Lorentz invariant quantum dynamics in the multi-time formalism

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Hiermit erkläre ich an Eides statt, dass die Dissertation von mir selbstständig ohne unerlaubte Beihilfe angefertigt worden ist.
Die Abschnitte 1.3.5, 2.2, 4.2 und 4.3 enthalten Ergebnisse, die aus der Zusammenarbeit mit Mitautoren hervorgegangen sind. Verweise auf die entsprechenden Veröffentlichungen sind in den jeweiligen Abschnitten zu finden.

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Summary

The central idea of the multi-time formalism, first suggested by Dirac in 1932, is that relativistic quantum mechanics should be built around a wave function $\psi(x_1, ..., x_N)$ (for N particles) on space-time configurations $(x_1, ..., x_N)$, $x_i \in \mathbb{R}^4$ instead of on spatial configurations $(\mathbf{x}_1, ..., \mathbf{x}_N; t)$ at a common time t in a distinguished frame. This view has far-reaching consequences for both the physical and the mathematical structure of the theory. On the one hand, the question of the physical meaning of a wave function with many time coordinates t_i in $x_i = (t_i, \mathbf{x}_i)$ arises. On the other hand, one has to find suitable Lorentz invariant interacting wave equations to determine the time evolution of ψ in the many time coordinates. Both questions are related to two major conceptual difficulties in the foundations of relativistic quantum theory: (i) the measurement problem and (ii) the question of how to construct an interacting relativistic quantum theory.

In this work, we first give an overview over these questions, the present status towards their solution as well as the history of the multi-time formalism. Next, a conceptual framework for the abstract wave function level of the theory is developed. This includes a careful derivation of multi-time wave functions from single-time wave functions and Lorentz transformations of configurations in the Schrödinger picture as well as a comparison to different approaches, such as in the Heisenberg picture or using path integrals. Furthermore, a review of existence and uniqueness results for certain classes of multi-time evolution equations is given. A no-go theorem for potentials is shown to motivate the search for alternative mechanisms for relativistic interactions. Moreover, we study the question of appropriate conserved (tensor) currents, densities and respective continuity equations. A geometric formulation of probability conservation on space-like hypersurfaces using a differential form constructed from these tensor currents is developed. In addition, adequate Hilbert spaces are pointed out and it is shown that the (weak) uniqueness of solutions follows from probability conservation.

The next part deals with the question (i) of the physical meaning of multi-time wave functions. It is approached by studying realistic relativistic quantum theories, mainly the hypersurface Bohm-Dirac model, a relativistic generalization of Bohmian mechanics using a preferred foliation of space-time into space-like hypersurfaces, but also relativistic versions of objective collapse theories. Here, we work out which requirements on multi-time wave functions are needed to ensure the compatibility with the respective models. A great part of the work is focused on the development of a subsystem description for the hypersurface Bohm-Dirac model via a new relativistic conditional density matrix. Such a subsystem description is essential for Bohmian mechanics to derive statistical predictions. The properties of the density matrix are studied rigorously and a lifting to a density operator on the adequate Hilbert spaces is found. Furthermore, we take up an investigation by Bloch

of the physical meaning of multi-time wave functions in "measurements" and show that it supports the conjecture that the effective measurement formalism of the hypersurface Bohm-Dirac model is independent of the preferred foliation.

Then we come to the main part of the thesis, concerning point (ii), the construction of interacting multi-time wave equations. This goal is first approached for a relatively simple model for two mass-less Dirac particles in $1+1$ space-time dimensions. Interaction is introduced via boundary conditions on the space-time configurations where the two particles meet. This generalizes the idea of zero-range interactions to a relativistic (multi-time) setting. To prove the existence and uniqueness of solutions, the usual functional-analytic approach to zero-range interactions, namely implementing boundary conditions via the domain of the Hamiltonian, does not suffice anymore. Instead, the method of characteristics from the theory of partial differential equations is generalized to the multi-time case and applied to the existence and uniqueness proof. Furthermore, we show that physically reasonable classes of boundary conditions can be extracted which are compatible with probability conservation, Lorentz invariance and antisymmetry of the wave function. Moreover, it is demonstrated that the model is interacting in the sense that it generates entanglement.

Subsequently, the $1+1$-dimensional model is generalized with respect to various aspects. First, a new class of physically reasonable boundary conditions is studied. Then the model is extended to the N-particle case for a class of boundary conditions which allows for a compact formulation for all N. Furthermore, we answer the question of whether dynamics on a domain of space-like configurations with a minimal space-like distance exist, a question which is relevant to decide whether a generalization of the model to higher dimensions is possible.

Finally, we study a different interacting model for multi-time wave functions for $1+3$ dimensions: the Two-Body Dirac equations of constraint theory. These achieve a novel mechanism for relativistic interactions using additive interaction terms in the multi-time equations which involve arbitrary powers of the total momentum operator. Our work focuses on the question whether adequate tensor currents exist so that the model can be considered compatible with a probabilistic meaning of the wave function. We point out that the previous treatments of this question are incomplete. A criterion for the admissible interaction terms is developed and shown to be violated for certain interaction terms which haven been derived from quantum field theory in the literature. On the one hand, this raises doubts whether applications of the Two-Body Dirac equations, e.g. for the calculation of mesonic spectra, are justified. On the other hand, we demonstrate that it is, in fact, possible to extract a sub-class of the Two-Body Dirac equations for which the compatibility with the general framework for multi-time wave functions is ensured.

Zusammenfassung

Der Grundgedanke des Multi-time-Formalismus, wie zuerst von Dirac im Jahr 1932 vorgeschlagen, besagt, dass die relativistische Quantenmechanik auf einer Wellenfunktion $\psi(x_1, ..., x_N)$ aufgebaut sein sollte, die für N Teilchen als Argument eine Raumzeit-Konfiguration $(x_1, ..., x_N)$, $x_i \in \mathbb{R}^4$ enthält, anstelle einer räumlichen Konfiguration $(\mathbf{x}_1, ..., \mathbf{x}_N; t)$ zu einer Zeit t in einem ausgezeichneten Bezugssystem. Diese Auffassung hat weitreichende Konsequenzen, sowohl für die physikalische als auch für die mathematische Struktur der Theorie. Einerseits stellt sich die Frage nach der physikalischen Bedeutung einer Wellenfunktion mit vielen Zeitkoordinaten t_i in $x_i = (t_i, \mathbf{x}_i)$. Andererseits gilt es, geeignete Lorentz-invariante wechselwirkende Wellengleichungen zu finden, um die Zeitentwicklung von ψ in den vielen Zeitkoordinaten zu bestimmen. Die beiden Fragen stehen in engem Bezug zu zwei zentralen konzeptionellen Problemen in den Grundlagen der relativistischen Quantentheorie: (i) dem Messproblem sowie (ii) der Frage, wie man eine wechselwirkende, relativistisch invariante Quantentheorie konstruiert.

In dieser Arbeit wird zunächst ein Überblick über diese Fragen gegeben, über den bisherigen Fortschritt in Bezug auf ihre Lösung sowie die Geschichte des Multi-time-Formalismus. Anschließend wird ein konzeptioneller Rahmen für die abstrakte Wellenfunktionsebene der Theorie entwickelt. Dies schließt eine sorgfältige Herleitung von Multi-time-Wellenfunktionen aus Einzeit-Wellenfunktionen mittels der Lorentz-Transformation von Konfigurationen im Schrödingerbild ein, sowie einen Vergleich dieses Ansatzes mit anderen Zugängen, wie z.B. im Heisenbergbild oder mithilfe von Pfadintegralen. Außerdem fassen wir die bestehenden Existenz- und Eindeutigkeitsresultate für bestimmte Klassen von Multi-time-Entwicklungsgleichungen kurz zusammen. Es wird gezeigt, dass ein No-go-Theorem für Potentiale die Suche nach alternativen Mechanismen für relativistische Wechselwirkung motiviert. Darüber hinaus untersuchen wir die Frage nach angemessenen (Tensor-)Strömen, Dichten und entsprechenden Kontinuitätsgleichungen. Es wird eine geometrische Formulierung der Wahrscheinlichkeitserhaltung auf raumartigen Hyperflächen entwickelt, unter Benutzung einer Differentialform, die aus besagten Tensorströmen konstruiert ist. Außerdem zeigen wir auf, welche Hilberträume angemessen erscheinen, und beweisen, dass die (schwache) Eindeutigkeit von Lösungen aus der Wahrscheinlichkeitserhaltung folgt.

Der folgende Teil beschäftigt sich mit der Frage (i) nach der physikalischen Bedeutung von Multi-time-Wellenfunktionen. Wir nähern uns dieser Frage, indem wir realistische relativistische Quantentheorien untersuchen, hauptsächlich das Hypersurface-Bohm-Dirac-Modell, eine relativistische Erweiterung der Bohm'schen Mechanik mittels einer ausgezeichneten Blätterung der Raumzeit in raumartige Hyperflächen, aber auch relativistische Versionen Objektiver Kollapstheorien. Dabei stellen wir besonders die Anforderungen an die Multi-time-Wellenfunktionen heraus, die notwendig sind, um die Kompatibilität mit den

entsprechenden Modellen zu gewährleisten. Ein großer Teil unserer Arbeit zu dem Thema konzentriert sich auf die Entwicklung einer Subsystembeschreibung für das Hypersurface-Bohm-Dirac-Modell mithilfe einer neuartigen relativistischen bedingten Dichtematrix. Wir führen eine rigorose Analyse der Eigenschaften dieser Dichtematrix durch und geben ein Lifting zu einem Dichteoperator auf dem angemessenen Hilbertraum an. Darüber hinaus gehen wir auf eine Untersuchung Blochs der physikalischen Bedeutung von Multi-time-Wellenfunktionen in "Messungen" ein und zeigen, dass Blochs Untersuchung die Behauptung stützt, dass der effektive Messformalismus des Hypersurface-Bohm-Dirac-Modells unabhängig von der ausgezeichneten Blätterung sei.

Hierauf kommen wir zum Hauptteil der Arbeit, der die Frage (ii) betrifft: die Konstruktion wechselwirkender Multi-time-Wellengleichungen. Wir nähern uns diesem Ziel, indem wir ein stark vereinfachtes Modell für zwei masselose Diracteilchen in $1+1$ Raumzeitdimensionen betrachten. Die Wechselwirkung wird dann mittels Randbedingungen auf denjenigen Raumzeitpunkten eingeführt, an denen sich die beiden Teilchen zur gleichen Zeit am gleichen Ort befinden. Diese Idee stellt eine Verallgemeinerung des Gedankens der Kontaktwechselwirkungen für die relativistische (Multi-time-)Situation dar. Um die Existenz und Eindeutigkeit von Lösungen zu zeigen, reicht die übliche funktionalanalytische Herangehensweise für Kontaktwechselwirkungen, d.h. die Implementierung der Randbedingungen über den Definitionsbereich des Hamiltonoperators, nicht mehr aus. Stattdessen verallgemeinern wir die Methode der Charakteristiken aus der Theorie partieller Differentialgleichungen auf den Multi-time-Fall und wenden sie auf den Existenz- und Eindeutigkeitsbeweis an. Darüber hinaus wird gezeigt, dass sich physikalisch sinnvolle Klassen von Randbedingungen auffinden lassen, die mit den Aspekten der Wahrscheinlichkeitserhaltung, Lorentz-Invarianz und Antisymmetrie im Einklang stehen. Schließlich demonstrieren wir, dass das Modell wechselwirkend ist, nämlich in dem Sinne, dass es Verschränkung generiert.

Anschließend wird das $1+1$-dimensionale Modell in Hinblick auf verschiedene Aspekte verallgemeinert. Zunächst wird eine neue Klasse physikalisch vernünftiger Randbedingungen untersucht. Dann wird das Modell für eine Klasse von Randbedingungen, die sich besonders kompakt für jedes N formulieren lässt, auf den N-Teilchen-Fall ausgeweitet. Darüber hinaus beantworten wir die Frage nach der Existenz der Dynamik auf dem Definitionsbereich raumartiger Konfigurationen mit einem gewissen raumartigen Mindestabstand, eine Frage, die wichtig dafür ist, zu entscheiden, ob sich das Modell auch auf höhere Dimensionen verallgemeinern lässt.

Schließlich untersuchen wir noch ein weiteres wechselwirkendes Modell für Multi-time-Wellenfunktionen, das sich für $1+3$ Raumzeitdimensionen formulieren lässt: die Two-Body-Dirac-Gleichungen der Constraint-Theorie. Diese Gleichungen erreichen durch die Benutzung beliebiger Potenzen des Gesamtimpulsoperators in den Wechselwirkungstermen einen neuartigen Mechanismus für relativistische Wechselwirkung. Unsere Arbeit konzentriert sich dabei vor allem auf die Frage, ob angemessene Tensorströme existieren, über welche das Modell Kompatibilität mit der probabilistischen Bedeutung der Wellenfunktion erreicht. Wir weisen darauf hin, dass die bisherigen Untersuchungen dieser Frage unvollständig sind. Wir entwickeln ein Kriterium für zulässige Wechselwirkungsterme und zeigen, dass es in bestimmten Fällen verletzt ist, die in der Literatur aus der Quantenelektrodynamik hergeleitet wurden. Einerseits lässt dieser Befund Zweifel daran aufkommen, ob Anwendungen der Two-Body-Dirac-Gleichungen, z.B. zur Berechnung von Mesonenspektren, berechtigt sind. Andererseits zeigen wir, dass es in der Tat möglich ist, eine Unterklasse der Two-Body-

Dirac-Gleichungen zu finden, für welche die Kompatibilität mit dem allgemeinen Rahmen für Multi-time-Wellenfunktionen gewährleistet ist.

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Introduction

What is relativistic quantum mechanics? In order to find an adequate answer to this basic question, one has to reconsider the core aspects of special relativity (SR) and quantum mechanics (QM). In our opinion[1], the most important insights of SR can be considered (a) the necessity of a space-time description and (b) the Lorentz transformation law for a change between inertial frames (or equivalently, the space-time metric)[2]. The key structure of QM, on the other hand, is the wave function, a non-local object on configuration space which can be related to the statistical outcomes of experiments. It is crucial to note that the use of configuration space implies that QM is not independent of the underlying space-time theory. (Of course, there are more abstract ways of presenting QM but as long as they are equivalent this does not change the fact that genuinely relativistic generalizations of QM have to consider this aspect.)

Combining these key ingredients of SR and QM implies that relativistic QM should be based on a wave function on space-time configurations which obeys a Lorentz invariant law of motion. Furthermore, it should be related to the probability density for the statistics of experiments. This view constitutes the basic idea of the *multi-time formalism*, as first suggested by Dirac in 1932 in his article *Relativistic Quantum Mechanics* [38].

A brief history of the multi-time formalism: Dirac proposed that for two charged spin-$\frac{1}{2}$ particles interacting via a second-quantized field the wave function should be a map

$$\psi : \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathscr{F}, \quad (x_1, x_2) \longmapsto \psi(x_1, x_2), \tag{1}$$

where $\mathscr{F}$ is a suitable Fock space, including the spin components as well as the field degrees of freedom [38]. The presence of the many time coordinates in $x_i = (t_i, \mathbf{x}_i)$ explains the name of the multi-time formalism.

To determine the time evolution of ψ, Dirac prescribed simultaneous wave equations of

[1] Note, however, that there also exists the Lorentz-Poincaré view on SR (see e.g. [15, 25]). Its central position is that all special-relativistic effects can also be obtained using the concept of absolute time together with physical laws like Maxwell's equations which happen to have a symmetry under Lorentz transformations. As then there is no particular theoretical reason for this special form of physical laws (only compatibility with empiricism), laws for the quantum world do not necessarily have to be Lorentz invariant on the fundamental level, just statistically and in the classical limit.

