point-like objects in experiments like the famous double slit experiment: Use Schrödinger's wave function as a guiding field for point particles. Let us see how we can put this in a precise mathematical formulation.

2.1. Two equations govern the quantum world.

Suppose we have an N-particle system of spinless particles[2]. Then we assign to each particle $k = 1, \ldots, N$ a parameter m_k, called its *mass* and, most importantly, a position $\boldsymbol{Q}_k \in \mathbb{R}^3$. These positions altogether form a single element of the configuration space: $Q = (\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N) \in \mathbb{R}^{3N}$, $(a_1, \ldots, a_n)$ stands for the ordered n-tuple. From now on we will use uppercase letters like Q (or boldface $\boldsymbol{Q}$) to denote the actual state of the system in the configuration space (the *actual* particle position, respectively) and lowercase letters like q for a generic element of the configuration space (or again a boldface $\boldsymbol{q}$ for the *possible* particle positions). Particle positions in a quantum theory? Yes, this is what Bohmian mechanics tells us. We will see how these positions yield in a natural way all the predictions of usual quantum mechanics, as well as most of its axioms. Another ingredient to the theory, already known from standard quantum mechanics, where it is the sole description of the state of a quantum system, is the wave funtion ψ:

$$\begin{aligned} \psi\colon\ \mathbb{R}^{3N} \times \mathbb{R} &\to \mathbb{C} \\ (q,t) &\mapsto \psi(q,t)\,. \end{aligned}$$

We note that the wave function lives on the configuration space $\mathbb{R}^{3N}$ for an N-particle system. It is easy to talk about a configuration of N particles in Bohmian mechanics, where one actually has particles. However, in orthodox quantum mechanics, where the notion of particle is absent, this is not so clear: There are ways to find the configuration space by considering the results of experiments, but honest physicists will admit that they intuitively very well think of particles – they should not be ashamed of that, since it is possibly the most natural concept of all physics.

[2]The case with spin is similar with some conceptual changes here and there. As in [KW10] spinless particles are considered, we do not overflow the formalism with unneccessary complications through spin.

The wave function is a solution of the Schrödinger equation for the system:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(q,t) = -\sum_{k=1}^{N}\frac{\hbar^2}{2m_k}\Delta_k\psi(q,t) + V(q)\cdot\psi(q,t) \tag{1}$$

where Δ_k stands for the Laplace operator acting on the k-subspace of the configuration space, i.e. it acts on the three coordinates of the k^{th} particle. $V(q) \in \mathbb{R}$ is the potential funtion, e.g. a Coulomb potential for atomic problems. The second ingredient to the Bohmian description of nature is the guiding equation for the particles. This equation can be derived in many different ways, a simple approach using symmetry arguments can be found in [DT09][3], chapter eight. The dynamics of the k^{th} particle is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{Q}_k(t) = \frac{\hbar}{m_k}\Im\frac{\langle\psi,\nabla_k\psi\rangle}{\langle\psi,\psi\rangle}(Q(t)\,,t)\,. \tag{2}$$

We denote by $\Im$ the imaginary part, $\langle\cdot,\cdot\rangle$ stands for the usual L^2 scalar product,

$$\langle\varphi,\psi\rangle = \int\overline{\varphi(x)}\cdot\psi(x)\,\mathrm{d}^n x$$

for $\varphi,\psi \in L^2(\mathbb{R}^n,\mathrm{d}x)$ or different suitable domains, respectively. Note that the right hand side of equation (2) is evaluated at the actual particle positions at time t. It is in this simple fact that the nonlocality of Bohmian mechanics emerges, and it is this nonlocal structure which fascinated John Bell and inspired him to his famous inequalities[4].

Bohmian mechanics is realistic in the sense that it has a clear ontology, it spells out (in terms of the theory, not just talk!) what it is about. It is interesting to note that also orthodox quantum mechanics was intended to be realistic in the beginning. Heisenberg for example thought that the frequencies of emitted and absorbed light in an atomic experiment are quite real and should be taken as primary variables, substituting position and momentum. For more on that topic, see [Hei86].

2.2. **Positions and position measurements, PVMs.** In connection with Bohmian mechanics the notion of trajectories enters the stage. But we have to be very careful with this notion. Although a Bohmian would speak of the real motion of a real particle, this real movement need not be the result of a measurement. In Bohmian mechanics it is crucial to distinguish between reality and the outcomes of certain experiments, so-called measurements. Since the role of position and its measurement is most confusing for many we shall examine this subject a little closer. First, we will work on more intuitive footing and try to make clear why what follows fits well to our understanding of Nature and of measurements, thereby we will find that projector-valued measures (PVMs) are a very natural mathematical tool for the description of measurements. Once we arrived at this result, we will repeat the analysis on a more formal level which then leads us to the most general mathematical object in the toolbox of measurement analysis: that of positive operator-valued measures (POVMs).

In the following, when we talk of the 'wave function of the system', we will always mean the *effective wave function* of the system, see Appendix A. In a nutshell, the effective wave function is no more than that component of the total wave function that actually guides the particle(s). Most of the time, the effective wave function will be denoted by ψ.

Before we come to position measurements, we need to know what we mean by "measurement"[5]. Let us consider the discrete case first. Under "measurement" we understand an experiment in which a quantum mechanical system interacts with some macroscopic (big compared to the de-Broglie wavelength of the quantum system) device for a time T. After the measurement, a pointer on the measurement device points at some distinct value or the experimenter reads the result in a similar manner. Before we go on, let us agree on notation: The quantum mechanical system we wish to describe has a wave function φ and particle configuration $X \in \mathbb{R}^m$, the measurement device is described by a wave function Φ and configuration $Y \in \mathbb{R}^n$. Let all wave functions in this work be appropriately normalized. The measurement process is then formalized as follows: Before the measurement the apparatus is in some null state $\Phi_0(y)\,,\, Y \in \operatorname{supp}\Phi_0(y)$. Let the wave function of the system be φ_α where $\alpha \in A$ for an appropriate (countable) index set A. The system and the apparatus wave function couple and evolve together under the usual Schrödinger evolution U_T. We suppose that, before and after this evolution, the state of the combined system factorizes into the product of the system and

[3]This book gives a thorough introduction to Bohmian mechanics and discusses most of the relevant aspects.

[4]See the second article, 'On the Einstein-Podolsky-Rosen Paradox' in [Bel04].

[5]We follow the book [DT09].

[6]This is not neccessary, however. We only intend to keep things simple and this is the simplest case.

