Asymptotic Behavior of Bohmian Trajectories in Scattering Situations and Exit Statistics for Distant Surfaces

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1 Introduction

Bohmian mechanics [8, 5, 14, 16, 17] is a complete quantum theory about the motion of point particles from which the usual quantum formalism can be derived by an analysis of "measurement" [16].
The motion of spinless non-relativistic particles is defined by the ordinary differential equation (4) below, that depends on the quantum mechanical wave function of the system under consideration.
While particles in Bohmian mechanics in general have highly non-Newtonian trajectories, we will show how, in the special context of potential scattering theory, their long time asymptotes retain some "classical" features. We shall use those features to prove that for short range potentials $V(q)$ falling off like $|q|^{-4-\varepsilon}$ for $|q| \to \infty$ the exit statistics for a surface far away from the scattering center made by a wave function $\Psi = \Psi^{ac} + \Psi^{pp} \in \mathcal{H}_{ac}(H) + \mathcal{H}_{pp}(H)$ are up to the squared norm $\|\Psi^{pp}\|^2$ of the bound part of the wave function the same as those made solely by the scattering part Ψ^{ac}.
For $V = 0$ (i.e. for free particles) almost all Bohmian trajectories asymptotically behave like trajectories in classical mechanics; the particles asymptotically move with a uniform velocity (Section 3). For the related theory of Nelson, stochastic mechanics, the same was proved by Shucker [28].
In Section 4 we turn to the behavior of Bohmian trajectories if V is a short-range scattering potential. Then, for suitable potentials, there are two distinct classes of wave functions, scattering wave functions Ψ^{ac} (that belong to the absolute continuous spectral subspace $\mathcal{H}_{ac}(H)$ of the Hamiltonian H) and bound wave functions Ψ^{pp} (that belong to the pure point spectral subspace $\mathcal{H}_{pp}(H)$). That the former become asymptotically free for $t \to \infty$ (they tend in L_2 to asymptotic outgoing waves Ψ^{out} that evolve according to the free time evolution), is reflected in the behavior of their Bohmian trajectories. Almost all their long time asymptotes behave like those of the trajectories of Ψ^{out} and thus like trajectories in classical mechanics again (Subsection 4.1).
In Subsection 4.2 we deal with more general wave functions $\Psi = \Psi^{ac} + \Psi^{pp}$ that have both a scattering and a bound part. We show that the long time asymptotes of the Bohmian trajectories split according to the splitting of the wave function. The "scattering" part moves out to spatial infinity linear in time and becomes free and classical in the same sense as above. The situation is somewhat different for the "bound" part. Since a bound wave Ψ^{pp} stays in the sphere of influence of the potential V even in the long time limit, it should depend on the exact form of the potential V (how "strong" it is) and on Ψ^{pp} itself wether "bound" Bohmian trajectories behave like classical trajectories or not. This is an aspect of the classical limit for Bohmian mechanics that we will not deal with here. However, we prove that, under certain conditions

on $\nabla\Psi^{pp}$ and regardless of the exact form of V, almost all "bound" trajectories stay inside a ball around the origin with a radius growing only sublinear in time. So, while we cannot say that a "bound" Bohmian trajectory stays bound in the sense a classical trajectory would, it certainly stays bound in the weaker sense that it can move out to spatial infinity only on an much larger time scale than a "scattering" Bohmian trajectory.
While the main concern of Section 4 is the long time behavior of wave functions Ψ_t and their Bohmian trajectories, in Section 5 we will discuss its connection to the scattering cross section and thus to experiment.
For $\Psi_t^{ac} = e^{-iHt}\Psi_0^{ac} \in \mathcal{H}_{ac}(H)$ one such connection is given by Dollard's scattering into cones theorem [12]. Assuming asymptotic completeness of the wave operators it asserts that the probability of finding a particle in a cone $C \subset \mathbb{R}^3$ with vertex at the origin is in the long time limit the same as that of finding the quantum mechanical momentum of the asymptotic outgoing wave Ψ_0^{out} in the same cone,

$$\lim_{t\to\infty} \int_C |\Psi_t^{ac}(q)|^2\, d^3q = \int_C |\hat{\Psi}_0^{out}(k)|^2\, d^3k. \tag{1}$$

Since one can derive from it the expression of the differential cross section $\frac{d\sigma}{d\omega} = |f(\theta,\phi)|^2$ of time independent scattering theory (see e.g. [4]), (1) is regarded as fundamental for quantum mechanical scattering theory.
However, in a scattering experiment one will typically not look at the probability wether or not there is a particle in a given cone at a given time but rather at that wether or not a particle crosses a given distant (detector) surface in a given time interval. It was Combes, Newton and Shtokhamer [9] who first gave the heuristically clear notion that the latter should be given by integrating the quantum probability flux j^Ψ over the surface and the time interval in question the form of a mathematical rigorous theorem, the flux-across-surfaces theorem (FAST)

$$\lim_{R\to\infty} \int_0^\infty dt \int_{R\Sigma} j^{\Psi^{ac}}(q,t)\cdot \hat{n}\, d\sigma = \int_{C_\Sigma} |\hat{\Psi}_0^{out}(k)|^2\, d^3k, \tag{2}$$

where Σ is a measurable subset of S_1, the sphere with radius 1, $R\Sigma := \{Rq \in \mathbb{R}^3 \mid q \in \Sigma\}$ and $C_\Sigma := \{\lambda q \in \mathbb{R}^3 \mid q \in \Sigma,\ \lambda \geq 0\}$ is the cone spanned by Σ.
Meanwhile (2) has been proved for several classes of scattering wave functions Ψ^{ac} and potentials V [10, 19, 2, 3, 31, 11, 24, 23, 13]. With the help of (2) Dürr, Goldstein, Moser and Zanghì derived the scattering cross section for short-range potentials and pure scattering wave functions in a rigorous limit procedure [15](see also [22]).

Since a realistic scattering experiment is performed on a large but nevertheless finite scale there is however no reason to restrict oneself to pure scattering wave functions Ψ^{ac}. On the contrary, the preparation of a beam of states in a scattering experiment will in general produce wave functions with bound components (see [15]: Subsection 5.1 and Section 7). Thus in Section 5 we exploit the splitting of the asymptotic Bohmian trajectories into "scattering" and "bound" ones (described above) to prove a slightly modified version of (2) for general wave functions $\Psi = \Psi^{ac} + \Psi^{pp}$. Using this FAST we can deduce the following for the exit statistics through a surface at a distance R from the scattering center:

- For every wave function $\Psi = \Psi^{ac} + \Psi^{pp} \in \mathcal{H}_{ac}(H) + \mathcal{H}_{pp}(H)$ there is some time $t(R)$ with $t(R) \to \infty$ as $R \to \infty$ such that the exit statistics until this time $t(R)$ are – in the limit $R \to \infty$ – completely determined by the scattering part Ψ^{ac} of the wave function alone.
- Also, the exit statistics for all time are – up to an error of order $\|\Psi^{pp}\|^2$– induced solely by the scattering part Ψ^{ac} of the wave function.

Recall that states in a scattering experiment are usually prepared far away from the scattering center (if not at spatial infinity) so the bound component Ψ^{pp} of the wave function will be small in L_2-sense, $\|\Psi^{pp}\| < \varepsilon$. Then the difference in the exit statistics is at most of order ε^2.
We start with a brief account of Bohmian mechanics in Section 2.

2 Bohmian Mechanics

In Bohmian mechanics the state of a system of N spinless, non-relativistic particles is described by its quantum mechanical wave function $\Psi_t(q)$, where $q = (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n$ $(n = 3N)$, and by its actual configuration $Q = (Q_1, Q_2, \ldots, Q_n) \in \mathbb{R}^n$, where the Q_k are the positions of the particles. The wave function evolves according to the Schrödinger equation

$$i\hbar \frac{\partial \Psi_t}{\partial t} = H\Psi_t \tag{3}$$

and governs the motion of the particles by

$$\frac{dQ_k}{dt} = v_k^\Psi(Q,t) := \frac{\hbar}{m_k} \mathrm{Im}\Big(\frac{\nabla_k \Psi_t(Q)}{\Psi_t(Q)}\Big). \tag{4}$$

Here the m_k are the masses of the particles and $\nabla_k = \frac{\partial}{\partial q_k}$. In (3) H is the usual non-relativistic Schrödinger Hamiltonian

$$H = -\frac{1}{2}\sum_{k=1}^{N} \frac{1}{m_k} \triangle_k \ + V(q) =: H_0 + V(q) \tag{5}$$

with the non-relativistic interaction potential V[1].
The dynamical system defined by Bohmian mechanics is naturally associated with a family of finite measures $\mathbb{P}^{\Psi_t}$ given by the densities $\rho^{\Psi_t}(q) := |\Psi_t(q)|^2$ on configuration space $\mathbb{R}^n$[2]. If at some time t_0 we start with a random distribution for the configuration q of the system given by $\rho_{t_0} = \rho^{\Psi_{t_0}}$, for any other time t the density which this is transported to by (4) will be given by $\rho_t = \rho^{\Psi_t}$. This property is called equivariance [17]. More precisely, let $\Phi_{t,t_0} : \mathbb{R}^n \to \mathbb{R}^n$ be the flow map of (4), i.e., if q is the initial configuration at time t_0, $\Phi_{t,t_0}(q)$ is the configuration at time t which q is transported to by (4). Then the density ρ_{t_0} is transported to $\rho_t = \rho_{t_0} \cdot \Phi^{-1}_{t,t_0} = \rho_{t_0} \cdot \Phi_{t_0,t}$. We say that the functional $\Psi_t \mapsto \mathbb{P}^{\Psi_t}$, from wave functions to the finite measures $\mathbb{P}^{\Psi_t}$ (given by the densities ρ^{Ψ_t}) on configuration space, is equivariant if the diagram

$$\begin{array}{ccc} \Psi_{t_0} & \xrightarrow{U_{t-t_0}} & \Psi_t \\ \downarrow & & \downarrow \\ \rho^{\Psi_t(t_0)} & \xrightarrow[\mathcal{F}_{t,t_0}]{} & \rho^{\Psi_t} \end{array}$$

[1]More rigorously: H is a self-adjoint extension of $H|_{C_0^\infty(\mathbb{R}^n)} = H_0 + V$ (with H_0 as above and $V : \mathbb{R}^n \to \mathbb{R}$) on the Hilbertspace $L_2(\mathbb{R}^n)$ with domain $\mathcal{D}(H)$

[2]Of course, for $\rho^{\Psi_t}(q)$ to define a measure Ψ_t must be normalized, i.e. the L_2 - norm $\|\Psi_t\| = \big(\int_{\mathbb{R}^{3N}} |\Psi_t(q)|^2\, dq\big)^{\frac{1}{2}}$ must be equal to 1.

commutes. Here $U_t = e^{-iHt}$ is the solution map for the Schrödinger equation (3) and $\mathcal{F}_{t,t_0}$ is the solution map for the natural evolution on densities arising from (4), i.e. $\mathcal{F}_{t,t_0}(\rho^{\Psi_{t_0}}) = \rho^{\Psi_{t_0}} \cdot \Phi_{t_0,t}$ (see above).
Equivariance follows from comparing the classical continuity equation

$$\frac{\partial}{\partial t}\rho_t(q) \; + \nabla \cdot \rho_t(q) v(q,t) = 0 \tag{6}$$

with the quantum continuity equation

$$\frac{\partial}{\partial t}|\Psi_t(q)|^2 \; + \nabla \cdot j^{\Psi}(q,t) = 0, \tag{7}$$

where the quantum probability flux j^{Ψ} is given by

$$j_k^{\Psi}(q,t) := \frac{\hbar}{m_k}\mathrm{Im}\big(\Psi_t(q)^* \nabla \Psi_t(q)\big) = v_k^{\Psi}(q,t)|\Psi_t(q)|^2. \tag{8}$$

On the family of measures $\mathbb{P}^{\Psi_t}$ we bestow the role usually played by the stationary "equilibrium measure"[3]. Thus $\mathbb{P}^{\Psi_t}$ defines our notion of typicality [17], which by equivariance is time independent. Let $A \subset \mathbb{R}^n$ be measurable. Then by equivariance

$$\begin{aligned}
\mathbb{P}^{\Psi_{t_1}}(A) &= \int\limits_{\mathbb{R}^n} \chi_A(q)|\Psi_{t_1}(q)|^2\, dq = \int\limits_{\mathbb{R}^n} \big(\chi_{\Phi_{t_2,t_1}(A)} \cdot \Phi_{t_1,t_2}\big)(q)|\Psi_{t_1}(q)|^2\, dq = \\
&= \int\limits_{\mathbb{R}^n} \chi_{\Phi_{t_2,t_1}(A)}(q)\big(|\Psi_{t_1}|^2 \cdot \Phi_{t_2,t_1}\big)(q)\, dq = \\
&= \int\limits_{\mathbb{R}^n} \chi_{\Phi_{t_2,t_1}(A)}(q)|\Psi_{t_2}(q)|^2\, dq = \mathbb{P}^{\Psi_{t_2}}\big(\Phi_{t_2,t_1}(A)\big),
\end{aligned} \tag{9}$$

where χ_A is the characteristic function of A that is one on A and zero elsewhere.
From now on we will set $\hbar = m_k = 1$ (without real loss of generality).

2.1 Global Existence of Bohmian Mechanics

Up to now we tacitly assumed that Bohmian mechanics exist globally (i.e. the particle trajectories are well defined for all times) for every Hamiltonian H as in (5), every wave function Ψ and all initial times t_0 and initial configurations

[3]Since in most cases the velocity field defined in (4) will be explicitly time dependent one cannot expect to find a stationary measure.

$q_{t_0} \in \mathbb{R}^n$. Of course this is not true. The velocity field as defined in (4) need not be well defined at singularities of the potential V (since there Ψ need not be differentiable) and surely is ill defined at the nodes of Ψ. A trajectory could also escape to infinity in finite time. However, Berndl et al. [6] showed $\mathbb{P}^{\Psi}$-almost sure global existence of Bohmian mechanics for suitable potentials and initial wave functions. While their proof is for spinless non-relativistic particles only, Teufel and Tumulka [32] recently gave an alternative proof that can be applied to any Bohm-type dynamics (e.g. Bohm-Dirac theory) and for spinless non-relativistic particles uses conditions on potential and wave function that are somewhat more general than those in [6].
Those conditions are:

A 1. *The potential V is locally in L_2 outside at most finitely many singularities: $V \in L_2^{loc}(\Omega)$, where Ω is the configuration space, $\Omega = \mathbb{R}^n \backslash \{q \in \mathbb{R}^n \mid V(q) \text{is singular}\}$.*

A 2. *The initial wave function Ψ_0 is in the domain of H, $\Psi_0 \in \mathcal{D}(H)$, and is normalized, $\|\Psi_0\| = 1$. Moreover $\Psi_t = e^{-iHt}\Psi_0$ is two times continuous differentiable, $\Psi \in C^2(\Omega \times \mathbb{R})$.*

A 3. *For all $0 < T < \infty$ there is some $C_T < \infty$ such that $\sup\limits_{|t| \leq T} \|\nabla \Psi_t\| < C_T$.*

Remark 1. $\Psi \in C^2(\Omega \times \mathbb{R})$ implies that v^{Ψ} is a C^1-function on $(\Omega \times \mathbb{R}) \backslash \mathcal{N}$, where $\mathcal{N} = \{(q,t) \in (\Omega \times \mathbb{R}) \mid \Psi_t(q) = 0\}$ is the set of nodes of Ψ.

Proposition 1. *Assume A1 - A3. Then for $\mathbb{P}^{\Psi_0}$-almost all $q \in \Omega$ the solution* $\mathrm{Q}(q,t)$ *of* (4) *starting at $Q(0) = q$ exists for all times $t \in \mathbb{R}$.*

The proof can be found in [32].

Remark 2. Since the set of singularities of the potential V consists of at most finitely many points, it has Lebesgue measure zero and thus also $\mathbb{P}^{\Psi_0}$-measure zero. So Proposition 1 holds also with "$\mathbb{P}^{\Psi_0}$-almost all $q \in \Omega$" replaced by "$\mathbb{P}^{\Psi_0}$-almost all $q \in \mathbb{R}^n$" (recall $\Omega = \mathbb{R}^n \backslash \{q \in \mathbb{R}^n \mid V(q) \text{ is singular}\}$).

Remark 3. If $V = V_1 + V_2 \in C^{\infty}(\Omega)$ (with Ω as in A1), V_1 is bounded from below and V_2 is H_0-bounded with relative bound < 1, then the form sum $H = H_0 + V$ is a self adjoint extension of $H|_{C_0^{\infty}(\mathbb{R}^n)}$. Moreover for $\Psi_t = e^{-iHt}\Psi_0$ with $\Psi_0 \in C^{\infty}(H) = \bigcap\limits_{l=1}^{\infty} \mathcal{D}(H^l)$ and $\|\Psi_0\| = 1$ A2 and A3 hold (Ψ is even in $C^{\infty}(\Omega \times \mathbb{R})$). Thus the Bohmian trajectories $\mathrm{Q}(q,t)$ exist globally in time for almost all $q \in \mathbb{R}^n$. For a proof see [6] (Corollary 3.2) or [32] (Corollary 4).

2.2 Some Notation

By Ω_{t_0} we denote the configuration space Ω without the "bad" points q for which the solution of (4) starting at q at time t_0 does not exist for all times. Since they differ only by sets of $\mathbb{P}^{\Psi_0}$-measure zero (Proposition 1 and Remark 2 above) we shall mostly not discern between $\mathbb{R}^n$, Ω and Ω_{t_0} and call all three "the configuration space". It will prove convenient to define most quantities on the whole of $\mathbb{R}^n$.

Further we adopt the following conventions for the solutions of (4).

$$\mathrm{Q}_{t_0}(q,t) := \Phi_{t,t_0}(q)$$

and

$$\mathrm{Q}(q,t) := \mathrm{Q}_0(q,t) = \Phi_{t,0}(q) =: \Phi_t(q)$$

for all t, $t_0 \in \mathbb{R}$ and $q \in \Omega_{t_0}$.

3 Bohmian Trajectories for Free Particles

In classical non-relativistic mechanics particles in systems with potential zero move with uniform velocity v (possibly zero), i.e. their trajectories are straight lines. We show that in the long time limit the same is true for $\mathbb{P}^{\Psi}$-almost all free trajectories in Bohmian Mechanics. In addition we find in accordance with orthodox quantum theory that the density of the probability distribution for the asymptotic velocity v_∞ is given by $|\hat{\Psi}_0|^2$ whenever $|\Psi_0(q)|^2$ is the probability density for finding a particle at q at time 0. Here Fourier transformation is denoted by $\widehat{\ }$.
In our proof we follow Shucker [28] who showed essentially the same thing for stochastic mechanics.
We consider a system of N freely moving spinless non-relativistic particles. Then the Hamiltonian of the system is given by $H_0 := -\frac{1}{2}\triangle$ where $\triangle = \nabla^2$ is the Laplace operator in $n = 3N$ dimensions. Let $\Psi_t = e^{-iH_0t}\Psi_0$ with $\Psi_0 \in S(\mathbb{R}^n)$, the set of Schwartz functions, such that A2 holds. Note that for $H = H_0$ $\Omega = \mathbb{R}^n$, so A1 is automatically fulfilled. Moreover $\Psi_0 \in S(\mathbb{R}^n)$ guarantees A3, i.e. we have almost sure global existence of Bohmian mechanics.
We state our main result as two corollaries to a rather technical theorem (Theorem 1).

Corollary 1. *Let $\Psi_0 \in S(\mathbb{R}^n)$ and $\Psi_t = e^{-iH_0t}\Psi_0$. Assume A2. Then $v_\infty(q) := \lim\limits_{t\to\infty} \frac{\mathrm{Q}(q,t)}{t}$ exists for $\mathbb{P}^{\Psi}$-almost all $q \in \mathbb{R}^n$ and the distribution of v_∞ has density $|\hat{\Psi}_0|^2$.*

Corollary 2. *Let $\Psi_0 \in S(\mathbb{R}^n)$ and $\Psi_t = e^{-iH_0t}\Psi_0$. Assume A2. Then for all $\varepsilon > 0$ and $\delta > 0$ there exists some $T^{\Psi}_{\epsilon\delta} > 0$ such that*

$$\mathbb{P}^{\Psi_0}\big(\{q \in \mathbb{R}^n \mid \sup_{t\geq T^{\Psi}_{\epsilon\delta}} |v^{\Psi}\big(\mathrm{Q}(q,t),t\big) - v_\infty(q)| < \delta\}\big) > 1 - \varepsilon. \tag{10}$$

The structure of the proof is the following. First we establish some estimates on the wave function Ψ and the velocity field v^{Ψ} (Proposition 2 and Lemma 1). Next we will find subsets of initial configurations q at some large time T that have $\mathbb{P}^{\Psi_T}$-measure arbitrarily close to 1 and guarantee that the velocity field v^{Ψ} is well behaved for all times $t \geq T$ in the sense that $\big|v^{\Psi}(\mathrm{Q}(q,t),t) - \frac{\mathrm{Q}(q,t)}{t}\big| < \delta$ for arbitrary small $\delta > 0$ and all $t \geq T$ (Theorem 1). Then Corollary 2 and Corollary 1 are indeed easy consequences of Theorem 1.

Proposition 2. *Let* $\Psi_0 \in S(\mathbb{R}^n)$ *and* $\Psi_t = e^{-iH_0 t}\Psi_0$*. Then*

$$\begin{aligned}\Psi_t(q) &= (2\pi i t)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} e^{i\frac{|q-y|^2}{2t}} \Psi_0(y)\, d^n y = \\ &= (it)^{-\frac{n}{2}} e^{\frac{iq^2}{2t}} \hat{\Psi}_0(\frac{q}{t}) + (2\pi i t)^{-\frac{n}{2}} e^{\frac{iq^2}{2t}} \int\limits_{\mathbb{R}^n} e^{-i\frac{q\cdot y}{t}} (e^{\frac{iy^2}{2t}} - 1)\Psi_0(y)\, d^n y = \\ &=: \varphi_1(q,t) + \varphi_2(q,t)\end{aligned} \tag{11}$$

and for all $r \in \mathbb{R}_0^+$ *there is a* $c_r < \infty$ *such that*

$$|\varphi_2(q,t)| \le c_r t^{-\frac{n}{2}-1} (\frac{t}{|q|})^r \quad \textit{and} \quad |\nabla\varphi_2(q,t)| \le c_r t^{-\frac{n}{2}-1} (\frac{t}{|q|})^r \tag{12}$$

for all $t > 0$ *and* $q \in \mathbb{R}^n \backslash \{0\}$.

The proof can be found in [19] (proof of 2.7).