[2] A particularly clear derivation of this core structure of SR *and nothing more* was given by Whitehead [103]. His investigation shows that locality, in the sense that distant objects can only influence each other with a certain time delay, is not a central *postulate* of this "core SR". This fact is particularly important as locality in this sense would, according to Bell [14, 16], conflict with experimental findings (and quantum mechanics).

the form

$$F_1\psi(x_1,x_2) \;=\; 0, \qquad F_2\psi(x_1,x_2) \;=\; 0, \tag{2}$$

where the operators F_i are the operators in the covariant form of the Dirac equation with operator-valued 4-potential $\hat{A}_\mu(x)$. The latter was supposed to satisfy a Lorentz invariant, source-free wave equation. Remarkably, Dirac was able to show that these elements nevertheless lead to an interacting, manifestly covariant theory.

Shortly after, Dirac, Fock and Podolsky presented a first multi-time QED model for a finite number of particles [40]. Subsequently, Bloch investigated the mathematical consistency of the multi-time wave functions, finding that the system (2) can only possess a solution on the set of space-like configurations [20]. Furthermore, he analyzed the implications of certain assumptions about the time evolution operators together with the usual measurement formalism, i.e. for the wave function $\varphi(\mathbf{x}_1,\mathbf{x}_2,t)=\psi(t,\mathbf{x}_1,t,\mathbf{x}_2)$ evaluated at equal times.

In 1943[3], Tomonaga worked out the reasons for the necessity of a multi-time formulation and generalized it to matter fields [96]. He replaced the multi-time wave function by a wave functional $\Psi[C]$ on space-like hypersurfaces C, obeying a functional wave equation for each point $P\in C$. It takes the form

$$\left[-i\frac{\delta}{\delta C_P}+H(P)\right]\Psi[C] \;=\; 0, \tag{3}$$

where $H(P)$ is the Hamiltonian at the point $P\in C$ and $\frac{\delta}{\delta C_P}$ is the functional derivative at $P\in C$ (see [96] for details).

A similar idea was developed by Schwinger in 1948 [93]. Accordingly, the term "Tomonaga-Schwinger picture" was coined for the resulting formalism. For their fundamental work on QED, the two researchers were, together with Feynman, awarded the Nobel prize in 1965. However, at the same time this great success of a manifestly Lorentz invariant formulation of QED was achieved, it was immediately overshadowed by the realization of the UV-divergencies in quantum field theory (compare [96]). The situation in QED bears great similarity to the problem of self-interaction in classical electrodynamics (see e.g. [52], [51, chap. 2.4]). Even though renormalization procedures could be found to circumvent the UV-divergencies in practical calculations, the underlying problem has never been solved. It is no great secret that till present no rigorous formulation of quantum field theory in $1+3$ space-time dimensions could be found (see e.g. [53]). Consequently, there does not exist a finite time evolution for QED. Instead, one focuses mainly on the calculation of S-matrix elements. However, the S-matrix only describes a highly idealized transition of quantum states separated infinitely in space and time, for which the invariance requirements are greatly restricted. Perhaps because of this fact, the manifestly Lorentz invariant formulation of QED in the Tomonaga-Schwinger picture appears to have been largely forgotten. Instead, it is common practice to employ formalisms primarily designed to quickly derive efficient computational methods – but for which the Lorentz invariance on all levels of the theory cannot be seen as readily as for the multi-time formalism[4].

[3] An English translation was published only in 1946 after World War II.

[4] See chap. 1, sec. 1.1 for a justification of the claim, where the various approaches to relativistic quantum physics are compared in detail.

Main conceptual problems in relativistic quantum theory: In our discussion of relativistic quantum mechanics we have so far only considered the abstract wave function level. However, in order to make contact with empiricism, one also has to take into account the statistical postulates about "measurements" of orthodox quantum theory. Doing so, one immediately realizes that relativistic quantum (field) theory inherits the same conceptual problems associated with the measurement formalism as are present in non-relativistic QM. Since, for example, the collapse postulate of orthodox QM presupposes a distinguished frame, it becomes clear that the formulation of any seriously Lorentz invariant theory requires to take a position towards (or, better, avoid) this postulate and the associated problems.
We formulate the major classes of difficulties following Dirac and Bell [6, 11]:

1. *First-class difficulties:* the measurement problem[5] or, more generally, the question of how the abstract level of quantum theory (wave functions, density matrices, Hilbert spaces, operator observables) can be related to the physical world of everyday experience without invoking imprecise, subjective or external elements like "classical measurement devices" or "observers" (as required by the collapse postulate of orthodox QM).

2. *Second-class difficulties:* the UV divergencies or, more generally, the problem of how to find an interacting relativistic quantum theory.

The present progress towards a solution of these problems can be summarized as follows. The second-class difficulties could so far only be overcome for special models in low space-time dimensions, e.g. in $1+1$ dimensions [53]. For multi-time wave functions, however, there does not, to our best knowledge, exist any rigorous interacting relativistic model yet.

The first-class difficulties, on the other hand, have successfully been avoided by realistic relativistic quantum theories such as Bohmian mechanics [49, 51] and objective collapse or Ghirardi-Rimini-Weber (GRW) theories [5, 12] in the case of non-relativistic QM. For Bohmian mechanics and objective collapse theories, this was achieved by introducing quantities (and respective laws of motion) which can be thought of as being located in space and time and out of which physical systems are assumed to consist. These quantities have been termed the "beables" [18] or the "primitive ontology" of a theory [2, 3]. Important examples include point particles (in the case of Bohmian mechanics), mass densities and discrete events, so-called "flashes" (in the case of objective collapse theories).

For the very precision of theories with a primitive ontology, a relativistic extension then requires a more accurate specification of the term "relativistic". The reason is that one can, in fact, find models which are statistically Lorentz invariant but for which the primitive ontology is not (see [21, chap. 12], [19]), showing that a comprehensive discussion of relativistic quantum theory must consider the first-class difficulties. A good discussion of the associated questions was given by Maudlin [73]. He arrives at the following criterion for a relativistic theory[6]:

> "**Relativistic constraint:** a theory is compatible with Relativity if it can be formulated without ascribing to space-time more or different intrinsic structure than the (special or general) relativistic metric."

[5] See e.g. [11] for a particularly clear introduction.
[6] Note that the criterion fits well with the core structure of SR as outlined before.

This constraint, of course, entails that the primitive ontology has to obey a Lorentz invariant law of motion. Theories which can in this sense be considered "fundamentally Lorentz invariant" have been proposed for both Bohmian mechanics [42, 43] and objective collapse theories [10,97]. In all these cases, the Lorentz invariance of the primitive ontology requires multi-time wave functions, providing another reason for their study. However, as there does not seem to exist a set of rigorous, interacting and relativistic multi-time wave equations, a *simultaneous solution* of both first and second class difficulties has not been achieved yet.

Goals and scope of the thesis: The goal of our work is to provide an exemplary solution of the above-mentioned problems. The main challenges in this enterprise consist in

1. rigorously constructing interacting relativistic multi-time wave equations, and
2. ensuring the compatibility of these wave equations with a realistic relativistic quantum theory. Of course, the term "quantum theory" implies that the usual statistical predictions should follow from these theories in a suitable sense. To establish this for a given model may be a further challenge, amounting to a *derivation* of the quantum formalism, analogous to the way thermodynamics is derived from classical mechanics via statistical physics.

In order to achieve both points, we make several assumptions and simplifications. First of all, we focus on a finite number $N \geq 2$ of directly interacting particles. One can regard the assumption of a fixed particle number either as restricting the domain of applicability of the model or as requiring the Dirac sea picture to explain particle creation and annihilation. While the meaning of negative energies thus depends on the underlying realistic relativistic model, their appearance does not render models for finite particles generally impossible. These models rather work for any wave function, be it of positive or negative (or indefinite) energy, and imply a certain behavior of matter which may or may not be adequate to describe certain experimental situations. This, in turn, may restrict their practical applicability – but not the possibility to logically conceive such a model[7].

The concept of direct interaction can be seen as particularly appealing because the UV-divergencies in classical field theory can exactly be avoided by a theory of this type, Wheeler-Feynman electrodynamics [52, 101, 102]. One of the discussed models will furthermore assume low space-time dimensions. This constitutes a real restriction. To devise more realistic models will be a highly challenging step.

While our personal motivation derives from the above-mentioned conceptual questions, we emphasize that the interacting models which will be developed in this thesis are just as interesting for a reader who is solely interested in a Lorentz invariant formulation of quantum dynamics on the wave function level.

[7] Claims to the contrary, e.g. that the "Klein paradox" or the "Zitterbewegung" or the non-existence of a canonical 4-position operator in relativistic QM would logically rule out relativistic N-particle theories, rather result from a "naive realism about operators" (see [48, sec. 9]), together with overly strong preconceptions about the structure of a relativistic measurement formalism.

Chapter 1

The multi-time formalism

> "The method of Schrödinger seems indeed more correctly conceived than that of Heisenberg, and yet it is hard to place a function in coordinate space and view it as an equivalent for a motion. But if one could succeed in doing something similar in four-dimensional space, then it would be more satisfying."
>
> EINSTEIN, 1926 [59, p. 83]

The goal of this chapter is to provide a conceptual framework for the abstract wave function level of the theory. First, we show that various different approaches towards relativistic quantum mechanics lead to the necessity to consider multi-time wave functions (sec. 1.1). Furthermore, possible structures for evolution equations for multi-time wave functions are introduced and the difficulty to implement interaction is discussed (sec. 1.2). Next, the implications for conserved currents, probability density and Hilbert spaces are worked out (sec. 1.3). A geometric formulation of probability conservation involving a differential form constructed from the conserved currents is developed and a proof that probability conservation implies the uniqueness of solutions is given.

1.1 Approaches towards Lorentz invariant quantum theories

An attentive student entering the subject of relativistic quantum physics cannot but be confused about the variety of conflicting approaches, terminologies and claims regarding their status with respect to Lorentz invariance. Apart from the widespread but non-serious tendency to call any theory "relativistic" which contains corrections or spin terms that are not described by the Schrödinger-Pauli equation, it is sometimes held that the Schrödinger picture is not Lorentz invariant while the Heisenberg picture and/or the path integral are. Of course, these claims cannot be reconciled with the common understanding that all these different "pictures" are equivalent. It would seem that some of the claims are mistaken or that it is common practice to employ inequivalent notions of Lorentz invariance. A clarification seems in order.

1.1.1 Schrödinger picture

It is striking that the Schrödinger picture is universally used by those quantum theories which seriously attempt to explain the measurement formalism of quantum mechanics, such as Bohmian mechanics [49, 51], GRW (objective collapse) theories [5, 12] and also the many worlds interpretation [99]. This may be due to the fact that the problematic collapse postulate, which is required by orthodox QM for empirical adequacy, entails that the wave function changes upon "measurement". As a "measurement" is to be located in time, this implies that the wave function changes with time, even in the Heisenberg picture. This somewhat breaks the conceptual integrity of the Heisenberg picture. Furthermore, relativistic single-particle wave equations, such as the Klein-Gordon equation and the Dirac equation, are (and have historically been) derived in the Schrödinger picture as a matter of course. The likely reason for this is that then a space-time point appears naturally as an argument of the wave function.

Yet, the Schrödinger picture is often claimed to be inferior to the Heisenberg picture when it comes to relativistic multi-particle theories:

> "We have seen that in the Schrödinger picture the state is specified in terms of the results of possible measurements on the system at time t. Such a description picks out a particular Lorentz frame and is, therefore, not covariant. In the Heisenberg picture, on the other hand, the state of the system is the same for all time. For the discussion of the relativistic invariance the Heisenberg picture has, therefore, decided advantages."
>
> Schweber [92, p. 164]

In this subsection, we first take up the criticism of the usual (single-time) Schrödinger picture, following Tomonaga. Then we show how these problems can be overcome naturally and clearly in the multi-time Schrödinger picture. The question of the equivalence of the Schrödinger and Heisenberg pictures, implicitly raised by Schweber's quote, will be discussed in section 1.1.2.

1.1.1.1 Single-time Schrödinger picture

For simplicity, we focus on the case of $N \geq 2$ particles first.

Summary of Schrödinger picture: The single-time Schrödinger picture is based on a wave function

$$\varphi : \underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{N \text{ times}} \times \mathbb{R} \longrightarrow \mathcal{S}, \quad (\mathbf{x}_1, ..., \mathbf{x}_N, t) \longmapsto \varphi(\mathbf{x}_1, ..., \mathbf{x}_N, t), \tag{1.1}$$

where d is the number of spatial dimensions and $\mathcal{S}$ is a suitable spin space.
φ is supposed to satisfy a single-time wave equation

$$i\frac{\partial \varphi}{\partial t} = H\varphi, \tag{1.2}$$

where H is the Hamiltonian on the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}^{Nd}) \otimes \mathcal{S}. \tag{1.3}$$

The solution of eq. (1.2) for initial data

$$\varphi(\mathbf{x}_1, ..., \mathbf{x}_N, 0) \;=\; \varphi_0(\mathbf{x}_1, ..., \mathbf{x}_N) \tag{1.4}$$

is given by $\varphi(\cdot, t) = e^{-iHt}\varphi_0(\cdot)$, i.e. by letting the unitary group e^{-iHt} generated by H act on the initial data.

The connection with statistical experiments derives from the Born rule, i.e. by the postulate that

$$\rho(\mathbf{x}_1, ..., \mathbf{x}_N, t) \;=\; |\psi|^2(\mathbf{x}_1, ..., \mathbf{x}_N, t) \tag{1.5}$$

is the probability density to find particle 1 at $\mathbf{x}_1$, particle 2 at $\mathbf{x}_2$, ..., and particle N at $\mathbf{x}_N$ in an experiment at time t. As emphasized by Bell [17, p. 166], more complicated experiments can in principle always be reduced to position measurements, so the Born rule can indeed be considered the essential part of the (non-relativistic) measurement formalism.

Note that the form of ρ is not independent of the wave equation but has to be chosen according to the continuity equation which is implied by the latter. Furthermore, the Hilbert space (1.3) is chosen to make spatial integrals over the probability density $\rho = |\psi|^2$ well-defined. Its utility as a mathematical tool derives from the facts that (a) the wave equation (1.2) is linear and (b) the scalar product obtains physical meaning by the Born rule.

Criticism: The main points of criticism of the single-time Schrödinger picture with respect to relativistic invariance are (compare Tomonaga [96]):

1. The single-time wave function $\varphi(\cdot, t)$ and the Hilbert space $\mathcal{H}$ which contains it (at a particular time) treat space and time variables on an essentially different basis.Tomonaga does not explicitly refer to this fact but instead criticizes the use of equal-time commutation relations.

2. Relatedly, a single-time wave equation distinguishes a particular frame and therefore cannot be covariant.

3. The Born rule makes use of a particular frame. The measurement formalism based on it is therefore not covariant.

The lack of covariance of the single-time Schrödinger picture could not be clearer.

1.1.1.2 Multi-time Schrödinger picture

Instead of switching to the more formal Heisenberg picture and hoping for a solution of the problems with Lorentz invariance along different lines, we analyze the consequences of the view that the multi-particle wave function should transform according to a representation of the Lorentz group. This leads to the multi-time picture as outlined in the introduction. The course of the argument follows the paper [67, sec. 1] by the present author.

Derivation of multi-time wave functions from single-time wave functions and Lorentz transformations: From basic representation theory, we know that under a Lorentz transformation

$$\Lambda : \mathbb{R}^{1+d} \to \mathbb{R}^{1+d}, \quad x \mapsto \Lambda x \tag{1.6}$$

a single-particle wave function $\varphi(x)$ transforms as

$$\varphi(x) \;\overset{\Lambda}{\longmapsto}\; \varphi'(x) = S[\Lambda]\varphi(\Lambda^{-1}x), \tag{1.7}$$

where the matrices $S[\Lambda]$ form a representation of the proper Lorentz group $\mathcal{L}_+^\uparrow$.

