# A Statistical View on Quantum Electrodynamics

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München

30. Juni 2009

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# Contents

# Chapter 1

# Introduction

The concise content of this work is the following. It is a fact that in general a second quantized time evolution of the Dirac equation is impossible with the usual quantization rules. In fact this is impossible as soon as an external magnetic field is present. The "established" theory gives us a formalism only for scattering situations, in general a time evolution cannot be written down. This problem is understood immediately if Dirac's original idea, the introduction of a "Dirac sea", is taken serious. The problem is therefore not just a mathematical but mainly a physical one. The crucial question is: What is the vacuum?

That shall be investigated in this work. One could say, it is about quantum electrodynamics from a statistical point of view. That means the starting point will be the assumption that we live in an N-particle universe.[1] A relativistic quantum theory has to explain several new phenomena (compared to non-relativistic quantum mechanics which has the Schrödinger equation as its fundamental law). In particular it has to explain the appearance of anti-particles and (electron-positron) pair creation and annihilation. The Dirac equation is a good candidate for a fundamental law describing the dynamics of the particles (see chapter 2.1).[2] Therefore chapter 2 is devoted to the N-particle theory with the Dirac equation as its fundamental law. Several consequences, the main one being the occurrence of negative energy states have to be discussed. The short version is the following.

The one-particle free Dirac equation is

$$i\hbar\frac{\partial}{\partial t}\psi(t,\boldsymbol{x}) = H^0\psi(t,\boldsymbol{x}) \tag{1.1}$$

($\boldsymbol{x} \in \mathbb{R}^3, t \in \mathbb{R}$) with the free Dirac operator

$$H^0 = \hbar c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^2 \tag{1.2}$$

for a wavefunction $\psi(t,\boldsymbol{x}) \in L^2(\mathbb{R}^3,\mathbb{C}^4)$. $H^0$ has the energy spectrum $\sigma = (-\infty,-mc^2] \cup [+mc^2,+\infty)$, thus allowing for negative energy states, i.e. wavefunctions associated with the negative part of the spectrum. We then have a natural splitting of the one-particle Hilbert space into two spectral subspaces: $L^2(\mathbb{R}^3,\mathbb{C}^4) = \mathcal{H}_- \oplus \mathcal{H}_+$. It must be explained what a negative energy state (in $\mathcal{H}_-$) means as there don't seem to be negative energy particles in

[1]The reader may object that in relativistic quantum mechanics it is not clear if a particle ontology makes sense (although e.g. no experimental *particle* physicist would object). Those who don't like particles may replace the word "particle" with "degrees of freedom" or "wavefunction describing a particle". However, if one takes for granted that our universe is made up of particles the above statement is in a way obvious. There may be *a lot* of them but this nevertheless remains a finite number.

[2]Here we only deal with spin-$\frac{1}{2}$ particles.

nature (see chapter 2.2). Furthermore there is no mechanism preventing a positive energy state from making transitions to states of negative energy (thus it could radiate an infinite amount of energy).

Therefore Dirac had the idea that all negative energy states are occupied (by electrons). These occupied states constitute the “Dirac sea”. With that the above problems can be addressed. The exclusion principle prevents transitions to the negative part of the spectrum and “holes” in the Dirac sea are interpreted as anti-particles. Pair creation then means that a particle from the sea makes a transition to the positive spectrum leaving behind a “hole” in the sea. However if we introduce a Dirac sea like that (*all* negative energy states are occupied) we do not have an N-particle problem anymore but we have to deal with infinitely many particles in the sea. This is neither a physical nor a mathematical meaningful statement. (What should infinitely many particles even be?) In standard textbooks on QED one usually doesn't speak about a Dirac sea anymore but introduces a vacuum vector $|0\rangle$ on which creation and annihilation operators act (procedure of second quantization). The physical meaning of this is described in chapter 3.3.

In this work I investigate the possibility that the fundamental *microscopic* theory is indeed an N-particle theory. Of course new questions arise then: How can one think of pair creation in an N-particle universe? Why aren't there any transitions to negative energy states as there still are unoccupied ones? Why don't we “see” most of the particles, i.e. why don't we “see” the Dirac sea?[3] These questions shall be addressed in chapter 2.2. I will argue that the transition amplitude of a positive energy state to an unoccupied negative energy state is very small and gets smaller the more particles we add to the sea (chapter 2.5).

However as long as we have a finite number of particles in the sea the transition amplitude remains non-zero. So in principle it would be possible to observe an electron with negative kinetic energy. This has not been observed though. It could happen, but because the number of particles N is so large, it is never observed. If for our description we want the transition amplitude to be exactly zero, then this can be achieved by performing the limit $N \to \infty$. So our description becomes “sharp” only in this limit. The situation is quite similar to the description of phase transitions (and other phenomena like spontaneous symmetry breaking or Brownian motion) in statistical mechanics. There divergences in one of the derivatives of a free energy functional *define* a phase transition (classification according to Ehrenfest, see e.g. [10]). But in an N-particle theory there is no phase transition.[4] E.g. in experiments a specific heat or susceptibility may become very large, but never actual infinity. So to get the transition “sharp” one has to perform the thermodynamic limit $N \to \infty$. Again: every “real” system one considers in statistical mechanics is of finite size[5] so according to theory there shouldn't be any phase transition.[6] To explain this phenomenon nevertheless one considers the thermodynamic limit. So the basic idea that we live in an N-particle universe but use a “thermodynamic” description to explain phenomena in nature is widely used and very common in modern physics. In chapter 3.2, I explain the consequences of taking this limit in our case.

Furthermore we simply don't have enough information about the particles in the sea, so it doesn't make sense (or is not possible at all) to solve the equation of motion for all N

[3]Of course one has to deal with this question also in the case that the Dirac sea is made up of infinitely many particles.

[4]A nice discussion of this fact can be found in [9].

[5]Actually statistical mechanics was also applied to systems with not just finite but comparably very small size, e.g. the nucleus of an atom.

[6]At least none which involves a spontaneous symmetry breaking, see the exception of the liquid-vapor transition of water.

particles (similar to statistical mechanics). But it may be intuitively clear that interesting physics happens only "at the surface" of the sea, i.e. only for states with very high negative energy we can hope something interesting (e.g. pair creation) to happen. Therefore the approach is to ignore what goes on "deep down" in the sea and a statistical description with the limit $N \to \infty$ seems to make sense.

There is another cause why this is reasonable. It is known from experiments and theory that pair creation can only take place in the presence of an external (time-dependent) electromagnetic field. One can imagine that the time evolution with external field not only affects the positive energy particles and maybe some particles "at the surface" of the Dirac sea but also all particles "deep down" in the sea. This is indeed a great problem in the N-particle theory as at first one doesn't know what "deep down" in the sea means. One doesn't know which particles still belong to the sea after the time evolution. To achieve this one would have to specify a mechanism that tells us how the sea particles interact and how we can decide which particles still belong to the sea (and are thus not observable) and which not. Luckily the situation is better if we consider the limit $N \to \infty$. We know that e.g. transition amplitudes between two states have to be finite (to give at least probabilities for the number of particles that have been created or annihilated). However the "stirring" in the sea that external fields produce, is in general so great that the original and the final Dirac sea state are not "comparable" anymore, i.e. transition amplitudes (given by scalar products) inevitably diverge. In order to get the transition amplitudes finite one may think that one can introduce operators that undo the "stirring". In the N-particle theory there are indeed lots of operators that could describe how this happens but we have no chance to decide which one describes the correct physics. If we perform the limit $N \to \infty$ though, we have another situation. Unlike in the N-particle case where transition amplitudes are always finite, we can now find exactly one operator that leaves the transition amplitudes finite. It is unique up to a phase factor.

This is the mathematical setup proposed by Deckert, Dürr, Merkl and Schottenloher in [1] which is described in chapter 4. It appears quite naturally as an "effective" description in the limit $N \to \infty$. It can be interpreted as the *macroscopic* theory of the underlying N-particle theory. This setup accordingly resolves the problem of the external field which appears in the standard formulation of quantum electrodynamics: One cannot (in general) write down the Dirac time evolution in the presence of an external magnetic field as infinitely many particles would be created (the one-particle Dirac time evolution cannot in general be lifted to Fock space). This is the content of the Shale-Stinespring theorem and the work of Ruijsenaars (see [1] for references). Pictorially speaking the Dirac sea spinor states are rotated by the external field, thus the infinitely many particles from the negative spectrum grow into the positive part. The catastrophe of infinitely many particle creation happens as soon as a magnetic field is present. Usually this problem is not discussed in standard textbooks on QED. There one mostly discusses scattering situations. Indeed the S-matrix can be lifted to Fock space as in this case one deals with asymptotically free in- and out-going states. In a way one neglects what happened with the Dirac sea in between, i.e. sea wavefunctions may be rotated out of the sea at intermediate times during the scattering but in the end when the particles are free, most states are rotated back. However, if one regards pair creation in strong electromagnetic fields and wants to calculate the rates of pair creation one cannot circumvent the above problem. Although the existence of pair creation[7] has been rigorously proved in [6] and [7], it is still an unresolved problem to calculate the exact rates of pair

[7] I.e. that in strong fields there indeed *are* transitions from the negative to the positive spectral subspace and that those states can remain in the positive spectrum.

creation in the presence of external fields. This problem of the external field is resolved in [1] as a well-defined time evolution is found there. As already implied above, the idea is to consider the time evolution between time varying Fock spaces (the so called infinite wedge spaces, see chapter 4.2). The time evolution is concretely written down in chapter 4.7.

This time evolution is only unique up to a phase though. This may be important because another quantity one would like to define is the vacuum current (in the presence of an external field). Therefore the phase is probably needed which in the above formulation can be chosen arbitrarily. However, the phase cancels out when calculating transition amplitudes.

As a last introductory remark it should be noted that this work (or the paper [1]) may not only be interesting for mathematical physics but also for modern fields of theoretical physics. The effective formalism of chapter 3 is rather the playground for further applications. In principle it should be applicable to every Dirac theory as there one unavoidably has to deal with infinitely many degrees of freedom. One may for example think of applications in solid state physics (e.g. graphene).

# Chapter 2

# An N-Particle Theory of QED

This chapter deals with introducing an N-particle theory for quantum electrodynamics. First I will describe the equation of motion for the particles, which is the Dirac equation. Then I try to give a physical explanation of what the Dirac sea could be. Thereby I attempt to argue close to Dirac's original idea which he presented in his papers in the early 30s.

## 2.1 The Equation of Motion for N Particles

A good candidate for the equation of motion for relativistic spin-$\frac{1}{2}$ particles is the Dirac equation. To begin with, I briefly summarize its basic features.[1] In the non-relativistic case the substitutions $E \to i\hbar\frac{\partial}{\partial t}$ and $p \to -i\hbar\nabla$ (motivated by considering plane waves) in the classical Hamiltonian $H = \frac{\boldsymbol{p}^2}{2m} + V$ lead to the free Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(t,\boldsymbol{x}) = \left(-\frac{\hbar^2}{2m}\Delta + V\right)\psi(t,\boldsymbol{x}) \tag{2.1}$$

with $t \in \mathbb{R}$, $\boldsymbol{x} \in \mathbb{R}^3$. Note that the Schrödinger equation is Galilei-invariant and furthermore can easily be generalized to an N-particle equation for a wavefunction on configuration space:

$$i\hbar\frac{\partial}{\partial t}\psi(t,\boldsymbol{x}_1,\dots,\boldsymbol{x}_N) = \Big(\sum_{i=1}^{N} -\frac{\hbar^2}{2m_i}\frac{\partial^2}{\partial \boldsymbol{x}_i^2} + V\Big)\psi(t,\boldsymbol{x}_1,\dots,\boldsymbol{x}_N). \tag{2.2}$$

The same substitutions applied to the relativistic energy-momentum relation

$$E = \sqrt{c^2p^2 + m^2c^4} \tag{2.3}$$

give the square-root Klein-Gordon equation

$$i\hbar\frac{\partial}{\partial t}\psi(t,\boldsymbol{x}) = \sqrt{-c^2\hbar^2\Delta + m^2c^4}\ \psi(t,\boldsymbol{x}). \tag{2.4}$$

As time and space derivatives occur asymmetrically in (2.4) it seems impossible to include external fields in a relativistic invariant way. To include the description of spin and to keep an equation of first order in time derivatives, Dirac linearized the energy-momentum relation by writing

$$E = c\sum_{i=1}^{3}\alpha^i p_i + \beta mc^2 = c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^2, \tag{2.5}$$

[1] The description will closely follow [11] at the beginning.

with $\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$ and $\beta$ being the $\mathbb{C}^{4\times 4}$ Dirac matrices, which are determined by comparing with the energy-momentum relation (2.3). They satisfy

$$\alpha^i\alpha^k + \alpha^k\alpha^i = 2\delta^{ik}, \qquad \alpha^i\beta + \beta\alpha^i = 0, \qquad \beta^2 = 1, \qquad (i,k = 1,2,3). \tag{2.6}$$

Thus the one-particle free Dirac equation reads (from now on setting $c = \hbar = 1$):

$$i\frac{\partial}{\partial t}\psi(t, \boldsymbol{x}) = H^0\psi(t, \boldsymbol{x}) \tag{2.7}$$

with the free Dirac operator

$$H^0 = -i\boldsymbol{\alpha}\cdot\nabla + \beta m. \tag{2.8}$$

It is Lorentz-invariant and can easily be extended to include the description of external electromagnetic fields by writing

$$H^A = \boldsymbol{\alpha}\cdot(-i\nabla - e\boldsymbol{A}) + \beta m + eA^0, \tag{2.9}$$

with the four-vector potential $A = (A^\mu)_{\mu=0,1,2,3} = (A^0, \boldsymbol{A})$. The Hilbert space for $\psi(t, \boldsymbol{x})$ is $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$, i.e. the space of $\mathbb{C}^4$ valued square integrable functions on $\mathbb{R}^3$. We thus write $\psi$ as a 4-component vector $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^\top$.