[7]In the rest of the work, we will always assume that the apparatus is able to display the states α, it would be quite a shabby apparatus if it could not anyhow. It is possible to describe measurements impulsive, i.e. we can neglect the change of the wave functions due to the usual Schrödinger evolution and just consider the effect of the act of measurement.

apparatus wave functions[6,7]:

$$\varphi_\alpha \otimes \Phi_0 \xrightarrow{U_T} \varphi_\alpha \otimes \Phi_\alpha .$$

If the system is in a superposition state $\psi = \sum_\alpha c_\alpha \varphi_\alpha$ with $\sum_\alpha |c_\alpha|^2 = 1$, the measurement yields what is commonly called "measurement problem", a superposition of pointer states:

$$\psi \otimes \Phi_0 = \sum_\alpha c_\alpha \varphi_\alpha \otimes \Phi_0 \xrightarrow{U_T} \sum_\alpha c_\alpha \varphi_\alpha \otimes \Phi_\alpha .$$

Since we are thinking in terms of Bohmian mechanics, this "problem" is not a problem at all. The pointer configuration is guided by equation (2) to one and only one of the different final states. Since this final state is determined by the initial conditions, it is possible to know exactly which component of the wave function is "chosen" in a measurement. Accepting the premise of $|\psi|^2$ distributed initial conditions, we may talk about probabilites in a classical sense: Averaging over a statistical ensemble of possible initial conditions. Then we can use the quantum equilibrium hypothesis[8] to determine the probability of the pointer pointing at some value $k \in A$:

$$\mathbb{P}^\psi(Y_T \mathrel{\widehat{=}} k) = \mathbb{P}^\psi(Y_T \in \operatorname{supp} \Phi_k) = \\ = \int \left| \sum_\alpha c_\alpha \varphi_\alpha(x)\, \Phi_\alpha(y) \right|^2 \chi_{\operatorname{supp} \Phi_k}(y)\, \mathrm{d}^m x\, \mathrm{d}^n y$$

with the characteristic function χ:

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} .$$

Now, what can we see from this equation? Obviously, the square of the sum will contain terms without Φ_k and some mixed terms. The mixed terms vanish because the pointer positions of our measurement device are macroscopically distinct, such that the overlap of the supports of the apparatus wave functions for different pointer positions are negligible[9]: $\operatorname{supp} \Phi_a \cap \operatorname{supp} \Phi_b \approx \emptyset$ for $a \neq b$. We are left with only square terms, multiplied by the characteristic function that sets all terms but one to zero for the same reason. Then the single term we need to consider is:

$$\mathbb{P}^\psi(Y_T \in \operatorname{supp} \Phi_k) = \int_{\{x,y \mid y \in \operatorname{supp} \Phi_k\}} |c_k|^2 \, |\varphi_k(x)|^2 \, |\Phi_k(y)|^2 \, \mathrm{d}^m x\, \mathrm{d}^n y \\ = |c_k|^2$$

This equality is of course not exactly precise, but the error we make is extremely small: A good estimate for the width of the support for one particle is obtained by the uncertainty relation: $\Delta x \approx \operatorname{diam}(\operatorname{supp} \varphi) \approx \frac{\hbar}{p}$ where diam denotes the diameter of a set and p is the particle momentum. Let us assume the pointer moves during the duration of the measurement $T \approx 1ms$ a distance of about $100\mu m$, which we take as the distance of distinct pointer states, then each particle has a momentum $p \approx 10^{-29} \frac{kg\, m}{s}$ where we assumed the particle to be of mass $10^{-27} kg \approx 1\mathrm{u}$. Thus we find $\operatorname{diam}(\operatorname{supp} \varphi) \approx 10^{-6} m = 1\mu m$. This is obviously much smaller than the distance of the different pointer states. But let us assume we made an error of order $\epsilon \approx \frac{\operatorname{diam}(\operatorname{supp} \varphi)}{m}$. Then this error is multiplied for each particle, so for one gramme of material in the pointer we will find approximately $\epsilon^{10^{23}} \approx 10^{-6 \cdot 10^{23}}$ as error, which is extremely small. This estimate is no formal proof, but it shows intuitively very well that our assumptions are justified – experiment as the only valuable measure of success in physics supports Bohmian mechanics correspondingly well.

So far, so good. We derived the result one usually learns as an axiom in Schrödinger quantum mechanics, all we had to do was exploiting the fact that in a measurement a pointer points at a value. But this calculation demands for abbreviations, since it is quite long, especially compared to the usual approach in standard quantum mechanics. If we were to perform computations similar to this for every description of an experiment, we would soon run out of paper and time. Certainly, as a physicist one likes to extract the structure and results of this computation to a mathematical scheme, which then provides a handle for the complete process and which simplifies our calculation. Of course, the idea that comes to mind is to use operators for the description of such measurements. Note that we do not need to postulate operators as primary objects, we just want to use them as "bookkeeping devices" to encode the information of an experiment in one single mathematical object. This analysis will lead us to the notion of PVMs, projector-valued measures, and we will later generalize that to POVMs (positive operator valued measures). First we look again at the previous calculation. We can not only calculate the probability of an outcome k but also the norm (i.e. the probabilty of the whole base space[10])

[8] This hypothesis gives us the reason why $|\psi|^2$ is the correct distribution, see [DT09].

[9] This is the task a good experimenter can complete: Arrange your experiment in such a way that the supports of the wave function for different results are disjoint.

[10] Since the Schrödinger evolution preserves the norm $\|\cdot\|^2$ by equivariance (see [DT09]), we can calculate the norm before or after the Schrödinger evolution.

of the product wave funtion of system and apparatus:

$$\begin{aligned}\|\psi \otimes \Phi\|^2 &= \int |\psi(x)\ \Phi(y)|^2\, \mathrm{d}^m x\, \mathrm{d}^n y \\ &= \int \Big|\sum_\alpha c_\alpha\, \varphi_\alpha(x)\ \Phi(y)\Big|^2\, \mathrm{d}^m x\, \mathrm{d}^n y \\ &= \sum_\alpha |c_\alpha|^2\end{aligned}$$

by the same argumentation as above. If we write out the second line of that equation in detail and carry out the y-integration, we find the following:

$$\begin{aligned}&\int \Big|\sum_\alpha c_\alpha\, \varphi_\alpha(x)\ \Phi(y)\Big|^2\, \mathrm{d}^m x\, \mathrm{d}^n y \\ &= \int \sum_\alpha |c_\alpha|^2\, |\varphi_\alpha(x)|^2\ \mathrm{d}^m x + \\ &\quad + \int \sum_{\alpha\neq\beta} \overline{c_\alpha}\, c_\beta\, \overline{\varphi_\alpha(x)}\, \varphi_\beta(x)\, \mathrm{d}^m x \\ &= \sum_\alpha |c_\alpha|^2\end{aligned}$$

which means that the mixed terms have to vanish. Since the coefficients are quite arbitrary, we find for $\alpha \neq \beta$ the stronger statement

$$\begin{aligned}0 &= \int \overline{\varphi_\alpha(x)}\, \varphi_\beta(x)\ \mathrm{d}^m x \\ &= \langle \varphi_\alpha, \varphi_\beta \rangle.\end{aligned}$$

We also saw that $\langle \varphi_\alpha, \varphi_\alpha \rangle = 1$, so in conclusion we find orthogonality for the φ_α:

$$\langle \varphi_\alpha, \varphi_\beta \rangle = \delta_{\alpha,\beta}.$$

With this result we can use the scalar product to write the orthogonal projector onto φ_α as follows:

$$\mathscr{P}_{\varphi_\alpha} \varphi = \varphi_\alpha \langle \varphi_\alpha, \varphi \rangle.$$

If we assume that the φ_α form a basis, we can use the family of orthogonal projectors $\{\mathscr{P}_{\varphi_\alpha}\}_\alpha$ to define an operator which encodes the results of our experiment. Let, for all $\alpha \in A$, $\lambda_\alpha \in \Lambda$ be the number the hand of our apparatus points at when it is in the state Φ_α with $Y \in \operatorname{supp} \Phi_\alpha$. Then we define the operator A by

$$\mathsf{A} := \sum_\alpha \lambda_\alpha \mathscr{P}_{\varphi_\alpha}$$

its domain being the obvious one.