The attempt to Lorentz transform an N-particle single-time wave function φ immediately fails because its argument $(\mathbf{x}_1, ..., \mathbf{x}_N, t)$ cannot directly be associated with a collection of 4-vectors. However, $(\mathbf{x}_1, ..., \mathbf{x}_N, t)$ does carry space-time information. It characterizes the spatial configuration $(\mathbf{x}_1, ..., \mathbf{x}_N)$ of N points at time t. The very same configuration is expressed by $(t, \mathbf{x}_1; ...; t, \mathbf{x}_N)$ which has a spatio-temporal meaning. Under Λ,

$$(t, \mathbf{x}_1; ...; t, \mathbf{x}_N) \;\overset{\Lambda}{\longmapsto}\; \left(\Lambda^{-1}(t, \mathbf{x}_1); ...; \Lambda^{-1}(t, \mathbf{x}_N)\right) \;\equiv\; (t'_1, \mathbf{x}'_1; ...; t'_N, \mathbf{x}'_N), \tag{1.8}$$

where in general $t_j \neq t_k$ for $j \neq k$.

The key idea now is to regard the single-time wave function as the special case of a more general object, the *multi-time wave function* ψ, i.e.

$$\varphi(\mathbf{x}_1, ..., \mathbf{x}_N, t) \;=\; \psi\left(t, \mathbf{x}_1; ...; t, \mathbf{x}_N\right). \tag{1.9}$$

Note that in the single-particle case (which is unproblematic with respect to Lorentz invariance) the multi-time wave function coincides with the single-time wave function. We now determine the domain[1] Ω on which ψ should be defined (and thereby the nature of the map) by demanding that the Lorentz-transformed multi-time wave function ψ' be also a wave function on Ω.

Assuming the straightforward generalization of eq. (1.7), i.e.

$$\psi(x_1, ..., x_N) \;\overset{\Lambda}{\longmapsto}\; \underbrace{S[\Lambda] \otimes \cdots \otimes S[\Lambda]}_{N \text{ times}} \psi(\Lambda^{-1}x_1, ..., \Lambda^{-1}x_N) \tag{1.10}$$

we obtain

$$\varphi(\mathbf{x}_1, ..., \mathbf{x}_N, t) \equiv \psi\left(t, \mathbf{x}_1; ...; t, \mathbf{x}_N\right) \;\overset{\Lambda}{\longmapsto}\; S[\Lambda] \otimes \cdots \otimes S[\Lambda]\psi(t'_1, \mathbf{x}'_1; ...; t'_N, \mathbf{x}'_N), \tag{1.11}$$

where in general $t_j \neq t_k$. Therefore, we arrive at the necessity to consider multi-time wave functions

$$\psi : \Omega \subset \underbrace{\mathbb{R}^{1+d} \times \cdots \times \mathbb{R}^{1+d}}_{N \text{ times}} \;\longrightarrow\; \mathcal{S}, \quad (x_1, ..., x_N) \;\longmapsto\; \psi(x_1, ..., x_N), \tag{1.12}$$

where the natural domain Ω is given by the set of *space-like configurations*

$$\mathscr{S} \;:=\; \{(x_1, ..., x_N) \in \mathbb{R}^{N(1+d)} | \, \forall j \neq k : (x_j - x_k)^2 < 0\}. \tag{1.13}$$

Our sign convention for the Minkowski metric is $\eta = \mathrm{diag}(1, -1, -1, -1)$.

[1] Here and in the following, the symbol Ω denotes the general domain of a multi-time wave function. Specific choices A of Ω will be introduced by $\Omega = A$.

Remarks:

1. Note that the above argument only yields $\Omega = \mathscr{S}$ for $N = 2$. For general N, the orbit of the simultaneous configurations under Lorentz transformations gives a set strictly contained in $\mathscr{S}$. However, because a collection of space-like 4-vectors is the most general notion of an N-particle configuration, we regard $\mathscr{S}$ as the natural domain for the multi-time wave function. The set $\mathscr{C}$ of points where the coordinates of at least a pair of particles coincide, are not contained in $\mathscr{S}$ because this would contradict the premise of an N-particle configuration[2].

2. One may also consider different domains than $\mathscr{S}$, either larger ones such as $\mathbb{R}^{N(1+d)}$, i.e. the whole of configuration space-time, or smaller ones such as the set Ω_α of space-like configurations with a minimum space-like distance α.
   In the case of larger domains than $\mathscr{S}$, the physical meaning of a wave function on time-like configurations is obscure. From the above argument, it is clear that this is not a natural generalization of non-relativistic QM. Furthermore, the envisaged statistical meaning of the wave function can only be assumed to hold on space-like hypersurfaces. Similarly to $\rho = |\psi|^2$, one would expect that a quadratic function of ψ yields the crossing probability for sets of the form Σ^N (see sec. 1.3). Assuming that particle trajectories are time-like, a crossing density can only be defined on sets of this form. Moreover, the density associated with a relativistic wave equation should not be expected to be integrable along time-like directions as this is not the case for the non-relativistic $|\psi|^2$-density, either. The case of smaller domains is possible but not natural.
   Historically, Dirac considered multi-time wave functions on $\mathbb{R}^{N(1+d)}$. The domain $\mathscr{S}$ was first proposed by Bloch [20] and taken up by Tomonaga [96], albeit mostly for less general reasons concerning only the consistency of particular multi-time QED models.

3. Instead of (1.10), one can also consider a more general notion of Lorentz transformations which is motivated by a general-relativistic view on multi-time wave functions (compare [54,97]). Let $(\mathcal{M}, g)$ be the space-time manifold with metric g. In this case, a multi-time wave function corresponds to a section of a spinor bundle over the set of space-like configurations $\mathscr{S}$, defined similarly as before by using g instead of η. Then it is natural to consider separate Lorentz transformations $\Lambda_j \in \mathcal{L}_+^\uparrow$ in the tangent spaces $T_{p_j}\mathcal{M}$, $j = 1, ..., N$. Together they comprise a transformation which we call a *multi-time Lorentz transformation*, i.e.

$$L : T_{p_1}\mathcal{M}\times\cdots\times T_{p_N}\mathcal{M} \longrightarrow T_{p_1}\mathcal{M}\times\cdots\times T_{p_N}\mathcal{M}, \quad (x_1, ..., x_N) \longmapsto (\Lambda_1 x_1, ..., \Lambda_N x_N). \tag{1.14}$$

   We set $\mathcal{M} = \mathbb{R}^{1+d}$. Then under L, ψ transforms as

$$\psi(x_1, ..., x_N) \overset{L}{\longmapsto} \psi'(x_1, ..., x_N) = S[\Lambda_1] \otimes \cdots \otimes S[\Lambda_N]\psi(\Lambda_1^{-1}x_1, ..., \Lambda_N^{-1}x_N). \tag{1.15}$$

   This notion of Lorentz transformations is slightly more general than (1.10) in that the latter is the special case of (1.15) for $\Lambda_j \equiv \Lambda$ for all j, corresponding to an

[2] This can best be seen for indistinguishable particles. The most natural view then is that wave functions are functions of unordered configurations, i.e. of subsets of configuration space with power N (see [44]).

identification of all the tangent spaces $T_{p_j}\mathcal{M}$. Multi-time Lorentz transformations can also be helpful in a special-relativistic setting for technical arguments and we shall employ them when appropriate (see e.g. sec. 2.2). They are not appropriate for terms involving differences of coordinates.
With this generalized notion of Lorentz transformations we briefly return to point 1, noting that $\mathscr{S}$ is indeed the orbit of the simultaneous configurations under multi-time Lorentz transformations.

Evolution equations: Starting from the non-relativistic theory, the most obvious possibility for multi-time evolution equations is to prescribe N simultaneous first-order equations which can be written in Hamiltonian form. Dirac's original suggestion [38] falls into this category (disregarding additional constraints on the quantum fields). The subject of multi-time evolution equations will be discussed in more detail in sec. 1.2. Here we only outline as much as needed for a brief discussion of Lorentz invariance and the connection with the Heisenberg picture (see sec. 1.1.2).

The class of *Hamiltonian multi-time equations* is given by:

$$\begin{aligned} i\frac{\partial}{\partial t_1}\psi &= H_1\psi, \\ &\vdots \\ i\frac{\partial}{\partial t_N}\psi &= H_N\psi, \end{aligned} \tag{1.16}$$

where the H_j are first order differential operators, called *partial Hamiltonians*. They contain a $k\times k$ matrix structure where k is the number of spin components of ψ. The whole system of equations (1.16) is supposed to transform covariantly under Lorentz transformations (1.10), despite the asymmetric way of writing the time derivatives.

Examples:

1. *Free multi-time Dirac theory* is defined by

$$H_k \equiv H_k^{\text{Dirac}} := -i\gamma_k^0\gamma_k^j\partial_{k,j} + m_k\gamma_k^0, \tag{1.17}$$

where γ_k^μ is the μ-th Dirac gamma matrix acting on the spin index of the k-th particle, m_k is the mass of the k-th particle and $\partial_{j,\mu} := \frac{\partial}{\partial x_j^\mu}$.
Then eqs. (1.16) can be recast into the manifestly covariant form

$$(i\gamma_j^\mu\partial_{j,\mu} - m_k)\psi(x_1,...,x_N) = 0, \quad j=1,...,N. \tag{1.18}$$

2. *Free multi-time Klein-Gordon theory:* Alternatively, one can consider second-order equations

$$(\Box_j + m_j^2)\psi = 0, \quad j=1,...,N. \tag{1.19}$$

Note that it is not as easy as in the single-particle theory to rewrite the multi-time Klein Gordon (KG) equations into Hamiltonian form (1.16) by introducing the time derivatives as new variables, i.e. by defining $\tilde{\psi} := (\psi, i\partial_{t_1}\psi, ..., i\partial_{t_N}\psi)$. This is because e.g. considering $i\partial_{t_1}\tilde{\psi}$ for $N=2$ yields a term in which $-\partial_{t_1}\partial_{t_2}\psi$ does not further simplify to an expression without time derivatives.

The connection of Hamiltonian multi-time wave equations with single-time wave equations is straightforward. Suppose that ψ satisfies eqs. (1.16). Consider the associated single-time wave function φ given by eq. (1.9). Using the chain rule, we find that it satisfies the single-time wave equation (1.2) with Hamiltonian

$$H \;=\; \sum_{j=1}^{N} H_j. \tag{1.20}$$

Statistical meaning: This point will be discussed in sec. 1.3 (on a formal level) and in chap. 2 (with respect to its physical meaning). Here we just mention that under reasonable assumptions on the wave equations of the theory, it is possible to obtain a formal crossing density for general space-like hypersurfaces, meeting the criticism of the non-relativistic picture. When it comes to the derivation of a generalized Born rule from a deeper ontological level, the arguments are more subtle (see chap. 2).

Conclusion: Assuming that the statistical meaning of ψ can be established in a relativistically satisfactory way, the multi-time Schrödinger picture indeed meets all the points of criticism of the single-time Schrödinger picture.

A different proposal involving wave functions with many time coordinates: A different approach towards relativistic quantum dynamics involving a wave function with many times has been suggested by Horwitz and Rohrlich [58].

The central object of their theory is a wave function

$$\chi : \underbrace{\mathbb{R}^5 \times \cdots \times \mathbb{R}^5}_{N \text{ times}} \longrightarrow \mathbb{C}, \quad (x_1, \tau_1; ...; x_N, \tau_N) \longmapsto \chi(x_1, \tau_1; ...; x_N, \tau_N). \tag{1.21}$$

Horwitz and Rohrlich regard $\chi(\cdot, \tau_1; ...; \cdot, \tau_N)$ as a function of the N parameters τ_j with values in $L^2(\mathbb{R}^{4N}, d^{4N}x)$.
They consider evolution equations

$$i\frac{\partial \chi}{\partial \tau_j} \;=\; \left[p_j^2 + m_j^2 + \phi_j(x_1, ..., x_N)\right] \chi, \quad j = 1, ..., N, \tag{1.22}$$

where $p_{j,\mu} = i\frac{\partial}{\partial x_j^\mu}$ (i.e. $p_j^2 = -\Box_j$) and ϕ_j is a potential.
Evidently, this differs from the multi-time approach outlined above by the presence of the N additional parameters τ_j and by the different choice of Hilbert space.

The approach is subject to severe criticism. Firstly, the τ parameters do not have any evident physical meaning. According to [58], the hope is to relate them to proper time parameters associated with particles. However, no such connection is actually given – and it is hard to imagine how this could be possible without the existence of trajectories in the theory. Secondly, the approach does not have the correct non-relativistic limit. The non-relativistic wave function simply does not contain τ-parameters. Even more importantly, it is not square integrable with respect to time which destroys the possibility of the wave function of Horwitz and Rohrlich to have statistical significance. Thirdly, the approach

does not encompass relativistic single-particle wave equations such as the Klein-Gordon equation.

In conclusion, the approach of Horwitz and Rohrlich seems only formally motivated and cannot be considered a generalization of non-relativistic QM.

Multi-time Schrödinger picture for a variable particle number: The relativistic Heisenberg picture and path integrals for relativistic multi-particle quantum theories make use of a Fock-space formulation. To have the possibility of direct comparison with these approaches, we now briefly summarize how to construct multi-time wave functions for a variable particle number.

By analogous arguments as in the N-particle case, manifest Lorentz invariance in the Schrödinger picture then requires to replace Fock space (which distinguishes a special frame) by a suitable set of functions on the set of space-like configurations of arbitrary length (see [81])[3]:

$$\mathscr{S}_{\text{var}} := \bigcup_{N=0}^{\infty} \{(x_1, ..., x_N) \in \mathbb{R}^{4N} : \forall j \neq k : (x_j - x_k)^2 < 0\}. \tag{1.23}$$

Let $\mathcal{S}^{(N)}$ denote the appropriate spin space for N particles. A multi-time wave function then is a map

$$\begin{aligned} \psi : \mathscr{S}_{\text{var}} &\longrightarrow \bigoplus_{N=0}^{\infty} \mathcal{S}^{(N)}, \\ (\emptyset, (x_1), (y_1, y_2), ...) &\longmapsto (\psi^{(0)}(\emptyset), \psi^{(1)}(x_1), \psi^{(2)}(y_1, y_2), ...). \end{aligned} \tag{1.24}$$

Here, $\psi^{(N)}$ is the N-particle sector part of ψ. This construction of variable particle number wave functions is the relativistic analog of the Fock space construction in position representation (see [81] and [92, chap. 6f]).

The dynamics for ψ can then be expressed by multi-time evolution equations for each of the $\psi^{(N)}$. The crucial difference to the N-particle case is that the rhs. of (1.16) may couple $\psi^{(N)}$ to $\psi^{(K)}$, e.g. $K = N - 1,\ N + 1$. An interacting, albeit ultraviolet-divergent and not yet fully Lorentz invariant model of this type has recently been presented in [81].

For the purpose of the comparison of the various approaches the simple case of a free neutral scalar field is sufficient. Then each ψ^N satisfies the free N-time KG equations (1.19).

1.1.2 Heisenberg picture

We now return to the questions raised at the beginning of sec. 1.1.1: Is the Heisenberg picture "more Lorentz invariant" than the Schrödinger picture? If so, how can it be equivalent to the Schrödinger picture? Furthermore, what is its relation to the multi-time Schrödinger picture?

First, we briefly summarize the well-known formalism using the example of a free neutral scalar field (sec. 1.1.2.1) to create a basis for the subsequent discussion of Lorentz invariance (sec. 1.1.2.2). Here, we also take into account subtle points which are often omitted in less

[3] Petrat and Tumulka also admit light-like configurations.

critical textbook discussions. Finally, we point out the relation of the Heisenberg picture with the single-time and multi-time Schrödinger pictures, resolving the tension between the conflicting equivalence and Lorentz invariance claims (sec. 1.1.2.3).