When one tries to generalize the Dirac equation to an N-particle equation the first guess would be

$$i\frac{\partial}{\partial t}\Psi(t, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = \left(\sum_{k=1}^N H_k^A\right)\Psi(t, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \tag{2.10}$$

with

$$H_k^A = \boldsymbol{\alpha}_k\cdot(-i\nabla_k - e\boldsymbol{A}(t, \boldsymbol{x}_k)) + \beta_k m + eA^0(t, \boldsymbol{x}_k) \tag{2.11}$$

($\boldsymbol{\alpha}_k, \beta_k$ act on the four indices that belong to the k-th particle). Equation (2.11) is not Lorentz-invariant though. This is clear from a relativistic point of view ("there is no absolute simultaneity") as the wavefunction depends on an absolute time $t$. It seems more natural to relate to each particle an individual time. One then arrives at the multi-time Dirac equation, here written as the set of N equations

$$i\frac{\partial}{\partial t_k}\Psi(t_1, \boldsymbol{x}_1, \ldots, t_N, \boldsymbol{x}_N) = H_k^A\Psi(t_1, \boldsymbol{x}_1, \ldots, t_N, \boldsymbol{x}_N), \tag{2.12}$$

$k = 1, \ldots, N$. This idea was developed in 1932 in a paper by Dirac, Fock and Podolsky ([5]). Note that so far no interaction potential is included in this description. It is not yet clear how this can be achieved without the usual problems of ultraviolet and infrared divergences. Next one can regard these N multi-time Dirac equations on a hypersurface $\Sigma_t$. For example one restricts $\Psi$ to a simultaneity hypersurface, i.e. one solves the equations for $\Psi|_{\Sigma_t} = \Psi|_{t=t_1=\ldots=t_N}$. Then $\Psi|_{\Sigma_t}$ is also a solution of (2.10).

Thus we have a law for describing N relativistic spin-$\frac{1}{2}$ particles in an external electromagnetic field. However, until now I didn't mention the most intriguing feature of the Dirac equation which is the occurrence of negative energy states and which led Dirac to the invention of his "sea". This shall be dealt with in detail in the next section.

## 2.2 What is the Dirac Sea?

Consider the free Dirac Hamiltonian in momentum representation

$$H^0(\boldsymbol{p}) = \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m \tag{2.13}$$

which is a (hermitian) $4 \times 4$-matrix. One can easily calculate its eigenvalues which are $\pm E(p)$ with $E(p) = \sqrt{p^2 + m^2}$. Thus the spectrum of the free Dirac operator is given by $\sigma = (-\infty, -m] \cup [+m, +\infty)$. Let $P_-$ and $P_+$ denote the projection operators onto the negative and positive spectral subspace, explicitly

$$P_\pm = \frac{1}{2}\left(1 \pm \frac{H^0(p)}{E(p)}\right). \tag{2.14}$$

Then we have a natural splitting of the one-particle Hilbert space into two spectral subspaces: $L^2(\mathbb{R}^3, \mathbb{C}^4) = P_- L^2(\mathbb{R}^3, \mathbb{C}^4) \oplus P_+ L^2(\mathbb{R}^3, \mathbb{C}^4) = \mathcal{H}_- \oplus \mathcal{H}_+$.[2] A state in $\mathcal{H}_+$ describes an electron (with positive energy) but it is not clear what a state in $\mathcal{H}_-$ (an electron with negative energy) means as negative energy particles were not observed in nature. Note that this would not be a problem in a classical theory as there is a spectral gap of $2mc^2$ and classical dynamic variables must always vary continuously. In a classical theory a state in $\mathcal{H}_+$ remains a positive energy state for all times (see [2]).

However in a quantum theory the negative energy states cannot be ignored as transitions from $\mathcal{H}_+$ to $\mathcal{H}_-$ can take place in an external (time varying) electromagnetic field. So let's take a closer look at those solutions. First note that "an electron with negative energy moves in an external field as though it carries a positive charge" (Dirac in [2]).[3] It cannot simply be interpreted as an anti-particle (a positron) though, for several reasons: A transition from a positive to a negative energy state would violate charge conservation, a positron would nevertheless produce a negative charged field (thus repelling electrons) and it would move faster the less energy it has.

Therefore Dirac had the idea that nearly all negative energy states are occupied by electrons. Positrons are then "holes" in the sea: they have a positive charge and positive energy (negative energy is missing). According to Dirac the "exclusion principle will operate to prevent a positive-energy electron ordinarily from making transitions to states of negative energy" (see [4]). In this picture pair creation means that an electron is pushed out of the sea leaving behind a hole.

So in order to explain certain phenomena in our world Dirac introduces a "uniformity hypothesis" which makes the Dirac sea inaccessible for our observation. In his own words:

> Admettons que dans l'Univers tel que nous le connaissons, les états d'energie négative soient presque tous occupés par des électrons, et que la distribution ainsi obtenue ne soit pas accessible à notre observation à cause de son uniformité dans toute l'etendue de l'espace. (Dirac in [3])

This idea that the sea particles are totally in equilibrium and thus hidden from us may seem very peculiar at first sight. But this isn't the case as one encounters similar hypotheses in various field of physics. For instance one makes the hypothesis that all gravitational effects

[2]Note that in the external field case (2.9) the splitting into a positive and negative spectral subspace is not at all straightforward, see chapter 2.4.2.

[3]This can easily be seen as the total energy H is given as $H = W + ef(A)$, with W independent of the elementary charge e (the coupling constant) and $f(A)$ being linear in A.

from far away galaxies cancel out somehow and thus do not contribute to physics here on earth ("spacetime is flat"). It is often the case that we ignore "the rest of the universe" in order to describe reasonable physics.

A similar thing happens in a formulation of electrodynamics proposed by Feynman and Wheeler in 1945 (see [8]). They consider a theory without fields and only with particles that interact directly on their light cones. According to this theory two single charges orbiting each other would not radiate at all (as there are no other particles that could absorb the radiation). Therefore "the absorber [is] an essential element in the mechanism of radiation" ([8, p. 160]) in that theory. The important point for us is that this theory admits solutions in which n particles interact which each other but no "radiation" goes outside. That is, there are solutions for which $\sum_{j=1}^{n} F_j(x) = 0$, $F_j(x)$ denoting the contribution of the j-th particle to the interaction. So a test charge feels no forces in the region where the above condition holds. The same should hold in the case of the free Dirac equation for a particle with positive energy in the sea of negative energy particles. The sea particles may interact with each other but they do not disturb the motion of a particle outside. The Dirac sea is unobservable for us "à cause de son uniformité" ([3]). The only thing we can observe is when this equilibrium is disturbed and a particle is pushed out of the sea. So the Dirac sea is also an "absorber" or "equilibrium" or "balance of forces" assumption. Note the connection between such an "absorber" hypothesis and particles with negative kinetic energy: In itself the "absorber" hypothesis is independent from the notion of negative energies but in the case of the free Dirac particle it makes sense to connect both.

Hence the Dirac sea should be taken serious. The idea is that we live in an N-particle universe and most of the particles are hidden in the sea.[4] They are inaccessible for us, except if an external field is present which opens the possibility for pair creation. Again, I want to emphasize that the Dirac sea is really needed: to deal with the problem of negative energy states and to explain the phenomenon of pair creation. Maybe Dirac rather had the latter in mind. Note that in this way it is possible to speak of pair "creation" also in a theory with constant particle number.

The next section is about how to construct Dirac sea states explicitly.

## 2.3 Properties of Dirac Sea States

### 2.3.1 Construction of Dirac Sea States I

First of all, an N-particle Dirac sea is described as an N-particle fermion state. Electrons obey Pauli's exclusion principle and are described by antisymmetric wavefunctions

$$\Psi(x_1, \ldots, x_k, \ldots, x_l, \ldots, x_N) = -\Psi(x_1, \ldots, x_l, \ldots, x_k, \ldots, x_N). \tag{2.15}$$

One can write down such a wavefunction explicitly by using Slater determinants. Take $\varphi = (\varphi_i)_{i \in \mathbb{N}}$ an orthonormal basis (ONB) in the one-particle Hilbert space $\mathcal{H}^{(1)}$ (for a "pure" Dirac sea state take $\mathcal{H}_-$). A one-particle wavefunction can be written in that basis as

$$\psi(x_1) = \sum_{n_1 \in \mathbb{N}} c_{n_1}^{(1)} \varphi_{n_1}(x_1) \tag{2.16}$$

[4]As a side remark: One encounters a similar situation in astrophysics and cosmology, which is that 96 % of the universe consists of dark matter and dark energy.

(with coefficients $c^{(1)}_{n_i}$ such that the state can be normalized). For N such one-particle solutions there is a unique antisymmetric N-particle wavefunction given by

$$\Psi(x_1,\dots,x_N)=\frac{1}{\sqrt{N!}}\sum_{n_1,\dots,n_N\in\mathbb{N}} c^{(1)}_{n_1}\cdot\ldots\cdot c^{(N)}_{n_N}\cdot det\begin{pmatrix}\varphi_{n_1}(x_1) & \dots & \varphi_{n_1}(x_N)\\ \vdots & & \vdots\\ \varphi_{n_N}(x_1) & \dots & \varphi_{n_N}(x_N)\end{pmatrix}. \quad (2.17)$$

This can be written more convenient. Consider only the basis vectors $(\varphi_i)_{i\in\mathbb{N}}$. Then the determinant can be written as a wedge product which has the same properties like a determinant:

$$\Phi_{n_1\dots n_N}=\varphi_{n_1}\wedge\varphi_{n_2}\wedge\ldots\wedge\varphi_{n_N}. \quad (2.18)$$

The basis for the N-particle Hilbert space is thus

$$B_N=\{\varphi_{n_1}\wedge\ldots\wedge\varphi_{n_N}|\ n_1,\dots,n_N\in\mathbb{N}\}. \quad (2.19)$$

The N-particle Hilbert space is then given as

$$\mathcal{H}^{(1)\wedge N}=\overline{span(B_N)}=\overline{\mathbb{C}^{(B_N)}}, \quad (2.20)$$

i.e. by taking formal finite linear combinations of elements of $B_N$. For simplicity of notation in the N-particle case, we write the basis elements as

$$\Phi_N=\varphi_1\wedge\varphi_2\wedge\ldots\wedge\varphi_N. \quad (2.21)$$

A scalar product on $\mathcal{H}^{(1)\wedge N}$ is defined straightforward. For another orthonormal basis $\psi=(\psi_i)_{i\in\mathbb{N}}$ we have a scalar product defined by the action on the basis vectors as

$$\langle\varphi,\psi\rangle=det(\langle\varphi_n,\psi_m\rangle)_{n,m=1,\dots,N}. \quad (2.22)$$

In fact we don't know much about Dirac sea states. The "absorber" hypothesis makes the Dirac sea inaccessible for us. Therefore later we will write "the" vacuum state simply as $\Omega=\varphi_0^-\wedge\varphi_1^-\wedge\ldots\wedge\varphi_N^-$ ($(\varphi_i^-)_{i\geq 0}$ an ONB in $\mathcal{H}_-$). Note that this is similar to a Hartree-Fock approximation. A notation with all the coefficients $c^{(j)}_{n_i}$ thus does not actually make sense.