This mathematical object is a very handy tool for the calculation of statistics of the experimental results, but in no way does it represent an actual physical quantity by itself. How to use this object to compute any statistical quantity is well known from orthodox quantum mechanics, since there it is all there is. Now we want to take the step to PVMs[11] which is most intuitively done by considering position measurements since their results are continuous rather than discrete. So we came full circle to the position measurements. It is crucial that in Bohmian mechanics the real positions need not be the result of a position measurement. Why is this? Obviously a measurement needs a measurement device. This is true for every physicist, but this simple statement has an enormous impact when taken seriously. While in orthodox quantum mechanics the apparatus is cut out of the picture by introducing the fuzzy notion of an observer and axiomatically postulating the facts we have shown so far, Bohmian mechanics takes the apparatus seriously and introduces it in the mathematical description of the measurement[12]. But we saw that whenever we performed an analysis of a measurement, the apparatus played a crucial role which lead us to introducing an operator for the description of the results. So instead of the actual position, given by equation (2), we rather measure a composite object which to a certain extent respects the act of measuring. What will this object be for a measurement of position? Let in the following Ω be a measurable subset of $\mathbb{R}$ (we denote the set of measurable subsets of $\mathbb{R}$ with $\mathcal{B}(\mathbb{R})$) and consider a single particle described by wave function ψ and generic position $x \in \mathbb{R}$. Then the probability of finding this particle in Ω is of course

$$\begin{aligned}\mathbb{P}^\psi(\Omega) &= \int |\psi(x)|^2\, \chi_\Omega(x)\ \mathrm{d}x \\ &=: \int_\Omega \langle \psi, \chi_{\{\mathrm{d}x\}} \psi \rangle.\end{aligned}$$

Now it is natural to define an operator O_Ω with $\Omega \in \mathcal{B}(\mathbb{R})$ by

$$\mathsf{O}_\Omega \varphi(x) := \chi_\Omega(x)\, \varphi(x) = \begin{cases} \varphi(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases}.$$

This operator is part of a family of operators indexed by the measurable sets Ω which is called a PVM as it fulfills the following definition[13]:

[11] Actually, the projectors $\mathscr{P}_{\varphi_\alpha}$ we defined form a PVM as well. But a discrete spectrum is not the most general case and would not lead to an intuitive definition, so we postpone the introduction of PVMs to the next step.

[12] This situation is somewhat analogous to the difference between Riemann and Lebesgue integration theory: The former forces the function, similar to what conventional quantum mechanics does to the system, into a tight corset to fulfill the demanded expectations, while the latter allows the function to behave freely and measures what is really there, very much like Bohmian mechanics does it with a quantum system and the apparatus. It is clear which integration theory is more useful, the reader is encouraged to draw his or her own conclusions regarding quantum theory.

[13] See e.g. [RS81]

Definition. *A family of projections* $\{\mathscr{P}_\Omega\}$ *with* $\Omega \in \mathcal{B}(\mathbb{R})$ *is called a* projection-valued measure (PVM) *if it obeys:*

(a) *Each* $\mathscr{P}_\Omega$ *in the family is an orthogonal projection.*
(b) $\mathscr{P}_\emptyset = 0$, $\mathscr{P}_\mathbb{R} = \mathbb{1}$.
(c) *If* $\Omega = \bigsqcup_{n=1}^{\infty} \Omega_n$ *(i.e.* Ω *is the union of pairwise disjoint subsets), then* $\mathscr{P}_\Omega = \operatorname*{s-lim}_{N\to\infty} \left(\sum_{n=1}^{N} \mathscr{P}_{\Omega_n} \right)$
(d) $\mathscr{P}_{\Omega_1} \mathscr{P}_{\Omega_2} = \mathscr{P}_{\Omega_1 \cap \Omega_2}$

The operator O_Ω inherits these properties from the characteristic function in its definition. Generalization to three dimensions is straightforward, we just use one operator for each component. To shorten the notation we define the "three-dimensional" operator O_Ω by

$$\mathsf{O}_\Omega \varphi(\boldsymbol{x}) := \chi_\Omega(\boldsymbol{x})\, \varphi(\boldsymbol{x})$$

for $\boldsymbol{x} \in \mathbb{R}^3$ and $\Omega \in \mathcal{B}(\mathbb{R}^3)$, using the same symbol O since it will always be clear from context in how many dimensions we work. With this definition we can write the probability of finding the particle in the set Ω as

$$\begin{aligned} \mathbb{P}^\psi(\Omega) &= \langle \psi, \mathsf{O}_\Omega \psi \rangle \\ &=: \int_\Omega \mathrm{d}\langle \psi, \mathsf{O}_{\boldsymbol{x}} \psi \rangle \\ &=: \int_\Omega \langle \psi, \mathrm{d}\mathsf{O}_{\boldsymbol{x}} \psi \rangle. \end{aligned}$$

It is interesting to see that in Dirac's bra-ket notation we can write $\mathrm{d}\mathsf{O}_{\boldsymbol{x}} = |\boldsymbol{x}\rangle\langle\boldsymbol{x}|\, \mathrm{d}^3x$.

Now, what is the physical meaning, if any, of the PVM O_Ω? It is analogous to that of the projectors $\mathscr{P}_{\varphi_\alpha}$ which we used to construct the operator $\mathsf{A} = \sum_\alpha \lambda_\alpha \mathscr{P}_{\varphi_\alpha}$ as a bookkeeping device for the statistics of a measurement with results $\lambda_\alpha \in \Lambda$. So, since we want to describe position measurements, we construct an operator for the statistics of such an experiment with the possible results of it, i.e. the generic particle position $\boldsymbol{x}$:

$$\mathsf{X} = \int \boldsymbol{x}\, \mathrm{d}\mathsf{O}_{\boldsymbol{x}} \tag{3}$$

$$= \int \boldsymbol{x}\, |\boldsymbol{x}\rangle\langle\boldsymbol{x}|\, \mathrm{d}^3x. \tag{4}$$

This is the famous position operator of quantum mechanics[14]. Now we see what describes position measurements in Bohmian mechanics: It is the same position operator as in usual quantum mechanics, albeit there is an exact particle position in the theory. This fact leads to confusion, which is to a certain degree understandable. On the other hand we can be sure that any "measurement" of an operator in orthodox quantum theory is also a prediction of Bohmian mechanics.