Remark 4. Recall that we want the long time asymptotes of the Bohmian trajectories $\mathrm{Q}(q,t)$ to become straight lines, i.e. $\frac{\mathrm{Q}(q,t)}{t}$ to be of order one (for $t \to \infty$). This desire is mirrored by our choice of wave functions. For $\Psi_0 \in S(\mathbb{R}^n)$ it turns out that relative to Ψ_t we can treat $\frac{q}{t}$, too, as if it was of order one in the sense that multiplication of Ψ_t or $\nabla\Psi_t$ by $\left(\frac{q}{t}\right)^r$ (where $r \in \mathbb{R}_0^+$ is arbitrary) does not alter how fast Ψ_t or $\nabla\Psi_t$ decays in t. This can be read of from Proposition 2 as follows.
In (11) $\Psi_t(q)$ is split into $\varphi_1(q,t)$ and $\varphi_2(q,t)$. While (12) ascertains the desired behavior for φ_2, for φ_1 we get it by noting that $\Psi_0 \in S(\mathbb{R}^n)$ and thus $\left(\frac{q}{t}\right)^m \hat{\Psi}_0(\frac{q}{t})$ is bounded for every $m \in \mathbb{N}_0$. So by (11)

$$|\varphi_1(q,t)| \le c_m t^{-\frac{n}{2}} \left(\frac{t}{|q|}\right)^m$$

for all $m \in \mathbb{N}_0$ and some $c_m < \infty$. But this can be easily extended to hold for all $r \in \mathbb{R}_0^+$; for $\frac{t}{|q|} \le 1$ respectively $\frac{t}{|q|} > 1$, $\left(\frac{t}{|q|}\right)^r$ is bounded by $\left(\frac{t}{|q|}\right)^m$ where m is the smallest integer larger than r resp. the largest integer smaller than r. In the same way we get bounds for $\nabla\varphi_1$.
Later on the ability to treat $\frac{q}{t}$ as essentially constant in certain circumstances will come in handy since it allows us to "translate" decay in time into spatial decay. So we keep the full statement of (12) although in this Section (and the next) we will use it with $r = 0$ only.

Lemma 1. *Let $\Psi_0 \in S(\mathbb{R}^n)$ and $\Psi_t := e^{-iH_0 t}\Psi_0$. Then*

$$\lim_{t\to\infty} \int\limits_{\mathbb{R}^n} \big| |\Psi_t|^2 - t^{-n} \big|\hat{\Psi}_0(\frac{q}{t})\big|^2 \big| \, d^n q = 0 \tag{13}$$

and there exists a $C_1 < \infty$ such that for all $q \in \mathbb{R}^n$ and all $t > 0$ with $\Psi_t(q) \neq 0$

$$\big| v^{\Psi}(q,t) - \frac{q}{t} \big| \leq C_1 t^{-1-\frac{n}{2}} |\Psi_t(q)|^{-1}. \tag{14}$$

Although the first part of Lemma 1 is well known (see e.g. [12, 14, 27]) its proof for $\Psi_0 \in S(\mathbb{R}^n)$ is so short we add it as a matter of completeness.

Proof of Lemma 1. Since $\Psi_0 \in S(\mathbb{R}^n)$

$$\begin{aligned} \|\varphi_2(\cdot,t)\| &= \|(2\pi t)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} e^{-i\frac{\cdot}{t}\cdot y}(e^{i\frac{y^2}{2t}} - 1)\Psi_0(y)\, d^n y\| = \\ &= \|\mathcal{F}\big((e^{i\frac{y^2}{2t}} - 1)\Psi_0(y)\big)(\cdot)\| = \|(e^{i\frac{\cdot^2}{2t}} - 1)\Psi_0(\cdot)\| \leq \\ &\leq \int\limits_{\mathbb{R}^n} \frac{|q|^4}{4t^2} |\Psi_0(q)|^2 \, d^n q \overset{\Psi_0 \in S}{\leq} Ct^{-2} \overset{t\to\infty}{\longrightarrow} 0, \end{aligned}$$

where $\mathcal{F}$ denotes Fourier transformation. Then, using $||a|^2 - |b|^2| \leq |a - b|^2 + 2|b||a-b|$ (where $a = \Psi_t(q)$ and $b = \varphi_1(q,t)$), (13) follows by Schwarz inequality and the normalization of Ψ_0.

Now let $q \in \mathbb{R}^n$, $t > 0$. Since $\Psi_0 \in S(\mathbb{R}^n)$, $\int |\nabla_q e^{i\frac{(q-y)^2}{2t}} \Psi_0(y)| \, d^n y = \int |\frac{|q-y|}{t} \Psi_0(y)| \, d^n y < \infty$. Therefore we can interchange integration and differentiation in

$$\begin{aligned} \nabla \Psi_t(q) &= (2\pi i t)^{-\frac{n}{2}} \nabla \int\limits_{\mathbb{R}^n} e^{i\frac{|q-y|^2}{2t}} \Psi_0(y)\, d^n y = \\ &= (2\pi i t)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} \nabla_q e^{i\frac{|q-y|^2}{2t}} \Psi_0(y)\, d^n y = \\ &= i\frac{q}{t}\Psi_t(q) - \frac{i}{t}(2\pi i t)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} y e^{i\frac{|q-y|^2}{2t}} \Psi_0(y)\, d^n y. \end{aligned} \tag{15}$$

Again using $\Psi_0 \in S(\mathbb{R}^n)$, this gives us with (4)

$$\begin{aligned} |v^\Psi(q,t) - \frac{q}{t}| &= \big|\mathrm{Im}\big(\frac{\nabla\Psi_t(q)}{\Psi_t(q)}\big) - \frac{q}{t}\big| = \\ &= \Big|\mathrm{Im}\Big(\frac{i}{t}(2\pi i t)^{-\frac{n}{2}}\Psi_t(q)^{-1}\int\limits_{\mathbb{R}^n} y e^{i\frac{|q-y|^2}{2t}}\Psi_0(y)\,d^n y\Big)\Big| \leq \\ &\leq \Big|\frac{i}{t}(2\pi i t)^{-\frac{n}{2}}\Psi_t(q)^{-1}\int\limits_{\mathbb{R}^n} y e^{i\frac{|q-y|^2}{2t}}\Psi_0(y)\,d^n y\Big| \leq \\ &\leq (2\pi)^{-\frac{n}{2}} t^{-1-\frac{n}{2}}|\Psi_t(q)|^{-1}\int\limits_{\mathbb{R}^n} |y||\Psi_0(y)|\,d^n y \leq C_1 t^{-1-\frac{n}{2}}|\Psi_t(q)|^{-1} \end{aligned}$$

for some $C_1 < \infty$. □

As mentioned above we wish to control $\big|v^\Psi(\mathrm{Q}(q,t),t) - \frac{\mathrm{Q}(q,t)}{t}\big|$ for T (and thus t) big enough. By (14) this is tantamount to finding a lower bound on $|\Psi_t(\mathrm{Q}(q,t))|$. While we do know by [6, 32] that $\mathbb{P}^\Psi$-almost all Bohmian trajectories $\mathrm{Q}(q,t)$ do not run into nodes of Ψ, this only implies $\big|v^\Psi(\mathrm{Q}(q,t),t) - \frac{\mathrm{Q}(q,t)}{t}\big| < \infty$. We need to do better than that. Indeed we can do better than that as in our special case it suffices to put a bound on $|t^{\frac{n}{2}}\Psi_t(\mathrm{Q}(q,t))|$. But for t big enough this is nearly the same as $|\hat{\Psi}_0(\frac{\mathrm{Q}(q,t)}{t})|$ and if the Bohmian trajectories really become straight lines asymptotically, $\frac{\mathrm{Q}(q,t)}{t}$ will be essentially constant and we should be able to control $\big|v^\Psi(\mathrm{Q}(q,t),t) - \frac{\mathrm{Q}(q,t)}{t}\big|$ by putting suitable conditions on $|\hat{\Psi}_0(\frac{q}{T})|$ alone. This considerations lead us to the following definition and to the formulation of Theorem 1.

Definition 1. *For $\Psi_0 \in L_2(\mathbb{R}^n)$ and $\delta_1 > 0$, $\delta_2 > 0$ define*

$$A^{\hat{\Psi}_0}_{\delta_1} := \{q \in \mathbb{R}^n \mid |\hat{\Psi}_0(q)| > \delta_1\}$$

and

$$A^{\hat{\Psi}_0}_{\delta_1\delta_2} := \{q \in \mathbb{R}^n \mid U_{\delta_2}(q) := \{y \in \mathbb{R}^n \mid |q-y| < \delta_2\} \subset A^{\hat{\Psi}_0}_{\delta_1}\}.$$

Theorem 1. *Let $\Psi_0 \in S(\mathbb{R}^n)$ and $\Psi_t = e^{-iH_0 t}\Psi_0$. Assume A2. Then for all $\varepsilon > 0$ there exist some $T_\varepsilon > 0$, $\delta_1 > 0$ and $\delta_2 > 0$ such that*

$$\mathbb{P}^{\Psi_{T_\varepsilon}}\big(\{q \in \mathbb{R}^n \mid \frac{q}{T_\varepsilon} \notin A^{\hat{\Psi}_0}_{\delta_1\delta_2}\}\big) < \varepsilon, \tag{16}$$

$$\sup_{q\in\mathbb{R}^n\setminus\{0\}} \big|t^{\frac{n}{2}}\big|\Psi_t(q)\big| - \big|\hat{\Psi}_0(\frac{q}{t})\big|\big| < \frac{\delta_1}{2} \qquad \forall\, t \geq T_\varepsilon \tag{17}$$

and for all $q \in \mathbb{R}^n\setminus\{0\}$ such that $\frac{q}{T_\varepsilon} \in A^{\hat{\Psi}_0}_{\delta_1\delta_2}$

$$\left.\begin{aligned} &\frac{\mathrm{Q}_{T_\varepsilon}(q,t)}{t} \in U_{\delta_2}(\frac{q}{T_\varepsilon}) \text{ and} \\ &\qquad \big|v^\Psi\big(\mathrm{Q}_{T_\varepsilon}(q,t),t\big) - \frac{\mathrm{Q}_{T_\varepsilon}(q,t)}{t}\big| = \mathcal{O}(t^{-1}) < \delta_2 \end{aligned}\right\} \quad \forall\, t \geq T_\varepsilon. \tag{18}$$

Proof of Theorem 1. Let $\varepsilon > 0$ and define

$$C^{\hat{\Psi}_0}_{\delta_1\delta_2}(T_\varepsilon) := \{q \in \mathbb{R}^n \mid \frac{q}{T_\varepsilon} \notin A^{\hat{\Psi}_0}_{\delta_1\delta_2}\}.$$

Then

$$\begin{aligned} \mathbb{P}^{\Psi_{T_\varepsilon}}\big(C^{\hat{\Psi}_0}_{\delta_1\delta_2}(T_\varepsilon)\big) = &\int\limits_{C^{\hat{\Psi}_0}_{\delta_1\delta_2}(T_\varepsilon)} |\Psi_{T_\varepsilon}(q)|^2\, d^n q \leq \\ \leq \int\limits_{C^{\hat{\Psi}_0}_{\delta_1\delta_2}(T_\varepsilon)} T_\varepsilon^{-\frac{n}{2}}\big|\hat{\Psi}_0(\frac{q}{T_\varepsilon})\big|^2\, d^n q &+ \int\limits_{C^{\hat{\Psi}_0}_{\delta_1\delta_2}(T_\varepsilon)} \big|\big|\Psi_{T_\varepsilon}(q)\big|^2 - T_\varepsilon^{-n}\big|\hat{\Psi}_0(\frac{q}{T_\varepsilon})\big|^2\big|\, d^n q \leq \\ \leq \int\limits_{(A^{\hat{\Psi}_0}_{\delta_1\delta_2})^c} \big|\hat{\Psi}_0(k)\big|^2\, d^n k &+ \int\limits_{\mathbb{R}^n} \big|\big|\Psi_{T_\varepsilon}(q)\big|^2 - T_\varepsilon^{-n}\big|\hat{\Psi}_0(\frac{q}{T_\varepsilon})\big|^2\big|\, d^n q, \end{aligned}$$

where $k := \frac{q}{T_\varepsilon}$.
To get a bound on the first term we note that

$$(A^{\hat{\Psi}_0}_{\delta_1\delta_2})^c = (A^{\hat{\Psi}_0}_{\delta_1})^c \cup \{k \in A^{\hat{\Psi}_0}_{\delta_1} \mid U_{\delta_2}(k) \not\subset A^{\hat{\Psi}_0}_{\delta_1}\}.$$

Then

$$\int\limits_{(A_{\delta_1}^{\hat{\Psi}_0})^c} |\hat{\Psi}_0(k)|^2 \, d^n k = \int\limits_{|\hat{\Psi}_0(k)| \leq \delta_1} |\hat{\Psi}_0(k)|^2 \, d^n k \leq$$

$$\leq \int\limits_{|\hat{\Psi}_0(k)| \leq \delta_1} |\hat{\Psi}_0(k)|^2 \chi_{B_R}(k) \, d^n k + \int\limits_{B_R^c} |\hat{\Psi}_0(k)|^2 \, d^n k \leq$$

$$\leq 4\pi R^3 \delta_1 + \int\limits_{B_R^c} |\hat{\Psi}_0(k)|^2 \, d^n k$$

for all $R > 0$. Since $\hat{\Psi}_0 \in L_2(\mathbb{R}^n)$ there exists some $R > 0$ such that

$$\int\limits_{B_R^c} |\hat{\Psi}_0(k)|^2 \, d^n k < \frac{\varepsilon}{8}.$$

Thus it is possible to choose $\delta_1 > 0$ small enough such that

$$\int\limits_{(A_{\delta_1}^{\hat{\Psi}_0})^c} |\hat{\Psi}_0(k)|^2 \, d^n k < \frac{\varepsilon}{4}.$$

Moreover, since $\Psi_0 \in S(\mathbb{R}^n)$ and thus also $\hat{\Psi}_0 \in S(\mathbb{R}^n)$, $\hat{\Psi}_0$ is continuous. Therefore $A_{\delta_1}^{\hat{\Psi}_0}$ is open and there exists some δ_2 small enough such that

$$\int\limits_{\{k \in A_{\delta_1}^{\hat{\Psi}_0} | U_{\delta_2}(k) \not\subset A_{\delta_1}^{\hat{\Psi}_0}\}} |\hat{\Psi}_0(k)|^2 \, d^n k < \frac{\varepsilon}{4}.$$

To get a bound on the second term we use that by (13)

$$\int\limits_{\mathbb{R}^n} \left| |\Psi_{T_\varepsilon}(q)|^2 - T_\varepsilon^{-n} |\hat{\Psi}_0(\frac{q}{T_\varepsilon})|^2 \right| d^n q < \frac{\varepsilon}{2}$$

for all T_ε big enough. Thus (16) holds for all T_ε big enough.
Moreover, noting that by (11)

$$\left| t^{\frac{n}{2}} \left| \Psi_t(q) \right| - \left| \hat{\Psi}_0(\frac{q}{t}) \right| \right| \leq t^{\frac{n}{2}} \left| \varphi_2(q,t) \right|,$$

(17) follows directly from (12) if one takes $r = 0$ and T_ε big enough.
Now let $\frac{q}{T_\varepsilon} \in A_{\delta_1 \delta_2}^{\hat{\Psi}_0}$ and suppose there exists some $t_1 > T_\varepsilon$ such that $\frac{Q_{T_\varepsilon}(q,t_1)}{t_1} \notin$

$U_{\delta_2}(\frac{q}{T_\varepsilon})$. Since $\mathrm{Q}_{T_\varepsilon}(q,t)$ is continuous in t (by Remark 1), this implies that the first exit time $t_{ex} := \max\{T_\varepsilon < s \mid \frac{\mathrm{Q}_{T_\varepsilon}(q,s)}{s} \notin U_{\delta_2}(\frac{q}{T_\varepsilon}) \ \wedge \ \frac{\mathrm{Q}_{T_\varepsilon}(q,t)}{t} \in U_{\delta_2}(\frac{q}{T_\varepsilon}) \ \ \forall T_\varepsilon \le t < s\}$ exists and $\left|\frac{\mathrm{Q}_{T_\varepsilon}(q,t_{ex})}{t_{ex}} - \frac{q}{T_\varepsilon}\right| = \delta_2$. Moreover $\frac{\mathrm{Q}_{T_\varepsilon}(q,\tau)}{\tau} \in U_{\delta_2}(\frac{q}{T_\varepsilon}) \subset A_{\delta_1}^{\hat\Psi_0}$, i.e. $|\hat\Psi_0\big(\frac{\mathrm{Q}_{T_\varepsilon}(q,\tau)}{\tau}\big)| > \delta_1$ for all $T_\varepsilon \le \tau < t_{ex}$. By (17) this implies

$$\big|\Psi_\tau\big(\mathrm{Q}_{T_\varepsilon}(q,\tau)\big)\big| \ge \frac{\delta_1}{2}\tau^{-\frac{n}{2}} \qquad \forall\, T_\varepsilon \le \tau < t_{ex}$$

and thus by (14)

$$\big|v^\Psi\big(\mathrm{Q}_{T_\varepsilon}(q,\tau),\tau\big) - \frac{\mathrm{Q}_{T_\varepsilon}(q,\tau)}{\tau}\big| \le \frac{2C_1}{\delta_1}\tau^{-1} \qquad \forall\, T_\varepsilon \le \tau < t_{ex}.$$

Therefore for T_ε big enough

$$\begin{aligned}
\big|\frac{\mathrm{Q}_{T_\varepsilon}(q,t_{ex})}{t_{ex}} - \frac{q}{T_\varepsilon}\big| \le \int_{T_\varepsilon}^{t_{ex}} \big|\frac{\partial}{\partial\tau}\frac{\mathrm{Q}_{T_\varepsilon}(q,\tau)}{\tau}\big|\, d\tau = \\
&= \int_{T_\varepsilon}^{t_{ex}} \frac{1}{\tau}\big|v^\Psi\big(\mathrm{Q}_{T_\varepsilon}(q,\tau),\tau\big) - \frac{\mathrm{Q}_{T_\varepsilon}(q,\tau)}{\tau}\big|\, d\tau \le \int_{T_\varepsilon}^{t_{ex}} \frac{1}{\tau^2}\frac{2C_1}{\delta_1}\, d\tau \le \\
&\le \frac{2C_1}{\delta_1}T_\varepsilon^{-1} < \delta_2,
\end{aligned}$$

which is a contradiction. Thus (18) holds. □

Remark 5. We used the domain $\Psi_0 \in S(\mathbb{R}^n)$ to simplify the proof. However, Theorem 1 and thus Corollary 1 and Corollary 2 hold also if this condition is replaced by

$$|\Psi_0(q)| \le C(1+|q|)^{-3-n-\varepsilon}$$

and

$$|\partial_q^\eta \Psi_0(q)| \le C(1+|q|)^{-2-n-\varepsilon} \qquad (|\eta| = 1)$$

for some $C < \infty$, $\varepsilon > 0$ and all $q \in \mathbb{R}^n$. Here η is a multi-index.
Then all our estimates hold except for (12) in Proposition 2, which remains valid for $r = 0$ only. But that was all we used.

With Theorem 1 it is now easy to show Corollary 1 and Corollary 2.

Proof of Corollary 1. Let $\varepsilon > 0$. Then there exist $\delta_1 > 0$, $\delta_2 > 0$ and $T_\varepsilon > 0$ such that (16), (17) and (18) hold.
Let $\frac{q}{T_\varepsilon} \in A^{\hat{\Psi}_0}_{\delta_1\delta_2}$. Then (18) implies that there is a $C > 0$ such that

$$\begin{aligned}|\frac{\mathrm{Q}_{T_\varepsilon}(q,t_1)}{t_1} - \frac{\mathrm{Q}_{T_\varepsilon}(q,t_2)}{t_2}| \leq \int_{t_1}^{t_2} \frac{1}{\tau}|v^\Psi(\mathrm{Q}_{T_\varepsilon}(q,\tau),\tau) - \frac{\mathrm{Q}_{T_\varepsilon}(q,\tau)}{\tau}|\, d\tau < \\ < C(t_1^{-1} - t_2^{-1}) \leq C t_1^{-1}\end{aligned} \tag{19}$$

for all $t_2 \geq t_1 \geq T_\varepsilon$.
Now let $\delta > 0$ and define $T_{\varepsilon\delta} := \max\{T_\varepsilon, \frac{C}{\delta}\}$. Then (19) implies

$$\frac{q}{T_\varepsilon} \in A^{\hat{\Psi}_0}_{\delta_1\delta_2} \quad \Rightarrow \quad \sup_{t_1,t_2 \geq T_{\varepsilon\delta}} |\frac{\mathrm{Q}_{T_\varepsilon}(q,t_1)}{t_1} - \frac{\mathrm{Q}_{T_\varepsilon}(q,t_2)}{t_2}| < \delta$$

and by (16) we get

$$\mathbb{P}^{\Psi_{T_\varepsilon}}\big(\{q \in \mathbb{R}^n \mid \sup_{t_1,t_2 \geq T_{\varepsilon\delta}} |\frac{\mathrm{Q}_{T_\varepsilon}(q,t_1)}{t_1} - \frac{\mathrm{Q}_{T_\varepsilon}(q,t_2)}{t_2}| < \delta\}\big) > 1 - \varepsilon.$$

Note however, that $\mathrm{Q}_{T_\varepsilon}(q,t) = \Phi_{t,T_\varepsilon}(q)$ and $\mathrm{Q}(q,t) = \Phi_{t,0}(q) = \Phi_t(q)$ (see Subsection 2.2), so by equivariance (see (9))

$$\begin{aligned}&\mathbb{P}^{\Psi_{T_\varepsilon}}\big(\{q \in \mathbb{R}^n \mid \sup_{t_1,t_2 \geq T_{\varepsilon\delta}} |\frac{\mathrm{Q}_{T_\varepsilon}(q,t_1)}{t_1} - \frac{\mathrm{Q}_{T_\varepsilon}(q,t_2)}{t_2}| < \delta\}\big) = \\ &= \mathbb{P}^{\Psi_0}\Bigg(\Phi_{0,T_\varepsilon}\big(\{q \in \mathbb{R}^n \mid \sup_{t_1,t_2 \geq T_{\varepsilon\delta}} |\frac{\Phi_{t_1,T_\varepsilon}(q)}{t_1} - \frac{\Phi_{t_2,T_\varepsilon}(q)}{t_2}| < \delta\}\big)\Bigg) = \\ &= \mathbb{P}^{\Psi_0}\big(\{q \in \mathbb{R}^n \mid \sup_{t_1,t_2 \geq T_{\varepsilon\delta}} |\frac{\Phi_{t_1}(q)}{t_1} - \frac{\Phi_{t_2}(q)}{t_2}| < \delta\}\big) = \\ &= \mathbb{P}^{\Psi_0}\big(\{q \in \mathbb{R}^n \mid \sup_{t_1,t_2 \geq T_{\varepsilon\delta}} |\frac{\mathrm{Q}(q,t_1)}{t_1} - \frac{\mathrm{Q}(q,t_2)}{t_2}| < \delta\}\big)\end{aligned}$$

and thus

$$\mathbb{P}^{\Psi_0}\big(\{q \in \mathbb{R}^n \mid \sup_{t_1,t_2 \geq T_{\varepsilon\delta}} |\frac{\mathrm{Q}(q,t_1)}{t_1} - \frac{\mathrm{Q}(q,t_2)}{t_2}| < \delta\}\big) > 1 - \varepsilon. \tag{20}$$