1.1.2.1 Summary

For the relativistic Heisenberg picture, one considers Fock space $\mathscr{F}$ as Hilbert space. Elements $|\psi_H\rangle \in \mathscr{F}$ are called Heisenberg state vectors. They do not evolve in time[4].

The dynamics is carried by the field operators $\Phi(x)$, $x \in \mathbb{R}^4$:

$$i\partial_t \Phi(t, \mathbf{x}) \;=\; [\Phi(t, \mathbf{x}), H]. \tag{1.25}$$

For suitable Hamiltonians H, these equations of motion can be written in a manifestly Lorentz covariant form. For example, for the free neutral scalar field, one has [92, chap. 7]:

$$(\Box_j + m_j^2)\Phi(x) \;=\; 0. \tag{1.26}$$

The question of the general statistical significance of the second-quantized Heisenberg picture theory is often discussed only in limited situations like scattering in textbooks, not on a general level (compare e.g. [79]). One of the few books containing a statement about the situation is Schweber's[5]. For $|\psi_H\rangle \in \mathscr{F}$ let $\psi_H^{(N)}$ be the N-particle amplitude of $|\psi_H\rangle$ in position (i.e. space-time) representation, i.e. informally $\psi_H^{(N)}(x_1, ..., x_N) = \langle x_1, ..., x_N | \psi_H \rangle$, or

$$\psi_H^{(N)}(x_1, ..., x_N) \;:=\; (N!)^{-\frac{1}{2}} \langle 0 | \Phi(x_1) \cdots \Phi(x_N) | \psi_H \rangle, \tag{1.27}$$

where $|0\rangle$ is the "vacuum" state. Then[6]:

> "The physical interpretation of the Fock space component $\psi_H^{(N)}(x_1, ..., x_N)$ when $x_1^0 = x_2^0 = \cdots = x_N^0$ is the probability amplitude for finding N particles at time $x^0 = x_1^0 = \cdots = x_N^0$."
>
> Schweber, [92, p. 171]

Along with other points, this quote strongly motivates a discussion of Lorentz invariance.

1.1.2.2 Lorentz invariance

To check the covariance of the theory, one defines a unitary operator $U(\Lambda)$ which implements the action of a Lorentz transformation $\Lambda \in \mathcal{L}_+^\uparrow$ on Fock space (see [92, chap. 7b]).
The Heisenberg state transforms as

$$|\psi_H\rangle \;\overset{\Lambda}{\longmapsto}\; U(\Lambda)|\psi_H\rangle. \tag{1.28}$$

The vacuum state is defined to be invariant, i.e.

$$U(\Lambda)|0\rangle \;=\; |0\rangle, \tag{1.29}$$

[4] Note that it would be misleading to characterize the fixed state vector by the obviously non-covariant equation $\partial_t |\psi_H\rangle = 0$ which one can find in textbooks (see e.g. [92, p. 650]).

[5] Schweber uses the positive energy field operators $\Phi^{(+)}(x)$ instead of the total ones $\Phi(x)$. The reason for this is independent of the present discussion.

[6] Our notation conventions have been used in the quote, without changes in content.

and the field operators satisfy the relation

$$U(\Lambda)^{-1}\Phi(x)U(\Lambda) \;=\; \Phi(\Lambda^{-1}x). \tag{1.30}$$

Relations (1.28)-(1.30) imply:

$$\begin{aligned}\langle 0|\Phi(x_1)\cdots\Phi(x_N)|\psi_H\rangle \;&\overset{\Lambda}{\longmapsto}\; \langle 0|U^{-1}(\Lambda)\Phi(x_1)\cdots\Phi(x_N)U(\Lambda)|\psi_H\rangle \\ &=\; \langle 0|U^{-1}(\Lambda)\Phi(x_1)U(\Lambda)\cdots U^{-1}(\Lambda)\Phi(x_N)U(\Lambda)|\psi_H\rangle \\ &\overset{(1.30)}{=}\; \langle 0|\Phi(\Lambda^{-1}x_1)\cdots\Phi(\Lambda^{-1}x_N)|\psi_H\rangle \end{aligned} \tag{1.31}$$

and thus the N-particle amplitude (1.27) transforms as (compare eq. (1.10)):

$$\psi_H^{(N)}(x_1,...,x_N) \;\overset{\Lambda}{\longmapsto}\; \psi_H^{(N)}(\Lambda^{-1}x_1,...,\Lambda^{-1}x_N). \tag{1.32}$$

The above information suffices to check the points required for relativistic invariance.

1. *Invariance of the basic objects:* According to eqs. (1.28), (1.30) both the Heisenberg state as well as the field operators transform according to a representation of the Lorentz group. Nevertheless, one may criticize the fact the Heisenberg state is an element of Fock space. The latter is derived from the one-particle Hilbert space $L^2(\mathbb{R}^3, d^3x)$ which distinguishes a particular frame. This fact depends, in turn, on the envisaged statistical meaning.

2. *Invariance of the equations of motion:* With the transformation rule (1.30), the Heisenberg equations of motion are Lorentz covariant.

3. *Statistical meaning:* Recall Schweber's quote about the statistical meaning of the N-particle amplitude. It is a direct translation of the Born rule into the Heisenberg picture and, therefore, by the same reasons as before, not covariant. Note, however, (1.27) is satisfactory from a formal point of view (as are expectation values of Heisenberg operators functions) because it only involves space-time points. One thus has reason to hope for a Lorentz covariant generalization – but such a generalization is usually not sought and would have to genuinely generalize the Born rule. We shall return to this question later.
   Note that the S-matrix formalism can, of course, be shown Lorentz invariant along standard lines [79, 92]. The notion of Lorentz invariance is then greatly reduced to invariance requirements only at two points separated infinitely in space and time. However, to restrict the formalism to this highly idealized situation (other than as an approximation) is not satisfactory from a conceptual point of view because (a) strictly speaking, no experiment satisfies the assumptions of infinite separation in space and time and (b) not all experimentally accessible situations can be treated as scattering processes, for example bound states.

Conclusion: For a comparison with the single-time Schrödinger picture, we exclude point 3. Also dropping the objection about the unequal treatment of space and time in point 1 for the moment, the Heisenberg picture indeed satisfies the covariance requirements of points 1 and 2 while the single-time Schrödinger picture does not. *They thus cannot be equivalent.* We shall elaborate on this important point in the next section.

1.1.2.3 Relation to the Schrödinger picture

Denote the state vector in the second-quantized single-time Schrödinger picture by $|\varphi(t)\rangle$ and the Schrödinger picture field operators by $\Phi(\mathbf{x})$. Then the standard relation between the two pictures is given by

$$|\psi_H\rangle \;=\; e^{iHt}|\varphi(t)\rangle \tag{1.33}$$

and

$$\Phi(t,\mathbf{x}) \;=\; e^{iHt}\Phi(\mathbf{x})e^{-iHt}. \tag{1.34}$$

We first aim at working out more clearly the inequivalence between the single-time Schrödinger picture and the Heisenberg picture in the relativistic context. The following lemma will prove useful in this respect.

Lemma 1.1.1 *Let* $|0(t)\rangle = e^{-iHt}|0\rangle$ *and let* $\langle \mathbf{x}_1,...,\mathbf{x}_N|\varphi(t)\rangle = (N!)^{-\frac{1}{2}}\langle 0(t)|\Phi(\mathbf{x}_1)\cdots\Phi(\mathbf{x}_N)|\varphi(t)\rangle$ *be the N-particle Fock space amplitude in the single-time Schrödinger picture. Then the following relation to the N-particle amplitude in the Heisenberg picture* $\psi_H^{(N)}(x_1,...,x_N)$ *holds:*

$$\langle \mathbf{x}_1,...,\mathbf{x}_N|\varphi(t)\rangle \;=\; \psi_H^{(N)}(t,\mathbf{x}_1;...;t,\mathbf{x}_N). \tag{1.35}$$

Proof:

$$\begin{aligned}(N!)^{-\frac{1}{2}}\langle 0(t)|\Phi(\mathbf{x}_1)\cdots\Phi(\mathbf{x}_N)|\varphi(t)\rangle &\overset{(1.33),(1.34)}{=} (N!)^{-\frac{1}{2}}\langle 0|e^{iHt}\Phi(\mathbf{x}_1)e^{-iHt}\cdots e^{iHt}\Phi(\mathbf{x}_N)e^{-iHt}|\psi_H\rangle \\ &= (N!)^{-\frac{1}{2}}\langle 0|\Phi(t,\mathbf{x}_1)\cdots\Phi(t,\mathbf{x}_N)|\psi_H\rangle \\ &\overset{(1.27)}{=} \psi_H^{(N)}(t,\mathbf{x}_1;...;t,\mathbf{x}_N). \qquad \square\end{aligned} \tag{1.36}$$

The inequivalence between the relativistic Heisenberg picture and the single-time Schrödinger picture can now be seen quite clearly. In the single-time Schrödinger picture, the Fock space amplitude (1.35) is limited to equal times. In contrast, in the Heisenberg picture $\psi_H^{(N)}(x_1,...,x_N)$ can, and in fact must, be considered for general space-time points[7] in order to achieve Lorentz invariance (see eq. 1.32). *The Heisenberg picture thus contains the single-time Schrödinger picture but not vice versa.*

Now only the question of the exact relation of the Heisenberg picture with the multi-time Schrödinger picture remains. Eq. (1.35) suggests that the connection is straightforward. Let $\psi(x_1,...,x_N)$ be the variable particle number wave function, as defined in eq. (1.24), and $\psi^{(N)}(x_1,...,x_N)$ its N-particle component[8]. Then one has (see also [81, assertion 3]):

$$\psi^{(N)}(x_1,...,x_N) \;=\; \psi_H^{(N)}(x_1,...,x_N), \tag{1.37}$$

where general space-time points are admitted on both sides.

It becomes evident that the Heisenberg picture and the multi-time Schrödinger picture are equivalent if and only if the dynamical equations of either picture imply the ones of the

[7] Note that arbitrary configurations $(x_1,...,x_N)$ can be considered. However, only space-like ones are natural.

[8] $\psi^{(N)}(x_1,...,x_N)$ coincides with the N-time wave function of sec. 1.1.1.2 if it has autonomous dynamics.

other. The fact that the Heisenberg equations of motion imply the multi-time equations can be seen as follows. Consider the action of $(\Box_j + m_j^2)$ on $\psi^{(N)}(x_1, ..., x_N)$:

$$(\Box_j + m_j^2)\psi^{(N)}(x_1, ..., x_N) \overset{(1.37),(1.27)}{=} (N!)^{-\frac{1}{2}} \langle 0|\Phi(x_1)\cdots\left[(\Box_j + m_j^2)\Phi(x_j)\right]\cdots\Phi(x_N)|\psi_H\rangle$$
$$\overset{(1.26)}{=} 0. \tag{1.38}$$

The converse is easily seen along standard lines. One first defines the (single-time) Schrödinger picture field operators $\Phi(\mathbf{x})$ and obtains the Heisenberg picture field operators $\Phi(t, \mathbf{x})$ from the unitary group e^{-iHt} via eq. (1.34). Then the Heisenberg equations of motion are satisfied by definition. Note that the additional freedom of the multi-time Schrödinger picture as compared to the single-time Schrödinger picture plays no role in this particular step.

We thus see that starting from the single-time Schrödinger picture, going to the Heisenberg picture and translating back the formalism obtained, one arrives at the multi-time Schrödinger picture. This shows that the (relativistic) Heisenberg picture tacitly generalizes the single-time to the multi-time Schrödinger picture. This transition, albeit indirect, constitutes a further reason, independent from the intuitive argument given in sec. 1.1.1.2, to consider multi-time wave functions. Note that the crucial ingredient in this second derivation of multi-time wave functions is the possibility to consider products of Heisenberg field operators at space-time points with different time coordinates.

Conclusion: We have seen that, contrary to widespread belief, the single-time Schrödinger picture and the Heisenberg picture for relativistic quantum field theory[9] are not quite equivalent. However, the generally assumed equivalence of Schrödinger and Heisenberg picture can be re-established as an equivalence between the Heisenberg picture and the *multi-time* Schrödinger picture.
We close the section with some comments on more subtle points.

Remarks:

1. *Effort to see the Lorentz invariance:* Recall Schweber's quote at the beginning of sec. 1.1.1. Comparing secs. 1.1.1.2 and 1.1.2.2, Schweber's claim that the Heisenberg picture has "decided advantages" for the discussion of Lorentz invariance appears unfounded. Especially when it comes to the question of the relation of the wave function with space-time in the two pictures, the multi-time Schrödinger picture is more natural. Viewed from this perspective, the equivalence with the multi-time Schrödinger picture rather alleviates the criticism that the Heisenberg state $|\psi_H\rangle$ is an element of $\mathscr{F}$ which uses a strict separation of space and time.

2. *Domain:* Similarly to the first point, the fact that the multi-time wave function (alias the N-particle amplitude (1.27)) is naturally only defined on $\mathscr{S}$ seems not to be reflected in the Heisenberg picture. This may seem only a minor subtlety, but we shall see later that the domain $\mathscr{S}$ instead of $\mathbb{R}^{4N}$ is indeed crucial for the construction of interacting theories.

[9] For non-relativistic quantum theory, one can self-consistently restrict the Heisenberg picture to equal times without violating symmetry requirements (such as Lorentz invariance). Then both pictures are indeed equivalent.

3. *Multi-time Lorentz transformations:* In the multi-time Schrödinger picture, general-relativistic considerations straightforwardly lead to multi-time Lorentz transformations (1.14). For the Heisenberg picture, it is hard to see how this generalization should be possible. A natural idea to obtain the transformation behavior (1.15) also for the N-particle amplitude (1.27) would be to replace $U(\Lambda)$ by an operator "$U(\Lambda_x)$", meaning that a Lorentz transformation Λ_x is chosen for each space-time point x and that (1.30) is replaced by $\Phi(y) \overset{\Lambda_x}{\longmapsto} U^{-1}(\Lambda_x)\Phi(y)U(\Lambda_x) = \Phi(\Lambda_y^{-1}y)$. However, it is unclear how this idea could be implemented in a Hilbert space setting, considering that $U(\Lambda_x)$ then would have to be on operator on $\mathcal{H}$, which does not contain time, and at the same time, informally speaking, would have to "get the y from $\Phi(y)$".

4. *Statistical meaning:* In order to generalize the obviously non-covariant interpretation of the N-particle amplitude by Schweber, one might be inclined to guess that $|\psi_H^{(N)}(x_1, ..., x_N)|^2$ is the probability density to find particles at space-time points $x_1, ..., x_N$, at least if all of them are space-like related. However, together with the analogous guess for the multi-time Schrödinger picture this will in general turn out wrong because it disregards the fact that probability conservation is only ensured when the probability density and the current are chosen according to the wave equations of the theory (see sec. 1.3 and chap. 2). The train of thought necessary for such a generalization is that the main role of the wave equations of the theory is to ensure the existence of conserved tensor currents (from which the connection with physics arises). This is much more natural in the multi-time Schrödinger picture than in the Heisenberg picture.

1.1.3 Path integrals

In this section, we analyze the question of whether or to what extent the path integral can be used as a fundamental basis of relativistic quantum theory, as distinct from a useful computational tool. We illustrate the concepts using the example of the free neutral scalar field.

It should first be mentioned that from a mathematical perspective it is as yet unclear how to even define the path integral for realistic models [53, 85]. The situation is different from the other two approaches where it is quite clear how to understand the time evolution (e.g. as solutions of PDE systems or as unitary groups generated by self-adjoint Hamiltonians), independently of whether these proofs have actually been given. To be fair, however, one should note that a consistent interacting multi-particle quantum theory has not been found for the correct number of space-time dimensions in either framework.