We need one more fact to complete our analysis of PVMs: The Heisenberg operator, i.e. in our case, the position operator at a time t. This operator is found easily, just consider as before the probability of finding a particle with coordinate $\boldsymbol{x}$ and wave function ψ in a measurable set Ω, but now at time t:

$$\begin{aligned} \mathbb{P}^{\psi_t}(\Omega) &= \langle \psi_t, \mathsf{O}_\Omega \psi_t \rangle \\ &= \langle U(t)\,\psi, \mathsf{O}_\Omega\, U(t)\,\psi \rangle \\ &= \langle \psi, U^*(t)\, \mathsf{O}_\Omega\, U(t)\,\psi \rangle \\ &=: \langle \psi, \mathsf{O}_\Omega(t)\,\psi \rangle. \end{aligned}$$

This is of course also a PVM since $U(t)$ is unitary. So we find that the Heisenberg position operator at time t is $\mathsf{X}(t) = U^*(t)\,\mathsf{X}\,U(t)$. This calculation goes through as in conventional quantum theory and the operator is exactly the same. This comes as no surprise, since all operators are associated to PVMs (or POVMs) and they are what Bohmian mechanics brings as description of measurements, the Schrödinger evolution being the same in orthodox and Bohmian quantum mechanics.

It is clear that our analysis focused on position and the position operator because the paper we wish to refute is based on a position measurement. But what is true for the position operator is also true for any other operator of quantum mechanics: For every experiment we can find a PVM (or, more generally, a POVM, see below) which, when applied to the system wave function, yields those components associated to the outcomes of the experiment. Then we construct an operator with the possible results as eigenvalues (or the spectrum) and the PVM, which is of course a simple application of the spectral theorem. This operator encodes the complete statistics of the experiment in one single mathematical object.

Let us see what the connection between the position operator and the actual particle positions is. It is clear that if we want to (and if we are able to) measure the Bohmian positions, this is described by "measuring" the position operator, to use conventional language. But does the position operator always describe genuine[15] measurements of position? Of course not.

[14]A word on the domain of this operator is in order: In the so-called position representation the operator boils down to a multiplication operator, and thus its domain is just maximal.
[15]This notion means no more than the measurement of a preexisting quantity. See [DGZ04].

All we have for certain is that a "measurement" of the position operator yields the same statistics as a genuine position measurement. Yet, who would infer from the same statistics of experimental results that the measured physical quantities were the same? The classic example of this problem is the two-dimensional harmonic oscillator, see [DGZ04]. There we find for the Bohmian position of a certain state a movement which is not periodic with the same period as the oscillator period. We conclude that the measurement of the position operator (which yields a periodicity equivalent to the harmonic oscillator) is not a measurement of the Bohmian position.

Now, after we have come to an intuitive understanding of the Bohmian notion of experiment and measurement, let us become more precise.

2.3. **POVMs.** Thus far we saw only PVMs, but we announced POVMs as the most general description of measurements in Bohmian mechanics. The structure of such a POVM is very similar to that of a PVM, we just drop the condition of the operators being projectors for the weaker condition that they are merely positive (bounded) operators. Here is the formal definition[16]:

Definition. *For measurable subsets Ω of $\mathbb{R}^n$, $\Omega \in \mathcal{B}(\mathbb{R}^n)$, we call a family of bounded linear operators $\{\mathsf{P}_\Omega\}_\Omega \in \mathscr{L}(\mathscr{H})$ on a Hilbert space $\mathscr{H}$ a* positive operator valued measure (POVM) *if:*

(a) *Each P_Ω is positive, i.e. $\langle\varphi, \mathsf{P}_\Omega\varphi\rangle \geq 0 \; \forall\varphi \in \mathscr{H}$.*
(b) $\mathsf{P}_\emptyset = 0$, $\mathsf{P}_{\mathbb{R}^n} = \mathbb{1}_{\mathscr{H}}$.
(c) *For $(\Omega_n)_{n\in\mathbb{N}}$ with $\Omega_n \cap \Omega_m = \emptyset$ if $n \neq m$ we have*
$$\mathsf{P}_{\bigsqcup\limits_{n=1}^{\infty}\Omega_n} = \operatorname*{s-lim}_{N\to\infty} \sum_{n=1}^{N} \mathsf{P}_{\Omega_n}.$$

Let us see why we would need this even more abstract notion compared to PVMs in Bohmian mechanics. For that, we go back again to the notion of measurement. We consider the discrete case. In the beginning, before the measurement starts, we have a combined wave function of system and apparatus[17]

$$\Psi_0 := \psi \otimes \Phi_0.$$

Then the coupled system will undergo a change during the time of the measurement T. This change is caught in the unitary time evolution U_T:

$$U_T \colon \mathscr{H} \otimes \Phi_0 \to \bigoplus_\alpha \mathscr{H} \otimes \Phi_\alpha$$
$$\Psi_0 = \psi \otimes \Phi_0 \mapsto \sum_\alpha \psi_\alpha \otimes \Phi_\alpha =: \Psi_T.$$

Above we made clear that at the end of the experiment the apparatus shows a macroscopically distinguishable result, say λ_α if the configuration of the apparatus at the end Y_T lies in the right region G_α of the configuration space. So there is a coarse-graining function (one may call it "calibration") F from configurations to numbers. This function need not be one-to-one, in general. But let us assume for now, F were a bijection. Then the probability of outcome α is:

$$\begin{aligned}\mathbb{P}^{\Psi_T}(\alpha) &= \int\limits_{F^{-1}(\lambda_\alpha)} |\Psi_T(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &= \int\limits_{G_\alpha} |\Psi_T(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &= \int \mathrm{d}x \int\limits_{G_\alpha} \mathrm{d}y \left|\sum_{\alpha'} \psi_{\alpha'}(x)\,\Phi_{\alpha'}(y)\right|^2,\end{aligned}$$

with $\Phi_{\alpha'}(y \in G_\alpha) = \delta_{\alpha,\alpha'}$ fapp :

$$\begin{aligned}&= \int |\psi_\alpha(x)|^2 \,\mathrm{d}x \\ &= \|\psi_\alpha\|^2.\end{aligned}$$

Everything is very similar to what we did before. But note that the key item of the computation lies in the single summands of the final wave function. We can write them as follows:

$$\psi_\alpha \otimes \Phi_\alpha = \mathscr{P}_{\Phi_\alpha}[U_T(\psi \otimes \Phi_0)].$$