But (20) is a sufficient condition for $\mathbb{P}^\Psi$-almost sure convergence of $\frac{\mathrm{Q}(q,t)}{t}$.
Let $\tilde{q} \in \mathbb{R}^n$ such that $\lim\limits_{t\to\infty} \frac{\mathrm{Q}(\tilde{q},t)}{t}$ does not exist. Then $\frac{\mathrm{Q}(\tilde{q},t)}{t}$ is not Cauchy, i.e. there is some $\delta > 0$ such that

$$\tilde{q} \notin \{q \in \mathbb{R}^n \mid \sup_{t_1,t_2 \geq T} |\frac{\mathrm{Q}(q,t_1)}{t_1} - \frac{\mathrm{Q}(q,t_2)}{t_2}| < \delta\}$$

for all $T \in \mathbb{R}$ and thus also for all $T_{\varepsilon\delta}$ where $\varepsilon > 0$ is arbitrary:

$$\tilde{q} \notin \big\{q \in \mathbb{R}^n \mid \sup_{t_1,t_2 \geq T_{\varepsilon\delta}} \Big|\frac{\mathrm{Q}(q,t_1)}{t_1} - \frac{\mathrm{Q}(q,t_2)}{t_2}\Big| < \delta\big\}$$

for all $\varepsilon > 0$. By (20) this implies that $\tilde{q}$ is in a set of measure zero.
It is left to show that the distribution of v_∞ has density $|\hat{\Psi}_0|^2$. Let $A \subset \mathbb{R}^n$ be measurable. Then by (13), the definition of v_∞ and using equivariance

$$\begin{aligned}
\mathbb{P}^{\Psi_0}\big(\big\{q \in \mathbb{R}^n \,|v_\infty \in A\big\}\big) &= \lim_{t\to\infty} \mathbb{P}^{\Psi_0}\big(\big\{q \in \mathbb{R}^n \mid \frac{\mathrm{Q}(q,t)}{t} \in A\big\}\big) = \\
&= \lim_{t\to\infty} \mathbb{P}^{\Psi_t}\big(\big\{q \in \mathbb{R}^n \mid \frac{q}{t} \in A\big\}\big) = \lim_{t\to\infty} \int\limits_{\frac{q}{t}\in A} |\Psi_t(q)|^2 \, d^n q = \\
&= \lim_{t\to\infty} \int\limits_{\frac{q}{t}\in A} t^{-n} |\hat{\Psi}_0(\frac{q}{t})|^2 \, d^n q = \int\limits_A |\hat{\Psi}_0(k)| \, d^n k,
\end{aligned}$$

where we substituted $k = \frac{q}{t}$ in the last step. $\square$

Proof of Corollary 2. Let $\varepsilon > 0$. Then there are $\delta_1 > 0$, $\delta_2 > 0$ and $T_\varepsilon > 0$ such that (16)-(18) hold. In particular, for q such that $\frac{q}{T_\varepsilon} \in A^{\hat{\Psi}_0}_{\delta_1\delta_2}$, (18) implies

$$\begin{aligned}
&|v^\Psi(\mathrm{Q}_{T_\varepsilon}(q,t),t) - \lim_{s\to\infty} \frac{\mathrm{Q}_{T_\varepsilon}(q,s)}{s}| \leq \\
&\quad \leq \lim_{s\to\infty} \Big|\frac{\mathrm{Q}_{T_\varepsilon}(q,t)}{t} - \frac{\mathrm{Q}_{T_\varepsilon}(q,s)}{s}\Big| + \Big|v^\Psi\big(\mathrm{Q}_{T_\varepsilon}(q,t),t\big) - \frac{\mathrm{Q}_{T_\varepsilon}(q,t)}{t}\Big| < \\
&\quad < 2Ct^{-1}
\end{aligned} \tag{21}$$

for all $t \geq T_\varepsilon$, where C is the same constant as in (19).
Now let $\delta > 0$ and define $T_{\varepsilon\delta} := \max\{T_\varepsilon, \frac{2C}{\delta}\}$. Then (21) implies

$$\frac{q}{T_\varepsilon} \in A^{\hat{\Psi}_0}_{\delta_1\delta_2} \quad \Rightarrow \qquad \sup_{t\geq T_{\varepsilon\delta}} \Big|v^\Psi(\mathrm{Q}_{T_\varepsilon}(q,t),t) - \lim_{s\to\infty} \frac{\mathrm{Q}_{T_\varepsilon}(q,s)}{s}\Big| < \delta.$$

This together with (16) and equivariance yields the desired result (see the proof of Corollary 1 for details):

$$
\begin{aligned}
1-\varepsilon &< \mathbb{P}^{\Psi_{T_\varepsilon}}\Big(\Big\{q\in\mathbb{R}^n \mid \frac{q}{T_\varepsilon}\in A^{\hat{\Psi}_0}_{\delta_1\delta_2}\Big\}\Big) \leq \\
&\leq \mathbb{P}^{\Psi_{T_\varepsilon}}\Big(\Big\{q\in\mathbb{R}^n \mid \sup_{t\geq T_{\varepsilon\delta}}\Big|v^\Psi(\mathrm{Q}_{T_\varepsilon}(q,t),t)-\lim_{s\to\infty}\frac{\mathrm{Q}_{T_\varepsilon}(q,s)}{s}\Big| < \delta\Big\}\Big) = \\
&= \mathbb{P}^{\Psi_0}\Big(\Big\{q\in\mathbb{R}^n \mid \sup_{t\geq T_{\varepsilon\delta}}\Big|v^\Psi(\mathrm{Q}(q,t),t)-v_\infty(q)\Big| < \delta\Big\}\Big).
\end{aligned}
$$

$\square$

4 Bohmian Trajectories in Scattering Situations

We will now look at a situation where a spinless, non-relativistic particle is scattered by a short-range potential, i.e. the Hamiltonian of the system is given by $H = H_0 + V(q)$, where H_0 is the free Hamiltonian from above (in three dimensions, $N = 1$) and the potential $V(q)$ decays sufficiently fast for $|q| \to \infty$. Then, if the particle is far away from the scattering center, it is quite natural to assume that it is almost free and again has a (virtually) straight trajectory. Also if the potential allows of bound wave functions one would expect to find trajectories staying in some sense close to the scattering center.
By a short-range potential we mean $V \in (V)_4$, which is defined as follows.

Definition 2 ($(\mathbf{V})_{\mathbf{m}}$). *For $m \geq 2$ the following conditions on the potential V will be denoted by $V \in (V)_m$.*

(i) $V \in L^2(\mathbb{R}^3, \mathbb{R})$

(ii) V is locally Hölder continuous[4] *except at a finite number of singularities.*

(iii) There exist $\varepsilon > 0$, $C_0 > 0$ and $R_0 > 0$ such that $|V(q)| \leq C_0|q|^{-m-\varepsilon}$ for all $|q| \geq R_0$.

For $n = 2$ those are the conditions of Ikebe [20] and the following holds.

Proposition 3 (Asymptotic Completeness). *Let $V \in (V)_2$. Then*

(i) V is H_0-bounded with arbitrary small relative bound and $H = H_0 + V$ is self adjoint on $\mathcal{D}(H_0)$.

(ii) The wave operators $W_\pm := \mathrm{s} - \lim_{t\to\pm\infty} e^{iHt}e^{-iH_0t}$ exist and are asymptotically complete[5].

(iii) The absolute continuous part of the spectrum is $[0, \infty)$ and there are no positive eigenvalues.

---

[4] $V : D \to \mathbb{R}$ is locally Hölder continuous if for all $q_0 \in D$ there is an open neighborhood $U(q_0) \subset D$ and some $C > 0$, $\alpha > 0$ such that $|V(q_0) - V(q)| \leq C|q_0 - q|^\alpha \quad \forall q \in U(q_0)$.

[5] $W_\pm$ are called asymptotically complete if $\mathrm{Ran}W_\pm = \mathcal{H}_c(H) = \mathcal{H}_{ac}(H)$.

For a proof of assertion (i) see also [26] (Theorem X.15).

So now we know that all wave functions orthogonal to all bound wave functions, i.e. all wave functions in $\mathcal{H}_{pp}^{\perp}(H)$, lie in $\mathcal{H}_{ac}(H)$, the absolutely continuous subspace for H. More importantly their long time asymptotes become free in the following sense (for a more detailed discussion of the following see [19]).

$$\lim_{t\to\infty} \|(W_+ - 1)\Psi_t^{ac}\| = 0,$$

i.e. $e^{-iHs}\Psi_t^{ac} \approx e^{-iH_0 s}\Psi_t^{ac}$ for t big enough and $\Psi_t^{ac} \in \mathcal{H}_{ac}(H)$. Note that apart from some obvious change of signs the same is true also for $t \to -\infty$. Thus those so-called scattering wave functions roughly speaking start off at $t = -\infty$ as "free" wave functions, evolve into something more complicated and finally end as "free" wave functions again at $t = \infty$.
But how does this fit in with our notion that a particle "far away" from the scattering center moves almost freely? As far as we know it was Enss [25] who first showed that the spatial support of scattering wave functions travels to infinity as $|t| \to \infty$. Then Born's statistical interpretation of Ψ_t^{ac} tells us that for $|t| \to \infty$, i.e. when Ψ_t^{ac} evolves according to the free time evolution, the particle is indeed far away from the scattering center.
In line with the above we show in Subsection 4.1 that the Bohmian trajectories for a large class of scattering wave functions become asymptotically free in the sense of Section 3.
But what about the bound wave functions? It is known (see e.g. [25]) that the spatial support of every bound wave function $\Psi_t^{pp} \in \mathcal{H}_{pp}(H)$ essentially stays in a bounded region around the scattering center for all times, in the sense that for all $\varepsilon > 0$ there is a $R > 0$ such that

$$\sup_{t\in\mathbb{R}} \int_{B_R^c} |\Psi_t^{pp}(q)|^2 \, d^3q < \varepsilon. \tag{22}$$

Note that although this shows that the probability of finding the particle outside a ball with a certain radius R is very small for all times, taken alone it does say nothing at all about the probability that the particle's trajectory leaves this ball at some time. Nevertheless, in Subsection 4.2 we show that under certain conditions on the decay of Ψ_t^{pp} and $\nabla\Psi_t^{pp}$ the probability of the trajectory leaving a ball with a radius that grows like $t^{\frac{1}{1+\gamma}}$ for some suitable $\gamma > 0$ can be made arbitrarily small.
More than that in Subsection 4.2 we not really look at a pure bound wave function alone but at the more general case of a mixed wave function $\Psi = \Psi^{ac} + \Psi^{pp}$. We show that for large times the set of all possible trajectories

splits up according to the splitting of the wave function and that, while the "bound" part of the trajectories stays inside a slowly growing ball (as described above), the "scattering" part stays outside a ball with radius growing linear in time and consists of nearly "free" trajectories (Theorem 3)[6]. Then the above statement about pure bound wave functions is an easy corollary and we can recover also our statement about pure scattering wave functions (Remark 11). Since the Bohmian velocity field $v^\Psi = \mathrm{Im}(\frac{\nabla\Psi}{\Psi})$ is not linear in Ψ this is in no way a trivial result.

4.1 Pure Scattering State

In the following let $H = H_0 + V$ with $V \in (V)_4$ and zero neither an eigenvalue nor a resonance of H [7]. Let Ω be $\mathbb{R}^3$ without the singularities of the potential V. Then A1 holds.
We show that the long time asymptotes of $\mathbb{P}^\Psi$-almost all Bohmian trajectories of pure scattering wave functions $\Psi_t = e^{-iHt}W_+\Psi_0^{out} \in \mathcal{H}_{ac}(H)$ such that A2 holds and with $\Psi_0^{out} \in S(\mathbb{R}^3)$ behave like free trajectories in classical mechanics. Here $\Psi_0^{out} = W_+^{-1}\Psi_0$ is the asymptotic outgoing wave and evolves according to the free time evolution $\Psi_t^{out} = e^{-iH_0t}\Psi_0^{out}$.

Remark 6. Analog to Section 3 we use $\hat{\Psi}_0^{out} \in S(\mathbb{R}^3)$ just for convenience (q.v Remark 5). To prove the results in this section it suffices that there exists some $C < \infty$ and some $\varepsilon > 0$ such that

$$|\Psi_0^{out}(q)| \le C(1+|q|)^{-6-\varepsilon}, \qquad |\partial_q^\eta \Psi_0^{out}(q)| \le C(1+|q|)^{-5-\varepsilon} \qquad (|\eta| = 1)$$

and

$$|\hat{\Psi}_0^{out}(k)| \le C(1+|k|)^{-4-\varepsilon}, \qquad |\partial_k^\eta \hat{\Psi}_0^{out}(k)| \le C(1+|k|)^{-4-\varepsilon} \qquad (|\eta| = 1)$$

for all q, $k \in \mathbb{R}^3$.
Note that the conditions on Ψ_0^{out} are the same as those on Ψ_0 in Remark 5 (with $n = 3$), so (12) in Proposition 2 (and thus (27) in Lemma 2) remain valid for $r = 0$ only, which however will be quite sufficient.
It would be of course preferable to have conditions on Ψ_0 instead of Ψ_0^{out} and

[6] Of course this is only true for $\mathbb{P}^\Psi$-almost all trajectories, but since all our results concerning trajectories are of that probabilistic form we will refrain from mentioning it every time.

[7] For $V \in (V)_4$ zero is said to be a resonance of H if there exists a solution $f \notin L_2(\mathbb{R}^3)$ of $(H_0 + V(q))f(q) = 0$ such that $(1+|q|^2)^{-\frac{s}{2}}(1-\triangle)^{\frac{1}{2}}f(q) \in L_2(\mathbb{R}^3)$ for some $\frac{1}{2} < s < (4+\varepsilon_0) - \frac{1}{2}$ ([21] p. 584). The occurance of a zero eigenvalue or resonance is an exceptional event ([21] p.589).

$\hat{\Psi}_0^{out}$. For the conditions on $\hat{\Psi}_0^{out}$ we can use mapping properties between Ψ_0 and $\hat{\Psi}_0^{out}$ Dürr, Moser and Pickl proved in Lemma 7 of [18]. Note however, that their conditions on $\hat{\Psi}_0^{out}$ and thus on Ψ_0 are stronger than what we need here.

Remark 7. Since $S(\mathbb{R}^3)$ is left invariant under the free time evolution the so-called intertwining property of the wave operators, $e^{-iHt}W_\pm = W_\pm e^{-iH_0 t}$, guarantees that the class of scattering wave functions defined above is left invariant under the full time evolution.

Remark 8. By Proposition 3 V is H_0-bounded with arbitrary small relative bound. Then A3 holds. This can be seen as follows (see e.g. Corollary 3.2 of [6] or Corollary 3 of [32]). For some $0 < a < 1$ and $0 < b$

$$\|\triangle\Psi_t\| \leq 2\|(V-H)\Psi_t\| \leq 2\big[\frac{a}{2}\|\triangle\Psi_t\| + b\|\Psi_t\| + \|H\Psi_t\|\big],$$

i.e.

$$\|\triangle\Psi_t\| \leq \frac{2\|H\Psi_t\| + 2b\|\Psi_t\|}{1-a} = \frac{2\|H\Psi_0\| + 2b\|\Psi_0\|}{1-a} =: C$$

But then also

$$\|\,|\nabla\Psi_t|\,\| = \langle\nabla\Psi_t, \cdot\nabla\Psi_t\rangle = -\langle\Psi_t, \triangle\Psi_t\rangle \leq \|\Psi_t\|\|\triangle\Psi_t\| \leq C.$$

Note that this together with what was said above gives us almost sure global existence of Bohmian mechanics (by Proposition 1).

As in Section 3 we start with some estimates on the wave function Ψ and the velocity field v^Ψ.

Proposition 4. *Let $\Psi_0^{out} \in S(\mathbb{R}^3)$. Then $\Psi_t = e^{-iHt}W_+\Psi_0^{out}$ is continuously differentiable except at the singularities of V and the following holds for*

$$\varphi_3(q,t) := \Psi_t(q) - \Psi_t^{out}(q). \tag{23}$$

There is a $R_0 > 0$ such that for every $T > 0$ there is a $C_2 < \infty$ such that

$$|\varphi_3(q,t)| \leq C_2 \frac{1}{|q|(t+|q|)} \qquad \forall |q| > 0 \tag{24}$$

and

$$|\nabla\varphi_3(q,t)| \leq C_2 \frac{1}{|q|(t+|q|)} \qquad \forall |q| > R_0 \tag{25}$$

for all $t \geq T$.

The proof can be found in [31] (Thm 2.1 and its proof; pp.5,6).

Lemma 2. *Let $\Psi_0^{out} \in S(\mathbb{R}^3)$, $\Psi_t = e^{-iHt}W_+\Psi_0^{out}$. Then*

$$\lim_{t\to\infty} \int_{\mathbb{R}^3} \left| \big|\Psi_t(q)\big|^2 - t^{-3}\big|\hat{\Psi}_0^{out}(\frac{q}{t})\big|^2 \right| d^3q = 0. \tag{26}$$

Moreover there exists a $R_0 > 0$ such that for every $r \geq 0$ there exists a $c_r < \infty$ and for every $T > 0$ there exist $C_i < \infty$ ($i = 2, 3, 4$) such that for all $t \geq T$ and $|q| > 0$

$$\left| \big|\Psi_t(q)\big| - t^{-\frac{3}{2}}\big|\hat{\Psi}_0^{out}(\frac{q}{t})\big| \right| \leq \frac{C_2}{|q|(t+|q|)} + c_r t^{-\frac{5}{2}} \big(\frac{t}{|q|}\big)^r \tag{27}$$

and for all $|q| > R_0$ and all $t \geq T$ with $\Psi_t(q) \neq 0$ and $\Psi_t^{out}(q) \neq 0$

$$\begin{aligned} &\big|v^\Psi(q,t) - \frac{q}{t}\big| \leq \mathrm{f}_1(q,t) t^{-2} |\Psi_t(q)|^{-1}, \\ &\textit{where} \quad \mathrm{f}_1(q,t) := C_3\left(1 + C_4 \frac{t}{|q|} \frac{t^{-\frac{5}{2}}}{|\Psi_t^{out}(q)|}\right) + C_2 \frac{t}{|q|}. \end{aligned} \tag{28}$$

Remark 9. Note that the C_i ($i = 2, 3, 4$) in Proposition 4 and Lemma 2 depend on T. However we shall not denote this dependence for reasons of simplicity of notation.
Regarding (27), it suffices for the moment to keep it in mind with $r = 0$ (see also Remark 4).

Proof of Lemma 2. Since $\Psi_0^{out} \in S(\mathbb{R}^3)$ and Ψ_t^{out} obeys the free time evolution Proposition 2 and Lemma 1 hold for Ψ_t^{out}. Moreover by Proposition 3

$$\lim_{t\to\infty} \|\Psi_t - \Psi_t^{out}\| = \lim_{t\to\infty} \|\Psi_0 - e^{iHt}e^{-iH_0t}\Psi_0^{out}\| = \|\Psi_0 - W_+\Psi_0^{out}\| = 0.$$

Since $||a|^2 - |b|^2| \leq |a-b|^2 + 2|b||a-b|$, where $a = \Psi_t(q)$ and $b = \Psi_t^{out}(q)$, by the Schwarz inequality and the normalization of Ψ_t^{out} we get

$$\lim_{t\to\infty} \int_{\mathbb{R}^3} ||\Psi_t(q)|^2 - |\Psi_t^{out}(q)|^2| \, d^3q = 0 \tag{29}$$

Then (13) (with $n = 3$) and (29) yield (26).
Now let $R_0 > 0$ as in Proposition 4 and $T > 0$. Noting that by (11) and (23)

$$\begin{aligned}\big||\Psi_t(q)| - t^{-\frac{3}{2}}|\hat{\Psi}_0^{out}(\frac{q}{t})|\big| &\leq |\Psi_t(q) - \Psi_t^{out}(q) + \Psi_t^{out}(q) - \varphi_1(q,t)| \leq \\ &\leq |\varphi_3(q,t)| + |\varphi_2(q,t)|\end{aligned}$$

we get (27) from (24) and (12)(with $n = 3$).
Finally, to prove (28) we use

$$\big|v^\Psi(q,t) - \frac{q}{t}\big| \leq \big|v^\Psi(q,t) - v^{\Psi^{out}}(q,t)\big| + \big|v^{\Psi^{out}}(q,t) - \frac{q}{t}\big|. \tag{30}$$

Then by Lemma 1 there is a $C_1 < \infty$ such that

$$\begin{aligned}\big|v^{\Psi^{out}}(q,t) - \frac{q}{t}\big| &\leq C_1 t^{-\frac{5}{2}}\big|\Psi_t^{out}(q)\big|^{-1} = C_1 t^{-\frac{5}{2}}\big|\Psi_t(q)\big|^{-1}\frac{|\Psi_t(q)|}{|\Psi_t^{out}(q)|} \leq \\ &\leq C_1 t^{-\frac{5}{2}}\big|\Psi_t(q)\big|^{-1}\frac{|\Psi_t^{out}(q)| + |\varphi_3(q,t)|}{|\Psi_t^{out}(q)|} = \\ &= C_1 t^{-\frac{5}{2}}\big|\Psi_t(q)\big|^{-1}\big(1 + \frac{|\varphi_3(q,t)|}{|\Psi_t^{out}(q)|}\big),\end{aligned} \tag{31}$$

where we again used (23). To get a bound on the first term in (30) we will use that

$$\big|\nabla\Psi_t^{out}(q)\big| \leq \frac{|q|}{t}\big|\Psi_t^{out}(q)\big| + Ct^{-\frac{5}{2}} \tag{32}$$

for some $C < \infty$.
This is an immediate consequence of (15) (with $n = 3$) and $\Psi_0^{out} \in S(\mathbb{R}^3)$.
Then using the definition of v^Ψ and (23)

$$\begin{aligned}&\big|v^\Psi(q,t) - v^{\Psi^{out}}(q,t)\big| = \big|\mathrm{Im}\big(\frac{\nabla\Psi_t(q)}{\Psi_t(q)}\big) - \mathrm{Im}\big(\frac{\nabla\Psi_t^{out}(q)}{\Psi_t^{out}(q)}\big)\big| \leq \\ &\leq \left|\frac{\Psi_t^{out}(q)\big(\nabla\Psi_t^{out}(q) + \nabla\varphi_3(q,t)\big) - \nabla\Psi_t^{out}(q)\big(\Psi_t^{out}(q) + \varphi_3(q,t)\big)}{\Psi_t(q)\Psi_t^{out}(q)}\right| = \\ &= |\Psi_t(q)|^{-1}\big|\nabla\varphi_3(q,t) - \nabla\Psi_t^{out}(q)\frac{\varphi_3(q,t)}{\Psi_t^{out}(q)}\big| \leq \\ &\leq \big|\Psi_t(q)\big|^{-1}\left(|\nabla\varphi_3(q,t)| + |\nabla\Psi_t^{out}(q)|\frac{|\varphi_3(q,t)|}{|\Psi_t^{out}(q)|}\right) \leq \\ &\leq \big|\Psi_t(q)\big|^{-1}\left(|\nabla\varphi_3(q,t)| + \frac{|q|}{t}|\varphi_3(q,t)| + Ct^{-\frac{5}{2}}\frac{|\varphi_3(q,t)|}{|\Psi_t^{out}(q)|}\right).\end{aligned}$$