We can simplify the notation even more, in particular to compare with the formalism of chapter 4. Let $\ell$ be another (finite dimensional) Hilbert space which plays the role of an index space (here think of $\mathbb{C}^N$). Then one can encode the basis of $\mathcal{H}^{(1)\wedge N}$ in a linear map $\Phi:\ell\to\mathcal{H}$. E.g. think of a $\Phi$ that is defined by the action on the canonical basis $(e_n)_{n=1,\dots,N}$ of $\ell$ as $\Phi e_n=\varphi_n$. Then the scalar product between two (Dirac sea) states can be written as $\langle\Phi,\Psi\rangle=det(\Phi^*\Psi)$.

### 2.3.2 k-pair States and the Charge

For N particles we have the total electrical charge $eN$ which is conserved by the Dirac equation. Pair creation in the N-particle case has nothing to do with changing the total charge. Nevertheless we want to express the fact that only "pairs" can be created, i.e. one particle escapes from the "absorber" leaving behind a hole in the sea. Thus it makes sense to speak of a relative charge. Say a negative spectral subspace $\mathcal{H}_-$ with ONB $(\varphi_i^-)_{i\geq 0}$ (say all particle states in $\mathcal{H}_-$ fulfill an "absorber" condition) and a positive spectral subspace $\mathcal{H}_+$ with ONB $(\varphi_i^+)_{i>0}$ is given. Then one can write a state with one electron as $\Phi=\varphi_1^+\wedge\varphi_0^-\wedge\varphi_1^-\wedge\ldots\wedge\varphi_{N-2}^-$, i.e. $charge(\Phi)=1$. A state with $k$ additional electron-positron pairs ($k\ll N$) could be $\Psi=\varphi_{k+1}^+\wedge\ldots\wedge\varphi_1^+\wedge\varphi_k^-\wedge\ldots\wedge\varphi_{N-2}^-$, i.e. $charge(\Psi)=(k+1)-k=1=charge(\Phi)$. Later we express "charge conservation" in the formula that the relative charge between two states is zero. This will be denoted in the following way. Say $range\Phi=V$ and $range\Psi=W$. Then we demand $charge(V,W)=0$.

## 2.4 The Dirac Time Evolution

### 2.4.1 One-Particle Time Evolution

Let $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$. The one-particle free Dirac equation (2.7) gives rise to a family of unitary operators $(U^0(t_1, t_0))_{t_0,t_1 \in \mathbb{R}}$ on $\mathcal{H}$ with $U^0(t_1, t_0) = exp(-iH^0(t_1 - t_0))$ for all $t_0, t_1 \in \mathbb{R}$. Then the wavefunction at time $t_1$ can be written as $\psi^0(t_1) = U^0(t_1, t_0)\psi^0(t_0)$. This is the unique solution of the Cauchy problem (2.7). The same for the Dirac equation with external field (2.9): One gets a family of unitary operators $(U^A(t_1, t_0))_{t_0,t_1 \in \mathbb{R}}$, such that for the wavefunction at time $t_1$ one has $\psi(t_1) = U^A(t_1, t_0)\psi(t_0)$. $U^A$ can be obtained from the fixed point form of the Dirac equation:

$$U(t_1, t_0) = U^0(t_1, t_0) + \int_{t_0}^{t_1} U^0(t_1, t)Z(t)U(t, t_0)dt \tag{2.23}$$

with $Z(t)$ defined by $H^{A(t)} = H^0 + iZ(t)$.

In section 2.5 it is dealt with transitions between states in the negative and the positive spectral subspaces. Therefore it should be explained how the time evolution acts on both subspaces. For a free particle the splitting into a positive and a negative subspace is clear. With the projectors given by (2.14) we can define $\mathcal{H}_\pm = P_\pm\mathcal{H}$ with $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$. Any linear operator $U$ on $\mathcal{H}$ can then be split into an even and an odd part:

$$U = (U_{++} + U_{--}) + (U_{+-} + U_{-+}) \equiv U_{even} + U_{odd} \tag{2.24}$$

with $U_{\pm\pm} = P_\pm U P_\pm$ and $U_{\pm\mp} = P_\pm U P_\mp$. If this is applied to the unitary one-particle Dirac time evolution one gets the following map:

$$U^A : \begin{pmatrix} P_+\mathcal{H} \\ P_-\mathcal{H} \end{pmatrix} \to \begin{pmatrix} P_+\mathcal{H} \\ P_-\mathcal{H} \end{pmatrix}, \psi \mapsto U^A\psi = \begin{pmatrix} U^A_{++} & U^A_{+-} \\ U^A_{-+} & U^A_{--} \end{pmatrix} \begin{pmatrix} P_+\psi \\ P_-\psi \end{pmatrix}. \tag{2.25}$$

Here one can see that the diagonal part of $U^A$ is "harmless": $U^A_{\pm\pm}$ maps states in the positive (negative) subspace into states in the same subspace. But $U^A_{+-}$ causes transitions from the Dirac sea into $\mathcal{H}_+$ and $U^A_{-+}$ entails that positive energy states make transitions into the sea. So to investigate how "stable" positive energy states are one has to regard the properties of $U^A_{-+}$.

### 2.4.2 N-Particle Time Evolution

Given the one-particle time evolution one can construct the N-particle time evolution. First note that it is not really clear how to speak of *a time evolution* in a relativistic sense. This becomes especially a problem when regarding the multi-time Dirac equation (2.12). Therefore here we only deal with solutions restricted to a simultaneity hypersurface such that there is only one time denoted by $t$.

Then the one-particle unitary time evolution $U^A(t_1, t_0)$ on $\mathcal{H}^{(1)}$ can be "lifted" to a unitary time evolution $\tilde{U}^A(t_1, t_0)$ on the N-particle Hilbert space $\mathcal{H}^{(1)\wedge N}$, which is defined by the action on the basis vectors as

$$\tilde{U}^A(t_1, t_0)\Phi_N = U^A(t_1, t_0)\varphi_1 \wedge U^A(t_1, t_0)\varphi_2 \wedge \ldots \wedge U^A(t_1, t_0)\varphi_N. \tag{2.26}$$

For the Dirac equation with external field the question arises, what the splitting into a positive and negative spectral subspace is, i.e. which particles can be regarded to be in the sea and which not. As mentioned in section 2.2 the splitting is clear for the free case. This splitting must in some way correspond to an "absorber" condition that tells us which particles are to be regarded as being in the sea (in equilibrium or balance of forces). Consider e.g. the following situation. We start with a vacuum at time $t_0$ with zero external field. The vacuum state can for example be written as $\Omega = \varphi_0^- \wedge \varphi_1^- \wedge \ldots \wedge \varphi_N^-$ with $(\varphi_i^-)_{i \geq 0}$ an orthonormal basis of the subspace $\mathcal{H}_-$. Then an external (time dependent) electromagnetic field is switched on (e.g. an experiment is performed). At time $t_1$ we observe what has happened. We could (in principle) find a unitary operator $\tilde{U}^A(t_1, t_0)$ such that the state at time $t_1$ is

$$(\tilde{U}^A \Omega)(t_1) = U^A \varphi_0^- \wedge U^A \varphi_1^- \wedge \ldots \wedge U^A \varphi_N^-. \tag{2.27}$$

*A lot* can have happened now. Thousands of electron-positron pairs may have been created and some annihilated again. The question is: Which particles still belong to the Dirac sea? One cannot simply write down a new splitting $\mathcal{H} = \mathcal{H}_-^{U^A(t_1,t_0)} \oplus \mathcal{H}_+^{U^A(t_1,t_0)}$ and then differentiate between positive and negative energy states. How should this be done? As mentioned above one rather has to scrutinize the "absorber" condition. What possibly could be done is to write done another state $\Psi$ and to calculate a transition amplitude as

$$W = |\langle \Psi, \tilde{U}^A \Omega \rangle|^2. \tag{2.28}$$

To what extent this makes sense is another question. The time evolution may have changed all the states "deep down" in the sea in such a way that the "absorber" condition is still fulfilled. This gives rise to contributions in the scalar product which we do not want to consider. Luckily the situation is better if we consider the limit $N \to \infty$. There we have only one choice (except for a phase $e^{i\varphi}$) to make transition amplitudes finite and thus there are (nearly) no contributions from "deep down" in the sea.

## 2.5 Probability for Negative Energy Particles

I want to give a brief (heuristic) argument why transitions from positive energy states to unoccupied "deep" Dirac sea states (not holes in the sea, that would be usual pair annihilation) are very unlikely. The idea is that the probability $W_{-+}$ for such a transition gets smaller the more particles are in the sea. Given $(\varphi_i^-)_{i>0}$ an ONB of $\mathcal{H}_-$ and $\psi$ a state in $\mathcal{H}_+$, then the probability for a transition to any negative energy state is

$$W_{-+} = \sum_{n \in \mathbb{N}} |\langle \varphi_n^-, U_{-+}\psi \rangle|^2 = \sum_{n \in \mathbb{N}} \langle U_{-+}\psi, \varphi_n^- \rangle \langle \varphi_n^-, U_{-+}\psi \rangle = ||U_{-+}\psi||^2. \tag{2.29}$$

If N states $\varphi_n^-$ are occupied the sum $\sum_{n \in \mathbb{N}}$ is replaced by $\sum_{n > N}$. It is heuristically clear that for great $N$, the probability $W_{-+}$ becomes small. One would need a very special time evolution to map positive energy states to "deep" Dirac sea states.

In our world it seems to be the case that electrons do not suddenly jump into a state of very high negative energy. All the typical negative energy states are occupied. So for the description of our world it seems reasonable to make the probability $W_{-+}$ vanish. This happens in the limit $N \to \infty$. One could also say that the conditions for applying this limit are excellently confirmed by experiments like it is the case in statistical mechanics (see the introduction).

That is why in chapter 3, I investigate the consequences of performing this limit. From our point of view it seems clear why some physical quantities inevitably diverge. The "right" limit has to be a scaling limit. A formalism which can be regarded as an effective formalism in the limit $N \to \infty$ is afterwards presented in chapter 4. The main idea will be to ignore what goes on "deep down in the sea" as one neither knows enough about the Dirac sea wavefunction nor about the (approximate) number of particles in the sea (except that there are *a lot*).

## 2.6 The N-particle QED

So what is achieved up to now? An N-particle theory of quantum electrodynamics was written down which only includes the description of external fields and no interactions yet. We made an "absorber" or "equilibrium" assumption as an explanation of the Dirac sea. However, a correct explanation of which particles belong to the sea and which not can only be given if the absorber condition is known better. Only with such a concrete condition we can hope to distinguish between particles that are still in the sea (still in equilibrium) and pairs we can observe. This is a difficulty that has to be worked out. Of course therefore one needs a QED that includes relativistic interactions (to describe the interaction between the sea particles and thus to specify the "equilibrium" condition). This is closely related to the problem of defining a proper current. Nevertheless, if properly worked out this can be a well-defined theory of quantum electrodynamics which seems to be close to Dirac's original idea. It is a fundamental *microscopic* theory. There should be no divergences and in particular no renormalization should be needed. However, it may be impossible to calculate practically with this formalism, exactly as it is impossible in statistical mechanics to solve Newton's equations for $10^{23}$ particles. In fact in statistical mechanics we *don't have to* solve the equations as we have a consistent thermodynamical description for such a system. In our case the "thermodynamic" *macroscopic* description is not yet consistent. A first approach is the formalism presented in chapter 4. We use the limit $N \to \infty$ to get a "sharp" theory, i.e. to gain a perfectly fulfilled absorber condition and zero probability for the observation of negative energy states.