This again allows for the definition of a family of operators R_α by:

$$\mathscr{P}_{\Phi_\alpha}[U_T(\psi \otimes \Phi_0)] =: (\mathsf{R}_\alpha\psi) \otimes \Phi_\alpha.$$

Note that we did not require the ψ_α to be orthogonal. With help of the new operators, the probability of outcome α is simply

$$\mathbb{P}^{\Psi_T}(\alpha) = \|\psi_\alpha\|^2 = \|\mathsf{R}_\alpha\psi\|^2 = \langle\psi, \mathsf{R}_\alpha^*\mathsf{R}_\alpha\psi\rangle.$$

Since these probabilites add up to 1 we find

$$\sum_\alpha \mathsf{R}_\alpha^*\mathsf{R}_\alpha = \mathbb{1}.$$

Since we already know what we are headed to, it is clear to introduce the (family of) operator(s) $\mathsf{O}_\alpha = \mathsf{R}_\alpha^*\mathsf{R}_\alpha$. Now we are there: We define the POVM $\mathsf{O}(\Lambda)$ by:

$$\mathsf{O}(\Lambda) := \sum_{\lambda_\alpha\in\Lambda} \mathsf{O}_\alpha.$$

And with this we are able to abstractly describe the experiment in question[18]: The probability of finding

[16] A less formal approach to the topic of POVMs can be found in [LB06], we follow closely [DGZ04].
[17] Notation is as above.
[18] We shall from now on restrict ourselves to calling only those experiments measurements which can be described by a self-adjoint operator, i.e. we use standard quantum mechanics terminology. Experiments without this property can still be described by a POVM, we will simply refer to them as experiments.

a result in Λ is given by $\langle\psi, \mathsf{O}(\Lambda)\,\psi\rangle$, after the experiment is done at time T, the system wave function is effectively transformed to $\psi \to \psi_\alpha = \mathsf{R}_\alpha\psi$.

We briefly remark on the case when F is not a bijection. Then there will be several λ_α that are equal to a certain λ. Hence we define the projector associated with the result λ as:

$$\mathscr{P}^{\mathsf{A}}(\lambda) := \sum_{\alpha:\ \lambda_\alpha=\lambda} \mathscr{P}_{\Phi_\alpha}$$

where A is the operator constructed from the projectors and the eigenvalues: $\mathsf{A} := \sum \lambda_\alpha \mathscr{P}_{\Phi_\alpha}$.

Now we know how to treat any experiment: using POVMs. These are the most general means of description for quantum mechanical experiments, regarding statistics. And since we derived their application for quantum mechanics above using the main feature of Bohmian mechanics, namely positions, POVMs are a prediction of Bohmian mechanics in the sense that it is extremely natural to use them to describe an experiment. This will come in very handy in the analysis of [KW10] in the following section.

Let us finish our analysis of Bohmian mechanics by regarding the detailed description of a genuine position measurement – note again, that a measurement of the position operator is in general no genuine position measurement. Assume for simplicity that the particle lives in one dimension. Clearly, if we had a measurement device actively taking part in the measurement process, we could not detect the actual position of the particle. Thus we have the case of a passive, a so-called "no y, no Φ"[19] experiment. The calibration function of our passive apparatus is then simply $F(x) = x$, so the result of the "measurement" is just the actual configuration. The position PVM we found above serves as POVM in this case, since PVMs are a subset of POVMs. In our new notation, this PVM is denoted by $\mathscr{P}^{\mathsf{X}}(\Delta)$, where Δ is the set for which we want to calculate the probability of finding the particle in it. The chance of measuring the particle's position to be in Δ then is

$$\mathbb{P}^\psi(X \in \Delta) = \langle\psi, \mathscr{P}^{\mathsf{X}}(\Delta)\,\psi\rangle,$$

where we assumed $U = \mathbb{1}$, an instantaneous measurement. The actual process of measuring takes in fact place after this formal measurement – in a detection of the initial position of the particle. This detection is again to be described together with the measurement device and it transforms the effective wave function of the particle to the one associated with the result, according to the general way to handle experiments we derived above. This fact is the key to disproving [KW10].

2.4. **Concluding remarks on Bohmian mechanics.** What did we learn about Bohmian mechanics thus far? First and foremost we saw that a realistic quantum theory can exist, since Bohmian mechanics is one and it exists. This disproves many physicists or mathematicians who proved that no such theory could exist. All these proofs have in common that additional assumptions about the theory are made, which are not part of – and in fact, violated by – Bohmian mechanics. It is then shown that these assumptions together with Bohmian mechanics yield false results. Most of these works are rigorous and the mathematics is correct - but they scarcely deal really with Bohmian mechanics. Of course, most important is that Bohmian mechanics agrees with Nature. We learned that the operator calculus of orthodox quantum theory is exactly reproduced (and in fact even derived) in Bohmian mechanics. All this is achieved without the unsharp notion "observer" or axiomatically stating what happens and what can be measured in a measurement. In the field of statistical predictions, Bohmian mechanics thus seems to describe Nature as well as conventional quantum mechanics, where the latter actually makes predictions. Yet there are differences: While orthodox quantum mechanics postulates a collapse of the wave function, no such thing happens in Bohmian mechanics. In principle it may be possible to detect the "empty" wave functions of Bohmian mechanics, i.e. the components of the wave function which do not actually guide the particle. Bohmian mechanics also yields results for problems which cannot, in fact, properly be dealt with in conventional QM and which thus does not provide testable predictions. Most often, these are experiments involving measurements of timespans.

Now we know enough about the underlying Bohmian theory to confront the paper [KW10] which once more tries to disprove Bohmian mechanics.

3. Refutation of Kiukas' and Werner's article [KW10]

Before we show that Kiukas and Werner went wrong in their paper, let us discuss some common objections to Bohmian mechanics.

3.1. **General misunderstandings of Bohmian mechanics.** When taken seriously, Bohmian mechanics is a realistic explanation of quantum mechanics in the sense that it leads to many of the axioms of orthodox quantum theory like Born's rule or the operator calculus. The ingredients of the theory are only the wave function and the particle positions[20].

[19] [DGZ04].

[20] Actually, we mean *all* the particles in the universe and the single wave function of the universe. But it is possible to reduce these quantities to the system under consideration, see Appendix A

Yet one often finds animosity towards Bohmian mechanics, which most of the time is unjustified and irrational. It is true, of course, that in conventional QM one only needs the wave function as a physical object, all the predictions follow from the wave function and a collection of rules which seem quite reasonable. And since one equation is less than two equations, many physicists tend to think that Bohmian mechanics needs more input to the theory than orthodox quantum mechanics, while it does not yield more results than the latter. But this is nonsensical because the axioms of quantum mechanics, without which the theory could not describe anything, are all predictions of Bohmian mechanics. The most important impact might be that in Bohmian mechanics it is no longer neccessary to postulate a collapse of the wave function which is not described mathematically in a dynamic law but is essentially only talk. Furthermore, Bohm's theory provides an actual physical theory of the quantum world, whereas the orthodox description consists to a great deal of talk, remember for example the many notions like "observer" or "measuring an operator". At the very least, in Bohmian mechanics one does not encounter a measurement problem.