If we plug this together with (31) into (30) we get

$$\begin{aligned}&\big|v^\Psi(q,t)-\frac{q}{t}\big|\le\\&\le\big|\Psi_t(q)\big|^{-1}\big[C_1t^{-\frac{5}{2}}+\frac{|q|}{t}|\varphi_3(q,t)|+\frac{|\varphi_3(q,t)|}{|\Psi_t^{out}(q)|}t^{-\frac{5}{2}}(C_1+C)+|\nabla\varphi_3(q,t)|\big].\end{aligned}$$

Noting that $\frac{1}{|q|(t+|q|)}\le\frac{t}{|q|}t^{-2}$ by (24) and (25) this finally yields

$$\begin{aligned}&\big|v^\Psi(q,t)-\frac{q}{t}\big|\le\\&\quad\le t^{-2}\big|\Psi_t(q)\big|^{-1}\big[(C_1t^{-\frac{1}{2}}+C_2)+C_2(C_1+C)\frac{t}{|q|}\frac{t^{-\frac{5}{2}}}{|\Psi_t^{out}(q)|}+C_2\frac{t}{|q|}\big]\le\\&\qquad\le t^{-2}\big|\Psi_t(q)\big|^{-1}\big[(C_1T^{-\frac{1}{2}}+C_2)+C_2(C_1+C)\frac{t}{|q|}\frac{t^{-\frac{5}{2}}}{|\Psi_t^{out}(q)|}+C_2\frac{t}{|q|}\big]\end{aligned}$$

for all $t\ge T$ and $|q|>R_0$. $\square$

As in Section 3 we wish to control $\big|v^\Psi(\mathrm{Q}(q,t),t)-\frac{\mathrm{Q}(q,t)}{t}\big|$ for times t greater as or equal to some big time T. Looking at (28) one sees that this can be done by putting suitable conditions on $t^{\frac{3}{2}}|\Psi_t(\mathrm{Q}(q,t))|$, $t^{\frac{3}{2}}|\Psi_t^{out}(\mathrm{Q}(q,t))|$ and $\frac{\mathrm{Q}(q,t)}{t}$. Since the first two terms both tend to $|\hat{\Psi}_0^{out}(\frac{\mathrm{Q}(q,t)}{t})|$ as $t\to\infty$, if the asymptotic velocities really are constant, i.e. if $\frac{\mathrm{Q}(q,t)}{t}=\mathcal{O}(1)$, as in Section 3 it should suffice to have a lower bound for $|\hat{\Psi}_0^{out}(\frac{q}{T})|$. Contrary to Section 3 here we have to put a lower bound also on $\frac{\mathrm{Q}(q,t)}{t}$ or rather $\frac{q}{T}$ itself. But this should come as no surprise. We cannot expect a particle to move "freely" if it is still close to the scattering center, i.e. within the sphere of influence of the potential. Since we wait only for a finite time T before we start to look at the particle's trajectory, we have to allow for some sort of minimal "averaged velocity" $a<\frac{q}{T}$ so that the particle is already sufficiently far away from the spatial support of the potential at time T. This consideration leads us to the following definition and Theorem 2.

Definition 3. *For $\Psi_0^{out}\in L_2(\mathbb{R}^3)$ and for $\delta_1>0$, $\delta_2>0$ and $a>0$ define*

$$B_{\delta_1 a}^{\hat{\Psi}_0^{out}}:=\{q\in\mathbb{R}^3\mid q\in A_{\delta_1}^{\hat{\Psi}_0^{out}}\wedge|q|>a\}$$

and

$$B_{\delta_1\delta_2 a}^{\hat{\Psi}_0^{out}}:=\{q\in\mathbb{R}^3\mid U_{\delta_2}(q)\subset B_{\delta_1 a}^{\hat{\Psi}_0^{out}}\}.$$

Theorem 2. *Let $\Psi_0^{out} \in S(\mathbb{R}^3)$, $\Psi_t = e^{-iHt}W_+\Psi_0^{out}$. Assume A2. Then for all $\varepsilon > 0$ there exist some $a > 0$, $\delta_1 > 0$, $\delta_2 > 0$ and $T_\varepsilon > 0$ such that*

$$\mathbb{P}^{\Psi_{T_\varepsilon}}\big(\{q \in \mathbb{R}^3 \mid \frac{q}{T_\varepsilon} \notin B^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a}\}\big) < \varepsilon. \tag{33}$$

Moreover, for all $q \in \mathbb{R}^3\backslash\{0\}$ and all $t \geq T_\varepsilon$ such that $\frac{q}{t} \in B^{\hat{\Psi}_0^{out}}_{\delta_1 a}$

$$t^{\frac{3}{2}}|\Psi_t^{out}(q)| > \frac{\delta_1}{2} \qquad \text{and} \qquad t^{\frac{3}{2}}|\Psi_t(q)| > \frac{\delta_1}{2} \tag{34}$$

and for all $q \in \mathbb{R}^3\backslash\{0\}$ such that $\frac{q}{T_\varepsilon} \in B^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a}$

$$\left.\begin{array}{l} \dfrac{\mathrm{Q}_{T_\varepsilon}(q,t)}{t} \in U_{\delta_2}(\dfrac{q}{T_\varepsilon}) \ \text{and} \\ \qquad \big|v^\Psi\big(\mathrm{Q}_{T_\varepsilon}(q,t),t\big) - \dfrac{\mathrm{Q}_{T_\varepsilon}(q,t)}{t}\big| = \mathcal{O}(t^{-\frac{1}{2}}) < \dfrac{\delta_2}{2} \end{array}\right. \qquad \forall\, t \geq T_\varepsilon. \tag{35}$$

As the proof of Theorem 2 is quite similar to that of Theorem 1 we put it into the appendix.
Now we can state the main result of this Subsection.

Corollary 3. *Let $\Psi_0^{out} \in S(\mathbb{R}^3)$, $\Psi_t = e^{-iHt}W_+\Psi_0^{out}$. Assume A2. Then*

(i) $v_\infty(q) := \lim\limits_{t\to\infty} \frac{\mathrm{Q}(q,t)}{t}$ exists for $\mathbb{P}^\Psi$-almost all $q \in \mathbb{R}^3$ and the distribution of v_∞ has density $|\hat{\Psi}_0^{out}|^2$.

(ii) For all $\varepsilon > 0$ and $\delta > 0$ there exists a $T^\Psi_{\varepsilon\delta} > 0$ such that

$$\mathbb{P}^{\Psi_0}\big(\{q \in \mathbb{R}^3 \mid \sup_{t \geq T^\Psi_{\varepsilon\delta}} |v^\Psi\big(\mathrm{Q}(q,t),t\big) - v_\infty(q)| > \delta\}\big) < \varepsilon. \tag{36}$$

Using Theorem 2 instead of Theorem 1 the proof is exactly the same as that of Corollary 1 respectively Corollary 2.

4.2 Scattering State + Bound State

In the following let $H = H_0 + V$ as in Subsection 4.1. Let $\Psi_t = \Psi_t^{pp} + \Psi_t^{ac}$ with $\Psi_t^{pp} \in \mathcal{H}_{pp}(H)$ and $\Psi_t^{ac} = e^{-iHt}W_+\Psi_0^{out} \in \mathcal{H}_{ac}(H)$ with $\Psi_0^{out} \in S(\mathbb{R}^3)$. Assume A2. Then $\mathbb{P}^{\Psi}$-almost all Bohmian trajectories exist globally in time (Remark 8).
Assume further

A 4. *There exist $R_1 > 0$, $\alpha > 0$ and $C_5 < \infty$ such that Ψ_t^{pp} is continuous differentiable in $B_{R_1}^c$, $\Psi_t \in C^1(B_{R_1}^c)$, for all $t \in \mathbb{R}$. Moreover*

$$|\Psi_t^{pp}(q)| \leq C_5|q|^{-\frac{3}{2}-\alpha} \qquad \textit{and} \qquad |\nabla\Psi_t^{pp}(q)| \leq C_5|q|^{-\frac{3}{2}-\alpha} \tag{37}$$

for all $t \in \mathbb{R}$ and $|q| > R_1$.

Remark 10. Note that A2 already implies $\Psi_t \in C^1(B_{R_1}^c)$ for all $t \in \mathbb{R}$. Nevertheless, as we will use A4 separately, we added it here.
More importantly (37) is supposedly not too strong an assumption. Indeed there is a huge amount of literature on the exponential decay of eigenfunctions of Schrödinger operators, although results for the gradient of eigenfunctions are rather rare (see [29, 30] for an overview). We wish to state two results on eigenfunctions $u \in L_2(\mathbb{R}^3)$, i.e. solutions of $Hu = Eu$ with H as above and $E < 0$.

(i) There exist $R > 0$ and $C < \infty$ such that

$$|u(q)| \leq C|q|^{-1}e^{-|E|^{\frac{1}{2}}|q|}$$

in B_R^c (see e.g. [1]).

(ii) If in addition to the above $V \in K_3^{(1)}$ (where we use the notation of [29], p. 467), i.e. if the singularities of V are not too bad, $u \in C^1(\Omega)$ and for every $q_0 \in \Omega$

$$\sup_{\{q\in\Omega \mid |q_0-q|\leq 1\}} |\nabla u(q)| \leq C \int\limits_{|q_0-q|\leq 2} |u(q)|\, dq$$

for some (possibly E-dependent) positive constant C (q.v. [29]: Theorems C.2.4. and C.2.5.). Using (i), we particulary get $|\nabla u(q)| = \mathcal{O}(|q|e^{-|E|^{\frac{1}{2}}|q|})$ for $|q| \to \infty$.

A4 severely curtails the probability that the modulus of the velocity $|v^{\Psi^{pp}}|$ of a "bound" particle far away from the scattering center is large. If $|v^{\Psi^{pp}}(q)| \leq \frac{|\nabla\Psi_t^{pp}(q)|}{|\Psi_t^{pp}(q)|}$ is large, either $|\nabla\Psi_t^{pp}(q)|$ is large or $|\Psi_t^{pp}(q)|$ is small or both. But A4 implies that $|\nabla\Psi_t^{pp}(q)|$ gets ever smaller for growing $|q|$, so the only possibility left is that $|\Psi^{pp}(q)|$ and thus also $|\Psi^{pp}(q)|^2$, the probability of the particle to be at q, is small and gets ever smaller for growing $|q|$. Therefore it is most likely (with respect to $\mathbb{P}^{\Psi^{pp}}$) that a particle far away from the scattering center has got only a small velocity and thus stays inside a ball around the origin with radius growing only slowly in time.
But let's turn back to the full wave function $\Psi = \Psi^{ac} + \Psi^{pp}$. We will prove that there exists a natural decomposition of all possible paths in the following sense. If one waits long enough, most (with respect to $\mathbb{P}^{\Psi}$) initial configurations q are such that the trajectory $\mathrm{Q}(q,t)$ either stays inside a sphere around the origin with a radius increasing slowly like $t^{\frac{1}{1+\gamma}}$ (where γ needs to be only arbitrarily smaller than 2α) or is a nearly straight line and stays outside a faster moving sphere (also centered around the origin) with a radius that grows proportional to t.

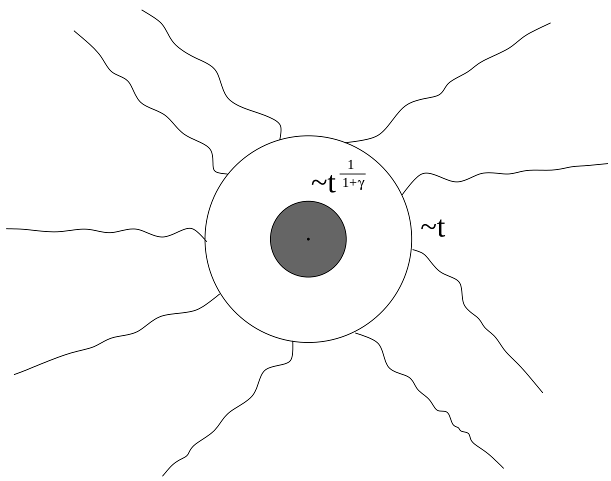

Figure 1: Splitting of the Bohmian trajectories for $\Psi = \Psi^{ac} + \Psi^{pp}$.

Moreover, the decomposition of paths corresponds quite clearly to the decomposition of $\Psi = \Psi^{ac} + \Psi^{pp}$ into scattering wave function and bound wave function.

Theorem 3. *Let* $\Psi_t = \Psi_t^{pp} + \Psi_t^{ac}$ *with* $\Psi_t^{pp} \in \mathcal{H}_{pp}(H)$ *and* $\Psi_t^{ac} = e^{-iHt}W_+\Psi_0^{out}$

$\in \mathcal{H}_{ac}(H)$ with $\Psi_0^{out} \in S(\mathbb{R}^3)$. Assume A2 and A4. Define $v_\infty(q) := \lim_{t\to\infty} \frac{\mathrm{Q}(q,t)}{t}$ and (for $R > 0$, $T > 0$, $\gamma > 0$ and $\delta > 0$)

$$\begin{aligned}
&\mathrm{K}_\gamma^<(R,T) := \Big\{q \in \mathbb{R}^3 \mid |\mathrm{Q}(q,t)| \le R\big(\frac{t}{T}\big)^{\frac{1}{1+\gamma}} \quad \forall t \ge T\Big\},\\
&\mathrm{K}_\delta^>(R,T) :=\\
&\quad = \Big\{q \in \mathbb{R}^3 \mid |\mathrm{Q}(q,t)| > R\frac{t}{T} \wedge \big|v^\Psi(\mathrm{Q}(q,t),t) - v_\infty(q)\big| < \delta \quad \forall t \ge T\Big\}.
\end{aligned}$$

Then for all $\varepsilon > 0$, $\delta > 0$ and all $0 < \gamma < 2\alpha$ there exist $R_{\varepsilon\gamma\delta} > 0$ and $T_{\varepsilon\gamma\delta} > 0$ such that

$$\mathbb{P}^{\Psi_0}\big(\mathrm{K}_\gamma^<(R_{\varepsilon\gamma\delta}, T_{\varepsilon\gamma\delta}) \cup \mathrm{K}_\delta^>(R_{\varepsilon\gamma\delta}, T_{\varepsilon\gamma\delta})\big) > 1 - \varepsilon. \tag{38}$$

In fact

$$\big|\mathbb{P}^{\Psi_0}\big(\mathrm{K}_\gamma^<(R_{\varepsilon\gamma\delta}, T_{\varepsilon\gamma\delta})\big) - \|\Psi_0^{pp}\|^2\big| < \varepsilon \tag{38a}$$

and

$$\big|\mathbb{P}^{\Psi_0}\big(\mathrm{K}_\delta^>(R_{\varepsilon\gamma\delta}, T_{\varepsilon\gamma\delta})\big) - \|\Psi_0^{ac}\|^2\big| < \varepsilon. \tag{38b}$$

We prove Theorem 3 in two steps. First we focus on the conditions under which trajectories become asymptotically free (Theorem 4 below). Those are the same as for pure scattering wave functions (i.e. $\frac{q}{t} \in B_{\delta_1\delta_2 a}^{\hat{\Psi}_0^{out}}$): if a particle at some big time T is sufficiently far away from the scattering center ($\frac{|q|}{T}$ is bigger than some minimal "averaged velocity" a) and its "momentum" $\frac{|q|}{T}$ (recall $\hbar = m = 1$) is not too close to a node of $\hat{\Psi}_0^{out}$ ($|\hat{\Psi}_0^{out}| > \delta_1$ in some neighborhood of $\frac{q}{t}$), the velocity of the particle for times t bigger than or equal to T is well behaved (in the sense of (48) in Theorem 4 below, resp. of (35) in Theorem 2) and the particles trajectory becomes asymptotically free (proof of Theorem 3, resp. Corollary 3).
There are, however, two differences. The first concerns how fast a trajectory becomes free. While for a pure scattering wave function the convergence of the "real" velocity $v^\Psi(\mathrm{Q}_T(q,t),t)$ along the trajectory $\mathrm{Q}_T(q,t)$ to the "free" velocity $\frac{\mathrm{Q}_T(q,t)}{t}$ is of order $t^{-\frac{1}{2}}$ ((35) again), for the more general wave function $\Psi = \Psi^{pp} + \Psi^{ac}$ it is of order $t^{-\beta}$ ((48)below), where $\beta := \min\{\alpha, \frac{1}{2}\}$ depends also on how fast Ψ^{pp} and $\nabla\Psi^{pp}$ decay spatially (q.v. A4). The convergence speed cannot be faster than for a pure scattering wave function but it might very well be slower if Ψ^{pp} is spread out too much. The second difference concerns how likely it is that a trajectory becomes free. While

for a pure scattering wave function the set for which a trajectory becomes free ($\frac{q}{t} \in B_{\delta_1\delta_2 a}^{\hat{\Psi}_0^{out}}$) has (depending on a, δ_1, δ_2, T) nearly full measure ((33) in Theorem 2), for $\Psi = \Psi^{pp} + \Psi^{ac}$ this is not true. The spatial support of the bound part Ψ^{pp} stays concentrated around the origin for all times (see (22)), so the probability that a particle is found near the origin and thus does not fulfill $\frac{q}{t} \in B_{\delta_1\delta_2 a}^{\hat{\Psi}_0^{out}}$ (which would entail $|q| > aT$) is not negligible, not even if the minimal "averaged velocity" a is so small that (nearly all of) the spatial support of the scattering part has already left the ball with radius aT. Thus to get a set of nearly full measure we have to add those initial configurations (at time T) that are still inside the ball with radius aT (Definition 4 and (45) in Theorem 4). Note however, that this radius aT is not so big that the ball encloses the spatial support of the whole wave function Ψ (in which case the statement of (45) would be rather trivial) but just that of the bound part Ψ^{pp} ((46) in Theorem 4).
So now we know that at T there are two sets of initial configurations, those that are outside and those that are inside a ball with a certain radius. But while we know what kind of trajectories are made by the initial configurations outside the ball (namely asymptotically free ones) we do not yet know what the trajectories made by the initial configurations inside the ball look like. This is what we concern ourselves with in the second step of the proof of Theorem 3. We show that, for T big enough, the probability that a particle starting inside the above mentioned ball leaves another ball with a slowly growing radius (in the sense of Theorem 3) can be made arbitrary small.

Lemma 3. *Let $\Psi_t = \Psi_t^{pp} + \Psi_t^{ac}$ with $\Psi_t^{pp} \in \mathcal{H}_{pp}(H)$ and $\Psi_t^{ac} = e^{-iHt}W_+\Psi_0^{out} \in \mathcal{H}_{ac}(H)$ with $\Psi_0^{out} \in S(\mathbb{R}^3)$. Assume A2 and A4. Define (for $R > 0$, $T > 0$, $\gamma > 0$)*

$$\begin{aligned}\mathrm{A}_\gamma(R,T) :&= \left\{q \in \mathbb{R}^3 \mid |\mathrm{Q}(q,T)| \leq R \wedge \exists t \geq T : \ |\mathrm{Q}(q,t)| > R\left(\frac{t}{T}\right)^{\frac{1}{1+\gamma}}\right\} = \\ &= \{q \in \mathbb{R}^3 \mid |\mathrm{Q}(q,T)| \leq R\} \cap \left(\mathrm{K}_\gamma^<(R,T)\right)^c.\end{aligned}$$

Then for all $\varepsilon > 0$ and $0 < \gamma < 2\alpha$ there exist $a > 0$ and $T_{\varepsilon\gamma} > 0$ such that Theorem 4 holds and

$$\sup_{t \geq T_{\varepsilon\gamma}} \mathbb{P}^{\Psi_0}\big(\mathrm{A}_\gamma(at,t)\big) < \varepsilon. \tag{39}$$

Remark 11. If $\Psi = \Psi^{pp}$, (38a) tells us that for $t \to \infty$ the probability of finding a particle outside every ball with radius growing like $t^{\frac{1}{1+\gamma}}$ (where γ

needs to be only arbitrarily smaller than 2α) is zero.
If $\Psi = \Psi^{ac}$, (38b) is a sharper form of (36). It adds that for $t \to \infty$ the probability of finding a particle inside every ball with radius growing linear in time is zero.

We proceed as in Subsection 4.1 and start with some estimates on the wave function Ψ and the velocity field v^{Ψ}.

Lemma 4. *Let $\Psi_t = \Psi_t^{pp} + \Psi_t^{ac}$ with $\Psi_t^{pp} \in \mathcal{H}_{pp}(H)$ and $\Psi_t^{ac} = e^{-iHt}W_+\Psi_0^{out} \in \mathcal{H}_{ac}(H)$ with $\Psi_0^{out} \in S(\mathbb{R}^3)$. Assume A4. Then*

(i) for all $\varepsilon > 0$ there is a $R > 0$ such that (22) holds.

(ii) There exist $R_2 > 0$ and $C_5 < \infty$ such that for all $r \in \mathbb{R}_0^+$ there are $c_r < \infty$ and for all $T > 0$ there are $C_i < \infty$ $(i = 2, 3, 4, 6, 7,)$ such that for all $|q| > R_2$ and $t \geq T$

$$\begin{aligned} \big||\Psi_t(q)| - t^{-\frac{3}{2}}|\hat{\Psi}_0^{out}(\frac{q}{t})|\big| &\leq \\ &\leq \frac{C_2}{|q|(t+|q|)} + c_r t^{-\frac{5}{2}}\big(\frac{t}{|q|}\big)^r + C_5|q|^{-\frac{3}{2}-\alpha} \end{aligned} \tag{40}$$

and, if neither $\Psi_t(q) = 0$ nor $\Psi_t^{ac}(q) = 0$ nor $\Psi_t^{out}(q) = 0$,

$$\big|v^{\Psi}(q,t) - \frac{q}{t}\big| \leq \big[\mathrm{f}_1(q,t)t^{\beta-\frac{1}{2}} + \mathrm{f}_2(q,t)t^{\beta-\alpha}\big]\frac{t^{-\frac{3}{2}-\beta}}{|\Psi_t(q)|} \tag{41}$$

with

$$\beta := \min\{\alpha, \frac{1}{2}\}, \quad \mathrm{f}_1(q,t) \quad \text{as in Lemma 2 and}$$
$$\mathrm{f}_2(q,t) := C_5\big(\frac{t}{|q|}\big)^{\frac{3}{2}+\alpha}\Big[1 + \frac{t^{-\frac{3}{2}}}{|\Psi_t^{ac}(q)|}\Big(\mathrm{f}_1(q,t)t^{-\frac{1}{2}} + C_6\big(1 + C_7\frac{t}{|q|}\big)\Big)\Big].$$

Proof of Lemma 4.