In order to understand the connection of the microscopic to the macroscopic theory it should be worked out how to perform the limit $N \to \infty$ explicitly (in analogy to the justification of the thermodynamical laws by statistical mechanics). It is clear that this has to be a scaling limit and thus it is not surprising that the formalism of renormalization enters into the standard description of quantum electrodynamics. One has to perform this limit in exactly the right way to arrive at meaningful results (like it is the case in statistical mechanics).

# Chapter 3

# The Limit $N \to \infty$

The crucial exercise would now be to construct the limit $N \to \infty$ explicitly. This cannot be done correctly yet. Here I rather want to point out the consequences of this limit. In section 3.1, Dirac seas will explicitly be constructed as infinite alternating product states. Then, in section 3.2, the problems that arise from this limit are discussed. Before I start to explain the effective formalism for $N \to \infty$, I briefly review some features of the standard procedure of second quantization in section 3.3.

## 3.1 Construction of Dirac Sea States II

Dirac's original idea was to construct "sea" states as infinite wedge products (see [3]). First of all this is a straightforward generalization of section 2.3.1 but it leads to some important consequences discussed in section 3.2.

Let $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$ (or any suitable complex separable Hilbert space) with an orthonormal basis $\varphi = (\varphi_j)_{j\in\mathbb{Z}}$. Then we can define an infinite form as

$$\psi = \varphi_{j_1} \wedge \varphi_{j_2} \wedge \ldots = \bigwedge_{n\in\mathbb{N}} \varphi_{j_n}, \tag{3.1}$$

with $(j_i)_{i\in\mathbb{N}}$ a (strictly) increasing sequence for which $j_{n+1} = j_n + 1$ holds for suitably large $n$. That means that only finitely many $\varphi_j$ with $j < 0$ and all except for finitely many $\varphi_j$ with $j \geq 0$ occur. The charge is then defined as $c = n - j_n$ (for suitably large $n$). To be concrete we define the following notation. For a given basis $\varphi$ of the Hilbert space $\mathcal{H}$, we can define the splitting $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$ in that way that $\mathcal{H}_+$ is the closed subspace generated by $\{\varphi_j|\ j < 0\} := \{\varphi_j^+|\ j > 0\}$ and $\mathcal{H}_-$ the one generated by $\{\varphi_j|\ j \geq 0\} := \{\varphi_j^-|\ j \geq 0\}$.

Let's give an example. We can write a vacuum state as $\Omega = \varphi_0^- \wedge \varphi_1^- \wedge \ldots$. Then a state with k electron-positron pairs is for example $\Phi = \varphi_1^+ \wedge \ldots \wedge \varphi_k^+ \wedge \varphi_k^- \wedge \varphi_{k+1}^- \wedge \ldots$. For this $\Phi$ the charge is defined as $c = k - k = 0$.

The definition of the charge for $N \to \infty$ is thus the same as for finite N. The charge plays an important role as indeed the Dirac time evolution conserves the charge. This has to be kept in mind when defining a suitable space for the Dirac time evolution in chapter 4.2.

## 3.2 What happens in the Limit $N \to \infty$?

Which problems can arise in the limit $N \to \infty$?

First of all it is not clear how Dirac seas (e.g. written down in a different basis) can be compared. A scalar product like (2.22) is in general not well-defined (it could diverge the greater N gets). As mentioned before the idea is to look for interesting physics only at the "surface" of the sea and to ignore what goes on "deep down" in the sea hoping that nothing physical relevant happens there. What we thus need is a notion of stating that two Dirac seas are equal "deep down" in the sea. We will gather all those seas in equivalence classes defined by the $\sim$ equivalence relation.

Something else is needed. The problem is that a splitting of the Hilbert space into a positive and negative spectral subspace (which will be called a polarization) is not clear in the presence of external fields. The first guess would be to take $\mathcal{H} = U^A(t_1,t_0)\mathcal{H}_- \oplus U^A(t_1,t_0)\mathcal{H}_+$. I already mentioned that a correct distinction between sea-particles and observable particles may only be possible if a correct "absorber" condition is specified. As this isn't clear for the finite case it also won't be clear for $N \to \infty$. But in this limit something else can happen. In principle it is possible that infinitely many particles are created out of the sea. That we do have to exclude. In fact this happens as soon as a magnetic field enters the Dirac Hamiltonian (the problem of the external field). We thus have to make sure that the non-diagonal parts of the unitary time evolution $U^A_{\pm\mp}$ behave in such a way that the transition amplitudes $|\langle\Psi, \tilde{U}^A\Phi\rangle|^2$ remain finite. In particular the probability to create a pair from the Dirac sea must be finite, i.e.

$$\sum_{n\in\mathbb{N}} ||U^A_{+-}(t_1,t_0)\varphi_n||^2 < \infty \tag{3.2}$$

($(\varphi_n)_{n\in\mathbb{N}}$ an ONB of $\mathcal{H}_-$). Mathematically the above expression equals the Hilbert-Schmidt norm of $U^A_{+-}$. Therefore a lift from the one-particle Hilbert space to an infinite particle Hilbert space (Fock space or infinite wedge space) is only possible if the non-diagonal parts of $U^A$ are Hilbert-Schmidt operators (have a finite Hilbert-Schmidt norm, i.e. a square integrable kernel). This result is known as the Shale-Stinespring condition. According to Ruijsenaars it is fulfilled if and only if the magnetic component of the four-vector field $A$ vanishes. This was a catastrophic result. Usually one says that infinitely many virtual pairs are created. From our point of view it is somehow understandable that this could happen.

To see how this problem can be approached, recall the example from chapter 2.4.2. Suppose that at time $t_0$ there is no external field, i.e. the splitting $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$ is defined with the projectors from equation (2.14). Then an external field $A$ is switched on and we consider the situation at time $t$. A first guess for the negative spectral subspace at time $t$ would be $U^A(t,t_0)\mathcal{H}_-$. But now consider another field $\tilde{A}$ with the property that $\tilde{A}(t_0) = 0 = A(t_0)$ and $\tilde{A}(t) = A(t)$. Then the projectors onto the subspaces $U^A(t,t_0)\mathcal{H}_-$ and $U^{\tilde{A}}(t,t_0)\mathcal{H}_-$ are in general not the same. Of course they can't be the same as a lot of different physics can have happened in between. So the choice $U^A(t,t_0)\mathcal{H}_-$ does not only depend on the times $t_0$ and $t$ but also on the whole history of the field. This may in particular be a problem when considering gauge transformations of the field $A$ which shouldn't change the physics. What one finds is that the difference of the orthogonal projectors onto $U^A(t,t_0)\mathcal{H}_-$ and $U^{\tilde{A}}(t,t_0)\mathcal{H}_-$[1] differ only by a Hilbert-Schmidt operator. Therefore first one considers only classes of polarizations for the time evolution. The polarization classes are defined by the property that the difference of the orthogonal projectors is a Hilbert-Schmidt operator. One also finds a simple representative $e^{Q^{A(t)}}$ for the time evolution between polarization classes such that one can set $\mathcal{H}_-^{A(t)} = e^{Q^{A(t)}}\mathcal{H}_-$, i.e. the orthogonal projector onto that subspace differs from the projectors from above only by a Hilbert-Schmidt operator. Note that $e^{Q^{A(t)}}$ depends only on

[1]Explicitly: $P_-^{U^A(t,t_0)} = U^A(t,t_0)P_-U^A(t_0,t)$ and $P_-^{U^{\tilde{A}}(t,t_0)} = U^{\tilde{A}}(t,t_0)P_-U^{\tilde{A}}(t_0,t)$.

the time $t$ and not on the whole history, therefore it is much simpler to handle, especially later in proofs (see chapter 4.5, furthermore $Q^{A(t)}$ is linear in $A$). To summarize, setting

$$P_{\pm}^{A(t)} = e^{Q^{A(t)}} P_{\pm} e^{-Q^{A(t)}} \tag{3.3}$$

one finds the Shale-Stinespring condition fulfilled. That is, the non-diagonal parts of the time evolution operator $U^A(t, t_0)$ which in this setting are $P_{\pm}^{A(t)} U^A(t, t_0) P_{\mp}$, are indeed Hilbert-Schmidt operators.

To summarize, we have to ensure two things:

- The time evolution may only go from a space associated with certain polarizations to another space associated with certain polarizations which expresses that at most finitely many particles have been created. This will lead to equivalence classes of polarizations defined by the $\approx$ equivalence relation. The relation $\approx$ is defined with the help of the Hilbert-Schmidt norm. We define an operator (the left-operation) that takes this into account.

- Finite transition amplitudes between two states can only be calculated if the two corresponding Dirac seas are equal "deep down". This is expressed via the $\sim$ equivalence relation and leads to the definition of the right-operation.

Again, compared to the case of an N-particle QED the problem of defining proper polarizations persists but is in a way shifted to the problem of specifying an "absorber" condition, whereas the second problem of comparing Dirac seas and defining finite transition amplitudes does not occur there.

We also must take into account the fact that the Dirac time evolution conserves the charge, i.e. only electron-positron *pairs* (with zero total electrical charge) can be created. Thus the relative charge between the original and the time developed state must remain zero which leads to a finer distinction with the $\approx_0$ equivalence relation.

So now it is clear what has to be done in order to construct a well-defined Dirac time evolution for the $(N \to \infty)$-theory:

- The space of all possible polarizations and the space of all possible Dirac seas have to be ordered into the equivalence classes as mentioned above.

- With this in mind a suitable space for states describing infinitely many particles has to be constructed which will be the infinite wedge space.

- Then we can define operations on this space which are the operations from the left and from the right.

- With this at hand we investigate the conditions under which a lift of the one-particle time evolution is possible and it is shown that the Dirac time evolution satisfies these conditions.

- As mentioned above the time evolution will take place between spaces each associated with a certain class of polarizations. Such a time evolution would still be a purely mathematical construction if one could not identify the polarization classes with anything physical. It can be shown that indeed the polarization classes are uniquely determined by the magnetic component of the four-vector field $A$.

## 3.3 Standard Quantization of the Dirac Field

Before I go on describing this formalism for the Dirac time evolution with external field, I briefly want to recall some features of the procedure of second quantization of the Dirac equation. This standard procedure is described in any textbook on quantum field theory. Here I want to show how it is dealt with negative energy states and how the view of what the vacuum is changes.

One begins with introducing creation and annihilation operators $a_\varphi^{(\dagger)}$ and $b_\varphi^{(\dagger)}$ that fulfill certain anti-commutation relations. One also introduces a vacuum state $|0\rangle$. There are two definitions of how the operators act on the vacuum state depending on the conception of what the vacuum is.

(a) One usually begins with the view that the vacuum is made up of infinitely many particles. Then $a_\varphi^\dagger$ ($a_\varphi$) creates (annihilates) a positive energy particle. Accordingly $b_\varphi^\dagger$ ($b_\varphi$) creates (annihilates) a negative energy particle. Thus $b_\varphi|0\rangle =$"vacuum without one particle" and $b_\varphi^\dagger|0\rangle = 0$, as all negative energy states are occupied (no more can be created). Hence we have

$$a_\varphi|0\rangle = 0, \quad a_\varphi^\dagger|0\rangle = |\varphi\rangle, \quad b_\varphi|0\rangle \neq 0, \quad b_\varphi^\dagger|0\rangle = 0. \tag{3.4}$$

(b) Then one changes the perspective. One wants to get rid of negative energy states and the "hole" theory. One wants a vacuum that is free of particles. Therefore the operators $b_\varphi^{(\dagger)}$ are redefined. One introduces operators $d_\varphi^\dagger$ ($d_\varphi$) that create (annihilate) an anti-particle, i.e. $d_\varphi|0\rangle = 0$ as there are no anti-particles in the vacuum and $d_\varphi^\dagger|0\rangle = |\varphi\rangle$ (one anti-particle is created). Hence we have

$$a_\varphi|0\rangle = 0, \quad a_\varphi^\dagger|0\rangle = |\varphi\rangle, \quad d_\varphi|0\rangle = 0, \quad d_\varphi^\dagger|0\rangle = |\varphi\rangle. \tag{3.5}$$

The "trick" of redefining the operators in this way gives us a vacuum that is the state of lowest energy. Thus one might say that there is no longer a problem with negative energy states. But by this procedure the real problem is concealed. The vacuum causes a lot of problems. We end up with a theory full of divergences. There is the Shale-Stinespring condition which makes a second quantization of the Dirac field impossible in general. Only if one takes the Dirac sea serious one understands where the problems come from. The vacuum is definitely more than just "nothing". This perception is formulated mathematically precise in the next chapter.