This being said, for the present purpose of the discussion of Lorentz invariance we adopt an optimistic viewpoint, assuming that mathematical sense can be given to all the expressions used[10]. Note that in any case this implies that the path integral measure "$\mathcal{D}\phi$" and the integrand cannot be regarded as separate mathematical objects.

[10] This is the case for the free neutral scalar field.

1.1.3.1 Summary

The path integral is often claimed to give one or even *the* Lorentz invariant formulation of relativistic quantum (field) theory. One then usually focuses on the Lorentz invariance of particular formulas. Only rarely one finds an outline of the whole framework which arises from taking the path integral as a fundamental basis of quantum theory. We briefly attempt to give such a summary here.

Let $\hat{T}$ denote the time ordering symbol. The path integral can be used to express correlation functions, such as the N-point Feynman functions

$$\tau_N(x_1, ..., x_N) \;:=\; \langle 0|\hat{T}\{\Phi(x_1)\cdots\Phi(x_N)\}|0\rangle \tag{1.39}$$

in terms of the Lagrangian density L ($= \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)$) of a corresponding classical field theory (see [79, p. 284]):

$$\tau_N(x_1, ..., x_N) \;=\; \lim_{T\to(1-i\varepsilon)\infty} \frac{\int \mathcal{D}\phi(x)\, \phi(x_1)\cdots\phi(x_N) \exp\left[i\int_{-T}^{T} dx^0 \int_{\mathbb{R}^3} d^3\mathbf{x}\, \mathcal{L}(\phi(x))\right]}{\int \mathcal{D}\phi(x)\, \exp\left[i\int_{-T}^{T} dx^0 \int_{\mathbb{R}^3} d^3\mathbf{x}\, \mathcal{L}(\phi(x))\right]}. \tag{1.40}$$

While this formula is indeed Lorentz invariant, it should be noted that the notion of Lorentz invariance is very formal, as the measure "$\mathcal{D}\phi(x)$" and the integrand do not exist as separate mathematical objects. Contrary to the suggestive way of writing, there is no direct relation to the Lorentz transformations of fields on space-time.

Of course, correlation functions *per se* do not constitute relativistic quantum physics. However, as Wightman showed [104], the set of all N-point Wightman functions

$$W_N(x_1, ..., x_N) \;=\; \langle 0|\Phi(x_1)\cdots\Phi(x_N)|0\rangle \tag{1.41}$$

allows for a reconstruction of both the Fock space and the Heisenberg field operators. At this point, some people claim (e.g. Roepstorff [85, p. 218]) that because of this relation, the path integral yields a complete description of quantum field theory. This is, however, not correct. Only the kinematical framework of Fock space is reconstructed. The fact that not all information of field theory can be reconstructed by the path integral can easily be seen from the non-existence of a path integral expression for the Heisenberg state $|\psi_H\rangle$. The crucial formula (1.37) which allows to reconstruct the multi-time wave equations from the Heisenberg equations of motion thus cannot be obtained. Moreover, a reconstruction of Fock space is not even desirable from a relativistic standpoint, as it treats space and time on an essentially different basis.

At this point, we briefly pause to consider figure 1.1. The framework obtained so far corresponds to the upper part of the flow chart till the dashed horizontal line. In order to discuss the Lorentz invariance of the path integral formalism, one has to give a *dynamical principle*, to be expressed solely in terms of the path integral.
(Note that the Heisenberg equations of motion which the reconstructed Heisenberg field operators satisfy do not provide a *useful* dynamical principle without the Heisenberg state. To add the state to the theory would mean to depart from a pure path integral approach and to switch to the Heisenberg picture.)

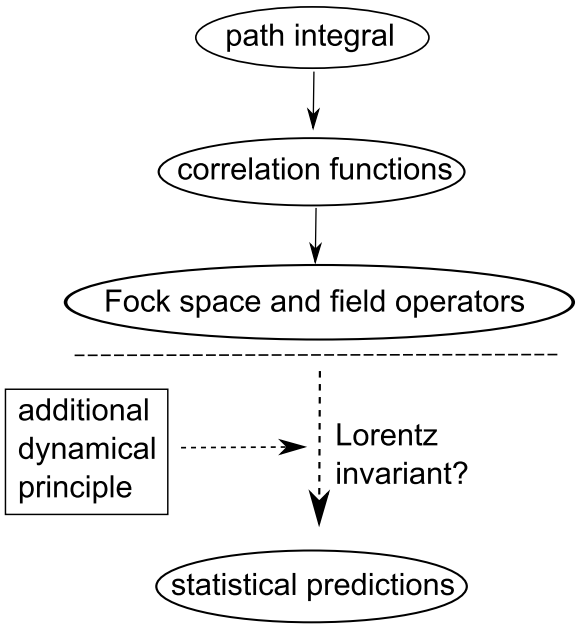

Figure 1.1: Conceptual scheme of the path integral

1.1.3.2 Infinite times: scattering processes

One possibility for a new dynamical principle (for scattering situations) is to postulate the LHZ reduction formula. This formula allows to express the S-matrix elements and therefore cross sections which can directly be compared with experiments by correlation functions (see e.g. [79, p. 307]). These can in turn be written as path integrals using eq. (1.40). As all the formulas involved in this process are Lorentz invariant, the resulting theory also possesses this property.

However, the fact that scattering processes can be expressed Lorentz invariantly by the path integral constitutes no particular advantage of the latter. The same result would have been obtained even in the single-time Schrödinger picture which is not Lorentz invariant in any other respect. It thus becomes evident that the notion of Lorentz invariance in scattering processes (i.e. idealized transitions between two free states separated infinitely in space and time) is too restricted to allow for a definite judgment about the Lorentz invariance of the overall theory.

1.1.3.3 Finite times

For a significant discussion of Lorentz invariance it is indispensable to consider finite time evolution. Analogously to the path integrals for propagators of the single-time Schrödinger equations, one can find an expression for matrix elements of the finite time evolution operator e^{-iHt} on the reconstructed Fock space of QFT by yet another path integral [79, p. 282]:

$$\langle \phi_b(\mathbf{x}) | e^{-iHt} | \phi_a(\mathbf{x}) \rangle \;=\; \int \mathcal{D}\phi(x)\, \exp\left[i \int_0^t dx^0 \int_{\mathbb{R}^3} d^3\mathbf{x}\, \mathcal{L}(\phi(x)) \right], \tag{1.42}$$

where the functions $\phi(x)$ in the path integral are constrained to configurations $\phi_a(\mathbf{x})$ at $x^0 = 0$ and $\phi_b(\mathbf{x})$ at $x^0 = t$.

Peskin and Schröder add in their explanation of the formula:

> "The time integral in the exponent of (1.42) goes from 0 to t as determined by our choice of what transition function to compute; in all other respects this formula is manifestly Lorentz invariant." [79, p. 282/283]

In other words: the formula (1.42) is not Lorentz invariant. This is evident from the fact that a single time t cannot be sufficient to characterize a multi-particle quantum theory. Of course, the lack of Lorentz covariance is inherent to the formulation of the problem, namely the search for the single-time evolution operator e^{-iHt}.

Conclusion: We have seen that (if mathematically well-defined) the path integral yields Lorentz covariant scattering statistics. However, finite time evolution seems to require[11] a distinguished time coordinate t. The notion of time evolution is thus similar to (and similarly problematic as in) the single-time Schrödinger picture.

Of course, this criticism concerns only the usual approaches found in textbooks. The next paragraph sketches how the new dynamical principle of fig. 1.1 might be obtained by a path integral for the multi-time propagator.

1.1.3.4 Outlook: multi-time path integrals

One may wonder whether the problems with the Lorentz invariance for finite time evolution can be overcome by using many times also for the path integral, similarly to the way the multi-time Schrödinger picture solves the problems of the single-time Schrödinger picture.

Consider the Hamiltonian multi-time system (1.16) for an N-particle wave function $\psi(x_1, ..., x_N)$. As will be shown in sec. 1.2, in certain situations the solution of (1.16) can be obtained by letting the operator

$$U(t_1, ..., t_N) \;:=\; e^{-iH_1 t_1} \cdots e^{-iH_N t_N} \tag{1.43}$$

act on initial data ψ_0 at $t_1 = \cdots = t_N = 0$. U then contains all dynamics. It satisfies the multi-time equations, i.e.

$$i\frac{\partial}{\partial t_k} U(t_1, ..., t_N) \;=\; H_k U(t_1, ..., t_N), \quad k = 1, ..., N. \tag{1.44}$$

Furthermore, it is completely characterized by its position representation

$$\langle \mathbf{y}_1, ..., \mathbf{y}_N | U(t_1, ..., t_N) | \mathbf{x}_1 ... \mathbf{x}_N \rangle. \tag{1.45}$$

In the non-interacting case[12], the operators H_k can be regarded as acting only on the tensor factor of Hilbert space associated with the k-th particle. Denote the corresponding operators on the single-particle Hilbert space by $\tilde{H}_k$. Then (1.45) becomes

$$\langle \mathbf{y}_1 | e^{-i\tilde{H}_1 t_1} | \mathbf{x}_1 \rangle \cdots \langle \mathbf{y}_N | e^{-i\tilde{H}_N t_N} | \mathbf{x}_N \rangle. \tag{1.46}$$

[11] Extensive literature research in standard textbooks on path integrals such as [62, 85] confirms that no Lorentz invariant treatment of finite multi-particle time evolution via the path integral seems to exist.

[12] As will be shown in sec. 1.2.1.2, this may be more or less the only possibility for Hamiltonian multi-time systems.

For suitable Hamiltonians $H_j,\ j = 1, ..., N$ each factor $\langle \mathbf{y}_k | e^{-i\tilde{H}_k t_k} | \mathbf{x}_k \rangle$ has its own single-particle path integral representation. It will be Lorentz invariant if the partial Hamiltonians H_k are suitably chosen.

This train of thought shows that there is hope to express relativistic (multi-time) propagators by path integrals. The problems with Lorentz invariance thus need not be inherent in the path integral formalism. However, the usual path integral approaches do not take this way. We think that a suitably constructed multi-time path integral formalism would be equivalent to the Heisenberg and multi-time Schrödinger pictures, at least for a Hamiltonian theory. The common dynamical structure then is the unitary time-evolution operator $U(t_1, ..., t_N)$, the subject of the next section.

We have thus seen that all major and genuinely relativistic approaches to quantum physics involve a multi-time wave function or are related to it. This suggests the systematical study of its evolution equations.

1.2 Multi-time evolution equations

In this section, we give an overview of various possibilities for multi-time evolution equations. This is most naturally done in the Schrödinger picture. For simplicity and for conceptual clarity, we focus on the case of N directly interacting particles. Firstly, we summarize mathematical results relevant for the class of Hamiltonian evolution equations introduced in sec. 1.1.1.2. Secondly, we discuss a no-go theorem for interaction potentials by Petrat and Tumulka. The conceptual implications are drawn and several alternative mechanisms for interactions are pointed out, some of which will be analyzed in detail in the following chapters.

1.2.1 Hamiltonian multi-time equations

We now return to the class of Hamiltonian evolution equations introduced in sec. 1.1.1.2, eq. (1.16). Its study is important because of several reasons: (a) It is the one proposed historically by Dirac [38] and further analyzed by Bloch [20] and Tomonaga [96]. (b) It is the only one for which precise and general results on the existence and uniqueness of solutions are available. (c) It straightforwardly generalizes the functional-analytic way of understanding time evolution as a unitary group on a Hilbert space.

1.2.1.1 Existence and uniqueness theorems

The main idea is to regard the partial Hamiltonians H_j in

$$i\partial_{t_j}\psi(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N) \ = \ H_j\psi(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N), \quad j = 1, ..., N \tag{1.47}$$

as self-adjoint operators on Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{3N}) \otimes \mathcal{S}$ where the spin space $\mathcal{S}$ is given by $\mathcal{S} = \mathbb{C}^k$ for some $k \in \mathbb{N}$.
The multi-time wave function is then viewed as a map

$$\psi : \mathbb{R}^N \to \mathcal{H}, \quad (t_1, ..., t_N) \mapsto \psi(t_1, ..., t_N). \tag{1.48}$$

Given initial data

$$\psi(0,...,0) \;=\; \psi_0, \tag{1.49}$$

where $\psi_0 \in \mathcal{H}$, one aims at obtaining the corresponding solution of the initial data problem via

$$\psi(t_1,...,t_N) \;=\; U(t_1,...,t_N)\psi_0, \tag{1.50}$$

where $U(t_1,...,t_N)$ was defined in eq. (1.43). The following theorem by Reed and Simon [82, thm. VIII.12] on unitary N-parameter groups (slightly abbreviated here) shows when exactly this idea works.

Theorem 1.2.1 (Reed/Simon) *Let $\mathbf{t} \to U(\mathbf{t}) = U(t_1,...,t_N)$ be a strongly continuous map of $\mathbb{R}^N$ into the unitary operators on a separable Hilbert space $\mathcal{H}$ satisfying $U(\mathbf{t}+\mathbf{s}) = U(\mathbf{t})U(\mathbf{s})$ and $U(0) = \mathrm{Id}$. Let $\mathcal{D}$ be the set of finite linear combinations of vectors of the form*

$$\varphi_f \;=\; \int_{\mathbb{R}^N} f(\mathbf{t})U(\mathbf{t})\varphi \, d\mathbf{t}, \quad \varphi \in \mathcal{H}, \quad f \in C_0^\infty(\mathbb{R}^N). \tag{1.51}$$

Then $\mathcal{D}$ is a domain of essential self-adjointness for each of the generators H_j of the one-parameter subgroups $U(0,...,0,t_j,0,...,0)$, each H_j: $\mathcal{D} \to \mathcal{D}$ and the H_j commute, $j = 1,...,N$.

Corollary 1.2.2 *There exists a strongly continuous unitary N-parameter group $U(t_1,...,t_N)$ if and only if all generators H_j of the one-parameter subgroups $U(0,...,0,t_j,0,...,0)$ are self-adjoint on a common domain $\mathcal{D}$ and commute pairwise (in the spectral sense), i.e.*

$$[H_j, H_k] \;=\; 0 \quad \forall j,k. \tag{1.52}$$

Proof: "$\Rightarrow$": See thm. 1.2.1.
"$\Leftarrow$": As every H_k is self-adjoint, we can define a strongly continuous unitary N-parameter group $U(\mathbf{t})$ by eq. (1.43). The commutability and the common domain of the generators H_k ensure the well-definedness of $U(\mathbf{t})$. □

Corollary 1.2.3 *There exists a unique $\mathcal{H}$-valued solution ψ of the Hamiltonian multi-time equations* (1.47) *with initial data* (1.49) *and conserved norm if the requirements of corollary 1.2.2 are satisfied.*

Proof: As $U(\mathbf{t})$ is strongly continuous one can differentiate with respect to t_j to obtain the generator H_j. The multi-time equations are satisfied because $U(\mathbf{t})\mathcal{D} = \mathcal{D}$ and thus for $\psi_0 \in \mathcal{D}$, we have $\psi(t) \in \mathcal{D}$. (If the domains of the H_k were different, it could happen that e^{-iH_kt} does not leave the domain $\mathcal{D}(H_j)$ for $j \neq k$ invariant.)
The solution is unique and has conserved norm because of the unitarity of $U(\mathbf{t})$. □

Remarks:

1. Note that the norm (and thus the Hilbert space) is not independent of the form of the wave equation. Presupposing a special norm, as necessary when one wants to view the Hilbert space as primary, may thus unreasonably restrict the set of possible wave equations. We therefore advocate the view of the wave equation as primary and

the Hilbert space formalism as secondary. The question of the adequate probability density, currents and their relation to the wave equations as well as the Hilbert space formalism will be analyzed comprehensively in sec. 1.3.