(i) As mentioned above (22) is a well known feature of all bound wave functions. It holds true if V fulfills the so-called Enss condition (Definition 4.1 in [25]): V is H_0- bounded with relative bound less than 1 and the bounded, monotone decreasing function $\|V(H_0 + i)^{-1}F(|q| \geq R)\|$ is integrable on $(0, \infty)$. Here $F(|q| \geq R)$ denotes the operator of multiplication by the characteristic function of the set $\{q \in \Omega \mid |q| \geq R\}$.
For $n = 3$ the Enss condition is necessarily fulfilled if $V = \mathcal{O}(|q|^{-1-\varepsilon})$

for $|q| \to \infty$ and some $\varepsilon > 0$ (see Example 2.2 in [25] or the discussion after (3.15) in [19]).
Since we assumed $V \in (V)_4$ (22) holds. Moreover in our case it follows directly from A4.

(ii) Let $T > 0$. Noting that

$$\big||\Psi_t(q)| - t^{-\frac{3}{2}}\big|\hat{\Psi}_0^{out}(\frac{q}{t})\big|\big| \le \big||\Psi_t^{ac}(q)| - t^{-\frac{3}{2}}\big|\hat{\Psi}_0^{out}(\frac{q}{t})\big|\big| + \big|\Psi_t^{pp}(q)\big|$$

(27) and A4 immediately give (40).
To get (41) we proceed as follows.
With (28) and A4 we get

$$\begin{aligned}\big|v^{\Psi}(q,t) - \frac{q}{t}\big| &\le \big|v^{\Psi^{ac}}(q,t) - \frac{q}{t}\big| + \big|v^{\Psi}(q,t) - v^{\Psi^{ac}}(q,t)\big| \le \\ &\le \mathrm{f}_1(q,t)\frac{t^{-2}}{\big|\Psi_t(q)\big|}\big(1 + \frac{\big|\Psi_t^{pp}(q)\big|}{\big|\Psi_t^{ac}(q)\big|}\big) + \\ &\qquad + \big|\Psi_t(q)\big|^{-1}\big[\big|\nabla\Psi_t^{pp}(q)\big| + \big|\Psi_t^{pp}(q)\big|\frac{\big|\nabla\Psi_t^{ac}(q)\big|}{\big|\Psi_t^{ac}(q)\big|}\big] \le \\ &\le \frac{t^{-\frac{3}{2}-\beta}}{\big|\Psi_t(q)\big|}\big[\mathrm{f}_1(q,t)t^{\beta-\frac{1}{2}} + \\ &\qquad + C_5\big(\frac{t}{|q|}\big)^{\frac{3}{2}+\alpha}\big(1 + \frac{\mathrm{f}_1(q,t)t^{-2} + \big|\nabla\Psi_t^{ac}(q)\big|}{\big|\Psi_t^{ac}(q)\big|}\big)t^{\beta-\alpha}\big]\end{aligned} \tag{42}$$

for all $t \ge T$ and all $|q| > R_1$.
To get a bound on the second term we note that by (11) (with $n = 3$) and (23)

$$\big|\nabla\Psi_t^{ac}(q)\big| \le \big|\nabla\varphi_1(q,t)\big| + \big|\nabla\varphi_2(q,t)\big| + \big|\nabla\varphi_3(q,t)\big|$$

and thus by (11), (12) and (25) there exist $R_0 > 0$, $c_r < \infty$ $(r \ge 0)$ and $C_2 < \infty$ such that

$$\begin{aligned}\big|\nabla\Psi_t^{ac}(q)\big| \le \frac{|q|}{t}t^{-\frac{3}{2}}\big|\hat{\Psi}_0^{out}(\frac{q}{t})\big| + t^{-\frac{3}{2}}\big|\nabla\hat{\Psi}_0^{out}(\frac{q}{t})\big| + \\ + \frac{C_2}{|q|(t+|q|)} + c_r t^{-\frac{5}{2}}\big(\frac{t}{|q|}\big)^r\end{aligned} \tag{43}$$

for all $t \ge T$ and $|q| > R_0$.
Since Ψ^{out} and thus also $\frac{q}{t}\hat{\Psi}_0^{out}(\frac{q}{t})$ and $\nabla\hat{\Psi}_0^{out}(\frac{q}{t})$ are Schwartz functions

and $\frac{1}{|q|(t+|q|)} \leq \frac{t}{|q|}t^{-2}$ we get with $r = 0$

$$\begin{aligned}
&\left|\nabla\Psi_t^{ac}(q)\right| \leq \\
&\qquad \leq t^{-\frac{3}{2}} \sup_{\frac{q}{t}\in\mathbb{R}^3} \left|\frac{q}{t}\hat{\Psi}_0^{out}(\frac{q}{t})\right| + t^{-\frac{3}{2}} \sup_{\frac{q}{t}\in\mathbb{R}^3} \left|\nabla\hat{\Psi}_0^{out}(\frac{q}{t})\right| + \\
&\qquad\qquad + C_2\frac{t}{|q|}t^{-2} + c_0 t^{-\frac{5}{2}} \leq \\
&\qquad \leq C_6 t^{-\frac{3}{2}}\big(1 + C_7\frac{t}{|q|}\big)
\end{aligned} \tag{44}$$

for some $C_6 < \infty$, $C_7 < \infty$ and all $t \geq T$ and $|q| > R_0$.
Plugging (44) into (42) and taking $R_2 := \max\{R_0, R_1\}$ gives the desired result.

□

Since the spatial support of the bound part Ψ^{pp} of the wave function stays concentrated around the scattering center we adjust Definition 3 and Theorem 2 of subsection 4.1 as described above and get:

Definition 4. *For $\Psi_0^{out} \in L_2(\mathbb{R}^3)$ and $\delta_1 > 0$, $\delta_2 > 0$ and $a > 0$ define*

$$\tilde{B}_{\delta_1 a}^{\hat{\Psi}_0^{out}} := B_{\delta_1 a}^{\hat{\Psi}_0^{out}} \cup \{q \in \mathbb{R}^3 \mid |q| \leq a\}$$

and

$$\tilde{B}_{\delta_1\delta_2 a}^{\hat{\Psi}_0^{out}} := \{q \in \mathbb{R}^3 \mid U_{\delta_2}(q) \subset \tilde{B}_{\delta_1(a+\delta_2)}^{\hat{\Psi}_0^{out}}\} = B_{\delta_1\delta_2 a}^{\hat{\Psi}_0^{out}} \cup \{q \in \mathbb{R}^3 \mid |q| \leq a\}.$$

Theorem 4. *Let $\Psi_t = \Psi_t^{pp} + \Psi_t^{ac}$ with $\Psi_t^{pp} \in \mathcal{H}_{pp}(H)$ and $\Psi_t^{ac} = e^{-iHt}W_+\Psi_0^{out} \in \mathcal{H}_{ac}(H)$ with $\Psi_0^{out} \in S(\mathbb{R}^3)$. Assume A2 and A4. Then for all $\varepsilon > 0$ there are $a > 0$, $\delta_1 > 0$, $\delta_2 > 0$ and $T_\varepsilon > 0$ such that*

$$\mathbb{P}^{\Psi_{T_\varepsilon}}\big(\{q \in \mathbb{R}^3 \mid \frac{q}{T_\varepsilon} \notin \tilde{B}_{\delta_1\delta_2 a}^{\hat{\Psi}_0^{out}}\}\big) < \varepsilon \tag{45}$$

and

$$\sup_{t\geq T_\varepsilon} \left|\mathbb{P}^{\Psi_t}\big(\{q \in \mathbb{R}^3 \mid |q| \leq at\}\big) - \|\Psi_0^{pp}\|^2\right| < \varepsilon. \tag{46}$$

Moreover, for all $t \geq T_\varepsilon$ and $q \in \mathbb{R}^3$ such that $\frac{q}{t} \in B_{\delta_1 a}^{\hat{\Psi}_0^{out}}$

$$t^{\frac{3}{2}}\left|\Psi_t^{out}(q)\right| > \frac{\delta_1}{2}, \quad t^{\frac{3}{2}}\left|\Psi_t^{ac}(q)\right| > \frac{\delta_1}{2} \quad \text{and} \quad t^{\frac{3}{2}}\left|\Psi_t(q)\right| > \frac{\delta_1}{2} \tag{47}$$

and for all $q \in \mathbb{R}^3$ *such that* $\frac{q}{T_\varepsilon} \in B^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a}$

$$\left.\begin{array}{l} \frac{\mathrm{Q}_{T_\varepsilon}(q,t)}{t} \in U_{\delta_2}(\frac{q}{T_\varepsilon}) \quad and \\ \qquad \big|v^\Psi\big(\mathrm{Q}_{T_\varepsilon}(q,t),t\big) - \frac{\mathrm{Q}_{T_\varepsilon}(q,t)}{t}\big| = \mathcal{O}(t^{-\beta}) < \delta_2 \end{array}\right\} \forall\, t \geq T_\varepsilon. \tag{48}$$

The proof of Theorem 4 is essentially the same as that of Theorem 2 (resp. Theorem 1) and can be found in the appendix.

To prove Lemma 3 we use that roughly speaking the quantum probability flux j^Ψ across a surface is a measure for the probability that a particle crosses this surface. Thus we first establish some estimates on the flux.

Lemma 5. *Let* $\Psi_t = \Psi_t^{pp} + \Psi_t^{ac}$ *with* $\Psi_t^{pp} \in \mathcal{H}_{pp}(H)$ *and* $\Psi_t^{ac} = e^{-iHt}W_+\Psi_0^{out} \in \mathcal{H}_{ac}(H)$ *with* $\Psi_0^{out} \in S(\mathbb{R}^3)$. *Assume A4 and let* $T > 0$, $r_i \in \mathbb{R}_0^+$ $(i = 1, 2, \ldots, 4)$. *Then there exist* $C_8 < \infty$, $C_9 < \infty$ *and* $c_{r_i} < \infty$ *such that*

$$\sup_{t\in\mathbb{R}} \big|j^{\Psi^{pp}}(q,t)\big| \leq C_8 |q|^{-3-2\alpha}, \tag{49}$$

$$\begin{aligned} \big|j^{\Psi^{ac}}(q,t)\big| \leq{}& c_{r_1}\big(\frac{t}{|q|}\big)^{r_1} t^{-3} + c_{r_2}\big(\frac{t}{|q|}\big)^{r_2} \frac{t^{-\frac{3}{2}}}{|q|(t+|q|)} + c_{r_3}\big(\frac{t}{|q|}\big)^{r_3} t^{-4} + \\ &+ c_{r_3}\big(\frac{t}{|q|}\big)^{2r_3} t^{-5} + c_{r_3}\big(\frac{t}{|q|}\big)^{r_3} \frac{t^{-\frac{5}{2}}}{|q|(t+|q|)} + C_9\big(\frac{1}{|q|(t+|q|)}\big)^2 \end{aligned} \tag{50}$$

and for $j_m = j^\Psi - j^{\Psi^{pp}} - j^{\Psi^{ac}}$

$$\big|j_m(q,t)\big| \leq |q|^{-\frac{3}{2}-\alpha}\big[c_{r_4}\big(\frac{t}{|q|}\big)^{r_4} t^{-\frac{3}{2}} + C_9 \frac{1}{|q|(t+|q|)} + c_{r_3}\big(\frac{t}{|q|}\big)^{r_3} t^{-\frac{5}{2}}\big] \tag{51}$$

for all $t \geq T$ *and* $|q| > R_2$. *Here* R_2 *is as in Lemma 4.*

Proof of Lemma 5. (49) we get directly from A4. Since $\Psi_0^{out} \in S(\mathbb{R}^3)$, by the same argument we used in Remark 4 we get that $\big(\frac{q}{t}\big)^{\tilde{r}}\hat{\Psi}_0^{out}(\frac{q}{t})$ and $\big(\frac{q}{t}\big)^{\tilde{r}}\nabla\hat{\Psi}_0^{out}(\frac{q}{t})$ are bounded for every $\tilde{r} \in \mathbb{R}_0^+$.
Then (27) (which holds for Ψ^{ac}) implies that for every r, $\tilde{r} \in \mathbb{R}_0^+$ there exist $c_r < \infty$, $c_{\tilde{r}} < \infty$ and for every $T > 0$ there exists some $C_2 < \infty$ such that

$$\begin{aligned} \big|\Psi_t^{ac}(q)\big| &\leq t^{-\frac{3}{2}}\big|\hat{\Psi}_0^{out}(\frac{q}{t})\big| + \frac{C_2}{|q|(t+|q|)} + c_r t^{-\frac{5}{2}}\big(\frac{t}{|q|}\big)^r \leq \\ &\leq c_{\tilde{r}} t^{-\frac{3}{2}}\big(\frac{t}{|q|}\big)^{\tilde{r}} + \frac{C_2}{|q|(t+|q|)} + c_r t^{-\frac{5}{2}}\big(\frac{t}{|q|}\big)^r \end{aligned} \tag{52}$$

for all $|q| > 0$ and $t \geq T$.
In exactly the same way (43) yields: For every $r, \tilde{r} \in \mathbb{R}_0^+$ there exist $c_r < \infty$, $c_{\tilde{r}} < \infty$ and for every $T > 0$ there exists some $C_2 < \infty$ such that

$$\left|\nabla\Psi_t^{ac}(q)\right| \leq c_{\tilde{r}} t^{-\frac{3}{2}} \Big(\frac{t}{|q|}\Big)^{\tilde{r}} + \frac{C_2}{|q|(t+|q|)} + c_r t^{-\frac{5}{2}} \Big(\frac{t}{|q|}\Big)^{r} \tag{53}$$

for all $t \geq T$ and $|q| > R_0$, where R_0 is as in Proposition 4.
Then (50) follows immediately from (52) and (53), (51) follows from (52), (53) and A4. □

Now we can prove Lemma 3.

Proof of Lemma 3. Let $\varepsilon > 0$ and $0 < \gamma < 2\alpha$, let $a > 0$ and $T_\varepsilon > 0$ such that Theorem 4 holds.
Let $t \geq T_\varepsilon$. Since $\mathrm{Q}(q,s)$ is continuous in s (Remark 1) $q \in \mathrm{A}_\gamma(at,t)$ implies that $\mathrm{Q}(q,s)$ crosses the moving sphere $S_{R_t(s)}$ (with $R_t(s) := (at)\big(\frac{s}{t}\big)^{\frac{1}{1+\gamma}}$) at least once and outwards in $[t,\infty)$. Therefor $\mathbb{P}^{\Psi_0}\big(\mathrm{A}_\gamma(at,t)\big)$ is bounded from above by the probability that some trajectory crosses $S_{R(s)}$ in any direction in $[t,\infty)$. The latter is given by (see e.g. [7, 15])

$$\int_t^\infty ds \int_{S_{R_t(s)}} \left|j^\Psi(q,s)\cdot\hat{n}\right| d\sigma =: \mathrm{P}_\gamma(at,t).$$

Therefore it suffices to show that $\sup\limits_{t\geq T_\varepsilon} \mathrm{P}_\gamma(at,t) < \varepsilon$ for a small and T_ε big enough[8]. To do this we split $\mathrm{P}_\gamma(at,t)$ according to the splitting of j^Ψ implicit

[8]Note that while T_ε depends on a (T_ε typically has to be increased if one decreases a) a is independent of T_ε. So we won't run into trouble even with terms like $a^{-1-2\alpha}t^{-2\alpha} \leq a^{-1-2\alpha}T_\varepsilon^{-2\alpha}$. In fact we will suppress the dependence on a of such terms and just write $\mathcal{O}(t^{-2\alpha})$.

in Lemma 5,

$$\mathrm{P}_\gamma(at,t) = \mathrm{P}_\gamma^{pp}(at,t) + \mathrm{P}_\gamma^{ac}(at,t) + \mathrm{P}_\gamma^{m}(at,t)$$

with

$$\mathrm{P}_\gamma^{pp/ac}(at,t) = \int_t^\infty ds \int_{S_{R_t(s)}} \left| j^{\Psi^{pp}/\Psi^{ac}}(q,s)\cdot \hat{n}\right| d\sigma$$

and

$$\mathrm{P}_\gamma^{m}(at,t) = \int_t^\infty ds \int_{S_{R_t(s)}} \left| j_m(q,t)\cdot \hat{n}\right| d\sigma,$$

and show
$\mathrm{P}_\gamma^{pp}(at,t) = \mathcal{O}(t^{-2\alpha})$, $\mathrm{P}_\gamma^{m}(at,t) = \mathcal{O}(t^{-\alpha})$ and $\mathrm{P}_\gamma^{ac}(at,t) = \mathcal{O}(a^2) + \mathcal{O}(t^{-\frac{1}{2}})$.
By (49)

$$\mathrm{P}_\gamma^{pp}(at,t) \le 4\pi C_8 \int_t^\infty R_t(s)^{-1-\alpha}\, ds = 4\pi C_8 (at)^{-1-2\alpha}\frac{1+\gamma}{2\alpha-\gamma} t = \mathcal{O}(t^{-2\alpha}).$$

In exactly the same way we get the desired bounds on $\mathrm{P}_\gamma^{m}(at,t)$ and $\mathrm{P}_\gamma^{ac}(at,t)$ since for $s \ge t \ge T_\varepsilon$ and $|q| \ge R_{T_\varepsilon}(T_\varepsilon)$ big enough, i.e. for T_ε big enough, (50) implies

$$\begin{aligned}
\left| j^{\Psi^{ac}}(q,s)\right| &\le \\
&\le c_0\big(s^{-3} + \frac{1}{|q|(s+|q|)} s^{-\frac{3}{2}} + s^{-4} + s^{-5} + \frac{1}{|q|(s+|q|)} s^{-\frac{5}{2}}\big) + \\
&\qquad\qquad + C_9\big(\frac{1}{|q|(s+|q|)}\big)^2 \le \\
&\le \tilde{C}_1 s^{-3} + \tilde{C}_2 |q|^{-2} s^{-\frac{3}{2}}
\end{aligned}$$

and (51) implies

$$\begin{aligned}
\left| j_m(q,s)\right| &\le |q|^{-\frac{3}{2}-\alpha}\big(c_0 s^{-\frac{3}{2}} + C_9 |q|^{-1} s^{-1} + c_0 s^{-\frac{5}{2}}\big) \le \\
&\le \tilde{C}_3 |q|^{-\frac{3}{2}-\alpha} s^{-\frac{3}{2}} + C_9 |q|^{-\frac{5}{2}-\alpha} s^{-1}.
\end{aligned}$$

□

Remark 12. In the proof of Lemma 3 we used Lemma 5 with $r_i = 0$ and thus (12) in Proposition 2 with $r = 0$ only. Thus as in subsection 4.1 we can replace the condition $\hat{\Psi}_0^{out} \in S(\mathbb{R}^3)$ by the weaker ones in Remark 6 and still attain our main result, Theorem 3.

Finally we prove our main result, Theorem 3.

Proof of Theorem 3. Let $\varepsilon > 0$ and $0 < \gamma < 2\alpha$. Let $a > 0$, $\delta_1 > 0$, $\delta_2 > 0$ and $T_{\varepsilon\gamma} > 0$ such that Theorem 4 and Lemma 3 hold.
Since

$$
\begin{aligned}
&\{q \in \mathbb{R}^3 \mid |q| \le aT_{\varepsilon\gamma}\} = \\
&\quad = \big\{q \in \mathbb{R}^3 \mid |q| \le aT_{\varepsilon\gamma} \ \wedge\ \exists t \ge T_{\varepsilon\gamma} :\ |\mathrm{Q}_{T_{\varepsilon\gamma}}(q,t)| > aT_{\varepsilon\gamma}\big(\frac{t}{T_{\varepsilon\gamma}}\big)^{\frac{1}{1+\gamma}}\big\} \cup \\
&\qquad\qquad \cup \big\{q \in \mathbb{R}^3 \mid |\mathrm{Q}_{T_{\varepsilon\gamma}}(q,t)| \le aT_{\varepsilon\gamma}\big(\frac{t}{T_{\varepsilon\gamma}}\big)^{\frac{1}{1+\gamma}} \quad \forall \ge T_{\varepsilon\gamma}\big\} = \\
&\quad =: \mathrm{A}_{\gamma T_{\varepsilon\gamma}}(aT_{\varepsilon\gamma}, T_{\varepsilon\gamma}) \cup \mathrm{K}^{<}_{\gamma T_{\varepsilon\gamma}}(aT_{\varepsilon\gamma}, T_{\varepsilon\gamma})
\end{aligned}
$$

and

$$
\tilde{B}^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a} = B^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a} \cup \{q \in \mathbb{R}^3 \mid |q| \le a\}
$$

we get

$$
\begin{aligned}
&\mathbb{P}^{\Psi_{T_{\varepsilon\gamma}}}\big(\big\{q \in \mathbb{R}^3 \mid \frac{q}{T_{\varepsilon\gamma}} \in \tilde{B}^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a}\big\}\big) = \\
&\quad = \mathbb{P}^{\Psi_{T_{\varepsilon\gamma}}}\Bigg(\big\{q \in \mathbb{R}^3 \mid \frac{q}{T_{\varepsilon\gamma}} \in B^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a}\big\} \cup \mathrm{A}_{\gamma T_{\varepsilon\gamma}}(aT_{\varepsilon\gamma}, T_{\varepsilon\gamma}) \cup \mathrm{K}^{<}_{\gamma T_{\varepsilon\gamma}}(aT_{\varepsilon\gamma}, T_{\varepsilon\gamma})\Bigg) \le \\
&\quad \le \mathbb{P}^{\Psi_{T_{\varepsilon\gamma}}}\big(\mathrm{A}_{\gamma T_{\varepsilon\gamma}}(aT_{\varepsilon\gamma}, T_{\varepsilon\gamma})\big) + \\
&\qquad\qquad \mathbb{P}^{\Psi_{T_{\varepsilon\gamma}}}\big(\mathrm{K}^{<}_{\gamma T_{\varepsilon\gamma}}(aT_{\varepsilon\gamma}, T_{\varepsilon\gamma}) \cup \big\{q \in \mathbb{R}^3 \mid \frac{q}{T_{\varepsilon\gamma}} \in B^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a}\big\}\big).
\end{aligned}
$$