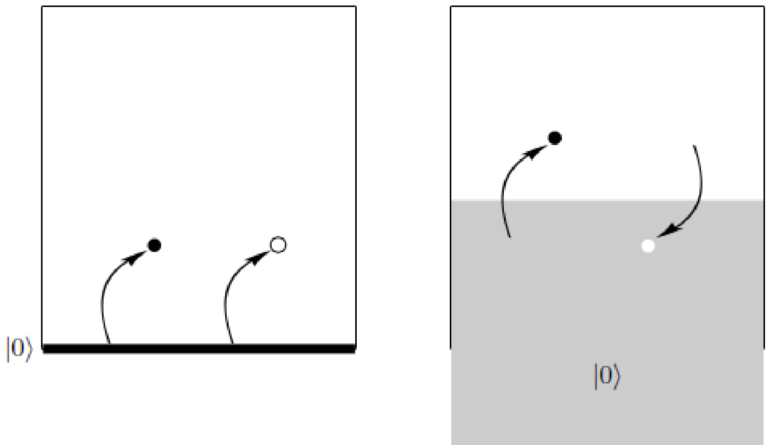


Figure 3.1: The two conceptions: on the left the vacuum is empty, on the right the vacuum is full of particles. The arrows should illustrate the action of the creation and annihilation operators (left: definition (b), right: definition (a)).

# Chapter 4

# A Setup for the Second Quantized Time Evolution

## 4.1 Polarization and Dirac Sea Classes

Let's begin to introduce the notions of polarization and Dirac sea equivalence classes properly. For a deeper mathematical understanding and rigorous proofs I refer the interested reader to the original paper [1]. I tried to present the results always with the physics in mind. Nevertheless most part of this chapter will be very technical. The reader who is not that interested in the mathematics should at first only read this section and then go on with reading the summary in section 4.7.

In the following $\mathcal{H}$ and $\ell$ (and also $\mathcal{H}'$, $\ell'$) are infinite dimensional, separable, complex Hilbert spaces. $\mathcal{H}$ will be the one-particle Hilbert space ($L^2(\mathbb{R}^3, \mathbb{C}^4)$) equipped with a scalar product $\langle \cdot, \cdot \rangle$ and $\ell$ will play the role of an index space (one can e.g. think of $\ell^2(\mathbb{N})$). In order to define the equivalence classes we need to introduce two important types of operators: Trace class and Hilbert-Schmidt operators. The space of trace class operators is

$$I_1(\ell) := \{T : \ell \to \ell,\ T \text{ bounded and } ||T||_{I_1} < \infty\} \qquad \text{with } ||T||_{I_1} := tr\sqrt{T^*T} \tag{4.1}$$

($T^*$ denoting the Hilbert space adjoint). For (bounded linear) operators that differ from the identity only by a trace class operator one can define a determinant (similar to (2.22)):

$$det(A) := \lim_{n\to\infty} det(A_{i,j})_{i,j=1,\dots,n} \qquad for\ A \in id_\ell + I_1(\ell). \tag{4.2}$$

The space of Hilbert-Schmidt operators is defined as

$$I_2(\ell, \mathcal{H}) := \{T : \ell \to \mathcal{H},\ T \text{ bounded and } ||T||_{I_2} < \infty\} \qquad \text{with } ||T||_{I_2} := \sqrt{trT^*T}. \tag{4.3}$$

We abbreviate $I_2(\mathcal{H}, \mathcal{H}) = I_2(\mathcal{H})$. The set of all unitary operators from $\mathcal{H}$ to $\mathcal{H}'$ will be denoted by $U(\mathcal{H}, \mathcal{H}')$, where again $U(\mathcal{H}) := U(\mathcal{H}, \mathcal{H})$.

Now we introduce the notion of polarizations and polarization classes. A polarization denotes the concrete splitting of the one-particle Hilbert space into a positive and a negative spectral subspace. The most general condition is that both subspaces are infinite dimensional.

**Definition 4.1** (Polarizations and Polarization Classes). *(a) The set of all polarizations is $Pol(\mathcal{H}) = \{V \subset \mathcal{H},\ V \text{ closed subspace and } dim(V) = \infty,\ dim(V^\perp) = \infty\}$. The orthogonal projection of $\mathcal{H}$ onto a polarization $V \in Pol(\mathcal{H})$ is denoted by $P_V : \mathcal{H} \to V$.*

(b) *For two polarizations $V, W \in Pol(\mathcal{H})$, the relation $\approx$ is defined by: $V \approx W \ :\Leftrightarrow\ P_V - P_W \in I_2(\mathcal{H})$.*

Indeed $\approx$ is an equivalence relation on $Pol(\mathcal{H})$. A polarization class is denoted by $C \in Pol(\mathcal{H})/\approx\ \subset Pol(\mathcal{H})$. Next the relative charge is defined in order to get a finer classification of polarization classes. This is necessary because as mentioned above, the Dirac time evolution conserves the total charge (only pairs with zero total electrical charge can be created). In a mathematically correct way the charge is defined by using the Fredholm index of operators. First note (without proof) that from the above definition one can show that $V \approx W$ is equivalent to the statement that $P_W|_{V\to W}$ is a Fredholm operator, where $|_{V\to W}$ means the restriction to the map $V \to W$.

**Definition 4.2** (Relative Charge)**.** *For $V, W \in C$ (i.e. $V \approx W$) the relative charge is defined as the Fredholm index of $P_W|_{V\to W}$, i.e.*

$$charge(V, W) = ind(P_W|_{V\to W}) = dim(ker(P_W|_{V\to W})) - dim(ker(P_W|_{V\to W})^*). \tag{4.4}$$

This charge definition has some desired properties which are collected in

**Lemma 4.3** (Properties of the Relative Charge)**.** *For $C \in Pol(\mathcal{H})/\approx$ and $V, W, X \in C$ the following properties hold:*

(a) *$charge(V, W) = -charge(W, V)$*

(b) *$charge(V, W) + charge(W, X) = charge(V, X)$*

(c) *If $U \in U(\mathcal{H}, \mathcal{H}')$ then $charge(V, W) = charge(UV, UW)$.*

(d) *If $U \in U(\mathcal{H})$ such that $UC = C$ then $charge(V, UV) = charge(W, UW)$.*

With this at hand a more useful relation can be defined which again is indeed an equivalence relation (because of the above properties).

**Definition 4.4** (Equal Charge Equivalence Classes)**.** *For $V, W \in Pol(\mathcal{H})$ the relation $\approx_0$ is defined by: $V \approx_0 W \ :\Leftrightarrow\ V \approx W$ and $charge(V, W) = 0$.*

So far, note how this formalism grasps the physics we want to describe. If we choose a specific polarization $V \in Pol(\mathcal{H})$ we clearly define where we cut our spectrum into parts (which particles belong to the Dirac sea). How can another $W \approx_0 V$ be understood? We demanded the minimal requirements $P_V - P_W \in I_2(\mathcal{H})$ and $charge(V, W) = 0$. Morally, the first requirement ensures that a state that belongs to $W$ differs from the one that belongs to $V$ only by *finitely many* particles, whereas the second guarantees that only *pairs* have been created (or annihilated).

Next we introduce Dirac seas in a handy way. Like in the finite case we encode an orthonormal basis in a map $\Phi$. The rigorous definition is the following:

**Definition 4.5** (Dirac seas)**.** (a) *Let $Seas_\ell(\mathcal{H}) = Seas(\mathcal{H})$ denote the set of all linear bounded operators $\Phi : \ell \to \mathcal{H}$, such that $range\Phi \in Pol(\mathcal{H})$ and $\Phi^*\Phi \in id_\ell + I_1(\ell)$, i.e. $\Phi^*\Phi : \ell \to \ell$ has a determinant.*

(b) *Let $Seas^\perp_\ell(\mathcal{H}) = Seas^\perp(\mathcal{H}) := \{\Phi \mid \Phi \in Seas(\mathcal{H})$ and $\Phi$ a linear isometry$\}$*

As example consider $\ell = \ell^2(\mathbb{N})$ with the canonical basis $(e_n)_{n\in\mathbb{N}}$. Then an orthonormal basis $(\varphi_n)_{n\in\mathbb{N}}$ of $V$ (take range$\Phi = V$) is encoded in the map $\Phi$ by setting $\Phi e_n = \varphi_n$ for all $n \in \mathbb{N}$. To identify Dirac seas that belong to a certain polarization class we define sets called *Oceans*:

**Definition 4.6** (Dirac Sea Oceans)**.** *Let $C \in Pol(\mathcal{H})/\approx_0$. Then $Ocean_\ell(C) = Ocean(C) := \{\Phi \mid \Phi \in Seas^\perp(\mathcal{H})$ and $range\Phi \in C\}$.*

All seas $\Phi \in Oceans(C)$ thus lead to polarizations in the same class. We also need a way to say which Dirac seas are "comparable" in the sense that a scalar product between two (sea) states is defined. We saw in chapter 2.3.1 that this is done via determinants. That's why we define the following splitting of the set of all Dirac seas.

**Definition 4.7** (Dirac Sea Equivalence Classes)**.** *For $\Phi, \Psi \in Seas(\mathcal{H})$, the relation $\sim$ is defined by: $\Phi \sim \Psi \;:\Leftrightarrow\; \Phi^*\Psi \in id_\ell + I_1(\ell)$, i.e. $\Phi^*\Psi$ has a determinant.*

Again it can be shown that $\sim$ is indeed an equivalence relation. We denote by $S(\Phi) \subset Seas(\mathcal{H})$ the equivalence class of $\Phi$ with respect to $\sim$, i.e. $S(\Phi) = [\Phi]_\sim$. In the above definitions one may wonder about the difference between $Seas^\perp(\mathcal{H})$ and $Seas(\mathcal{H})$. In fact in most cases one can work with Dirac seas in $Seas^\perp(\mathcal{H})$.[1] We introduced $Seas^\perp(\mathcal{H})$ (only isometries) as we want to work with orthonormal bases. It may also be interesting to note that $S(\Phi)$ is an affine space.[2]

To summarize, an $S(\Phi)$ contains all the Dirac seas which are "comparable" or "equal deep down in the sea". In a way it would be more suggestive to think of an $S(\Phi)$ as being *one* Dirac sea while the different $\Psi \sim \Phi$ are only varying "modes" (or "moods") of the same sea. Every $\Psi \sim \Phi$ can represent a different physical state. E.g. $\Psi$ may be a state with n electron-positron pairs while $\Phi$ is a state with no particles at all, but in such a way that the underlying Dirac seas are equal "deep down".

What is meant by "equal deep down in the sea" may be depicted by representing the Dirac sea operators as infinite dimensional matrices. Consider two Dirac seas $\Phi$ and $\Psi \sim \Phi$, i.e. $\Phi^*\Psi \in id_\ell + I_1(\ell)$ or $\Phi^*\Psi$ has a determinant. Then one can morally think of $\Phi^*\Psi$ as being "nearly" the identity except for a small $I_1$ "perturbation":

$$\Phi^*\Psi \text{ ``}{=}\text{''} \begin{pmatrix} 1 & & \overline{\underline{|I_1|}} & \\ & 1 & & \\ & & 1 & 0 \\ & & 0 & \ddots \end{pmatrix}. \tag{4.5}$$

The following argument shows how much two Dirac seas in the same equivalence class may differ. Consider $\Psi \sim \Phi$, i.e. $\Phi^*\Phi \in id + I_1$ and $\Phi^*\Psi \in id + I_1$ (or $\Psi^*\Phi \in id + I_1$). This implies (subtracting both conditions) that $\Phi^*(\Phi - \Psi) \in I_1$ and also $(\Phi - \Psi)^*\Phi \in I_1$. Therefore one also has $(\Phi - \Psi)^*(\Phi - \Psi) \in I_1$. Now observe that for $T^*T \in I_1$ one has

$$||T^*T||_{I_1} = tr\sqrt{(T^*T)^*T^*T} = tr\sqrt{T^*TT^*T} = trT^*T = ||T||^2_{I_2}. \tag{4.6}$$

That means $\Phi - \Psi \in I_2$, i.e. two Dirac seas which are equal "deep down" differ only by an $I_2$ "perturbation". Morally, this "perturbation" represents finitely many particles.