A great many physicists believe that locality should be an important feature of any physical theory. Of course, Bohmian mechanics in nonlocal. So it cannot be a feasible theory, correct? Not at all so! First of all, Bohmian mechanics is a nonrelativistic quantum theory and thus does not, as well as conventional quantum mechanics, need to respect any locality assumption. Secondly, it is not possible in Bohmian mechanics to send a signal faster than the speed of light. This is so because of the quantum equilibrium hypothesis – the results of distant measurements of entangled particles are random and thus worthless until the experimenters can communicate about their respective findings. This communication clearly takes longer than light would need to travel the distance between the laboratories, thus no signal travelled faster than light. Furthermore, locality is often held to be equivalent to causality and thus it is stated that Bohmian mechanics is acausal. This is wrong, however, since Bohmian mechanics is nonlcal only in the "hidden variables", the wave function and the Bohmian positions, and these quantities are never observed in any setting where causality plays a role. The results of any experiment on the other hand have to respect causality, since they are classical quantities, numbers on a screen, and the classical world seems to respect causality. But as we just said the quantum equilibrium hyptothesis protects randomness of observables and thus Bohmian mechanics is causal in all quantities which can be well defined in orthodox quantum mechanics. In short, it is obvious that any acausal theory is nonlocal, but the reverse is not true in general – again, Bohmian mechanics is the counterexample to such statements.

The biggest confusion one encounters lies in the ambivalency of the term 'position' in Bohmian mechanics. On the one hand, there are particles in the theory which do occupy actual positions and follow actual trajectories. On the other hand, due to the training in conventional quantum mechanics, people tend to refer to the position operator as 'position'. However, it is totally clear in the Bohm theory that the measurement of the position of a particle (or two particles, or n, at any time or at different times) is described by the suitable position operator, be it in the Heisenberg picture time-dependent or not. This position operator which we showed to agree with that from standard quantum mechanics incorporates naturally the same statistical predictions and thus *there cannot be a difference between Bohmian and orthodox quantum theory in the measurement of positions.* It is also often stated that the Bohmian positions were classical variables. This is not true: As we will see, in most cases the Bohmian positions behave in a very unclassical, nonlocal fashion, which is obvious to anyone who took the pains to calculate a Bohmian trajectory. These trajectories carry very little information. It is interesting to see them, but for the actual workings of the theory they are of little or even no importance.

3.2. **'Maximal violations of Bell inequalities with position measurements'.** In their paper, Kiukas and Werner consider a system of two entangled particles, say a and b. Similar to the well-known EPR experiment, there are two laboratories (A and B) which are spacelike separated[21]. Then the positions of the particles are measured twice for each particle, each measurement taking place at a different time[22]. From what we learned above we know how to deal with the situation of a two-time position measurement of one particle: We have a passive measurement of position, then the detection transforms the effective wave functions to that actually guiding the particle. After that, a unitary time evolution takes place until the second measurement. Now we have two entangled particles, so we need to describe both particles. Because of entanglement, the system wave function does not factor out into the

[21]Actually, this fact does not matter at all in Bohmian mechanics. Since the entire dynamics is governed by the wave function which lives on the configuration space, it is not so much important how far the sytems are remote in physical space, but how far the supports of the wave functions are seperated in cofiguration space.

[22]The actual description of what happens is not included, however. Instead, we find a reference to [CM02], which deals with the problem much more thoroughly, but this paper is equally wrong.

single-particle wave functions. Hence the first measurement transforms the entire wave function to the component which guides both the measured and the non-measured particle. Nota bene that the effective wave function transforms to the actual guiding one, which resembles the orthodox notion of a collapsed wave function. However, this transformation, or collapse if you wish, does in no way influence the actual motion of the particle, since it has been guided by the collapsed part of the wave function since the beginning. But the transformation changes the statistics very much! This answers directly to the statements of Kiukas and Werner:

> "The simplest position is to include the collapse of the wave function into the theory. Then the first measurement instantaneously collapses the wave function. So if agreement with quantum mechanics is to be kept, the probability distribution changes suddenly. There is no way to fit this with continuous trajectories: When the guiding field collapses, the particles must jump."
> ([KW10], page 3.)

The collapse needs not to, and it must not, be included to Bohmian mechanics. There is an effective transformation of the wave function that does the job – only better, since it is formulated in the theory and takes place continuously (unitary, in other words). Here is the explicit description with obvious notation:

At first, the two instruments are in their respective null state: $\Phi_0^A(y_A)$ and $\Phi_0^B(y_B)$. Since the two particles are entangled, we need to describe them in one single, non-factorizing wave function $\psi^{a,b}(x_a, x_b)$. Then $\Psi_0 = \psi^{a,b} \otimes \Phi^A \otimes \Phi^B$. Now, suppose without loss of generality that we measure first the position of particle a at time t_1 and after that the position of particle b at t_2, each in its respective laboratory. We shall call the results of the measurements α and β, respectively. Hence we describe the measurement as follows:

$$\Psi_0 \to \Psi_{\beta,\alpha} = \left(\mathscr{P}_{\Phi_\beta^B} U_{\mathcal{M}_2}\right) U_{t_2-t_1} \left(\mathscr{P}_{\Phi_\alpha^A} U_{\mathcal{M}_1}\right) U_{t_1} \Psi_0$$

where $\mathcal{M}$ stands for measurement. We put the parentheses very suggestively, indicating each act of measurement. We can 'zoom into' this description and talk solely about what happens to the system wave function:

$$\psi^{a,b} \to \psi_{\beta,\alpha}^{a,b} = \mathsf{R}_\beta U_{t_2-t_1} \mathsf{R}_\alpha U_{t_1} \psi^{a,b}.$$