Moreover, $\mathrm{Q}_{T_{\varepsilon\gamma}}(q,t) = \Phi_{t,T_{\varepsilon\gamma}}(q)$ and $\mathrm{Q}(q,t) = \Phi_{t,0}(q)$ (see Subsection 2.2), so as in the proof of Corollary 1 by equivariance we get

$$
\mathbb{P}^{\Psi_{T_{\varepsilon\gamma}}}\big(\mathrm{A}_{\gamma T_{\varepsilon\gamma}}(aT_{\varepsilon\gamma}, T_{\varepsilon\gamma})\big) = \mathbb{P}^{\Psi_0}\big(\mathrm{A}_{\gamma}(aT_{\varepsilon\gamma}, T_{\varepsilon\gamma})\big).
$$

Thus Lemma 3 and (45) in Theorem 4 yield

$$
\begin{aligned}
&\mathbb{P}^{\Psi_{T_{\varepsilon\gamma}}}\big(\mathrm{K}^{<}_{\gamma T_{\varepsilon\gamma}}(aT_{\varepsilon\gamma}, T_{\varepsilon\gamma}) \cup \big\{q \in \mathbb{R}^3 \mid \frac{q}{T_{\varepsilon\gamma}} \in B^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a}\big\}\big) \ge \\
&\qquad \ge \mathbb{P}^{\Psi_{T_{\varepsilon\gamma}}}\big(\big\{q \in \mathbb{R}^3 \mid \frac{q}{T_{\varepsilon\gamma}} \in \tilde{B}^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a}\big\}\big) - \mathbb{P}^{\Psi_0}\big(\mathrm{A}_{\gamma}(aT_{\varepsilon\gamma}, T_{\varepsilon\gamma})\big) > \\
&\qquad > 1 - 2\varepsilon.
\end{aligned} \tag{54}
$$

Now let $\frac{q}{T_{\varepsilon\gamma}} \in B^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a}$.
Then (48) in Theorem 4 implies that there is a $C > 0$ such that

$$\begin{aligned} \Big|\frac{\mathrm{Q}_{T_{\varepsilon\gamma}}(q,t_1)}{t_1} - \frac{\mathrm{Q}_{T_{\varepsilon\gamma}}(q,t_2)}{t_2}\Big| &\le \int_{t_1}^{t_2} \frac{1}{\tau}\Big|v^{\Psi}(\mathrm{Q}_{T_{\varepsilon\gamma}}(q,\tau),\tau) - \frac{\mathrm{Q}_{T_{\varepsilon\gamma}}(q,\tau)}{\tau}\Big|\, d\tau < \\ &< C\int_{t_1}^{t_2} \tau^{-1-\beta}\, d\tau = \frac{C}{\beta}\big(t_1^{-\beta} - t_2^{-\beta}\big) \le \frac{C}{\beta} t_1^{-\beta} \end{aligned} \tag{55}$$

for all $t_2 \ge t_1 \ge T_{\varepsilon\gamma}$.
For $\delta > 0$ define $T_{\varepsilon\gamma\delta} := \max\big\{T_{\varepsilon\gamma}, \big(\frac{C(1+\beta)}{\beta\delta}\big)^{\frac{1}{\beta}}\big\}$. Then (55) yields

$$\Big|\frac{\mathrm{Q}_{T_{\varepsilon\gamma}}(q,t_1)}{t_1} - \frac{\mathrm{Q}_{T_{\varepsilon\gamma}}(q,t_2)}{t_2}\Big| < \frac{C}{\beta} T_{\varepsilon\gamma\delta}^{-\beta} \le \delta$$

for all $t_1,\, t_2 \ge T_{\varepsilon\gamma\delta}$.
Since $\delta > 0$ was arbitrary this implies that $\frac{\mathrm{Q}_{T_{\varepsilon\gamma}}(q,t)}{t}$ is Cauchy, i.e. $\lim\limits_{t\to\infty} \frac{\mathrm{Q}_{T_{\varepsilon\gamma}}(q,t)}{t}$ exists.
Moreover, by (48) and (55),

$$\begin{aligned} &\Big|v^{\Psi}(\mathrm{Q}_{T_{\varepsilon\gamma}}(q,t),t) - \lim_{s\to\infty} \frac{\mathrm{Q}_{T_{\varepsilon\gamma}}(q,s)}{s}\Big| \le \\ &\le \Big|v^{\Psi}(\mathrm{Q}_{T_{\varepsilon\gamma}}(q,t),t) - \frac{\mathrm{Q}_{T_{\varepsilon\gamma}}(q,t)}{t}\Big| + \lim_{s\to\infty}\Big|\frac{\mathrm{Q}_{T_{\varepsilon\gamma}}(q,t)}{t} - \frac{\mathrm{Q}_{T_{\varepsilon\gamma}}(q,s)}{s}\Big| < \\ &< Ct^{-\beta} + \lim_{s\to\infty}\frac{C}{\beta}(t^{-\beta} - s^{-\beta}) \le \frac{1+\beta}{\beta} C T_{\varepsilon\gamma\delta}^{-\beta} \le \delta \end{aligned} \tag{56}$$

for all $t \ge T_{\varepsilon\gamma\delta}$. Since (48) also implies $\frac{\mathrm{Q}_{T_{\varepsilon\gamma}}(q,t)}{t} \in U_{\delta_2}\big(\frac{q}{T_{\varepsilon\gamma}}\big) \subset B^{\hat{\Psi}_0^{out}}_{\delta_1 a}$ and thus $|\mathrm{Q}_{T_{\varepsilon\gamma}}(q,t)| > at$ for all $t \ge T_{\varepsilon\gamma\delta} \ge T_{\varepsilon\gamma}$ (54) and (56) give

$$\begin{aligned} &\mathbb{P}^{\Psi_{T_{\varepsilon\gamma}}}\Big(\mathrm{K}^{<}_{\gamma T_{\varepsilon\gamma}}(aT_{\varepsilon\gamma}, T_{\varepsilon\gamma}) \cup \{q \in \mathbb{R}^3 \mid |\mathrm{Q}_{T_{\varepsilon\gamma}}(q,t)| > at \,\wedge \\ &\qquad \wedge\ \Big|v^{\Psi}(\mathrm{Q}_{T_{\varepsilon\gamma}}(q,t),t) - \lim_{s\to\infty}\frac{\mathrm{Q}_{T_{\varepsilon\gamma}}(q,s)}{s}\Big| < \delta \quad \forall t \ge T_{\varepsilon\gamma\delta}\}\Big) > \\ &> 1 - 2\varepsilon. \end{aligned}$$

But $T_{\varepsilon\gamma\delta} \ge T_{\varepsilon\gamma}$ and thus $\mathrm{K}^{<}_{\gamma T_{\varepsilon\gamma}}(aT_{\varepsilon\gamma}, T_{\varepsilon\gamma}) \subset \mathrm{K}^{<}_{\gamma T_{\varepsilon\gamma}}(aT_{\varepsilon\gamma\delta}, T_{\varepsilon\gamma\delta})$, so we also get

$$\begin{aligned} &\mathbb{P}^{\Psi_{T_{\varepsilon\gamma}}}\Big(\mathrm{K}^{<}_{\gamma T_{\varepsilon\gamma}}(aT_{\varepsilon\gamma\delta}, T_{\varepsilon\gamma\delta}) \cup \{q \in \mathbb{R}^3 \mid |\mathrm{Q}_{T_{\varepsilon\gamma}}(q,t)| > at \,\wedge \\ &\qquad \wedge\ \Big|v^{\Psi}(\mathrm{Q}_{T_{\varepsilon\gamma}}(q,t),t) - \lim_{s\to\infty}\frac{\mathrm{Q}_{T_{\varepsilon\gamma}}(q,s)}{s}\Big| < \delta \quad \forall t \ge T_{\varepsilon\gamma\delta}\}\Big) > \\ &> 1 - 2\varepsilon. \end{aligned}$$

By equivariance,

$$\begin{aligned}&\mathbb{P}^{\Psi_{T_{\varepsilon\gamma}}}\big(\mathrm{K}^{<}_{\gamma T_{\varepsilon\gamma}}(aT_{\varepsilon\gamma\delta},T_{\varepsilon\gamma\delta})\cup\{q\in\mathbb{R}^3\mid |\mathrm{Q}_{T_{\varepsilon\gamma}}(q,t)|>at\,\wedge\\&\qquad\wedge\,\Big|v^{\Psi}(\mathrm{Q}_{T_{\varepsilon\gamma}}(q,t),t)-\lim_{s\to\infty}\frac{\mathrm{Q}_{T_{\varepsilon\gamma}}(q,s)}{s}\Big|<\delta\quad\forall t\geq T_{\varepsilon\gamma\delta}\}\big)=\\&=\mathbb{P}^{\Psi_0}\big(\mathrm{K}^{<}_{\gamma}(aT_{\varepsilon\gamma\delta},T_{\varepsilon\gamma\delta})\cup\mathrm{K}^{>}_{\delta}(aT_{\varepsilon\gamma\delta},T_{\varepsilon\gamma\delta})\big),\end{aligned}$$

so finally

$$\mathbb{P}^{\Psi_0}\big(\mathrm{K}^{<}_{\gamma}(aT_{\varepsilon\gamma\delta},T_{\varepsilon\gamma\delta})\cup\mathrm{K}^{>}_{\delta}(aT_{\varepsilon\gamma\delta},T_{\varepsilon\gamma\delta})\big)>1-2\varepsilon. \tag{57}$$

Thus it is left to show (38a) and (38b).
Again using equivariance,

$$\mathbb{P}^{\Psi_0}\big(\{q\in\mathbb{R}^3\mid|\mathrm{Q}(q,t)|\leq at\}\big)=\mathbb{P}^{\Psi_t}\big(\{q\in\mathbb{R}^3\mid|q|\leq at\}\big),$$

by (46) in Theorem 4 we get

$$\sup_{t\geq T_{\varepsilon\gamma}}\Big|\mathbb{P}^{\Psi_0}\big(\{q\in\mathbb{R}^3\mid|\mathrm{Q}(q,t)|\leq at\}\big)-\|\Psi_0^{pp}\|^2\Big|<\varepsilon. \tag{58}$$

Noting that

$$\mathrm{K}^{<}_{\gamma}(aT_{\varepsilon\gamma\delta},T_{\varepsilon\gamma\delta})\subset\{q\in\mathbb{R}^3\mid|\mathrm{Q}(q,t)|\leq aT_{\varepsilon\gamma\delta}\}$$

and

$$\big(\mathrm{K}^{<}_{\gamma}(aT_{\varepsilon\gamma\delta},T_{\varepsilon\gamma\delta})\big)^c=\{q\in\mathbb{R}^3\mid|\mathrm{Q}(q,T_{\varepsilon\gamma\delta})|>aT_{\varepsilon\gamma\delta}\}\cup\mathrm{A}_{\gamma}(aT_{\varepsilon\gamma\delta},T_{\varepsilon\gamma\delta})$$

(58) gives us

$$\mathbb{P}^{\Psi_0}\big(\mathrm{K}^{<}_{\gamma}(aT_{\varepsilon\gamma\delta},T_{\varepsilon\gamma\delta})\big)\leq\mathbb{P}^{\Psi_0}\big(\{q\in\mathbb{R}^3\mid|\mathrm{Q}(q,t)|\leq aT_{\varepsilon\gamma\delta}\}\big)<\|\Psi_0^{pp}\|^2-\varepsilon$$

and

$$\begin{aligned}&\mathbb{P}^{\Psi_0}\big(\mathrm{K}^{<}_{\gamma}(aT_{\varepsilon\gamma\delta},T_{\varepsilon\gamma\delta})\big)=1-\mathbb{P}^{\Psi_0}\Big(\big(\mathrm{K}^{<}_{\gamma}(aT_{\varepsilon\gamma\delta},T_{\varepsilon\gamma\delta})\big)^c\Big)\geq\\&\geq1-\mathbb{P}^{\Psi_0}\big(\mathrm{A}_{\gamma}(aT_{\varepsilon\gamma\delta},T_{\varepsilon\gamma\delta})\big)-\mathbb{P}^{\Psi_0}\big(\{q\in\mathbb{R}^3\mid|\mathrm{Q}(q,T_{\varepsilon\gamma\delta})|>aT_{\varepsilon\gamma\delta}\}\big)=\\&=\mathbb{P}^{\Psi_0}\big(\{q\in\mathbb{R}^3\mid|\mathrm{Q}(q,T_{\varepsilon\gamma\delta})|\leq aT_{\varepsilon\gamma\delta}\}\big)-\mathbb{P}^{\Psi_0}\big(\mathrm{A}_{\gamma}(aT_{\varepsilon\gamma\delta},T_{\varepsilon\gamma\delta})\big)>\\&>\|\Psi_0^{pp}\|^2-\varepsilon-\mathbb{P}^{\Psi_0}\big(\mathrm{A}_{\gamma}(aT_{\varepsilon\gamma\delta},T_{\varepsilon\gamma\delta})\big).\end{aligned}$$

By Lemma 3 the latter gives

$$\mathbb{P}^{\Psi_0}\big(\mathrm{K}^{<}_{\gamma}(aT_{\varepsilon\gamma\delta},T_{\varepsilon\gamma\delta})\big)>\|\Psi_0^{pp}\|^2-2\varepsilon.$$

In the same way (58),

$$\mathrm{K}_\delta^{>}(aT_{\varepsilon\gamma\delta}, T_{\varepsilon\gamma\delta}) \subset \{q \in \mathbb{R}^3 \mid |\mathrm{Q}(q,t)| > aT_{\varepsilon\gamma\delta}\}$$

and again

$$\mathrm{K}_\gamma^{<}(aT_{\varepsilon\gamma\delta}, T_{\varepsilon\gamma\delta}) \subset \{q \in \mathbb{R}^3 \mid |\mathrm{Q}(q,t)| \le aT_{\varepsilon\gamma\delta}\}$$

yield

$$\begin{aligned}\mathbb{P}^{\Psi_0}\big(\mathrm{K}_\delta^{>}(aT_{\varepsilon\gamma\delta}, T_{\varepsilon\gamma\delta})\big) &\le \mathbb{P}^{\Psi_0}\big(\{q \in \mathbb{R}^3 \mid |\mathrm{Q}(q,t)| > aT_{\varepsilon\gamma\delta}\}\big) = \\ &= 1 - \mathbb{P}^{\Psi_0}\big(\{q \in \mathbb{R}^3 \mid |\mathrm{Q}(q,t)| \ge aT_{\varepsilon\gamma\delta}\}\big) < \\ &< 1 - \big(\|\Psi_0^{pp}\|^2 - \varepsilon\big) = \|\Psi_0^{ac}\|^2 + \varepsilon\end{aligned}$$

and

$$\begin{aligned}&\mathbb{P}^{\Psi_0}\big(\mathrm{K}_\delta^{>}(aT_{\varepsilon\gamma\delta}, T_{\varepsilon\gamma\delta})\big) \ge \\ &\quad \ge \mathbb{P}^{\Psi_0}\big(\{q \in \mathbb{R}^3 \mid |\mathrm{Q}(q,t)| \le aT_{\varepsilon\gamma\delta}\} \cup \mathrm{K}_\delta^{>}(aT_{\varepsilon\gamma\delta}, T_{\varepsilon\gamma\delta})\big) - \\ &\qquad\qquad - \mathbb{P}^{\Psi_0}\big(\{q \in \mathbb{R}^3 \mid |\mathrm{Q}(q,t)| \le aT_{\varepsilon\gamma\delta}\}\big) \ge \\ &\quad \ge \mathbb{P}^{\Psi_0}\big(\mathrm{K}_\gamma^{<}(aT_{\varepsilon\gamma\delta}, T_{\varepsilon\gamma\delta}) \cup \mathrm{K}_\delta^{>}(aT_{\varepsilon\gamma\delta}, T_{\varepsilon\gamma\delta})\big) - \\ &\qquad\qquad - \mathbb{P}^{\Psi_0}\big(\{q \in \mathbb{R}^3 \mid |\mathrm{Q}(q,t)| \le aT_{\varepsilon\gamma\delta}\}\big) > \\ &\quad > \mathbb{P}^{\Psi_0}\big(\mathrm{K}_\gamma^{<}(aT_{\varepsilon\gamma\delta}, T_{\varepsilon\gamma\delta}) \cup \mathrm{K}_\delta^{>}(aT_{\varepsilon\gamma\delta}, T_{\varepsilon\gamma\delta})\big) - \big(\|\Psi_0^{pp}\|^2 + \varepsilon\big).\end{aligned}$$

By (57) the latter gives

$$\mathbb{P}^{\Psi_0}\big(\mathrm{K}_\delta^{>}(aT_{\varepsilon\gamma\delta}, T_{\varepsilon\gamma\delta})\big) > 1 - \|\Psi_0^{ac}\|^2 - 3\varepsilon = \|\Psi_0^{pp}\|^2 - 3\varepsilon.$$

□

5 First Exit Statistics for Distant Surfaces and $\Psi \in \mathcal{H}_{ac}(H) + \mathcal{H}_{pp}(H)$

In this section we show that the first exit statistics for a distant surface made by a wave function $\Psi + \Psi^{ac} + \Psi^{pp}$ is mainly given by the scattering part Ψ^{ac} of the wave function and thus by the well known expression $\int\limits_C |\hat{\Psi}_0^{out}(k)|^2 \, d^3k$. Let R be the distance of the surface from the scattering center. Then there is some time $t(R)$ (with $t(R) \to \infty$ as $R \to \infty$) such that the first exit statistics made by Ψ until $t(R)$ are asymptotically the same as those made by Ψ^{ac} for all time. Moreover, the first exit statistics made by Ψ for all time differ from those made by Ψ^{ac} at most by $\|\Psi_0^{pp}\|^2$. So if $\|\Psi_0^{pp}\|^2$ is small, for distant surfaces we get essentially the same exit statistics no matter wether we use the whole wave function Ψ or only its scattering part Ψ^{ac} to compute it.
Let $\Sigma \subset \mathbb{R}^2$ be a smooth oriented surface and let $\Delta T = [T, T_f)$ be some time interval. We say a particle is detected on Σ during ΔT, if it crosses Σ in positive direction during ΔT, where we count first crossings only. Thus we are interested in the first exit statistics which are completely determined by the expectation value (with respect to $\mathbb{P}^\Psi$) of N_{det}^Ψ, the number of first crossings of Σ in positive direction during ΔT, respectively the number of particles detected on Σ during ΔT[9].
In a scattering situation the (detector) surfaces of interest are spherical and very far away from the scattering center, so we will consider surfaces $R\Sigma := \{Rq \in \mathbb{R}^3 \mid q \in \Sigma\}$, where $\Sigma \subset S_1$ is measurable and R tends to infinity.
Let $H = H_0 + V$ and $\Psi = \Psi^{ac} + \Psi^{pp}$ as in Subsection 4.2. Then in the sense of Theorem 3 all possible Bohmian trajectories split up into "bound" and "scattering" ones. Moreover, if the scattered particle has a "bound" trajectory it will reach a distant sphere S_R with radius $R \to \infty$ at the earliest at some time $t(R)$ proportional to $R^{1+\gamma}$ (see Figure 2). Therefore the probability that a particle crosses a spherical surface $R\Sigma$ before $t(R)$ should, for $R \to \infty$, be determined by the scattering part Ψ^{ac} of the wave function alone. More precisely, it should be determined by the flux-across-surfaces integral $\int\limits_T^{t(R)} dt \int\limits_{R\Sigma} j^{\Psi^{ac}} \cdot \hat{n} \, d\sigma$ (instead of $\int\limits_T^{t(R)} dt \int\limits_{R\Sigma} j^{\Psi} \cdot \hat{n} \, d\sigma$). Moreover, since "scattering" trajectories far away from the scattering center become straight lines pointing outwards radially, they should cross $R\Sigma$ at most once and outwards. Thus the first exit statistic for a particle crossing $R\Sigma$ before $t(R)$ should be given by the above crossing probability (by the above flux-across-

[9]Since N_{det}^Ψ takes on the values zero and one only, the expectation value indeed determines the whole statistics and is equal to the probability of $\mathrm{N}_{det}^\Psi = 1$.

surfaces integral).
This is presumably not true for the time after $t(R)$, since one would not expect a "bound" trajectory to become a straight line. Recall, however, that "scattering" trajectories move out linear in time, so what is left inside the sphere S_R at time $t(R)$ is of order $\|\Psi_0^{pp}\|^2$ (for $R \to \infty$; compare (38a) of Theorem 3 and recall $t(R) \sim R^{1+\gamma}$). But surely, if "what is left inside at time $t(R)$" is of order $\|\Psi_0^{pp}\|^2$, "what can come out afterwards" is of order $\|\Psi_0^{pp}\|^2$, too[10].

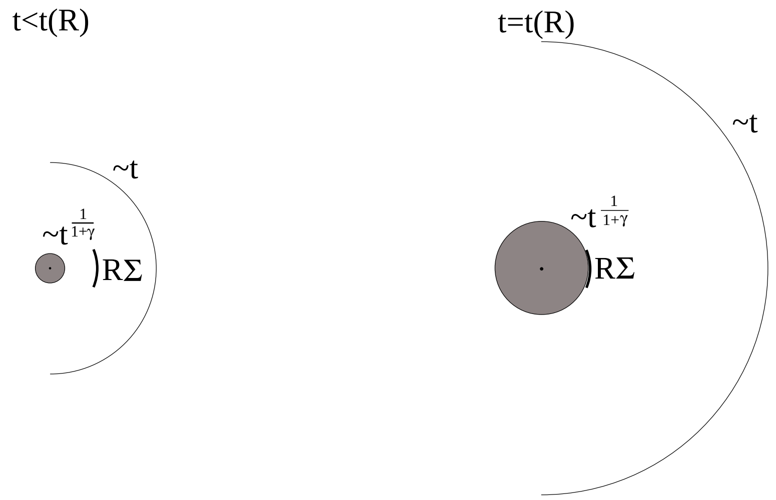

Figure 2: When do the bound trajectories reach $R\Sigma$?

In line with the above we shall establish the following. Let $\mathrm{N}_{det}^{\Psi}(T, T_f, R, \Sigma)$ be the number of particles detected on $R\Sigma$ during $[T, T_f)$ if the system under consideration is subject to the wave function $\Psi = \Psi^{ac} + \Psi^{pp}$. Then for distant surfaces the first exit statistics for a particle crossing $R\Sigma$ before $t(R)$ is determined by the scattering part Ψ^{ac} of the wave function alone[11] and

[10] Here it is important to note that we look at the *first* exit statistics, i.e. if a trajectory crosses a surface more than once we count it the first time only.