[1] The exact statement is that for every $\Psi \in Seas(\mathcal{H})$ there exist $\Upsilon \in Seas^\perp(\mathcal{H})$ and $R \in id_\ell + I_1(\ell)$ such that $\Psi = \Upsilon R$, $\Upsilon^*\Psi = R \geq 0$, $\Upsilon \sim \Psi$ and $R^2 = \Psi^*\Psi$.

[2] Affine space means that $S(\Phi) = \Phi + \mathcal{V}(\Phi)$ and for a $\Psi \sim \Phi$ one has $\mathcal{V}(\Phi) = \mathcal{V}(\Psi)$. The vector space $\mathcal{V}(\Phi)$ is defined as $\mathcal{V}(\Phi) := \{L : \ell \to \mathcal{H} \mid L \textit{ linear and bounded with } ||L||_\Phi := ||\Phi^*L||_{I_1} + ||L||_{I_2} < \infty\}$.

One may wonder if different Dirac sea classes that belong to the same Ocean differ so much. Luckily this is not so. One has the important statement that with every $S(\Phi)$ one can "reach" the whole polarization class $C$.

**Theorem 4.8** (Connection of $\sim$ and $\approx_0$). *For $C \in Pol(\mathcal{H})/\approx_0$ and $\Phi \in Ocean(C)$ we have*

$$\begin{aligned} C &= \{range\Psi \mid \Psi \in Seas^{\perp}(\mathcal{H}) \text{ such that } \Psi \sim \Phi\} \\ &= \{range\Psi \mid \Psi \in S(\Phi) \ \cap \ Seas^{\perp}(\mathcal{H})\}. \end{aligned} \tag{4.7}$$

Therefore it is "enough" to work with one $S$ out of $Ocean(C)$. One particular $S$ is rather a coordinate choice. The situation is depicted in Figure 4.1.

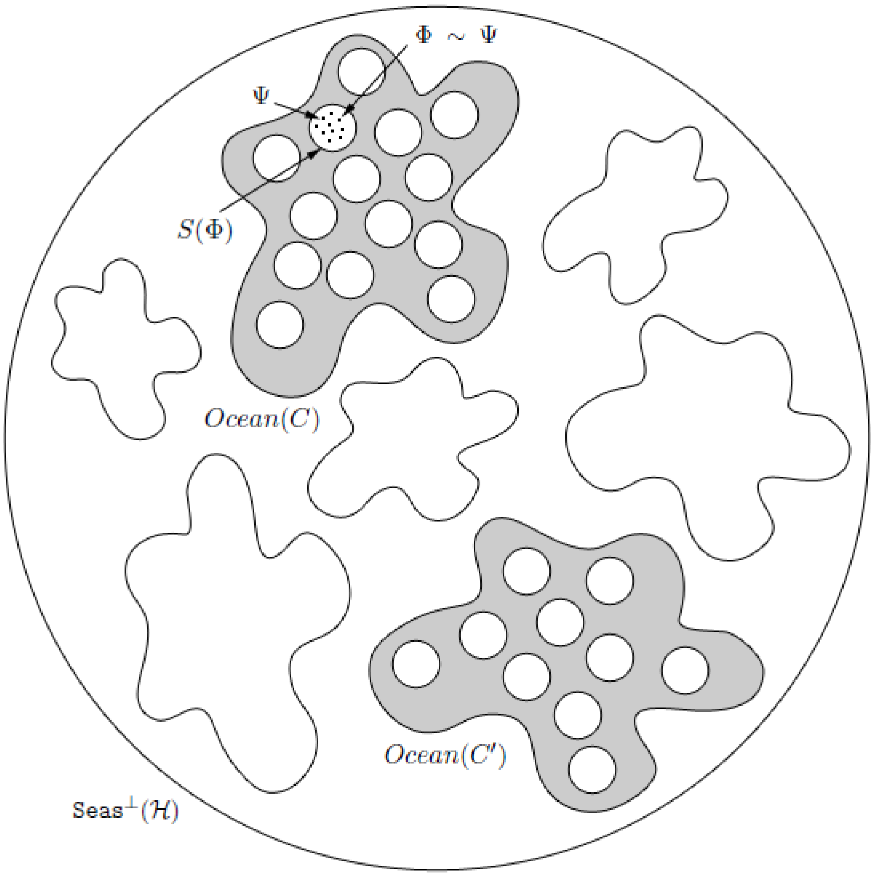


Figure 4.1: Seas that belong to one polarization class are gathered in $Ocean(C)$. One $Ocean(C)$ is made up of many Dirac sea equivalence classes $S$. One $S$ contains all Dirac seas which are "equal deep down".

Let's summarize the setup again. We have the one-particle Hilbert space $\mathcal{H}$, consider all polarizations $Pol(\mathcal{H})$ and order this set into equivalence classes $C \in Pol(\mathcal{H})/\approx_0$. The equivalence class is given *by nature* as it is shown in chapter 4.6. Now we want to take bases in $C$, therefore we introduced $Ocean_\ell(C)$. Ordering this set into equivalence classes again, we choose an $S \in Ocean_\ell(C)/\sim$, which contains all the information about the whole polarization class $C$. Note that a specific $S$ is chosen *by human*.

Before we go on and describe time evolutions we need two things. We need a suitable space we can work with (e.g. on which we can calculate scalar products). This will be the Hilbert space $\mathcal{F}_S$ that belongs to an equivalence class $S$. Then we have to define operators on that space which will be the operations from the left and from the right.

## 4.2 Infinite Wedge Spaces

The construction of the Hilbert space $\mathcal{F}_S$ is only sketched here, the mathematical interested reader may rather take a look at the paper [1]. In a first reading the next two sections may also be skipped, although the infinite wedge spaces are the key object for the time evolution. The essence is that we build a Hilbert space $\mathcal{F}_S$ for each $S$. The procedure is well known from linear algebra. First we take formal finite linear combinations of elements from $S$, then a semi-norm is defined and this space is completed with respect to that semi-norm.

The infinite formal linear combinations and a sesquilinear form thereon are defined in the following way.

**Definition 4.9** (Formal Linear Combinations). *(a) Let $\mathbb{C}^{(S)}$ denote the set of all maps $\alpha : S \to \mathbb{C}$ such that $\{\Phi \in S \mid \alpha(\Phi) \neq 0\}$ is finite. Equivalently $\mathbb{C}^{(S)}$ is the set of all finite formal linear combinations $\alpha = \sum_{\Psi \in S} \alpha(\Psi)[\Psi]$ of elements of $S$ with coefficients in $\mathbb{C}$. Hereby $[\Phi] \in \mathbb{C}^{(S)}$ is defined to be the map for which $[\Phi](\Phi) = 1$ and $[\Phi](\Psi) = 0$ (for $\Phi \neq \Psi \in S$).*

*(b) For $S \in Seas(\mathcal{H})/\sim$ we define the map $\langle\cdot,\cdot\rangle : S \times S \to \mathbb{C},\ (\Phi, \Psi) \mapsto \langle\Phi, \Psi\rangle := det(\Phi^*\Psi)$. This map is well-defined since $\Phi \sim \Psi$ implies that $\Phi^*\Psi$ has a determinant.*

*(c) The sesquilinear extension of this map is*

$$\langle\cdot,\cdot\rangle : \mathbb{C}^{(S)} \times \mathbb{C}^{(S)} \to \mathbb{C},\ (\alpha, \beta) \mapsto \langle\alpha, \beta\rangle := \sum_{\Phi \in S} \sum_{\Psi \in S} \overline{\alpha(\Phi)}\beta(\Psi) det(\Phi^*\Psi).$$

*The bar denotes the complex conjugate. Note that the sums consist of finitely many elements and that $\langle[\Phi], [\Psi]\rangle = \langle\Phi, \Psi\rangle$.*

One finds that the sesquilinear form $\langle\cdot,\cdot\rangle : \mathbb{C}^{(S)} \times \mathbb{C}^{(S)} \to \mathbb{C}$ is hermitian and positive semi-definite. Therefore one can define the following semi-norm.

**Definition 4.10** (Semi-norm on $\mathbb{C}^{(S)}$). *The semi-norm on $\mathbb{C}^{(S)}$ induced by $\langle\cdot,\cdot\rangle$ is denoted by $||\cdot|| : \mathbb{C}^{(S)} \to \mathbb{R},\ \alpha \mapsto ||\alpha|| = \sqrt{\langle\alpha, \alpha\rangle}$.*

So far we only have a semi-norm at hand, i.e. $||\alpha|| = 0$ does not in general imply $\alpha = 0$. In fact the null space $N_S = \{\alpha \in \mathbb{C}^{(S)} \mid ||\alpha|| = 0\}$ is quite large. One has that for $\Phi \in S$ and $R \in id_\ell + I_1(\ell)$ also $\Phi R \in S$ and $[\Phi R] - det(R)[\Phi] \in N_S$. In order to define a Hilbert space the null space has to be factored out. This step is important as it is exactly this what will make our time evolution unique only up to a phase. The definition of the infinite wedge spaces on which we will define the time evolution is

**Definition 4.11** (Infinite Wedge Spaces). *The infinite wedge space $\mathcal{F}_S$ (over $S$) is defined as the completion of $\mathbb{C}^{(S)}$ with respect to the semi-norm $||\cdot||$. Let the canonical map be denoted by $i : \mathbb{C}^{(S)} \to \mathcal{F}_S$. Then the sesquilinear form $\langle\cdot,\cdot\rangle : \mathbb{C}^{(S)} \times \mathbb{C}^{(S)} \to \mathbb{C}$ induces the scalar product $\langle\cdot,\cdot\rangle : \mathcal{F}_S \times \mathcal{F}_S \to \mathbb{C}$. Let the canonical map from $S \to \mathcal{F}_S$ be denoted by $\wedge : S \to \mathcal{F}_S,\ \Phi \mapsto \wedge\Phi = i([\Phi])$.*

Now the null space is automatically factored out: $i[N_S] = \{0\}$, in fact even $ker(i) = N_S$. Note that therewith we have for $\Phi \in S$ and $R \in id_\ell + I_1(\ell)$ that $\wedge(\Phi R) = det(R) \wedge \Phi$. One also finds that $\mathcal{F}_S$ is a separable Hilbert space.

Later we will define the Dirac time evolution on the wedge spaces $\mathcal{F}_S$. Therefore we need the operations from the next section.

## 4.3 Left and Right Operations

We introduce two types of operations on $\mathcal{F}_S$: the operation from the left and from the right. Both operations are defined according to our constructions in the foregoing chapters in four steps: first as operators acting on $Seas_\ell(\mathcal{H})$, then on $Seas_\ell(\mathcal{H})/\sim$, then on $\mathbb{C}^{(S)}$ and finally on the infinite wedge spaces $\mathcal{F}_S$.

**Definition 4.12** (Left Operation). *(a) A unitary operation from the left acting on elements in $Seas_\ell(\mathcal{H})$ is well-defined:*

$$U(\mathcal{H}, \mathcal{H}') \times Seas_\ell(\mathcal{H}) \to Seas_\ell(\mathcal{H}'),\ (U, \Phi) \mapsto U\Phi.$$

*(b) This operation is compatible with equivalence classes. For $U \in U(\mathcal{H}, \mathcal{H}')$ and $\Phi, \Psi \in Seas_\ell(\mathcal{H})$ one has $\Phi \sim \Psi \ \Leftrightarrow\ U\Phi \sim U\Psi$. Thus for $S \in Seas_\ell(\mathcal{H})/\sim$ one has*

$$US = \{U\Phi \mid \Phi \in S\} \in Seas_\ell(\mathcal{H}').$$

*(c) The induced operation $\mathcal{L}_U : \mathbb{C}^{(S)} \to \mathbb{C}^{(US)}$ is an isometry. It is given by*

$$\mathcal{L}_U\left(\sum_{\Phi \in S} \alpha(\Phi)[\Phi]\right) = \sum_{\Phi \in S} \alpha(\Phi)[U\Phi].$$

*(d) This induces the unitary map $\mathcal{L}_U : \mathcal{F}_S \to \mathcal{F}_{US}$ given by*

$$\mathcal{L}_U(\wedge\Phi) = \wedge(U\Phi)$$

*for $\Phi \in S$. One has for $U \in U(\mathcal{H}, \mathcal{H}')$ and $V \in U(\mathcal{H}', \mathcal{H}'')$ that $\mathcal{L}_U \mathcal{L}_V = \mathcal{L}_{UV} : \mathcal{F}_S \to \mathcal{F}_{UVS}$.*

The last definition is the one we will need.