Now we see that the system wave function is transformed to $\psi_\alpha^{a,b} = \mathsf{R}_\alpha U_{t_1} \psi^{a,b}$ after the first measurement, and that it is transformed to $\psi_{\beta,\alpha}^{a,b}$ after the second one. This transformation is only of significance for the computation of statistics, however. This is so because the particle has been guided by exactly that wave function component which is found as result of the measurement, and since the supports of the other components are disjoint, these components can be neglected. This is not a collapse, it is just tidying up the mathematics! The probability of finding the result β, α is then found as usual:

$$\begin{aligned}
\mathbb{P}^\Psi(\beta,\alpha) &= \langle \mathsf{R}_\beta U_{t_2-t_1} \mathsf{R}_\alpha U_{t_1} \psi^{a,b}, \mathsf{R}_\beta U_{t_2-t_1} \mathsf{R}_\alpha U_{t_1} \psi^{a,b} \rangle \\
&= \langle U_{t_2} \mathsf{R}_\beta U_{t_2-t_1} \mathsf{R}_\alpha U_{t_1} \psi^{a,b}, \\
&\qquad\qquad U_{t_2} \mathsf{R}_\beta U_{t_2-t_1} \mathsf{R}_\alpha U_{t_1} \psi^{a,b} \rangle \\
&=: \langle \mathsf{R}_\beta(t_2)\, \mathsf{R}_\alpha(t_1)\, \psi^{a,b}, \mathsf{R}_\beta(t_2)\, \mathsf{R}_\alpha(t_1)\, \psi^{a,b} \rangle \\
&= \langle \psi^{a,b}, \mathsf{R}_\alpha^*(t_1)\, \mathsf{R}_\beta^*(t_2)\, \mathsf{R}_\beta(t_2)\, \mathsf{R}_\alpha(t_1)\, \psi^{a,b} \rangle
\end{aligned}$$

where we used that U is a one-paramter group of unitary maps. If the operators R were projections and if they commuted, we would find a consistent family of joint distributions, but in general, we do not. The bookkeeping operator for such a measurement is then obviously $\mathsf{R}_\alpha^*(t_1)\, \mathsf{R}_\beta^*(t_2)\, \mathsf{R}_\beta(t_2)\, \mathsf{R}_\alpha(t_1)$.

We saw that treating the apparatūs seriously provided us with a powerful tool to describe a measurement. But although this is an all-honest approach using what we have in the theory, defenders of orthodox quantum mechanics attack this approach and call it 'contextual', meaning that the theory depends on the measurement setting. This position is schizophrenia: whereas Bohmian mechanics is a theory describing Nature and also, since it is a subset of Nature, the measurement process, conventional quantum mechanics on the contrary talks only about results of measurements. To overcome this weakness, certain postulations ensure that the results of measurements of the same quantity always yield the same statistics. A very cheap solution! The act of measuring the particle obviously changes the system, and what comes next is predetermined in Bohmian mechanics by the resulting wave function and configuration of the system – which is not the product of a collapse but is predicted by the theory.

> "The downside of this argument is that it also applies to single time measurements, i.e., the agreement between Bohm–Nelson configurational probabilities and quantum ones is equally irrelevant. The naive version of Bohmian theory holds "position" to be special, even "real," while all other measurement outcomes can only be described indirectly by including the measurement devices. Saving the Nelson–Bohm theory's failure regarding two-time two-particle correlations by going contextual also for position just means that the particle positions

> are declared *unobservable according to the theory itself*, hence truly hidden." ([KW10], page 3.)

A major drawback of Bohmian mechanics? Of course not. Simply stating this is worth nothing – if the authors had tried to justify their statement, they would have encountered the simple fact that they are wrong. We already showed above that it is very well possible to describe a position measurement using a passive apparatus. This is a rather delicate process, since it involves a very fine-tuned device. Apparatus and system have to be seperated very well, which is possible in principle, but hard to do experimentally. Supposed we were able to prepare our quantum system and the measurement devices in such a good way, nothing would keep us from measuring the Bohmian position of a particle very precisely. But after we have done so, the position and especially the wave function of the system will have changed due to the detection of the initial position. Thus it is a simple fact that the first of a sequence of measurements changes the result of the following measurements. The dynamics of the Bohmian positions changes after a measurement since it affects the wave function, and when dynamics change, the positions will be different from what they would have been without the first measurement. This process is reflected by applying the ordinary Heisenberg position operator to the system wave function in order to determine the statistics of many position measurements, so there is no difference to standard quantum mechanics. However, we see again that the contextual point of view explains *why* the Heisenberg operator describes the distribution of experimental results. It is remarkable that this fact, which lies in the heart of Bohmian mechanics, shall be stated as a weakness of the theory. We believe that this misunderstanding comes from the fact that the authors hold the Bohmian positions to be classical variables, which is mere nonsense as we already stated above. They are exactly right when they write:

> "This certainly explains the apparent paradox that Bohmians on the one hand place so much value on being able to say where the particles really are, but are, on the other hand, so remarkably uninterested in actually computing trajectories." ([KW10], page 3.)

Sadly, this correct insight is directly afterwards led to a wrong conclusion:

> "But when the interest in the real trajectories is gone, the only gain from the whole theory seems to be a pro forma justification for saying that the hand of a voltmeter is really somewhere. The mountain in labor gave birth to a mouse." ([KW10], page 3.)

It is baffling that there is such a widespread attitude to neglect the fact that in othodox quantum mechanics one talks about nothing. Not only is it conceptually important to have a theory of something, but we also saw how fruitful the positions of particles are when building the theory. Also, the "mountain in labor" was very small, actually it is very ironically to state this and then spend an article on disproving the "mouse" that was born. It seems that this unjustified attack on Bohmian mechanics was only included beacause the authors very well realized that the contextual theory that Bohmian mechanics is eludes their refuation.

Those were the arguments of Kiukas and Werner "refuting" Bohmian mechanics. We saw that they are not valid. But it is a remarkable fact that Bohmian mechanics even predicts its "refutations", as long as they agree with Nature. So we are going to show that the operator suggested in [KW10] encodes a POVM and thus is completely covered by Bohmian mechanics.

The position measurements each performed by 'Alice' and 'Bob', as the two experimenters in the seperated laboratories are usually called, are of course $\mathscr{P}_1^{A/B} = \chi_{\Delta_1}(\mathsf{Q}^{A/B})$ and $\mathscr{P}_2^{A/B} = \chi_{\Delta_2}\left(\mathsf{Q}_t^{A/B}\right)$, for A and B respectively. We already assumed, as Kiukas and Werner do in their work, that both experimenters choose the same position intervals, i.e. $\Delta_1^A = \Delta_1^B$ and $\Delta_2^A = \Delta_2^B$ and time t. As we follow their paper, we use a similar notation, thus Q is the position operator. As the labs are far removed, it suffices to speak about one experimenter only, so we choose Alice's laboratory and drop the index A. Now, each of these projectors is a PVM on $\mathbb{R}$. These projectors are then used to define operators for Alice and Bob, for example: $\mathsf{A}_1 = 2\chi_{\Delta_1}(\mathsf{Q}) - \mathbb{1}$ which is equivalent to $\mathsf{A}_1' = \chi_{\Delta_1}(\mathsf{Q}) - \chi_{\Delta_1^c}(\mathsf{Q}) = +1\mathscr{P}_1 + (-1)\,\mathscr{P}_1^c$, where $\cdot^c$ stands for the complement of $\cdot$. The structure of the definition of these operators is the same as above: we find a PVM and use the spectral theorem to construct an operator encoding the statistics. Since each experimenter conducts two measurements, and Kiukas and Werner are only interested in the violation of Bell inequalities, i.e. the commutators $\mathsf{A}_3 := \frac{1}{2\mathrm{i}}[\mathsf{A}_1, \mathsf{A}_2]$, they define an operator $\mathsf{C} := \mathbb{1} - \mathscr{P}_1 - \mathscr{P}_2 + \mathscr{P}_1\mathscr{P}_2 + \mathscr{P}_2\mathscr{P}_1$. Pulled back to the level of the projectors, the commutators of interest are then $[\mathscr{P}_1, \mathscr{P}_2]$ (and of course, as always, the analog for Bob), for which is found