[11] The term $\|\Psi_0^{ac}\|^2 \mathbb{E}^{\tilde{\Psi}^{ac}}\big(\mathrm{N}_{det}^{\tilde{\Psi}^{ac}}(T, \infty, R, \Sigma)\big)$ in Lemma 6 might at first come as a little surprise and seem unnecessarily unwieldy. One might rather expect to find something like $\mathbb{E}^{\Psi^{ac}}\big(\mathrm{N}_{det}^{\Psi^{ac}}(T, \infty, R, \Sigma)\big)$. However, since Ψ^{ac} is not normalized (unless $\Psi^{pp} = 0$) $\mathbb{P}^{\Psi^{ac}}$, is not a probability measure and thus $\mathbb{E}^{\Psi^{ac}}\big(\mathrm{N}_{det}^{\Psi^{ac}}(T, \infty, R, \Sigma)\big)$ is not an expectation value in the strict sense of the word. So we normalize Ψ^{ac}, $\tilde{\Psi}^{ac} := \frac{\Psi^{ac}}{\|\Psi_0^{ac}\|}$, take the well defined expectation value $\mathbb{E}^{\tilde{\Psi}^{ac}}\big(\mathrm{N}_{det}^{\tilde{\Psi}^{ac}}(T, \infty, R, \Sigma)\big)$ instead and account for the fact that we really want to look at the "statistics" made by Ψ^{ac} by scaling $\mathbb{E}^{\tilde{\Psi}^{ac}}\big(\mathrm{N}_{det}^{\tilde{\Psi}^{ac}}(T, \infty, R, \Sigma)\big)$ down with $\|\Psi_0^{ac}\|^2$.

thus by $\int\limits_{C_\Sigma} \|\hat{\Psi}_0^{out}(k)\|^2\, d^3k$, the probability to find the quantum mechanical momentum k in the cone C_Σ spanned by Σ (Lemma 6). Moreover, what comes out after $t(R)$ is bounded from above by $\|\Psi^{pp}\|^2$ (Lemma 7).

Lemma 6. *Let the potential $V \in (V)_4$ and let zero be neither a resonance nor an eigenvalue of H. Let $\Psi_t = \Psi_t^{ac} + \Psi_t^{pp}$ with $\Psi_t^{pp} \in \mathcal{H}_{pp}(H)$ and $\Psi_t^{ac} = e^{-iHt} W_+ \Psi_0^{out} \in \mathcal{H}_{ac}(H)$ with $\Psi_0^{out} \in S(\mathbb{R}^3)$. Assume A2 and A4. Let $\Sigma \subset S_1$ be measurable. Then for all $T \in \mathbb{R}$, $c > 0$ and $0 < \gamma < 2\alpha$*

$$\begin{aligned}
&\lim_{R\to\infty} \left|\mathbb{E}^{\Psi}\big(\mathrm{N}_{det}^{\Psi}(T, t_{c\gamma}(R), R, \Sigma)\big) - \|\Psi_0^{ac}\|^2 \mathbb{E}^{\tilde{\Psi}^{ac}}\big(\mathrm{N}_{det}^{\tilde{\Psi}^{ac}}(T, \infty, R, \Sigma)\big)\right| = \\
&= \lim_{R\to\infty} \Big[\mathbb{E}^{\Psi}\big(\mathrm{N}_{det}^{\Psi}(T, \infty, R, \Sigma)\big) - \int\limits_{C_\Sigma} |\hat{\Psi}_0^{out}(k)|^2\, d^3k\Big] = 0,
\end{aligned} \tag{59}$$

where $t_{c\gamma}(R) := \max\{T, cR^{1+\gamma}\}$ and $\tilde{\Psi}^{ac} := \frac{\Psi^{ac}}{\|\Psi_0^{ac}\|}$.

Lemma 7. *Let the potential $V \in (V)_4$ and let zero be neither a resonance nor an eigenvalue of H. Let $\Psi_t = \Psi_t^{ac} + \Psi_t^{pp}$ with $\Psi_t^{pp} \in \mathcal{H}_{pp}(H)$ and $\Psi_t^{ac} = e^{-iHt} W_+ \Psi_0^{out} \in \mathcal{H}_{ac}(H)$ with $\Psi_0^{out} \in S(\mathbb{R}^3)$. Assume A2 and A4. Let $\Sigma \subset S_1$ be measurable. Then for all $T \in \mathbb{R}$, $c > 0$ and $0 < \gamma < 2\alpha$*

$$\begin{aligned}
0 &\le \lim_{R\to\infty} \mathbb{E}^{\Psi}\big(\mathrm{N}_{det}^{\Psi}(t_{c\gamma}(R), \infty, R, \Sigma)\big) = \\
&= \lim_{R\to\infty} \big[\mathbb{E}^{\Psi}\big(\mathrm{N}_{det}^{\Psi}(T, \infty, R, \Sigma)\big) - \mathbb{E}^{\Psi}\big(\mathrm{N}_{det}^{\Psi}(T, t_{c\gamma}(R), R, \Sigma)\big)\big] \le \\
&\le \|\Psi_0^{pp}\|^2,
\end{aligned} \tag{60}$$

where $t_{c\gamma}(R) := \max\{T, cR^{1+\gamma}\}$.

Lemma 6 and Lemma 7 together immediately yield the main result of this section. The first exit statistics for a distant surface $R\Sigma$ of a wave function $\Psi = \Psi^{ac} + \Psi^{pp}$ and that of its scattering part Ψ^{ac}, which is given by $\int\limits_{C_\Sigma} |\hat{\Psi}_0^{out}(k)|^2\, d^3k$, differ only after some big time $t(R)$ and the difference is bounded from above by $\|\Psi^{pp}\|^2$.

Theorem 5. *Let the potential $V \in (V)_4$ and let zero be neither a resonance nor an eigenvalue of H. Let $\Psi_t = \Psi_t^{ac} + \Psi_t^{pp}$ with $\Psi_t^{pp} \in \mathcal{H}_{pp}(H)$ and $\Psi_t^{ac} = e^{-iHt} W_+ \Psi_0^{out} \in \mathcal{H}_{ac}(H)$ with $\Psi_0^{out} \in S(\mathbb{R}^3)$. Assume A2 and A4. Let $\Sigma \subset S_1$*

be measurable. Then for all $T \in \mathbb{R}$, $c > 0$ *and* $0 < \gamma < 2\alpha$

$$\begin{aligned} 0 &\leq \lim_{R\to\infty} \mathbb{E}^{\Psi}\big(\mathrm{N}^{\Psi}_{det}(t_{c\gamma}(R), \infty, R, \Sigma)\big) = \\ &= \lim_{R\to\infty} \big[\mathbb{E}^{\Psi}\big(\mathrm{N}^{\Psi}_{det}(T, \infty, R, \Sigma)\big) - \|\Psi^{ac}_0\|^2 \mathbb{E}^{\tilde{\Psi}^{ac}}\big(\mathrm{N}^{\tilde{\Psi}^{ac}}_{det}(T, \infty, R, \Sigma)\big)\big] = \\ &= \lim_{R\to\infty} \big[\mathbb{E}^{\Psi}\big(\mathrm{N}^{\Psi}_{det}(T, \infty, R, \Sigma)\big) - \int\limits_{C_\Sigma} |\hat{\Psi}^{out}_0(k)|^2 \, d^3k\big] \leq \\ &\leq \|\Psi^{pp}_0\|^2, \end{aligned} \tag{61}$$

where $t_{c\gamma}(R) := \max\{T, cR^{1+\gamma}\}$ *and* $\tilde{\Psi}^{ac} := \frac{\Psi^{ac}}{\|\Psi^{ac}_0\|}$.

To prove Lemma 6 and Lemma 7 we proceed as follows. First we give a rigorous definition of the number of detected particles N^{Ψ}_{det} and elaborate on the connection of $\mathbb{E}^{\Psi}(\mathrm{N}^{\Psi}_{det})$ to the flux-across-surfaces integrals (Definition 5, Definition 6, Proposition 5 and Proposition 6). This part of the proof is nothing new ([7, 15]). We will mainly follow [15] but adapt some of their definitions to our need of lesser generality and to conventions used earlier in this text. Next we prove a flux-across-surfaces theorem (FAST) for $\Psi = \Psi^{ac} + \Psi^{pp}$ to help us express $\mathbb{E}^{\Psi}\big(\mathrm{N}^{\Psi}_{det}(T, t(R), R, \Sigma)\big)$ in terms of $|\hat{\Psi}^{out}_0|^2$.

Lemma 8 (FAST). *Let the potential* V *be in* $(V)_4$ *and let zero be neither a resonance nor an eigenvalue of* H. *Let* $\Psi = \Psi^{ac}_t + \Psi^{pp}_t$ *with* $\Psi^{pp}_t \in \mathcal{H}_{pp}(H)$ *and* $\Psi^{ac}_t = e^{-iHt} W_+ \Psi^{out}_0 \in \mathcal{H}_{ac}(H)$ *with* $\Psi^{out}_0 \in S(\mathbb{R}^3)$. *Assume A4 and let* $\Sigma \subset S_1$ *be measurable. Then for all* $T \in \mathbb{R}$, $c > 0$ *and* $0 < \gamma < 2\alpha$

$$\begin{aligned} \lim_{R\to\infty} \int\limits_T^{t_{c\gamma}(R)} dt \int\limits_{R\Sigma} j^{\Psi}(q,t) \cdot \hat{n} \, d\sigma &= \lim_{R\to\infty} \int\limits_T^{t_{c\gamma}(R)} dt \int\limits_{R\Sigma} \big|j^{\Psi}(q,t) \cdot \hat{n}\big| \, d\sigma = \\ &= \int\limits_{C_\Sigma} |\hat{\Psi}^{out}_0(k)|^2 \, d^3k \end{aligned} \tag{62}$$

where $t_{c\gamma}(R) := \max\{T, cR^{1+\gamma}\}$.

This finally gives us the means to show Lemma 6 and Lemma 7.
We start with some definitions.

Definition 5. *Assume A1-A3. Let* $\Sigma \subset S_1$ *be measurable and* $T_f > T \in \mathbb{R}$. *Define* $R\Sigma := \{Rq \in \mathbb{R}^3 \mid q \in \Sigma\}$.
We define the number of positive (negative) crossings $\mathrm{N}^{\Psi}_{\pm}(T, T_f, R\Sigma)$ *of* $R\Sigma$ *during* $[T, T_f)$ *by*

$$\mathrm{N}^{\Psi}_{\pm}(T, T_f, R\Sigma) : \mathbb{R}^3 \longrightarrow \mathbb{N}_0$$

with

$$\mathrm{N}_{\pm}^{\Psi}(T,T_f,R\Sigma)(q) := \\ = |\{t \in [T,T_f) \mid \mathrm{Q}(q,t) \in R\Sigma \wedge \exists \varepsilon > 0 : \sup_{t-\varepsilon<s<t} |\mathrm{Q}(q,s) \lesseqgtr R\}|$$

if $q \in \Omega_T$, *and*

$$\mathrm{N}_{\pm}^{\Psi}(T,T_f,R\Sigma)(q) := 0$$

if $q \in \mathbb{R}^3 \backslash \Omega_T$. *Here* $|A|$ *denotes the cardinality of the set* A.
Then the number of total crossings is given by

$$\mathrm{N}^{\Psi}(T,T_f,R\Sigma) : \mathbb{R}^3 \longrightarrow \mathbb{N}_0, \\ \mathrm{N}^{\Psi}(T,T_f,R\Sigma) := \mathrm{N}_{+}^{\Psi}(T,T_f,R\Sigma) + \mathrm{N}_{-}^{\Psi}(T,T_f,R\Sigma)$$

and the number of signed crossings by

$$\mathrm{N}_{sig}^{\Psi}(T,T_f,R\Sigma) : \mathbb{R}^3 \longrightarrow \mathbb{Z}, \\ \mathrm{N}_{sig}^{\Psi}(T,T_f,R\Sigma) := \mathrm{N}_{+}^{\Psi}(T,T_f,R\Sigma) - \mathrm{N}_{-}^{\Psi}(T,T_f,R\Sigma).$$

Then one can show the following ([7] Lemma 4.2 and pp.34-37; see also Section 4 in [15]).

Proposition 5. *Assume A1-A3. Let* $\Sigma \subset S_1$ *be measurable and* $T_f > T \in \mathbb{R}$. *Define* $R\Sigma := \{Rq \in \mathbb{R}^3 \mid q \in \Sigma\}$. *Then* $\mathrm{N}^{\Psi}(T,T_F,R\Sigma)$ *and* $\mathrm{N}_{sig}^{\Psi}(T,T_F,R\Sigma)$ *are random variables on the space* $\mathbb{R}^3$ *of initial configurations.*
Moreover

$$\mathbb{E}^{\Psi}\big(\mathrm{N}_{sig}^{\Psi}(T,T_F,R\Sigma)\big) = \int_T^{T_f} dt \int_{R\Sigma} j^{\Psi}(q,t) \cdot \hat{n}\, d\sigma \tag{63}$$

and

$$\mathbb{E}^{\Psi}\big(\mathrm{N}^{\Psi}(T,T_F,R\Sigma)\big) = \int_T^{T_f} dt \int_{R\Sigma} |j^{\Psi}(q,t) \cdot \hat{n}|\, d\sigma. \tag{64}$$

Remark 13. In Definition 5 and Proposition 5 we use slightly different assumptions on H and Ψ than [7] and [15]. Note however, that what is used in their proof of Proposition 5 (and Proposition 6 below) is the regularity of the solutions $\mathrm{Q}(q,t)$ of (4) that follows in the context of the proof of almost sure global existence of Bohmian mechanics. So wether one uses their assumptions (which in fact are the assumptions of Remark 3) or A1-A3 is merely a question of preference.

Now we can define the number of detected particles N^{Ψ}_{det}.

Definition 6. *Assume A1-A3. Let $\Sigma \subset S_1$ be measurable and $T_f > T \in \mathbb{R}$. Define $R\Sigma := \{Rq \in \mathbb{R}^3 \mid q \in \Sigma\}$.*
We define the first exit time when a trajectory leaves the ball B_R (crosses the sphere S_R outwards) by

$$t_{ex}(T,R) : \Omega \longrightarrow [T,\infty],$$
$$t_{ex}(T,R)(q) := \max\{t \geq T \mid |\mathrm{Q}(q,t)| = R \wedge \sup_{T\leq s<t} |\mathrm{Q}(q,s)| < R\},$$

where we set $t_{ex}(T,R) = \infty$ if the above set is empty.
Then the number of particles detected on $R\Sigma$ during $[T,T_f)$ is defined by

$$\mathrm{N}^{\Psi}_{det}(T,T_f,R,\Sigma) : \mathbb{R}^3 \longrightarrow \{0,1\},$$
$$\mathrm{N}^{\Psi}_{det}(T,T_f,R,\Sigma)(q) := \begin{cases} 1 & \text{if } t_{ex}(T,R) \in [T,T_f) \text{ and } \mathrm{Q}(q,t_{ex}) \in R\Sigma, \\ 0 & \text{else.} \end{cases}$$

To connect $\mathbb{E}^{\Psi}(\mathrm{N}^{\Psi}_{det})$ with the flux-across-surfaces integrals of Proposition 5 we exploit that the difference between the number of first positive crossings N^{Ψ}_{det} and the number of signed crossings N^{Ψ}_{sig} for the closed surface S_R is bounded by the number of negative crossings $\mathrm{N}^{\Psi}_{-} = \frac{1}{2}(\mathrm{N}^{\Psi} - \mathrm{N}^{\Psi}_{sig})$ and get with the help of Proposition 5 (see [15] for details)

Proposition 6. *Assume A1-A3. Let $\Sigma \subset S_1$ be measurable and $T_f > T \in \mathbb{R}$. Define $R\Sigma := \{Rq \in \mathbb{R}^3 \mid q \in \Sigma\}$. Then*

$$\begin{aligned} &\left|\mathbb{E}^{\Psi}\big(\mathrm{N}^{\Psi}_{sig}(T,T_f,R\Sigma)\big) - \mathbb{E}^{\Psi}\big(\mathrm{N}^{\Psi}_{det}(T,T_f,R\Sigma)\big)\right| \leq \\ &\leq \frac{1}{2}\int_T^{T_f} dt \int_{R\Sigma} \big(|j^{\Psi}(q,t)\cdot\hat{n}| - j^{\Psi}(q,t)\cdot\hat{n}\big)\, d\sigma. \end{aligned} \tag{65}$$

With this and Lemma 8 we can prove Lemma 6 and Lemma 7.

Proof of Lemma 6. Let $T \in \mathbb{R}$, $c > 0$ and $0 < \gamma < 2\alpha$. Using (62) and (65) we get

$$\lim_{R\to\infty} \mathbb{E}^{\Psi}\big(\mathrm{N}^{\Psi}_{det}(T,t_{c\gamma}(R),R,\Sigma)\big) = \lim_{R\to\infty} \mathbb{E}^{\Psi}\big(\mathrm{N}^{\Psi}_{sig}(T,t_{c\gamma}(R),R\Sigma)\big).$$

Moreover (63) and again (62) yield

$$\begin{aligned} \lim_{R\to\infty} \mathbb{E}^{\Psi}\big(\mathrm{N}^{\Psi}_{sig}(T,t_{c\gamma}(R),R\Sigma)\big) &= \lim_{R\to\infty} \int_T^{t_{c\gamma}(R)} dt \int_{R\Sigma} j^{\Psi}(q,t)\cdot\hat{n}\, d\sigma = \\ &= \int_{C_\Sigma} |\hat{\Psi}^{out}_0(k)|^2\, d^3k. \end{aligned}$$

Thus

$$\lim_{R\to\infty} \mathbb{E}^{\Psi}\big(\mathrm{N}^{\Psi}_{det}(T, t_{c\gamma}(R), R, \Sigma)\big) = \int\limits_{C_\Sigma} |\hat{\Psi}^{out}_0(k)|^2 \, d^3k. \tag{66}$$

In exactly the same way we utilize the FAST for pure scattering wave functions ([31]: 2.1 Theorem, p.3)

$$\begin{aligned} \lim_{R\to\infty} \int\limits_T^\infty dt \int\limits_{R\Sigma} j^{\Psi^{ac}}(q,t)\cdot\hat{n}\, d\sigma &= \lim_{R\to\infty} \int\limits_T^\infty dt \int\limits_{R\Sigma} \big|j^{\Psi^{ac}}(q,t)\cdot\hat{n}\big|\, d\sigma = \\ &= \int\limits_{C_\Sigma} |\hat{\Psi}^{out}_0(k)|^2 \, d^3k \end{aligned} \tag{67}$$

to get

$$\lim_{R\to\infty} \mathbb{E}^{\tilde{\Psi}^{ac}}\big(\mathrm{N}^{\tilde{\Psi}^{ac}}_{det}(T, \infty, R, \Sigma)\big) = \int\limits_{C_\Sigma} |\hat{\tilde{\Psi}}^{out}_0(k)|^2 \, d^3k.$$

But by the definition of $\tilde{\Psi}_{ac}$ we have $|\hat{\tilde{\Psi}}^{out}_0(k)|^2 = \frac{1}{\|\Psi^{ac}_0\|^2}|\hat{\Psi}^{out}_0(k)|^2$ and thus

$$\lim_{R\to\infty} \mathbb{E}^{\tilde{\Psi}^{ac}}\big(\mathrm{N}^{\tilde{\Psi}^{ac}}_{det}(T, \infty, R, \Sigma)\big) = \frac{1}{\|\Psi^{ac}_0\|^2}\int\limits_{C_\Sigma} |\hat{\Psi}^{out}_0(k)|^2 \, d^3k. \tag{68}$$

This and (66) immediately give (59). □

Proof of Lemma 7. Let $T \in \mathbb{R}$, $c > 0$ and $0 < \gamma < 2\alpha$.
The equality in (60) is trivial. Since $\mathrm{N}^{\Psi}_{det}(t_{c\gamma}(R), \infty, R, \Sigma)(q) \in \{0,1\}$ for all $q \in \mathbb{R}^3$, so is the first inequality. Thus it is left to show

$$\lim_{R\to\infty} \mathbb{E}^{\Psi}\big(\mathrm{N}^{\Psi}_{det}(t_{c\gamma}(R), \infty, R, \Sigma)\big) \le \|\Psi^{pp}_0\|^2.$$

Since $\Sigma \subset S_1$, $\mathrm{N}^{\Psi}_{det}(t_{c\gamma}(R), \infty, R, \Sigma)(q) \le \mathrm{N}^{\Psi}_{det}(t_{c\gamma}(R), \infty, R, S_1)$ for all $q \in \mathbb{R}^3$ and thus also

$$\begin{aligned} &\lim_{R\to\infty} \mathbb{E}^{\Psi}\big(\mathrm{N}^{\Psi}_{det}(t_{c\gamma}(R), \infty, R, \Sigma)\big) \le \lim_{R\to\infty} \mathbb{E}^{\Psi}\big(\mathrm{N}^{\Psi}_{det}(t_{c\gamma}(R), \infty, R, S_1)\big) = \\ &\quad = \lim_{R\to\infty}\Big[\mathbb{E}^{\Psi}\big(\mathrm{N}^{\Psi}_{det}(T, \infty, R, S_1)\big) - \mathbb{E}^{\Psi}\big(\mathrm{N}^{\Psi}_{det}(T, t_{c\gamma}(R), R, S_1)\big)\Big] \le \\ &\quad \le 1 - \lim_{R\to\infty} \mathbb{E}^{\Psi}\big(\mathrm{N}^{\Psi}_{det}(T, t_{c\gamma}(R), R, S_1)\big), \end{aligned} \tag{69}$$

where in the last step we again used $\mathrm{N}^{\Psi}_{det}(T, \infty, R, S_1)(q) \in \{0,1\}$ for all $q \in \mathbb{R}^3$.

However, by (66) and Dollard's scattering into cones theorem (equation (1))we know that

$$\lim_{R\to\infty} \mathbb{E}^\Psi\left(\mathrm{N}^\Psi_{det}(T, t_{c\gamma}(R), R, S_1)\right) = \int\limits_{\mathbb{R}^3} |\hat{\Psi}^{out}_0(k)|^2\, d^3k =$$

$$= \lim_{t\to\infty} \int\limits_{\mathbb{R}^3} |\Psi^{ac}_t(q)|^2\, d^3q = \lim_{t\to\infty} \|\Psi^{ac}_t\|^2 = \|\Psi^{ac}_0\|^2.$$

Putting this into (69) yields the desired result. □

Finally it is left to show Lemma 8.