The operation from the right is defined analogously. In order to get some desired properties we consider the set $GL_-(\ell', \ell) = \{R : \ell' \to \ell \mid R\ linear,\ bounded,\ invertible\ and\ R^*R \in id_{\ell'} + I_1(\ell')\}$. We abbreviate $GL_-(\ell) := GL_-(\ell, \ell)$.

**Definition 4.13** (Right Operation). *(a) An operation from the right acting on elements in $Seas_\ell(\mathcal{H})$ is well-defined:*

$$Seas_\ell(\mathcal{H}) \times GL_-(\ell', \ell) \to Seas_{\ell'}(\mathcal{H}),\ (\Phi, R) \mapsto \Phi R.$$

*(b) This operation is compatible with equivalence classes. For $R \in GL_-(\ell', \ell)$ and $\Phi, \Psi \in Seas_\ell(\mathcal{H})$ one has $\Phi \sim \Psi \ \Leftrightarrow\ \Phi R \sim \Psi R$. Thus for $S \in Seas_\ell(\mathcal{H})/\sim$ one has*

$$SR = \{\Phi R \mid \Phi \in S\} \in Seas_{\ell'}(\mathcal{H}).$$

*(c) The induced operation $\mathcal{R}_R : \mathbb{C}^{(S)} \to \mathbb{C}^{(SR)}$ is an isometry up to scaling. It is given by*

$$\mathcal{R}_R\left(\sum_{\Phi\in S}\alpha(\Phi)[\Phi]\right) = \sum_{\Phi\in S}\alpha(\Phi)[\Phi R].$$

*More precisely one has*

$$\langle\mathcal{R}_R\alpha, \mathcal{R}_R\beta\rangle = det(R^*R)\langle\alpha,\beta\rangle.$$

*(d) This induces the bounded map $\mathcal{R}_R : \mathcal{F}_S \to \mathcal{F}_{SR}$ which is unitary up to scaling and given by*

$$\mathcal{R}_R(\wedge\Phi) = \wedge(\Phi R)$$

*for $\Phi \in S$. Again for $\Phi, \Psi \in S$:*

$$\langle\mathcal{R}_R\Phi, \mathcal{R}_R\Psi\rangle = det(R^*R)\langle\Phi,\Psi\rangle.$$

*One has for $R \in GL_-(\ell', \ell)$ and $Q \in GL_-(\ell'', \ell')$ that $\mathcal{R}_Q\mathcal{R}_R = \mathcal{R}_{RQ} : \mathcal{F}_S \to \mathcal{F}_{SRQ}$.*

From the simple associativity of composition one gets that the left and right operations commute:

$$\mathcal{L}_U\mathcal{R}_R = \mathcal{R}_R\mathcal{L}_U : \mathcal{F}_S \to \mathcal{F}_{USR}.$$

An important statement about the operations from the right which gives us the uniqueness up to a phase is

**Lemma 4.14** (Uniqueness up to a Phase). *(a) For all $R \in GL_-(\ell)$ and $S \in Seas_\ell(\mathcal{H})/\sim$ we have: $S = SR \Leftrightarrow R$ has a determinant. If $R$ has a determinant then $\mathcal{R}_R(\Psi) = (detR)\Psi$ for all $\Psi \in \mathcal{F}_S$.*

*(b) For all $Q, R \in GL_-(\ell', \ell)$ and $S \in Seas_\ell(\mathcal{H})/\sim$ we have: $SR = SQ \Leftrightarrow Q^{-1}R \in GL_-(\ell')$ has a determinant. Then $\mathcal{R}_R\Psi = det(Q^{-1}R)\mathcal{R}_Q\Psi$ for all $\Psi \in \mathcal{F}_S$.*

Note how strong this statement is. We have for every $R \in GL_-(\ell)$ that has a determinant that $\mathcal{F}_S = \mathcal{F}_{SR}$ and the action of the right operation $\mathcal{R}_R$ on an element $\Psi \in \mathcal{F}_S$ is just multiplication with a factor $det(R)$. With that at hand we are nearly finished to define the lift of the one-particle time evolution to a time evolution between two wedge spaces.

## 4.4 Lift Condition

We still have to investigate under which conditions one can lift a unitary operator between two (one-particle) Hilbert spaces to a unitary operator between two wedge spaces. Let us first consider the action of a unitary operator on polarization classes. This is essentially the same as in the case of Dirac sea equivalence classes.

**Lemma 4.15** (Action of U on Polarization Classes). *The following operation is well-defined:*

$$U(\mathcal{H},\mathcal{H}') \times Pol(\mathcal{H}) \to Pol(\mathcal{H}'),\ (U,V) \mapsto UV = \{Uv \mid v \in V\}.$$

*It is also compatible with the $\approx$ relation. For $U \in U(\mathcal{H},\mathcal{H}')$ and $V, W \in Pol(\mathcal{H})$ one has: $V \approx W \Leftrightarrow UV \approx UW$. Therefore the action of $U$ on polarization classes is straightforward:*

$$U(\mathcal{H},\mathcal{H}') \times Pol(\mathcal{H})/\approx \ \to\ Pol(\mathcal{H}')/\approx,\ (U,[V]_\approx) \mapsto [UV]_\approx.$$

With that at hand we can define a finer structure of the set of unitary operators which is the restricted set of unitary operators. These are all unitary operators that go from a certain polarization class to another polarization class and don't change the charge.

**Definition 4.16** (Restricted Set of Unitary Operators). *For two polarization classes $C \in Pol(\mathcal{H})/\approx_0$ and $C' \in Pol(\mathcal{H}')/\approx_0$ we define*

$$\begin{aligned} U^0_{res}(\mathcal{H}, C; \mathcal{H}', C') &= \{U \in U(\mathcal{H}, \mathcal{H}') \mid \textit{for all } V \in C \textit{ we have } UV \in C'\} \\ &= \{U \in U(\mathcal{H}, \mathcal{H}') \mid \textit{there exist } V \in C \textit{ such that } UV \in C'\}. \end{aligned}$$

*We abbreviate $U^0_{res}(\mathcal{H}, C) := U^0_{res}(\mathcal{H}, C; \mathcal{H}, C)$ (which is a group).*

Now we can state under which conditions a lift is possible. This is essentially the Shale-Stinespring theorem (see section 3.2) expressed in our formalism.

**Lemma 4.17** (Lift Condition). *For two polarization classes $C \in Pol(\mathcal{H})/\approx_0$ and $C' \in Pol(\mathcal{H}')/\approx_0$ let $S \in Ocean_\ell(C)/\sim$ and $S' \in Ocean_\ell(C')/\sim$ be two Dirac sea equivalence classes. Then for every $U \in U(\mathcal{H}, \mathcal{H}')$ we have that $U \in U^0_{res}(\mathcal{H}, C; \mathcal{H}', C')$ if and only if there is $R \in U(\ell)$ such that $USR = S'$ and therefore $\mathcal{R}_R\mathcal{L}_U : \mathcal{F}_S \to \mathcal{F}_{S'}$ (or equivalently there is an $R \in GL_-(\ell)$ such that $U\Phi R \sim \Phi'$).*

Note that this shows us what the distinction between different $S$ in the same Ocean is. Setting $U = id_{\mathcal{H}}$ we get

**Theorem 4.18** (Orbits in Ocean). *For $C \in Pol(\mathcal{H})/\approx_0$ and $S \in Ocean(C)/\sim$ we have*

$$Ocean(C)/\sim \;=\; \{SR \mid R \in U(\ell)\}. \tag{4.8}$$

Thus all the wedge spaces $\mathcal{F}_S$ such that $S \in Ocean(C)/\sim$ are related by unitary operations from the right. Indeed combining our results we have that, given two wedge spaces $\mathcal{F}_S$ and $\mathcal{F}_{S'}$, our lift is *unique* (except for a phase $e^{i\varphi}$).

**Lemma 4.19** (Uniqueness up to a Phase). *Let $U \in U^0_{res}(\mathcal{H}, C; \mathcal{H}', C')$ and $R \in U(\ell)$ such that $\mathcal{R}_R\mathcal{L}_U : \mathcal{F}_S \to \mathcal{F}_{S'}$. Then the set*

$$\{\mathcal{R}_Q\mathcal{R}_R\mathcal{L}_U \mid Q \in U(\ell) \cap (id_\ell + I_1(\ell))\} = \{e^{i\varphi}\, \mathcal{R}_R\mathcal{L}_U \mid \varphi \in \mathbb{R}\} \tag{4.9}$$

*contains* all *unitary maps from $\mathcal{F}_S$ to $\mathcal{F}_{S'}$ in the set $\{\mathcal{R}_T\mathcal{L}_U \mid T \in U(\ell)\}$.*

Note that if Q is unitary then $|det(Q)| = 1$, i.e. $det(Q) = e^{i\varphi}$. It must be emphasized how strong the statements 4.18 and 4.19 are. Recall why we made the whole construction. We wanted to compare different Dirac sea states, e.g. in order to calculate transition amplitudes given by scalar products. Suppose we describe an initial state in a wedge space $\mathcal{F}_S$. Then the time evolution acts and we want to compare the final state with another state, say in $\mathcal{F}_{S'}$. The time evolution will map the initial state to a state in $\mathcal{F}_{US}$ (which probably belongs to a different Ocean). So now we have two states we want to compare: one in $\mathcal{F}_{US}$ and one in $\mathcal{F}_{S'}$, which are both in the same Ocean. In general we will find that the scalar product between those states is not defined. This is clear because both Dirac seas are not in general equal "deep down". Now the above statement means that there is *exactly one* possibility to make the states comparable, except for a phase which doesn't play a role for transition amplitudes. Usually one would think that there are many different right operations that "translate" between the two states thus making the results of calculations arbitrary. Luckily this is not the case. The right operation is unique up to a phase. To summarize we have

$$\mathcal{F}_S \xrightarrow{\mathcal{L}_U} \mathcal{F}_{US} \xrightarrow{\mathcal{R}_R} \mathcal{F}_{USR} = \mathcal{F}_{S'} \circlearrowleft e^{i\varphi}. \tag{4.10}$$

The situation is depicted in Figure 4.2.

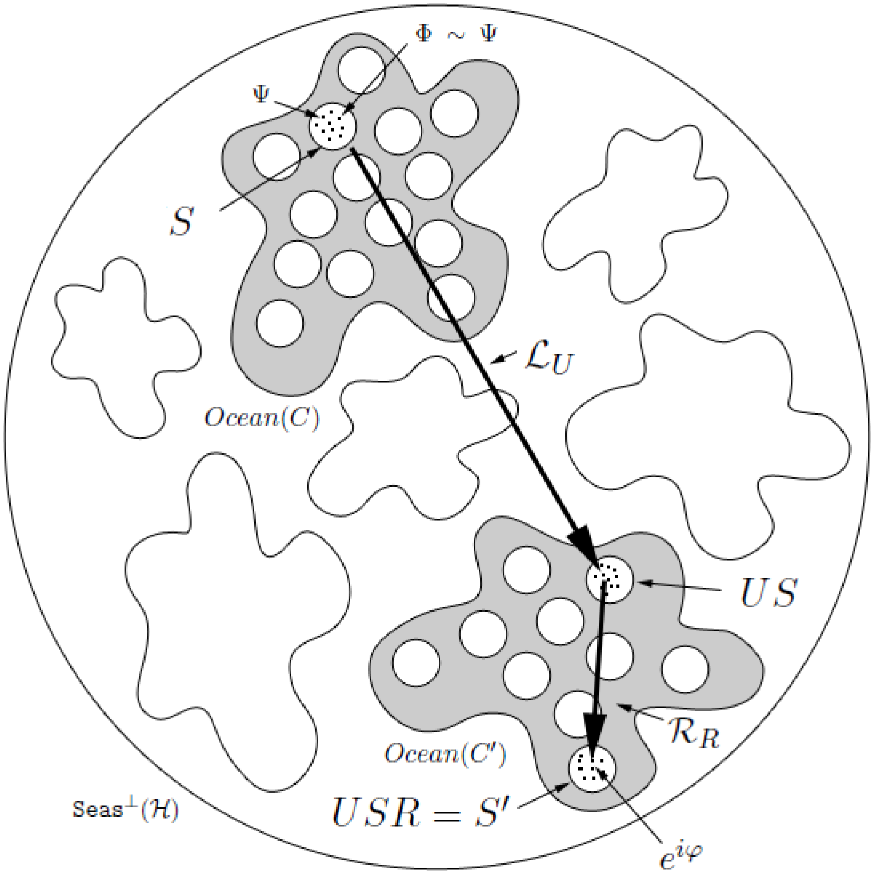


Figure 4.2: Illustration of the setup, with a slight abuse of notation: The maps $\mathcal{L}_U$ and $\mathcal{R}_R$ are in fact between wedge spaces like in equation (4.10). With a unitary right operation we can switch between any $S$ in a specific *Ocean*. If the $R$ from this right operation additionally has a determinant we dont't leave the class $S$.