2. The *consistency condition* (1.52) of the commutability of the partial Hamiltonians is of crucial importance to determine the consequences of the Hamiltonian viewpoint for the interaction terms in multi-time equations. It expresses that it should not matter in which order one time-evolves the initial wave function ψ_0 in the various time coordinates, first in the t_j-direction and then in the t_k-direction or the other way around. In other words, the following diagram has to commute:

$$\begin{array}{ccc} \psi(0,0) & \xrightarrow{e^{-iH_j t_j}} & \psi(t_j,0) \\ {\scriptstyle e^{-iH_k t_k}}\downarrow & & \downarrow{\scriptstyle e^{-iH_k t_k}} \\ \psi(0,t_k) & \xrightarrow{e^{-iH_j t_j}} & \psi(t_j,t_k). \end{array} \tag{1.53}$$

3. Note also that the partial Hamiltonians H_k have to be self-adjoint *on a common domain*. This is new compared to the results of Petrat and Tumulka [80] who assume the partial Hamiltonians to be defined on the whole of $\mathcal{H}$ (and therefore to be bounded). The common domain of self-adjointness plays a decisive role for the question of whether it is possible to formulate zero-range interactions in the functional-analytic framework (see chap. 3).

In the setting of eq. (1.47), the partial Hamiltonians are supposed to be fixed, i.e. time-independent operators. This is, of course, not sufficient to discuss Lorentz covariant interaction terms, added to the partial Hamiltonians H_j^{free} of the free theory, i.e.

$$H_j \;=\; H_j^{\text{free}} + V_j(x_1, ..., x_N), \tag{1.54}$$

because the V_j have to contain space and time variables on a symmetric basis. In order to treat terms of this kind, we quote a theorem of Petrat and Tumulka [80, thm. 2].

Theorem 1.2.4 (Petrat/Tumulka) *Let $\mathcal{H}$ be a Hilbert space, and let $H_1, ..., H_N$ be smooth functions on $\mathbb{R}^N$ with values in the bounded (symmetric) operators of $\mathcal{H}$. Then the system* (1.47) *possesses a solution $\psi : \mathbb{R}^N \to \mathcal{H}$ for every initial condition $\psi(0, ..., 0) \in \mathcal{H}$ if and only if*

$$[i\partial_{t_j} - H_j, i\partial_{t_k} - H_k] \;=\; 0 \quad \forall j, k. \tag{1.55}$$

Remarks:

1. The boundedness of the operators is, of course, unrealistic. This is a technical assumption which was used in [80] only to simplify the proof. As the assumption is not essential for the proof, we expect with Petrat and Tumulka that a similar theorem holds also for unbounded H_j's with values in a set of self-adjoint operators with common domain $\mathcal{D}$.

2. Non-Hamiltonian evolution equations are not covered by thm. 1.2.4, even if they can be written in first order form. General theorems which do not require the Hamiltonian setting of a unitary group (or propagator) on a fixed Hilbert space do not exist yet (to the best of our knowledge).

1.2.1.2 A no-go theorem for potentials

As we shall see now, the consistency condition (1.55) rules out certain types of interaction potentials. The following theorem collects the results of [80, thms. 3-6].

Theorem 1.2.5 (Petrat/Tumulka) *Let $\mathcal{H} = L^2(\mathbb{R}^{3N}) \otimes \mathbb{C}^k$. Suppose $H_j = H_j^{\text{Dirac}} + V_j(x_1, ..., x_N)$ where H_j^{Dirac} is the free Dirac Hamiltonian (1.17) acting on $\mathbf{x}_j$ and on the j-th spin index, and $V_j : \mathbb{R}^{4N} \to M(\mathbb{R}^{4^N}, \mathbb{R}^{4^N})$ is a smooth function with values in the symmetric matrices acting only on the spin index of the j-th particle. Then the consistency condition (1.55) is satisfied if and only if the evolution (1.47) is gauge-equivalent to a non-interacting one, i.e. there is a smooth real-valued function $\theta(x_1, ..., x_N)$ and there exist, for every $j \in \{1, ..., N\}$, smooth matrix-valued functions $\tilde{V}_j(x_j) : \mathbb{R}^4 \to M(\mathbb{R}^{4^N}, \mathbb{R}^{4^N})$ (also acting only on the spin index of the j-th particle) such that*

$$\tilde{\psi}(x_1, ..., x_N) \;:=\; e^{i\theta(x_1,...,x_N)}\psi(x_1, ..., x_N) \tag{1.56}$$

satisfies the equations

$$i\partial_{t_j}\tilde{\psi} \;=\; \left[H_j^{\text{Dirac}} + \tilde{V}_j(x_j)\right]\tilde{\psi}, \quad j = 1, ..., N. \tag{1.57}$$

Remarks:

1. Theorem 1.2.5 remains valid for a multi-time system (1.47) on the space-like configurations $\mathscr{S}$ [80].

2. The assumption that V_j only acts on the spin index of the j-th particle is important for the proof of the theorem. However, the interaction terms in the Hamiltonian of the semi-relativistic Breit equation, including for example $\sum_{a=1}^3 \gamma_j^0\gamma_j^a\gamma_k^0\gamma_k^a/|\mathbf{x}_j - \mathbf{x}_k|$, are not of this form [22]. More precisely, assuming the form of the interaction terms in thm. 1.2.5, one cannot recover such terms by considering $i\partial_t\psi(t, ..., t) = \sum_{j=1}^N H_j\psi(t, ..., t)$. Furthermore, the assumed form of the potentials cannot be made more plausible, either, by the self-consistent assumption[13] that the Dirac current $\overline{\psi}\gamma_1^{\mu_1} \cdots \gamma_N^{\mu_N}\psi$ should be conserved. (This assumption only leads to the requirement that V_j has to commute with $\gamma_k^0\gamma_k^i$ for $j \neq k$ which, for example, still allows V_j to depend on $\gamma_k^5 = i\gamma_k^0\gamma_k^1\gamma_k^2\gamma_k^3$.) Nevertheless, we strongly doubt that the remaining possibilities for potentials could lead to both consistent dynamics and physically adequate interactions.

3. The supposed impossibility of potential terms is not surprising for two more reasons. (a) Potentials are generally not regarded compatible with relativity and famous ones like the Breit or Coulomb potential are not Lorentz invariant, either. (b) Assuming the form (1.54), one would like to choose the potential terms V_j in such a way that they sum up to the corresponding single-time Hamiltonian (1.20). However, the exact distribution of these terms is then ambiguous (see also [80, sec. 1.4]). Furthermore, adding a Breit interaction term as above breaks the covariance of the multi-time equations solely because of the matrix structure.

[13] Note that the notion of a $\mathcal{H}$-valued solution, $\mathcal{H} = L^2(\mathbb{R}^{3N}) \otimes \mathbb{C}^k$, with conserved L^2-norm requires a conserved tensor current $j_\psi^{\mu_1\cdots\mu_N}$ to exist (see sec. 1.3), with $j_\psi^{0\cdots 0} = \psi^\dagger\psi$. A natural choice which satisfies this requirement is the Dirac current $j_\psi^{\mu_1\cdots\mu_N} = \overline{\psi}\gamma_1^{\mu_1} \cdots \gamma_N^{\mu_N}\psi$.

4. Theorem 1.2.5 bears some similarity with the so-called "no-interaction-theorem" by Currie, Jordan and Sudarshan [35] in that it rules out a certain mechanism for relativistic interactions. Assuming canonical quantization rules, generalized versions of the "no-interaction-theorem" may also have significance for interaction potentials in multi-time wave equations (see [27]). However, even then the "no-interaction-theorem" would only be valid for Klein-Gordon-like multi-time equations. Theorem 1.2.5 comes closer to the natural situations.

Conclusion: Together with the previous points, theorem 1.2.5 suggests that, at least for N directly interacting particles, the separable and Hamiltonian view that one can time-evolve a wave function separately in each time coordinate is too naive and too restrictive. The separate existence of the operators $e^{-iH_j t_j}$ is also implausible from two further points of view:

1. Let $q \in \mathscr{S}$ be a space-like configuration and $\psi_0 \in \mathcal{H}$. Then for any $t_j > 0$, $e^{-iH_j t_j}\psi_0(q) = \psi(q') \text{ with } q' \notin \mathscr{S}$. In other words, $e^{-iH_j t_j}$ does not respect the natural domain for multi-time wave functions (see sec. 1.1.1.2).

2. Relatedly, it is hard to imagine how the conservation of $\int d^{3N}\mathbf{x}\, \psi^\dagger\psi(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N)$ for all $t_1, ..., t_N$, which is implied by the existence of the unitary N-parameter group $U(\mathbf{t})$ for the case $\mathcal{H} = L^2(\mathbb{R}^{3N}) \otimes \mathbb{C}^k$, could attain physical significance. The reason for this is that crossing probabilities of time-like trajectories can in general only be defined for space-like hypersurfaces (see also sec. 2.1).

In view of the implausibility of the existence of the partial time evolution operators $e^{-iH_j t_j}$ and the fact that they do not respect the natural domain $\mathscr{S}$ for multi-time wave functions, one should aim only for a *time evolution between space-like hypersurfaces.* This option was first suggested by Tomonaga [96] in a QFT context, building on results by Bloch [20] who showed that a multi-time QED model of Dirac, Fock and Podolsky [40] was only consistent in the sense of eq. (1.55) on the set of space-like configurations (disregarding UV divergencies). The view that a multi-time wave function should only be defined on $\mathscr{S}$ (or $\overline{\mathscr{S}}$) has also been advocated by Petrat and Tumulka [80,81]. We shall adopt this view from now on. However, according to point 1 in the remark after thm. 1.2.5, the restriction to $\mathscr{S}$ alone does not help to find consistent potentials. The question of alternative mechanisms for relativistic interactions arises.

1.2.2 Alternative mechanisms of interaction

The above-mentioned situation constitutes the starting point for the present thesis. One can think of various options of how to modify the Hamilonian multi-time setting of eq. (1.47) of configuration space-time $\mathbb{R}^{4N}$ (in order of radicality):

1. One can introduce (quantum) fields in addition to the particles and allow for a variable particle number. This is the usual way, and it has been studied to great extent, although usually not as a multi-time theory. Recall, however, from the introduction that QED was initially formulated as a multi-time theory by Dirac, Fock, Podolsky, Bloch, Tomonaga and Schwinger [20,40,93,96]. More recently, a multi-time emission-absorption model involving two different species of particles was presented in [81]. The

main conceptual problem of all these QFT models is that up to now no satisfactory method[14] has been found to avoid the UV divergencies.

2. In the 1+1-dimensional case, the domain $\mathscr{S}$ together with an appropriate relativistic notion of probability conservation (see sec. 1.3) naturally leads to boundary conditions on the set of coincidence points in space-time. This includes the possibility of *relativistic contact interactions*, an idea which will be analyzed in chaps. 3 and 4.

3. Besides the domain $\mathbb{R}^{4N}$, also the possibility of a division between space and time derivatives such as in (1.47) is unnatural, or at least unnecessary. It can be relaxed by allowing *higher time derivatives* in the equations (but still assuming that only differential operators occur in the equations). If the time derivatives are of infinite order, one cannot reformulate the equations in the Hamiltonian way. (For finite order PDEs, this may be possible by introducing new variables for the time derivatives.) In this case, one does not expect a Cauchy data initial value problem anymore and one has to leave the well-explored mathematical territory. An example for this option, the *Two-Body Dirac equations*, will be analyzed in chap. 5.

4. One can consider *integral or integro-differential equations* such as the Bethe-Salpeter (BS) equation [87], [55, chap. 6]. The BS equation is indeed an equation for a multi-time wave function, describing two directly interacting particles. Apart from UV divergencies it does, however, encounter problems with negative-"norm" states (see [76]). Besides, because of the unusual mathematical structure, the question of appropriate initial data is normally ignored.
   Note that the option of integro-differential equations is put forward by the analogy with Wheeler-Feynman (WF) electrodynamics [101, 102]. This theory reformulates classical electrodynamics in terms of directly interacting particles, thereby avoiding the problems with self-interaction and UV-divergencies. It is, however, very challenging from a mathematical point of view and little is known about the existence and uniqueness theory. Because of this fact and the additional difficulty that there exists no clear and convincing (not to mention canonical) strategy to extend WF electrodynamics to the quantum case, despite attempts in this direction [4, 36, 60], the option of integral and integro-differential equations lies beyond the scope of this thesis.

5. Finally, there are suggestions in the literature to formulate equations for a multi-time wave function only on a distinguished foliation of space-time (see [4], [37, chap. 8]). However, these approaches do not seem compatible with a serious understanding of Lorentz invariance (compare Bell [13]), and we do not consider them here. Note, however, that this option may seem attractive to a proponent of the Lorentz-Poincaré view on special relativity (see [25]).

Remark: Note that for points 3-5, one cannot expect on *a priori* grounds that the (multi-time) Schrödinger and Heisenberg pictures are equivalent, as the equal-time evolution operator e^{-iHt}, which is evidently an element of the Hamiltonian theory, need not exist anymore.

[14] We regard renormalization as a way to extract partial information from the theory despite the occurrence of UV divergencies. In our view, it represents a circumvention rather than a solution of the second-class difficulties.

For point 1, the equivalence is evident. For point 2, one also expects a time evolution map analogous to e^{-iHt} to exist, as the equal-time hypersurfaces in a distinguished frame are a subclass of the space-like hypersurfaces (between which the time evolution exists).

1.2.3 Single-time reducible theories

The above-mentioned possibilities show that besides restrictions, multi-time theories also provide new possibilities to formulate a relativistically interacting theory. In order to better grasp these possibilities, we introduce the following (informal) terminology. A multi-time theory is called *single-time reducible* if and only if the following procedures yield the same set of solutions:

1. restricting the theory (multi-time equations, boundary conditions etc.) to equal times and then solving the resulting single-time theory, and

2. solving the full multi-time theory and restricting the solution to equal times.

Evidently, a multi-time theory can only be *single-time irreducible* if it cannot be formulated as a single-time theory. It is then understood that a single-time theory can only be obtained by making certain approximations, such as replacing light cones with simultaneity surfaces (see the appendix).

Examples:

1. Hamiltonian multi-time theories are single-time reducible and the corresponding single-time theory is also Hamiltonian, with H given by eq. (1.20).

2. The $1+1$-dimensional model of chap. 3 with boundary conditions at the set of coincidence points is single-time reducible but not Hamiltonian.

3. The Two-Body Dirac equations, which will be discussed in chap. 5, are single-time irreducible because they contain arbitrary powers of the total momentum operator and furthermore mix the time derivatives and spin components of the two particles.

4. The Bethe-Salpeter theory is also single-time irreducible because it involves convolutions of relativistic propagators with the multi-time wave function. The respective integrals cannot be reduced to simultaneous integrations in any frame. This fact is particularly evident in the position space formulation of the BS equation (see [55, chap. 6]).

Note that certain classes of single-time reducible multi-time theories can be treated by an inductive method developed by Petrat and Tumulka [80, sec. 7.2]. This requires the multi-time wave equations to have finite propagation speed. Furthermore, the solution theory of the corresponding single-time equation has to be known (which is taken for granted in [80]). The important case of Dirac particles is thus included (depending, of course, on the interaction terms).

On the conceptual value of single-time irreducible theories: Single-time irreducible theories are particularly interesting from a conceptual point of view because they fully use the freedom provided by the space-time formulation. As insisted on by Whitehead [103, chap. VIII], the role of laws in relativistic space-time is not (a) to characterize the change of the physical state from one space-like hypersurface to another but rather (b) to determine the overall spatio-temporal form of the physical structures. Of course, possibility (a) is an example for possibility (b). Whitehead makes the point that (a) is neither necessary nor natural from a relativistic standpoint.

However, so far no realistic theory of this kind has been studied mathematically in a comprehensive way, even though with Wheeler-Feynman electrodynamics and the Bethe-Salpeter equation there do exist prominent examples and research is ongoing [7–9, 37, and references therein]. As multi-time theories *per se* provide their own challenges and as little is known about them yet, we do not attempt to systematically analyze single-time irreducible multi-time equations here, even though this might well turn out necessary for the formulation of a realistic and mathematically consistent quantum theory. We will, however, make first steps in this direction by digressing from the Hamiltonian point of view in chap. 3 and furthermore by considering the Two-Body Dirac equations.