Proof of Lemma 8. It suffices to show (62) for some fixed $T > 0$ since the set for which (62) holds as well as the right hand side of (62) is invariant under finite time shifts[12]:

$$\lim_{R\to\infty} \int\limits_{\tilde{t}_0}^{t_{c\gamma}(R)} dt \int\limits_{R\Sigma} j^\Psi(q,t)\cdot \hat{n}\, d\sigma = \lim_{R\to\infty} \int\limits_{T}^{t_{c\gamma}(R)} dt \int\limits_{R\Sigma} j^\Psi(q, t+\tilde{t}_0 - T)\cdot \hat{n}\, d\sigma =$$

$$= \int\limits_{C_\Sigma} |e^{-i\frac{k^2}{2}(\tilde{t}_0 - T)} \hat{\Psi}^{out}_0(k)|^2\, d^3k = \int\limits_{C_\Sigma} |\hat{\Psi}^{out}_0(k)|^2\, d^3k.$$

Now let $T > 0$, $c > 0$. Since $t_{c\gamma}(R) \overset{R\to\infty}{\longrightarrow} \infty$ for all $0 < \gamma < 2\alpha$ (67), the FAST for pure scattering wave functions, yields

$$\lim_{R\to\infty} \int\limits_{T}^{t_{c\gamma}(R)} dt \int\limits_{R\Sigma} j^{\Psi^{ac}}(q,t)\cdot \hat{n}\, d\sigma = \lim_{R\to\infty} \int\limits_{T}^{t_{c\gamma}(R)} dt \int\limits_{R\Sigma} \left|j^{\Psi^{ac}}(q,t)\cdot \hat{n}\right| d\sigma =$$

$$= \int\limits_{C_\Sigma} |\hat{\Psi}^{out}_0(k)|^2\, d^3k.$$

To prove (62) it therefor suffices to show that $j^\Psi - j^{\Psi^{ac}}$ does not contribute to the flux across distant surfaces during $[T, t_{c\gamma}(R))$, i.e.

$$\lim_{R\to\infty} \int\limits_{T}^{t_{c\gamma}(R)} dt \int\limits_{S_R} \left|\left(j^\Psi(q,t) - j^{\Psi^{ac}}(q,t)\right)\cdot \hat{n}\right| d\sigma = 0. \tag{70}$$

[12]Invariance of $S(\mathbb{R}^3)$ was established in Remark 7, for A4 it is trivial.

Let $0 < \gamma < 2\alpha$ and $R > R_2$ such that $cR^{1+\gamma} \geq T$ (where R_2 is as in Lemma 4). Using $\left|\left(j^{\Psi} - j^{\Psi^{ac}}\right) \cdot \hat{n}\right| \leq \left|j^{\Psi^{pp}}\right| + \left|j_m\right|$ and the definition of $t_{c\gamma}(R)$, (49) and (51) (with $r_3 = r_4 = 1$ and $\frac{1}{|q|(t+|q|)} \leq |q|^{-1}t^{-1}$) yield

$$\int_T^{t_{c\gamma}(R)} dt \int_{S_R} \left|\left(j^{\Psi}(q,t) - j^{\Psi^{ac}}(q,t)\right) \cdot \hat{n}\right| d\sigma \leq$$

$$\leq \int_T^{t_{c\gamma}(R)} dt \int_{S_R} \left[C_8 R^{-3-2\alpha} + R^{-\frac{3}{2}-\alpha}\left(c_1 R^{-1}(t^{-\frac{1}{2}} + t^{-\frac{3}{2}}) + C_9 R^{-1} t^{-1}\right)\right] d\sigma =$$

$$= \int_T^{t_{c\gamma}(R)} dt \int_{S_R} \left[\mathcal{O}\left(R^{-3-2\alpha}\right) + \mathcal{O}\left(R^{-\frac{5}{2}-\alpha} t^{-\frac{1}{2}}\right)\right] d\sigma =$$

$$= \mathcal{O}\left(R^{-1-2\alpha} t_{c\gamma}(R)\right) + \mathcal{O}\left(R^{-\frac{1}{2}-2\alpha} t_{c\gamma}(R)^{\frac{1}{2}}\right) = \mathcal{O}(R^{-\frac{1}{2}(2\alpha-\gamma)})$$

Thus (70) holds and we are done. □

Remark 14. If we use the FAST for pure scattering wave functions under the condition it was formulated with in [18] (namely $\Psi_0^{ac} \in G$), we can replace the condition $\hat{\Psi}_0^{out} \in S(\mathbb{R}^3)$ in the results of this section by:

(i) $\Psi_0^{ac} \in G := \bigcup_{t \in \mathbb{R}} e^{-iHt} f$ with $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\begin{gathered} f \in \mathcal{H}_{ac}(H), \\ H^m f \in \mathcal{D}(H), \quad m \in \{0, 1, \ldots, 7\}, \\ (1 + |q|)^2 H^m f(q) \in L_2(\mathbb{R}^3), \quad m \in \{0, 1, \ldots, 8\}, \\ (1 + |q|)^4 H^m f(q) \in L_2(\mathbb{R}^3), \quad m \in \{0, 1, \ldots, 3\}. \end{gathered}$$

(ii) There is some $C < \infty$, $\varepsilon > 0$ such that

$$\begin{gathered} |\Psi_0^{out}(q)| \leq C(1 + |q|)^{-5-\varepsilon}, \\ |\partial_q^{\eta} \Psi_0^{out}(q)| \leq C(1 + |q|)^{-5-\varepsilon}, \quad |\eta| = 1, \\ |\partial_q^{\eta} \Psi_0^{out}(q)| \leq C(1 + |q|)^{-4-\varepsilon}, \quad |\eta| = 2. \end{gathered}$$

Here η is a multi-index.

The conditions on Ψ_0^{out} assure that (12) in Proposition 2 holds for $r = 1$ which was all we used it with (via (51) of Lemma 5).

6 Appendix

Proof of Theorem 2. Let $\varepsilon > 0$, $T_\varepsilon > 0$ and $C^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a}(T_\varepsilon) := \{q \in \mathbb{R}^3 \mid \frac{q}{T_\varepsilon} \notin B^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a}\}$. Then

$$\begin{aligned}
&\mathbb{P}^{\Psi_{T_\varepsilon}}\big(C^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a}(T_\varepsilon)\big) \le \\
&\le \int\limits_{C^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a}(T_\varepsilon)} T_\varepsilon^{-3}|\hat{\Psi}_0^{out}(\frac{q}{T_\varepsilon})|^2\, d^3q + \int\limits_{\mathbb{R}^3} |T_\varepsilon^{-3}|\hat{\Psi}_0^{out}(\frac{q}{T_\varepsilon})|^2 - |\Psi_{T_\varepsilon}(q)|^2|\, d^3q \le \\
&\le \int\limits_{(B^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a})^c} |\hat{\Psi}_0^{out}(k)|^2\, d^3k + \int\limits_{\mathbb{R}^3} |T_\varepsilon^{-3}|\hat{\Psi}_0^{out}(\frac{q}{T_\varepsilon})|^2 - |\Psi_{T_\varepsilon}(q)|^2|\, d^3q,
\end{aligned} \tag{71}$$

where $k := \frac{q}{T_\varepsilon}$.
To get a bound on the first term we note that

$$\begin{aligned}
(B^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a})^c &= (B^{\hat{\Psi}_0^{out}}_{\delta_1 a})^c \cup \{q \in \mathbb{R}^3 \mid q \in B^{\hat{\Psi}_0^{out}}_{\delta_1 a} \wedge U_{\delta_2} \not\subset B^{\hat{\Psi}_0^{out}}_{\delta_1 a}\} = \\
&=: (B^{\hat{\Psi}_0^{out}}_{\delta_1 a})^c \cup D^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a}
\end{aligned}$$

and

$$(B^{\hat{\Psi}_0^{out}}_{\delta_1 a})^c \subset (A^{\hat{\Psi}_0}_{\delta_1})^c \cup \{q \in \mathbb{R}^3 \mid |q| \le a\}.$$

Thus

$$\begin{aligned}
&\int\limits_{(B^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a})^c} |\hat{\Psi}_0^{out}(k)|^2\, d^3k \le \\
&\qquad \le \int\limits_{(A^{\hat{\Psi}_0}_{\delta_1})^c} |\hat{\Psi}_0^{out}(k)|^2\, d^3k + \int\limits_{|k|\le a} |\hat{\Psi}_0^{out}(k)|^2\, d^3k + \int\limits_{D^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a}} |\hat{\Psi}_0^{out}(k)|^2\, d^3k.
\end{aligned}$$

We note that $\hat{\Psi}_0^{out}$ is square integrable, so the first term can be made smaller than $\frac{\varepsilon}{4}$ by choosing δ_1 small enough (as in the proof of Theorem 1), the second by choosing a small enough. Since $\hat{\Psi}_0^{out} \in S(\mathbb{R}^3)$ also implies that $\hat{\Psi}_0^{out}(k)$ is continuous in k, $B^{\hat{\Psi}_0^{out}}_{\delta_1 a}$ is open and the third integral can be made smaller than $\frac{\varepsilon}{4}$ by choosing δ_2 small enough.
To get a bound on the second term in (71) we use that by (26)

$$\int\limits_{\mathbb{R}^3} |T_\varepsilon^{-3}|\hat{\Psi}_0^{out}(\frac{q}{T_\varepsilon})|^2 - |\Psi_{T_\varepsilon}(q)|^2|\, d^3q \le \frac{\varepsilon}{4}$$

for all T_ε big enough.
Thus (33) holds for all T_ε big enough.
Since Ψ_t^{out} obeys the free time evolution the first part of (34) follows immediately from (17). Noting that $\frac{q}{t} \in B_{\delta_1 a}^{\hat{\Psi}_0^{out}}$ implies $\frac{t}{|q|} < \frac{1}{a}$ and $|\Psi_0^{out}(\frac{q}{t})| > \delta_1$ the second part of (34) is a direct consequence of (27) (with $r = 0$ and T_ε big enough):

$$t^{\frac{3}{2}}|\Psi_t(q)| \geq |\hat{\Psi}_0^{out}(\frac{q}{t})| - \frac{C_2}{|q|(t+|q|)}t^{\frac{3}{2}} - c_0 t^{-1} >$$
$$> \delta_1 - C_2 t^{-\frac{1}{2}}\frac{t}{|q|} - c_0 t^{-1} > \delta_1 - \frac{C_2}{a}t^{-\frac{1}{2}} - c_0 t^{-1} > \frac{\delta_1}{2}$$

Now let $q \in \mathbb{R}^3 \backslash \{0\}$ such that $\frac{q}{T_\varepsilon} \in B_{\delta_1 \delta_2 a}^{\hat{\Psi}_0^{out}}$. Suppose there exists some $t_1 > T_\varepsilon$ such that $\frac{\mathrm{Q}_{T_\varepsilon}(q,t_1)}{t_1} \notin U_{\delta_2}(\frac{q}{T_\varepsilon})$. Since $\mathrm{Q}_{T_\varepsilon}(q,t)$ is continuous in t (Remark 1), this implies that the first exit time $t_{ex} := \max\{s > T_\varepsilon \mid \frac{\mathrm{Q}_{T_\varepsilon}(q,s)}{s} \notin U_{\delta_2}(\frac{q}{T_\varepsilon}) \quad \wedge \quad \frac{\mathrm{Q}_{T_\varepsilon}(q,t)}{t} \in U_{\delta_2}(\frac{q}{T_\varepsilon}) \quad \forall T_\varepsilon \leq t < s\}$ exists and $\left|\frac{\mathrm{Q}_{T_\varepsilon}(q,t_{ex})}{t_{ex}} - \frac{q}{T_\varepsilon}\right| = \delta_2$. Moreover $\frac{\mathrm{Q}_{T_\varepsilon}(q,\tau)}{\tau} \in B_{\delta_1 a}^{\hat{\Psi}_0^{out}}$ (i.e. (34) holds), and $\frac{\tau}{\mathrm{Q}_{T_\varepsilon}(q,\tau)} < \frac{1}{a}$ for all $T_\varepsilon \leq \tau < t_{ex}$. For T_ε big enough the latter also implies $\mathrm{Q}_{T_\varepsilon}(q,\tau) > R_0$, i.e. (28) holds (with R_0 as in Lemma 2).
Then by (28) and (34)

$$\left|v^\Psi\big(\mathrm{Q}_{T_\varepsilon}(q,t),t\big) - \frac{\mathrm{Q}_{T_\varepsilon}(q,t)}{t}\right| \leq$$
$$\leq \left[C_3\left(1 + \frac{2C_4}{a\delta_1}t^{-1}\right) + \frac{C_2}{a}\right]\frac{2}{\delta_1}t^{-\frac{1}{2}} \leq Ct^{-\frac{1}{2}} < \frac{\delta_2}{2} \qquad \forall T_\varepsilon \leq \tau < t_{ex}.$$

So finally for T_ε big enough

$$\left|\frac{\mathrm{Q}_{T_\varepsilon}(q,t_{ex})}{t_{ex}} - \frac{q}{T_\varepsilon}\right| \leq$$
$$\leq \int_{T_\varepsilon}^{t_{ex}} \frac{1}{\tau}\left|v^\Psi\big(\mathrm{Q}_{T_\varepsilon}(q,\tau),\tau\big) - \frac{\mathrm{Q}_{T_\varepsilon}(q,\tau)}{\tau}\right| d\tau \leq C' \int_{T_\varepsilon}^{t_{ex}} \tau^{-\frac{3}{2}}\, d\tau \leq 2C' T_\varepsilon^{-\frac{1}{2}} < \delta_2,$$

which is a contradiction.
Hence (35) holds. $\square$

Proof of Theorem 4. Let $\varepsilon > 0$.
Since

$$\begin{aligned}|\Psi_t(q)|^2 &= |\Psi_t^{ac}(q)|^2 + |\Psi_t^{pp}(q)|^2 + 2\mathrm{Re}(\Psi_t^{ac}(q)^*\Psi_t^{pp}(q)) \leq \\ &\leq |\Psi_t^{ac}(q)|^2 + |\Psi_t^{pp}(q)|^2 + 2|\Psi_t^{ac}(q)||\Psi_t^{pp}(q)|\end{aligned} \tag{72}$$

and

$$(\tilde{B}_{\delta_1\delta_2 a}^{\hat{\Psi}_0^{out}})^c = (B_{\delta_1\delta_2 a}^{\hat{\Psi}_0^{out}})^c \cap \{q \in \mathbb{R}^3 \mid |q| > a\}$$

we get by Schwarz inequality

$$\begin{aligned}&\mathbb{P}^{\Psi_{T_\varepsilon}}\Big(\Big\{q \in \mathbb{R}^3 \mid \frac{q}{T_\varepsilon} \notin \tilde{B}_{\delta_1\delta_2 a}^{\hat{\Psi}_0^{out}}\Big\}\Big) \leq \\ &\quad \leq \int\limits_{C_{\delta_1\delta_2 a}^{\hat{\Psi}_0^{out}}(T_\varepsilon)} |\Psi_{T_\varepsilon}^{ac}(q)|^2\, d^3q + \int\limits_{|q|>aT_\varepsilon} |\Psi_{T_\varepsilon}^{pp}(q)|^2\, d^3q + \\ &\qquad + 2\Big[\int\limits_{C_{\delta_1\delta_2 a}^{\hat{\Psi}_0^{out}}(T_\varepsilon)} |\Psi_{T_\varepsilon}^{ac}(q)|^2\, d^3q\Big]^{\frac{1}{2}}\Big[\int\limits_{|q|>aT_\varepsilon} |\Psi_{T_\varepsilon}^{pp}(q)|^2\, d^3q\Big]^{\frac{1}{2}},\end{aligned} \tag{73}$$

where $C_{\delta_1\delta_2 a}^{\hat{\Psi}_0^{out}}(T_\varepsilon) := \{q \in \mathbb{R}^3 \mid \frac{q}{T_\varepsilon} \notin B_{\delta_1\delta_2 a}^{\hat{\Psi}_0^{out}}\}$ is the same set as in the proof of Theorem 2.
In the same way we get for $t \geq T_\varepsilon$

$$\begin{aligned}&\mathbb{P}^{\Psi_t}\big(\{q \in \mathbb{R}^3 \mid |q| \leq at\}\big) \leq \\ &\quad \leq \int\limits_{|q|\leq at} |\Psi_t^{ac}(q)|^2\, d^3q + \int\limits_{\mathbb{R}^3} |\Psi_t^{pp}(q)|^2\, d^3q + \\ &\qquad + 2\Big[\int\limits_{|q|\leq at} |\Psi_t^{ac}(q)|^2\, d^3q\Big]^{\frac{1}{2}}\Big[\int\limits_{\mathbb{R}^3} |\Psi_t^{pp}(q)|^2\, d^3q\Big]^{\frac{1}{2}} \leq \\ &\quad \leq \int\limits_{|q|\leq at} |\Psi_t^{ac}(q)|^2\, d^3q + \|\Psi_t^{pp}\|^2 + 2\|\Psi_t^{pp}\|\Big[\int\limits_{|q|\leq at} |\Psi_t^{ac}(q)|^2\, d^3q\Big]^{\frac{1}{2}} \leq \\ &\quad \leq \|\Psi_0^{pp}\|^2 + \int\limits_{|q|\leq at} |\Psi_t^{ac}(q)|^2\, d^3q + 2\Big[\int\limits_{|q|\leq at} |\Psi_t^{ac}(q)|^2\, d^3q\Big]^{\frac{1}{2}}\end{aligned} \tag{74}$$

and

$$
\begin{aligned}
&\mathbb{P}^{\Psi_t}\big(\{q \in \mathbb{R}^3 \mid |q| \le at\}\big) \ge \\
&\qquad \ge \mathbb{P}^{\Psi_t}\big(\{q \in \mathbb{R}^3 \mid |q| \le aT_\varepsilon\}\big) \ge 1 - \mathbb{P}^{\Psi_t}\big(\{q \in \mathbb{R}^3 \mid |q| > aT_\varepsilon\}\big) \ge \\
&\qquad \ge 1 - \|\Psi_t^{ac}\|^2 - \int\limits_{|q|>aT_\varepsilon} |\Psi_t^{pp}(q)|^2\, d^3q - \\
&\qquad\qquad\qquad - 2\|\Psi_t^{ac}\| \Big[\int\limits_{|q|>aT_\varepsilon} |\Psi_t^{pp}(q)|^2\, d^3q \Big]^{\frac{1}{2}} \ge \\
&\qquad \ge \|\Psi_0^{pp}\|^2 - \int\limits_{|q|>aT_\varepsilon} |\Psi_t^{pp}(q)|^2\, d^3q - \Big[\int\limits_{|q|>aT_\varepsilon} |\Psi_t^{pp}(q)|^2\, d^3q \Big]^{\frac{1}{2}}.
\end{aligned} \tag{75}
$$

By (33) $\int\limits_{C^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a}(t)} |\Psi_t^{ac}(q)|^2\, d^3q$ can be made arbitrary small if a, δ_1 and δ_2 are small enough and T_ε (and thus $t \ge T_\varepsilon$) is big enough. Since $\{q \in \mathbb{R}^3 \mid |q| \le at\} \subset C^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a}(t)$ this implies, that $\int\limits_{|q|\le at} |\Psi_t^{ac}(q)|^2\, d^3q$, too, can be made arbitrary small (in fact it suffices to suitably decrease a and increase T_ε). By Lemma 4 (i) $\int\limits_{|q|>at} |\Psi_t^{pp}(q)|^2\, d^3q$ can be made arbitrary small if $at \ge aT_\varepsilon$ is big enough, i.e. by increasing T_ε appropriately. Thus (45) holds by (73) and (46) holds by (74) and (75).
Now let $t \ge T_\varepsilon$ and $q \in \mathbb{R}^3\backslash\{0\}$ such that $\frac{q}{t} \in B^{\hat{\Psi}_0^{out}}_{\delta_1 a}$. Then by (34) the first and second part of (47) hold.
Moreover, since $\frac{|q|}{t} > a$ (which for T_ε big enough implies $|q| > R_2$ with R_2 as in Lemma 4), $\big|\hat{\Psi}_0^{out}(\frac{q}{t})\big| > \delta_1$ and $\frac{1}{|q|(t+|q|)} \le \frac{t}{|q|} t^{-2}$, (40) with $r = 0$ gives us

$$
t^{\frac{3}{2}}\big|\Psi_t(q)\big| > \delta_1 - \frac{C_2}{a} t^{-\frac{1}{2}} - c_0 t^{-1} - \frac{C_5}{a^{\frac{3}{2}+\alpha}} t^{-\alpha} > \frac{\delta_1}{2}
$$

for T_ε big enough.
Now let $q \in \mathbb{R}^3\backslash\{0\}$ such that $\frac{q}{T_\varepsilon} \in B^{\hat{\Psi}_0^{out}}_{\delta_1\delta_2 a}$ and suppose that the first exit time $t_{ex} := \max\{s > T_\varepsilon \mid \frac{Q_{T_\varepsilon}(q,s)}{s} \notin U_{\delta_2}(\frac{q}{T_\varepsilon}) \quad \wedge \quad \frac{Q_{T_\varepsilon}(q,t)}{t} \in U_{\delta_2}(\frac{q}{T_\varepsilon}) \quad \forall T_\varepsilon \le t < s\}$ exists. Then, by continuity of $Q_{T_\varepsilon}(q,t)$ (Remark 1), $\big|\frac{Q_{T_\varepsilon}(q,t_{ex})}{t_{ex}} - \frac{q}{T_\varepsilon}\big| = \delta_2$. Moreover $\frac{Q_{T_\varepsilon}(q,t)}{t} \in B^{\hat{\Psi}_0^{out}}_{\delta_1 a}$ for all $T_\varepsilon \le t < t_{ex}$. By (41) and (47) this implies

that there is some $C < \infty$ such that for all $T_\varepsilon \le t < t_{ex}$

$$
\begin{aligned}
&\big|v^\Psi(\mathrm{Q}_{T_\varepsilon}(q,t),t) - \frac{\mathrm{Q}_{T_\varepsilon}(q,t)}{t}\big| \le \\
&\quad \le \frac{2}{\delta_1} t^{-\beta} \Big\{ t^{\beta-\frac{1}{2}} \Big[C_3\big(1 + \frac{2C_4}{a\delta_1} t^{-1}\big) + \frac{C_2}{a} \Big] + \\
&\qquad\quad + t^{\beta-\alpha} \frac{C_5}{a^{\frac{3}{2}+\alpha}} \Big[1 + \frac{2}{\delta_1} \Big(\big(C_3\big(1 + \frac{2C_4}{a\delta_1} t^{-1}\big) + \frac{C_2}{a}\big) t^{-\frac{1}{2}} + C_6\big(1 + \frac{C_7}{a}\big) \Big) \Big] \Big\} \le \\
&\quad \le C t^{-\beta}.
\end{aligned}
$$

Thus for T_ε big enough

$$
\begin{aligned}
\big|\frac{\mathrm{Q}_{T_\varepsilon}(q,t_{ex})}{t_{ex}} - \frac{q}{T_\varepsilon}\big| &\le \int_{T_\varepsilon}^{t_{ex}} \frac{1}{\tau} \big|v^\Psi(\mathrm{Q}_{T_\varepsilon}(q,\tau),\tau) - \frac{\mathrm{Q}_{T_\varepsilon}(q,\tau)}{\tau}\big|\, d\tau \le \\
&\le C \int_{T_\varepsilon}^{t^- ex} \tau^{-1-\beta}\, d\tau \le \frac{C}{\beta} T_\varepsilon^{-\beta} < \delta_2.
\end{aligned}
$$

But this is a contradiction. $\square$

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Hiermit erkläre ich, die vorliegende Arbeit selbstständig verfaßt und keine anderen als die genannten Quellen und Hilfsmittel verwendet zu haben.

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Sarah Römer