$$[\mathscr{P}_1, \mathscr{P}_2]^* [\mathscr{P}_1, \mathscr{P}_2] = \mathsf{C}\,(\mathbb{1} - \mathsf{C})\,,$$

so C carries all information about the commutators and with that about the Bell violations. C finally is restricted to the subspace of all φ with $\mathscr{P}_1\varphi = \varphi$, i.e. the states in Δ_1, but the kernel of the commutator is removed, since only non-commuting states can violate a Bell inequality. This restriction of C is called H and is the operator of interest in [KW10]. On the fundamental level, this experiment is, as we have seen, described by PVMs which are connected via unitary time evolution. This altogether yields a POVM for Alice and one for Bob. Hence it is a Bohmian experiment, which comes as no surprise, as all quantum measurements are Bohmian in a Bohmian universe.

4. Conclusion

We saw that the contextual understanding of quantum theory in the setting of Bohmian mechanics yields a consistent description of Nature and Its subsets, for example measurements. We also found that Bohmian mechanics not only agrees with orthodox quantum mechanics on an experimental level, it is clear that Bohmian mechanics predicts usual quantum mechanics in a very elegant manner. The paper [KW10] did not really attack Bohmian mechanics, since the authors themselves state what saves Bohmian mechanics, but they seemingly did not know that this point, to wit, the contextuality, is already a central part of Bohmian mechanics. The mathematical analysis they provide then does not apply any pressure to Bohmian mechanics. We hope that the elegance of Bohmian mechanics and its consistent description of Nature were convincing for the reader, so that Bohmian mechanics is at least appreciated as a strong alternative to orthodox quantum mechanics.

Appendix A. Effective wave function

Since the notion of an effective wave function is probably not known for readers unfamiliar with Bohmian mechanics, we shall review it here.

At first, there is only one wave function in Bohmian mechanics: that of the entire universe. This might be a little repelling, since we want to describe a small subsystem of the universe like an atom, not the universe. Additionally, we do not have the means to detect the state of the whole space. Hence, we need to find an object similar to the wave function which guides the particles and provides the correct statistics but takes only the coordinates of the system as arguments. In the optimal case this object also obeys the Schrödinger equation. This object will be the effective wave function.

Before we can introduce the effective wave function, we wish to show an object similar in meaning but somewhat weaker: the conditional wave function. It is obtained by simply replacing the generic positions of all particles we are not interested in by their actual values. We denote the conditional wave function by $\varphi^Y(x)$, as above x and y stand for the generic positions of the system and the rest of the universe, respectively, uppercase letters will then denote actual positions. Then this is the conditional wave function:

$$\varphi^Y(x) := \frac{\Psi(x, Y)}{\|\Psi(Y)\|}$$

with $\|\Psi(Y)\|^2 := \int |\Psi(x, Y)|^2 \, \mathrm{d}^m x$. Clearly, this function is not neccessarily a solution to the Schrödinger equation (1) except for special cases, and we will not know the actual configuration of the y-space.

Now, the effective wave function will be obtained similarly and it will obey Schrödinger's equation. Here is the definition:

Definition. A system has an *effective wave function* ψ if the combined wave function of system and environment can be written as

$$\Psi(x, y) = \psi(x)\,\Phi(y) + \Psi^{\perp}(x, y)$$

and $Y \in \operatorname{supp}\Phi$. Also, $\operatorname{supp}\Psi^{\perp}(x, \cdot) \cap \operatorname{supp}\Phi(\cdot) \approx \emptyset$ since the apparatus states are disjoint.

The effective wave function ψ then obeys equation (1) as long as the interaction term in the Schrödinger equation allows for the splitting of the wave function. Note that we did not need to know the positions of all particles in the universe, we just demand that they are restricted to $\operatorname{supp}\Phi$. For more on that topic, see [DGZ92].

References

[Bel04] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics: Collected Papers on Quantum Philosophy*, 2^{nd} ed., Cambridge University Press, 6 2004.

[Boh52] David Bohm, *A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. I and II*, Phys. Rev. **85** (1952), no. 2, 166–179 and 180–193.

[CHSH69] John F. Clauser, Michael A. Horne, Abner Shimony, and Richard A. Holt, *Proposed Experiment to Test Local Hidden-Variable Theories*, Phys. Rev. Lett. **23** (1969), no. 15, 880–884.

[CM02] M. Correggi and G. Morchio, *Quantum Mechanics and Stochastic Mechanics for Compatible Observables at Different Times*, Annals of Physics **296** (2002), no. 2, 371 – 389.

[DGTZ09] D. Dürr, S. Goldstein, R. Tumulka, and N. Zanghi, *Bohmian Mechanics*, ArXiv e-prints (2009), arXiv:0903.2601[quant-ph].

[DGZ92] Detlef Dürr, Sheldon Goldstein, and Nino Zanghí, *Quantum equilibrium and the origin of absolute uncertainty*, Journal of Statistical Physics **67** (1992), 843–907.

[DGZ04] Detlef Dürr, Sheldon Goldstein, and Nino Zanghì, *Quantum Equilibrium and the Role of Operators as Observables in Quantum Theory*, Journal of Statistical Physics **116** (2004), 959–1055.

[DT09] Detlef Dürr and Stefan Teufel, *Bohmian Mechanics: The Physics and Mathematics of Quantum Theory*, 1$^{\text{st}}$ ed., Springer, 5 2009.

[Hei86] Werner Heisenberg, *Quantentheorie und Philosophie: Vorlesungen und Aufsätze*, Reclam, Ditzingen, 1986.

[KW10] J. Kiukas and R. F. Werner, *Maximal violation of Bell inequalities by position measurements*, Journal of Mathematical Physics **51** (2010), no. 7, 072105.

[LB06] Michel Le Bellac, *Quantum physics*, Cambridge University Press, 4 2006.

[RS81] Michael Reed and Barry Simon, *I: Functional Analysis, Volume 1 (Methods of Modern Mathematical Physics)*, REVISED AND ENLARGED ed., Academic Press, 1 1981.

[Tum04] Roderich Tumulka, *Understanding Bohmian mechanics: A dialogue*, American Journal of Physics **72** (2004), no. 9, 1220–1226.

Hiermit versichere ich, dass ich die vorliegende Arbeit selbständig und nur mit den angegebenen Hilfsmitteln und Quellen verfasst habe.

Robin Schlenga · München, den 7. Juli 2011.

Ludwig-Maximilians-Universität München
E-mail address: robin.schlenga@physik.uni-muenchen.de
URL: http://www.rschlenga.de/