Note that field theory, while also offering new possibilities for the formulation of multi-time theories [40, 81, 93, 96], can (as emphasized by Feynman in his Nobel lecture) be considered an attempt to enforce the Hamiltonian method and therefore, in the sense of Whitehead, the "non-relativistic" way of thinking about laws of motion.

> "From the overall space-time view of the least action principle, the field disappears as nothing but bookkeeping variables insisted on by the Hamiltonian method."
>
> Feynman [52]

1.3 Tensor currents, probability conservation and Hilbert spaces

Recall that the statistical meaning of the wave function in non-relativistic QM is based on the fact that $\rho = |\psi|^2$ and $\mathbf{j} = \frac{1}{m}\mathrm{Im}\,\psi^*\nabla\psi$ satisfy a continuity equation. Therefore, the understanding that $\int d^{3N}q\,\rho(q)$ is the probability for the N-particle configuration to be somewhere in configuration space at time t is consistent.

It is clear that this central physical understanding, around which the mathematical Hilbert space formalism is built, has to be generalized in a multi-time context. This is the subject of the present section. The *justification* of the statistical meaning of the wave function is postponed till chap. 2. Parts of this section are based on the present author's papers [68, sec. 2.2] and [67, sec. 4.2].

1.3.1 Tensor currents and continuity equations

Before defining the framework, note that the question of conserved currents and densities is not independent of the considered type of multi-time evolution equations. The ideas are motivated by Hamiltonian evolution equations but also apply to more general types of differential equations on $\mathscr{S}$ (see points 1-3 in sec. 1.2.2). Whether an extension of the

ideas is possible also for multi-time integral or integro-differential equations is yet to be investigated.

The main assumption on which the rest of this section is based, is the existence of a *positive-definite tensor current*[15] $j_\psi^{\mu_1\cdots\mu_N}(x_1,...,x_N)$ as a function of the wave function. It is supposed to satisfy N *continuity equations*

$$\partial_{k,\mu_k} j_\psi^{\mu_1\cdots\mu_k\cdots\mu_N}(x_1,...,x_N) \;=\; 0, \quad k=1,...,N. \tag{1.58}$$

These continuity equations are assumed to follow from the multi-time evolution equations. Indeed, it can be considered the main role of the multi-time equations to provide a divergence-free j_ψ. For example, for the free multi-time N-particle Dirac theory (1.18), one has:

$$j_\psi^{\mu_1\cdots\mu_N}(x_1,...,x_N) \;=\; \overline{\psi}(x_1,...,x_N)\gamma_1^{\mu_1}\cdots\gamma_N^{\mu_N}\psi(x_1,...,x_N), \tag{1.59}$$

where $\overline{\psi}=\psi^\dagger\gamma_1^0\cdots\gamma_N^0$. The positive component of j_ψ then is $j_\psi^{0\ldots0}=\psi^\dagger\psi$, i.e. the usual $|\psi|^2$-density.

As will be shown in chap. 2, the adequate crossing probability density ρ_Σ for a space-like hypersurface with normal covector field $n(x)$ is given by

$$\rho_\Sigma(x_1,...,x_N) \;=\; j_\psi^{\mu_1\cdots\mu_N}(x_1,...,x_N)\, n_{\mu_1}(x_1)\cdots n_{\mu_N}(x_N). \tag{1.60}$$

Another example for j_ψ is the tensor current of the free multi-time Klein-Gordon theory (1.19) (see also [84, sec. 1B]):

$$j_\psi^{\mu_1\cdots\mu_N}(x_1,...,x_N) \;=\; \psi^*(x_1,...,x_N)(i\overset{\leftrightarrow}{\partial}{}_1^{\mu_1})\cdots(i\overset{\leftrightarrow}{\partial}{}_N^{\mu_N})\psi(x_1,...,x_N), \tag{1.61}$$

where $f(x_1,...,x_N)\overset{\leftrightarrow}{\partial}_{j,\mu_j} g(x_1,...,x_N) := f(x_1,...,x_N)\partial_{j,\mu_j}g(x_1,...,x_N) - [\partial_{j,\mu_j}f(x_1,...,x_N)]g(x_1)$.

Note that similarly to the one-particle KG theory (see e.g. [92, chap. 3]) the KG current (1.61) can become negative. It thus cannot describe the crossing statistics of time-like trajectories on space-like hypersurfaces.

Remarks:

1. Even though the existence of j_ψ may seem only a minimal operational requirement, it is in fact sufficient to formulate a full realistic quantum theory (see chap. 2).

2. The functional form of the tensor currents depends on the form of the multi-time equations and is not given *a priori*. A modified probability density also has impact on the predictions for experiments, for example for transition rates. In fact, such a situation will be encountered for the Two-Body Dirac equations (see chap. 5).

3. The tensor currents may not be unique. Given a j_ψ which satisfies (1.58), one can for example add a term with zero four-divergence in the various particle coordinates and obtain another possible choice of the current. It is a matter of physics to select the right current. Symmetry and limiting considerations may help in this respect.

[15] Similar tensor currents have been used in [42, 84, 90].

1.3.2 Probability conservation

We now give a plausibility argument for the form of the probability density (1.60) by generalizing the non-relativistic notion of probability conservation for arbitrary space-like hypersurfaces, making use of the tensor current j_ψ.

Let d denote the number of spatial dimensions. Our starting point is the usual non-relativistic notion of probability conservation, expressed via a single-time wave function φ:

$$\int d^d\mathbf{x}_1 \cdots \int d^d\mathbf{x}_N \; |\varphi|^2(\mathbf{x}_1, ..., \mathbf{x}_N; t) \;=\; 1, \quad \text{independently of t.} \tag{1.62}$$

Using the connection (1.9) between single-time and multi-time wave functions ψ, this equation reads:

$$\int d^d\mathbf{x}_1 \cdots \int d^d\mathbf{x}_N \; |\psi|^2(t, \mathbf{x}_1, ..., t, \mathbf{x}_N) \;=\; 1, \quad \text{independently of t.} \tag{1.63}$$

We now identify $|\psi|^2 = \psi^\dagger\psi$ as the component $j_\psi^{0\cdots0}$ of the Dirac tensor current (1.59) and rewrite eq. (1.63) as a hypersurface integral, making the geometric structure explicit:

$$\int_{\Sigma_t} d\sigma(x_1) \cdots \int_{\Sigma_t} d\sigma(x_N) \; j_\psi^{0\cdots0}(x_1, ..., x_N) \;=\; 1, \quad \text{independently of t,} \tag{1.64}$$

where $\Sigma_t := \{(\tau, \mathbf{x}) \in \mathbb{R}^{1+d} : \tau = t\}$.

It is now easily recognized that a special family of hypersurfaces, the equal time surfaces Σ_t in a distinguished Lorentz frame, are used in the non-relativistic formulation. This flaw can be overcome by demanding the corresponding condition for all space-like hypersurfaces[16] Σ. Let n denote the normal covector field at Σ. We propose the following condition[17]:

$$\int_\Sigma d\sigma_1(x_1) \cdots \int_\Sigma d\sigma_N(x_N) \; n_{\mu_1}(x_1) \cdots n_{\mu_N}(x_N) \, j_\psi^{\mu_1\cdots\mu_N}(x_1, ..., x_N) \;=\; 1, \quad \text{independently of } \Sigma. \tag{1.65}$$

This is justified as follows. Firstly, the condition is completely geometric and does not attribute significance to a special class of space-like hypersurfaces. Secondly, for Σ_t one has $n \equiv (1, 0, ..., 0)$, so eq. (1.65) correctly reduces to eq. (1.64). Thirdly, the meaning of eq. (1.65) as expressing probability conservation can be established rigorously by a relativistic Bohmian analysis (see chap. 2).

In the case of a domain $\Omega \subset \mathbb{R}^{N(1+d)}$ with boundary, such as $\mathscr{S}$, one should restrict the range of integration to values in the domain and use the condition

$$\int_{\Sigma^N \cap \Omega} d\sigma_1(x_1) \wedge \cdots \wedge d\sigma_N(x_N) \; n_{\mu_1}(x_1) \cdots n_{\mu_N}(x_N) \, j_\psi^{\mu_1\cdots\mu_N}(x_1, ..., x_N) \;=\; 1, \quad \text{indpt. of } \Sigma. \tag{1.66}$$

It is clear that eqs. (1.65), (1.66) can be used for arbitrary positive-definite tensor currents j_ψ, not just for the Dirac current. An example for tensor currents which differ from the Dirac current will be encountered in chap. 5.

[16] We generally assume that space-like hypersurfaces are smooth and possess a normal covector field at every point.

[17] A related idea was used in [37, p. 163] to define N-particle Hilbert spaces associated with a space-like hypersurface Σ.

1.3.3 The current form ω_j

In order to obtain a completely geometric formulation of probability conservation, we recognize

$$\omega_j(x_1,...,x_N) \;:=\; d\sigma_1(x_1)\wedge\cdots\wedge d\sigma_N(x_N)\; n_{\mu_1}(x_1)\cdots n_{\mu_N}(x_N)\, j_\psi^{\mu_1\cdots\mu_N}(x_1,...,x_N) \quad (1.67)$$

as an Nd-form[18]. It will be referred to as the *current form* in the following.

For applications involving the current form on concrete domains such as $\mathscr{S}$, it is useful to have an expression for ω_j in terms of the coordinate differentials dx_i^μ (see chap. 3). To this end, we make use of the results [63, p. 435]:

$$d\sigma_i(x_i) \;=\; \sum_{\mu=0}^{d}(-1)^\mu n_\mu(x_i)\, dx_i^0\wedge\cdots\widehat{dx_i^\mu}\cdots\wedge dx_i^d, \quad (1.68)$$

$$n_\mu\, d\sigma_i \;=\; (-1)^\mu\, dx_i^0\wedge\cdots\widehat{dx_i^\mu}\cdots\wedge dx_i^d, \quad (1.69)$$

where $\widehat{dx_i^\mu}$ denotes omission from the wedge product. Using eq. (1.69) in the expression for ω_j, we obtain:

Lemma 1.3.1 *1. The current form can be rewritten as*

$$\omega_j \;:=\; \sum_{\mu_1,...,\mu_N=0}^{d}(-1)^{\mu_1+\cdots+\mu_N} j_\psi^{\mu_1\cdots\mu_N}(dx_1^0\wedge\ldots\widehat{dx_1^{\mu_1}}\cdots\wedge dx_1^d)$$
$$\wedge\cdots\wedge(dx_N^0\wedge\ldots\widehat{dx_N^{\mu_N}}\cdots\wedge dx_N^d) \quad (1.70)$$

2. Probability conservation on domains $\Omega\subset\mathbb{R}^{N(1+d)}$ with boundary is expressed by the following condition on the current form:

$$\int_{\Sigma^N\cap\Omega}\omega_j \;=\; \int_{(\Sigma')^N\cap\Omega}\omega_j \quad (1.71)$$

for all pairs of space-like hypersurfaces Σ,Σ'.

The continuity equations for j yield:

Lemma 1.3.2 *The exterior derivative of ω_j vanishes, i.e. $d\omega_j=0$.*

Proof:

$$d\omega_j \;=\; \sum_{\mu_1,...,\mu_N=0}^{d}(-1)^{\mu_1+\ldots+\mu_N}\partial_{1,\mu_1} j_\psi^{\mu_1\cdots\mu_N}(-1)^{\mu_1} dx_1^0\wedge\cdots\wedge dx_1^d\wedge\cdots\wedge dx_N^0\wedge\cdots\widehat{dx_N^{\mu_N}}\cdots\wedge dx_N^d$$
$$+\ldots+\sum_{\mu_1,...,\mu_N=0}^{d}(-1)^{\mu_1+\ldots+\mu_N}\partial_{N,\mu_N} j_\psi^{\mu_1\cdots\mu_N} dx_1^0\wedge\cdots\widehat{dx_1^{\mu_1}}\cdots\wedge dx_1^d$$
$$\wedge\cdots\wedge(-1)^{d(N-1)+\mu_N} dx_N^0\wedge\cdots\wedge dx_N^d$$
$$\overset{\text{eq.}(1.58)}{=}\; 0. \qquad \square \quad (1.72)$$

This result will allow us to relate the hypersurface integrals in (1.71) using Stokes' theorem (see chap. 3).

[18] See [51, chap. 16.1] for a similar idea for the non-relativistic case.

1.3.4 Hilbert spaces

We now show how adequate N-particle Hilbert spaces $\mathcal{H}_\Sigma^{(N)}$ associated with space-like hypersurface Σ arise naturally as a byproduct of the tensor currents. Let

$$\mathcal{H}_\Sigma^{(N)} \;:=\; \Big(L^2(\Sigma^N)\otimes\mathbb{C}^k,\, \langle\cdot,\cdot\rangle_\Sigma\Big)\,. \tag{1.73}$$

An appropriate scalar product $\langle\cdot,\cdot\rangle_\Sigma$ can be defined if there exists a sesquilinear form of two wave functions which generalizes the tensor current j_ψ, i.e. if there exists

$$j^{\mu_1\cdots\mu_N}[\psi_1,\psi_2](x_1,...,x_N) \quad\text{with}\quad \partial_{k,\mu_k} j^{\mu_1\cdots\mu_k\cdots\mu_N}[\psi_1,\psi_2](x_1,...,x_N) \;=\; 0, \quad k=1,...,N \tag{1.74}$$

for all solutions ψ_1,ψ_2 of the multi-time equations, and with

$$j^{\mu_1\cdots\mu_N}[\psi,\psi] \;=\; j_\psi^{\mu_1\cdots\mu_N}. \tag{1.75}$$

For the free multi-time Dirac theory, we have:

$$j^{\mu_1\cdots\mu_N}[\psi_1,\psi_2](x_1,...,x_N) \;=\; \overline{\psi}_1(x_1,...,x_N)\gamma_1^{\mu_1}\cdots\gamma_N^{\mu_N}\psi_2(x_1,...,x_N). \tag{1.76}$$

Then the natural choice[19] for the scalar product is given by:

$$\langle\phi,\chi\rangle_\Sigma \;:=\; \int_{\Sigma^N} d\sigma(x_1)\wedge\cdots\wedge d\sigma(x_N)\, j^{\mu_1\cdots\mu_N}[\phi,\chi](x_1,...,x_N)\, n_{\mu_1}(x_1)\cdots n_{\mu_N}(x_N). \tag{1.77}$$

One can easily verify that for $j[\psi_1,\psi_2]$ given by (1.76) the scalar product reduces to the familiar expression $\langle\phi,\chi\rangle_{\Sigma_t} = \int d^3\mathbf{x}_1\cdots d^3\mathbf{x}_N\,\phi^\dagger\chi$ for an equal-time hypersurface Σ_t.

Furthermore, the scalar product has physical meaning by its connection with ρ_Σ from eq. (1.60). This can be seen as follows: Define

$$\|\psi\|_\Sigma \;:=\sqrt{\langle\psi,\psi\rangle_\Sigma}. \tag{1.78}$$

Let $A\subset\Sigma^N$ and let $\mathbb{1}_A$ denote the indicator function of the set A. If $\|\psi\|_\Sigma=1$, then by eq. (1.60), $\|\mathbb{1}_A\psi\|_\Sigma^2$ yields the probability for a spatio-temporal configuration of N particles on Σ to be in A (see chap. 2 for a justification of the claim).