## 4.5 Application to Dirac Time Evolution

What has been worked out so far would be an "empty" formalism if it could not be applied to the Dirac time evolution. Consider the one-particle Dirac time evolution from chapter 2.4.1. It is given by a unitary operator $U^A(t_1, t_0)$ such that $\psi(t_1) = U^A(t_1, t_0)\psi(t_0)$. We want to lift this operator to an operator acting between wedge spaces. According to Lemma (4.17) it has to be proven that $U^A(t_1, t_0) \in U^0_{res}(\mathcal{H}, C(t_0); \mathcal{H}, C(t_1))$ for two appropriate polarization classes $C(t_0)$ and $C(t_1)$. The rigorous proof can be found in [1]. It is a lengthy calculation one has to perform. Although this is in a way the most important part of this chapter I only want to state the theorem here. Again, the setup described so far would just be a mathematical gimmick if this theorem would not hold.

**Theorem 4.20** (Dirac Time Evolution with External Field)**.** *Let $\mathcal{C}^\infty_c(\mathbb{R}^4, \mathbb{R}^4)$ denote the set of infinitely often differentiable $\mathbb{R}^4$ valued functions on $\mathbb{R}^4$ with compact support. Let $A \in \mathcal{C}^\infty_c(\mathbb{R}^4, \mathbb{R}^4)$. Then for all $t_0, t_1 \in \mathbb{R}$ it holds that:*

$$U^A(t_1, t_0) \in U^0_{res}(\mathcal{H}, C(A(t_0)); \mathcal{H}, C(A(t_1))). \tag{4.11}$$

This theorem also holds for a greater class of four-vector potentials. However this class does not contain e.g. the Coulomb potential. The main ingredient in the proof is the operator $Q^{A(t)}$ (see also section 3.2). It is a simple representative with which we can define the time evolution of polarization classes. If $C(0) := [\mathcal{H}_-]_{\approx_0}$ denotes the polarization class that belongs to the negative spectral subspace of the free Dirac operator then the polarization class at time $t$ with field $A(t)$ is

$$C(A(t)) := e^{Q^{A(t)}} C(0) = \{e^{Q^{A(t)}} V \mid V \in C(0)\}. \tag{4.12}$$

## 4.6 Identification of Polarization Classes

The last important step to apply the setup to the Dirac time evolution is to identify the polarization classes. Indeed we have that the polarization classes are uniquely determined by the magnetic component $\boldsymbol{A}$ of the four-vector potential $A$.

**Theorem 4.21** (Identification of Polarization Classes)**.** *Let $A = (A^\mu)_{\mu=0,1,2,3} = (A^0, \boldsymbol{A})$ and $A, A' \in \mathcal{C}^\infty_c(\mathbb{R}^4, \mathbb{R}^4)$. Then*

$$C(A) = C(A') \;\Leftrightarrow\; \boldsymbol{A} = \boldsymbol{A'}. \tag{4.13}$$

Again the proof is a lengthy calculation and can be found in [1].

## 4.7 Summary: The Second Quantized Dirac Time Evolution

Now we summarize and concretely write down the second quantized Dirac time evolution. We consider the Dirac equation with an external four-vector potential $A \in \mathcal{C}^\infty_c(\mathbb{R}^4, \mathbb{R}^4)$. For any time $t$ we can determine the polarization class $C(t) \in Pol(\mathcal{H})/\approx_0$ uniquely by the magnetic component of $A$. Let $U^A(t_1, t_0)$ be the unitary one particle Dirac time evolution between times $t_0, t_1 \in \mathbb{R}$, such that $\psi(t_1) = U^A(t_1, t_0)\psi(t_0)$. For a chosen $\Phi \in Ocean(C(t_0))$ we have $S(t_0) = [\Phi]_\sim$ and $S(t_1) = [e^{Q^{A(t_1)}}\Phi]_\sim \in Ocean(C(t_1))$. As $U^A(t_1, t_0) \in U^0_{res}(\mathcal{H}, C(t_0); \mathcal{H}, C(t_1))$ we can define the second quantized Dirac time evolution as

$$\mathcal{R}_R \mathcal{L}_U : \mathcal{F}_{S(t_0)} \to \mathcal{F}_{S(t_1)} : \;\; \wedge \Psi_{t_0} \mapsto \wedge \Psi_{t_1} = \wedge(U \Psi_{t_0} R) \tag{4.14}$$

$(R \in U(\ell))$ which is unique up to a phase. This means that for two choices $R_1, R_2 \in U(\ell)$ with $\mathcal{R}_{R_1}\mathcal{L}_U = \mathcal{R}_{R_2}\mathcal{L}_U : \mathcal{F}_{S(t_0)} \to \mathcal{F}_{S(t_1)}$ we have

$$\mathcal{R}_{R_1}\mathcal{L}_U = e^{i\varphi}\mathcal{R}_{R_2}\mathcal{L}_U \tag{4.15}$$

with $e^{i\varphi} = det(R_1^{-1}R_2)$, $\varphi \in \mathbb{R}$.

## 4.8 Transition Amplitudes

A main application is the calculation of transition amplitudes, in particular the calculation of pair creation rates. Say we perform an experiment with the following setup. We prepare a chamber with a vacuum, i.e. we start with a state without particles. Then some external electric and magnetic fields are switched on, i.e. there may be pair creation and annihilation. At the end we measure how many particle pairs have been created. Say for simplicity that the external fields are switched off again at the end of the experiment at time $t_1$.[3]

The first question is how to describe the vacuum appropriately. We don't know which particular polarization we should choose, in fact we cannot know this without a specific "absorber" condition. We only know the polarization class $C$ that belongs to zero external magnetic field. That's all nature gives us. There is no distinguished vacuum state. One may depict this like in Figure 4.3.

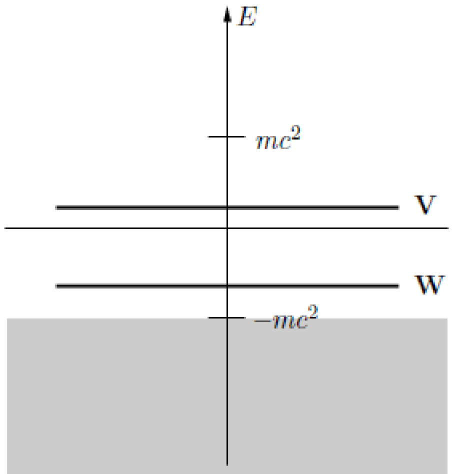


Figure 4.3: Attempt to draw the spectrum of the free Dirac operator. All states with energy less than $-mc^2$ are occupied but it is not clear where exactly the spectrum should be split into parts.

So we can choose one Dirac sea state $\Phi$ out of $Ocean(C)$ that we call a vacuum. Relative to this vacuum $\Phi$ we can write down how an N-pair state looks like:

$$\begin{aligned} \text{N-pair}(\Phi) \quad = \quad & \{\Psi \sim \Phi \mid \exists\, (e_n)_{n\in\mathbb{N}}\ ONB\ of\ \ell\ such\ that \\ & \Psi e_n \in range\Phi^{\perp}\ for\ 1 \le n \le N\ and \\ & \Psi e_n \in range\Phi\ for\ n > N\}. \end{aligned} \tag{4.16}$$

Note how this definition is independent of our particular choice of basis. One finds that for all unitary operators $R \in U(\ell)$ it is true that

$$\text{N-pair}(\Phi)R = \text{N-pair}(\Phi R), \tag{4.17}$$

[3]This is just to simplify the arguments in this section. The strength of the developed formalism is of course that it is applicable to situations with any external field.

i.e. the right operation doesn't change the physics we want to describe. Thus we may choose the final states we want to compare with in any appropriate basis. A more complete picture of our setup is Figure 4.4.

In general transition amplitudes are calculated in the following way. Take a state $\wedge\Psi^{in} \in \mathcal{F}_{S(t_0)}$ and $\wedge\Psi^{out} \in \mathcal{F}_{S(t_1)}$. Then the transition amplitude is given by

$$\begin{aligned} |\langle \wedge\Psi^{out}, \mathcal{R}_{R_1}\mathcal{L}_U \wedge \Psi^{in}\rangle|^2 &= |e^{i\varphi}|^2 |\langle \wedge\Psi^{out}, \mathcal{R}_{R_2}\mathcal{L}_U \wedge \Psi^{in}\rangle|^2 \\ &= |\langle \wedge\Psi^{out}, \mathcal{R}_{R_2}\mathcal{L}_U \wedge \Psi^{in}\rangle|^2. \end{aligned} \tag{4.18}$$

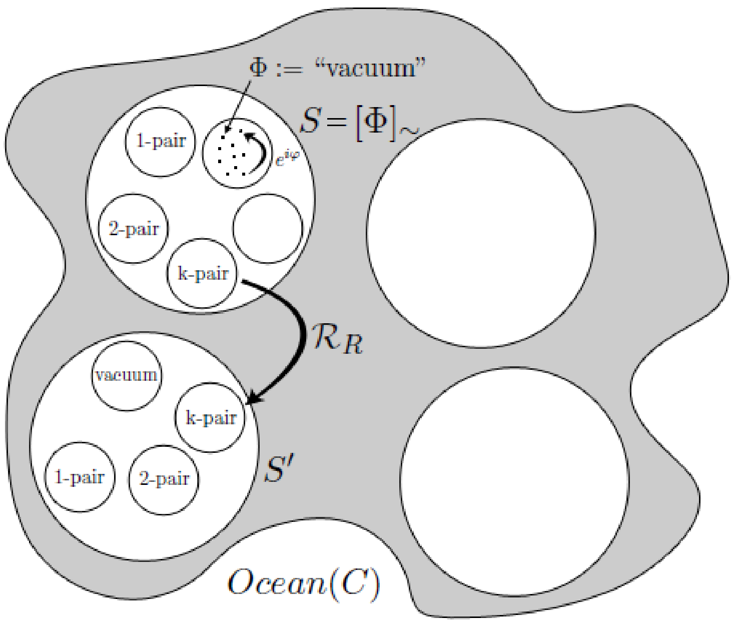


Figure 4.4: Every $S \in Ocean(C)$ is split into N-pair sectors which are defined relative to a specific choice of a vacuum state. The right operation "translates" between different bases and does not change the number of pairs. Within one N-pair sector we have the freedom to choose a phase factor $e^{i\varphi}$.

# Chapter 5

# Conclusion and Outlook

Quantum field theory is plagued by many problems and divergences. In this work I tried to point out where the problems come from and how they can be resolved. *It is not the case* that the Dirac sea is an out-dated idea and that everything becomes clear with creation and annihilation operators and a vacuum vector. This only hides the source of the problems, which is the vacuum. One should try to get rid of the divergences not with mathematical tricks but rather with physical intuition. This is possible like the setup of chapter 4 shows. I tried to reveal how much physics is behind the formalism of quantum electrodynamics and especially behind the notion of the "vacuum".

Next it would be interesting to see how the setup of chapter 4 can be applied concretely, e.g. to calculate pair creation rates. Furthermore this is only the first step towards a well-defined theory of quantum electrodynamics. It is for example not clear if the current can be defined without divergences in this setup.

**Acknowledgments:** I would like to thank Dirk Deckert, who was the best advisor I can imagine, and Detlef Dürr, who invested so much time in me. Without him I hadn't understood anything in physics.

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# Selbstständigkeitserklärung

Hiermit erkläre ich, die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet zu haben.

München, den 30. Juni 2009

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