We note that a similar expression for the scalar product was suggested by Rizov, Sazdjian and Torodorov [84,90]. They define:

$$(\phi,\chi) \;:=\; \int_{\Sigma_1} d\sigma(x_1)\cdots\int_{\Sigma_N} d\sigma(x_N)\, j^{\mu_1\cdots\mu_N}[\phi,\chi](x_1,...,x_N)\, n_{\mu_1}(x_1)\cdots n_{\mu_N}(x_N). \tag{1.79}$$

The crucial difference to eq. (1.77) is that Rizov *et al.* allow the space-like hypersurfaces $\Sigma_k, k=1,...,N$ to be different. This is problematic because according to the argument in sec. 1.1.1.2, a multi-time wave function is naturally only defined on the set $\mathscr{S}$ of space-like configurations. However, an element of $\Sigma_1\times\cdots\times\Sigma_N$ is in general not a space-like configuration. Furthermore, there is no reason why then $\sqrt{(\psi,\psi)}$ should be equal to

[19] Note that the natural range of integration is $\Sigma^N\cap\mathscr{S}$, expressing that the particles are always within the domain. However, as will be shown in sec. 3.8, for $d>1$ the difference to (1.77) is only a zero-measure set.

unity. This is because a crossing probability ρ_Σ can in general only be defined on sets of the form Σ^N. The physical reason for this is that a time-like configuration may actually correspond to two points on the world-line of the same particle. Besides these physical arguments, the mathematical consistency of the multi-time equations (1.55) on the whole of $\mathbb{R}^{4N}$ may lead to very restrictive conditions on the possible operators H_i, e.g. excluding a multi-time formulation of QED models which (disregarding UV divergencies) is possible on $\mathscr{S}$ [20,81,96].

Time evolution between space-like hypersurfaces: Returning to the Hilbert spaces $\mathcal{H}_\Sigma^{(N)}$, we can understand multi-time evolution equations on $\mathscr{S}$ to define a unitary evolution between different space-like hypersurfaces Σ, Σ' as follows (see also [81]). Denote by $\psi_{|\Sigma}$ the restriction of a solution of the multi-time equations to Σ. Then

$$U_{\Sigma\to\Sigma'} : \mathcal{H}_\Sigma^{(N)} \to \mathcal{H}_{\Sigma'}^{(N)}, \quad \psi_{|\Sigma} \mapsto \psi_{|\Sigma'} \tag{1.80}$$

defines a map with $\|\psi_{|\Sigma}\|_\Sigma = \|U_{\Sigma\to\Sigma'}\psi_{|\Sigma'}\|_{\Sigma'} = \|\psi_{|\Sigma'}\|_{\Sigma'}$. This way of formulating the time evolution puts the multi-time system of equations in the foreground and yields the Hilbert space picture as a byproduct. Note that reversing the train of thought and first defining Hilbert spaces $\mathcal{H}_\Sigma^{(N)}$ and the map $U_{\Sigma\to\Sigma'}$ does not in general yield a multi-time wave function on $\mathscr{S}$ (which may be necessary for a consistent physical meaning, see chap. 2). This can be seen as follows: Pick two space-like hypersurfaces $\Sigma \neq \Sigma'$ with $\Sigma \cap \Sigma' \neq \emptyset$. Then the Hilbert spaces $\mathcal{H}_\Sigma^{(N)}$ and $\mathcal{H}_{\Sigma'}^{(N)}$ are essentially different and for $\phi \in \mathcal{H}_\Sigma^{(N)}$ one may have $\phi(q) \neq (U_{\Sigma\to\Sigma'}\phi)(q)$.

The connection to the Hilbert space setting for the Schrödinger equation is given by restricting to equal-time hypersurfaces Σ_t in a distinguished frame and identifying all Hilbert spaces $\mathcal{H}_{\Sigma_t}^{(N)}$ for different t [81]. This is possible for flat hypersurfaces without changing the scalar product. Also, because of this identification, the above-mentioned problem cannot occur because trivially there do not exist $t \neq t'$ with $\Sigma_t \cap \Sigma_{t'} \neq \emptyset$.

Remark: One should only expect an evolution map $U_{\Sigma\to\Sigma'}$ to exist for multi-time evolution equations with a Cauchy data initial value problem. Out of the alternatives sketched in sec. 1.2.2, points 1 and 2 fall into this category, 3-5 do not.

1.3.5 Probability conservation implies uniqueness of solutions

This subsection is based on the article [69, sec. 4.3] by Lukas Nickel and the present author. The notion (1.65) of probability conservation is quite powerful. In this section, we show that $\int_{\Sigma^N\cap\Omega} \omega_j$ is a so-called *energy integral* and that therefore probability conservation implies uniqueness of solutions in a suitable sense.

Let Σ be a space-like hypersurface and let $\Omega \subset \mathbb{R}^{4N}$ the domain of the multi-time wave function. We define function spaces

$$\mathcal{H}_\Sigma^{(N)}(\Omega) \;:=\; L^2(\Sigma^N \cap \Omega) \otimes \mathbb{C}^k. \tag{1.81}$$

These function spaces slightly generalize the previous $\mathcal{H}_\Sigma^{(N)}$ by allowing for domains with boundaries.

Furthermore, we call the solution of multi-time evolution equations[20] (with possible additional conditions such as boundary conditions) *weakly unique* if and only if for every two solutions ψ, φ and every space-like hypersurface Σ the restrictions $\psi_{|\Sigma}$, $\varphi_{|\Sigma}$ of ψ, φ to arguments in $\Sigma^N \cap \Omega$ are equal as elements of $\mathcal{H}_\Sigma^{(N)}(\Omega)$.

In the concrete applications in chaps. 3 and 4, we will deal with classical, i.e. differentiable solutions. Then "weakly unique" simply means that the solutions are equal point-wise. In general, a more abstract (weak) understanding of a "solution" is conceivable, such as in sec. 1.2.1.1.

Theorem 1.3.3 *Let Σ_0 be a space-like hypersurface. Consider multi-time evolution equations which imply a conserved tensor current j_ψ (1.58), with boundary conditions ensuring probability conservation (1.71) and initial values*[21] *on $\mathcal{I} = (\Sigma_0)^N \cap \Omega$, i.e. $\psi_{|\mathcal{I}} \equiv g \in \mathcal{H}_{\Sigma_0}^{(N)}(\Omega)$. Then its solution is weakly unique.*

Proof: Consider the expression

$$\|\phi\|_\Sigma^2 \;:=\; \int_{\Sigma^N \cap \Omega} \omega_j(\phi), \tag{1.82}$$

where $\omega_j(\phi)$ is the current form constructed from j_ϕ according to eq. (1.70). Because the tensor current j_ϕ is assumed to be positive-definite and sesquilinear in the wave function, $\|\cdot\|_\Sigma$ defines a norm on $\mathcal{H}_\Sigma^{(N)}(\Omega)$.

Let ψ, φ be solutions of the initial boundary value problem. Then: $\psi_{|\Sigma_0} \equiv \varphi_{|\Sigma_0} \equiv g \in \mathcal{H}_{\Sigma_0}^{(N)}(\Omega)$ and therefore $\|\psi_{|\Sigma_0} - \varphi_{|\Sigma_0}\|_{\Sigma_0} = 0$. Now let Σ be an arbitrary space-like hypersurface. Probability conservation (1.71) yields:

$$\|\psi_{|\Sigma} - \varphi_{|\Sigma}\|_\Sigma \;=\; \|\psi_{|\Sigma_0} - \varphi_{|\Sigma_0}\|_{\Sigma_0} \;=\; 0 \tag{1.83}$$

and it follows that $\psi_{|\Sigma} \equiv \varphi_{|\Sigma}$ as elements of $\mathcal{H}_\Sigma^{(N)}(\Omega)$. □

Having reached this formal framework, we are prepared to analyze the physical role of multi-time wave functions (chap. 2) as well as the question of how to construct interacting wave equations (chaps. 3-5).

[20] We deliberately leave open the exact nature of these equations in order to arrive at a general result. A concrete example are the Hamiltonian multi-time equations (1.47) on $\Omega = \mathbb{R}^{4N}$. Further examples including boundary conditions will be presented in chap. 3.

[21] The theorem still holds if the multi-time evolution equations do not have a Cauchy data initial value problem. Initial values are then understood as one particular condition on the multi-time wave function.

Chapter 2

On the physical meaning of multi-time wave functions and their place in realistic relativistic quantum theories

> "[The usual quantum] paradoxes are simply disposed of by the 1952 theory of Bohm, leaving as *the* question, the question of Lorentz invariance. So one of my missions in life is to get people to see that if they want to talk about the problems of quantum mechanics – the real problems of quantum mechanics – they must be talking about Lorentz invariance."
>
> Bell, 1990, interview with philosopher Renée Weber

In this chapter, we treat the question of how a multi-time wave function attains physical significance in the overall structure of the theory. This includes a justification of the envisioned statistical meaning of the tensor current $j_\psi^{\mu_1 \cdots \mu_N}$ as well as its density component which were motivated by formal considerations in sec. 1.3.

We approach these questions using realistic quantum theories, such as Bohmian mechanics [21,51] and GRW theories [5,10,97]. Both have been used successfully in non-relativistic QM to avoid the measurement problem and to establish the statistical meaning of the wave function from a deeper (ontological) level[1]. An extension of these theories to the relativistic domain is thus desirable.

The chapter is structured as follows. We begin with a brief introduction of the most-discussed relativistic generalization of Bohmian mechanics, the *hypersurface Bohm-Dirac (HBD) model* (sec. 2.1). Particular emphasis is put on the resulting requirements on the multi-time wave function and the tensor currents. Subsequently, a subsystem description is developed for the model (sec. 2.2). Such a description is crucial to derive statistical predictions for typical subsystems in Bohmian mechanics. It furthermore answers the question of how the various wave function and/or density matrix descriptions of subsystems of a larger system are related and how they can be derived and justified in their meaning using the

[1] See the introduction for the notion of "primitive ontology".

equations of the larger system only. Large parts of the section concern a new *relativistic conditional density matrix*. In a brief outlook (sec. 2.2.5), we outline how the developed subsystem description could be used in further work on the statistical analysis of the HBD model. Besides, a comparison with a related analysis of Bloch is included (sec. 2.3).

The chapter ends with a brief presentation of alternative realistic relativistic models (sec. 2.4). The motivation for this is (i) to show that the assumption of a preferred foliation in the HBD model is not in principle necessary to formulate relativistic Bohmian laws of motion but rather to be able to perform a statistical analysis as in the non-relativistic case (sec. 2.4.1) and (ii) to demonstrate that also objective collapse (GRW) theories lead to similar requirements on a multi-time wave function as the HBD model (sec. 2.4.2).

2.1 The hypersurface Bohm-Dirac model

The HBD model aims at a Lorentz invariant generalization of non-relativistic Bohmian mechanics (BM, see e.g. [49,51]). It describes N point particles with world lines in Minkowski space-time. The law of motion for the k-th particle is determined by the multi-time wave function as well as the "simultaneous" configuration of all the other particles. Indeed, it can then be considered the primary role of the wave function to determine the law for the world lines. Recalling the ideal picture of a relativistic theory as sketched in the introduction, it may at first seem that any notion of "simultaneity" must conflict with any serious understanding of Lorentz invariance[2]. The key idea here is the realization that a *dynamical foliation $\mathcal{F}$ of space-time into space-like hypersurfaces* Σ can provide a seriously Lorentz invariant notion of simultaneity [43]. A dynamical foliation can, for example, be extracted from the (universal) wave function[3]. However, there is as yet no reason to pick a distinguished foliation. For the rest of the section, we therefore assume that an arbitrary foliation $\mathcal{F}$ has been chosen. Furthermore, the space-like hypersurfaces $\Sigma \in \mathcal{F}$, called "leaves" of the foliation, are assumed to be smooth.

Besides the generalization of the notion of simultaneity appearing in the law of motion of non-relativistic BM, there exists a further reason for a distinguished foliation:

> "There does not in general exist a probability measure P on N-paths for which the distribution of crossings ρ^Σ agrees with the corresponding quantum-mechanical distribution on all space-like hyperplanes Σ." [19]

Assuming that it is sufficiently clear what "the corresponding quantum-mechanical distribution" is[4], this statement suggests that either the way a statistical analysis proceeds has to be changed as compared to the one in the non-relativistic case (see [47]) or that a probability measure with the above properties can only be found on a special foliation. The HBD model focuses on the second possibility. (See sec. 2.4.1 for a more detailed discussion of the difficulties associated with the first possibility.)

Originally, the HBD model was formulated for N non-interacting but entangled Dirac (spin-$\frac{1}{2}$) particles [42], i.e. for j_ψ given by the free Dirac current (1.59). Here, we extend

[2] The term "serious Lorentz invariance" was coined by Bell, who considered subtleties about Lorentz invariance in [13, p. 179-180].

[3] This idea does, however, stand in conflict with a nomological understanding of the wave function (i.e. the wave function as defining the law of motion only), see [49, chaps. 12.3.8, 12.3.9].

[4] We take it to refer to ρ_Σ of eq. (1.60).

the model to encompass also the interacting case using the most general form of the tensor current $j_\psi^{\mu_1\cdots\mu_N}$ of sec. 1.3 which is compatible with the envisioned statistical role of ρ_Σ. From [42, sec. III], one can read off the necessary requirements on j_ψ. Let $n(x)$ be the future-oriented normal covector field associated with $\mathcal{F}$ (i.e. for every $\Sigma \in \mathcal{F}$ and $n(x)$ is the future-oriented normal covector at $x \in \Sigma$):

1. $j_\psi^{\mu_1\cdots\mu_N}(x_1,...,x_N)n_{\mu_1}(x_1)\cdots n_{\mu_N}(x_N) \geq 0$ for all space-like related $x_1,...,x_N$ and all multi-time wave functions ψ from a sufficiently general function space[5]. In words: the tensor current has to be *positive-definite.*

2. $\partial_{k,\mu_k} j_\psi^{\mu_1\cdots\mu_k\cdots\mu_N} = 0$ for all $k = 1,...,N$ and ψ as above. In words: the tensor current has to be *divergence-free* in all of its indices.

We assume that the multi-time wave equations are chosen as to ensure the existence of a j_ψ with these two properties. This is e.g. the case for the free multi-time Dirac theory (eq. (1.18)).

The law for the N jointly parametrized world lines $X_k(s)$, $k = 1,...,N$ is now readily formulated:

$$\frac{dX_k^{\mu_k}(s)}{ds} \;\propto\; j_\psi^{\mu_1\cdots\mu_k\cdots\mu_N}(x_1,...,x_N)\prod_{j\neq k} n_{\mu_j}(x_j)\Big|_{x_i=X_i(\Sigma),\ i=1,...,N.} \tag{2.1}$$

Here, $X_i(\Sigma)$ denotes the intersection point of the i-th world line with $\Sigma \in \mathcal{F}$, and $X_k(s) \in \Sigma$. The proportionality sign expresses that the tangent vector $\dot{X}_k(\Sigma)$ should be parallel to the rhs. (which is also a vector). This geometrical formulation implies the arbitrariness of the joint parametrization of the world lines via s.

The statistical import of the HBD model was analyzed in [42]: ρ_Σ of eq. (1.60) obeys the continuity equation for curved surfaces and thus is an equivariant density on the leaves of the foliation, generalizing the well-known $|\psi|^2$-distribution. Therefore, the crossing probability of $\Sigma \in \mathcal{F}$ is given by:

$$\text{Prob}\,(\text{particle } i \text{ crosses } \Sigma \text{ in } d\sigma_i,\ i = 1,...,N) \;=\; \rho_\Sigma(x_1,...,x_N)d\sigma_1\cdots d\sigma_N, \tag{2.2}$$

where (in slight abuse of notation) $d\sigma_i$ denotes both an infinitesimal area on Σ around x_i as well as its 3-volume.

The main goal of the HBD model has now been achieved: to find a manifestly Lorentz covariant law of relativistic particle motion which is consistent with the formal density on the wave function level.

Remarks:

1. Note that the HBD model only requires the multi-time wave function to be defined on configurations $(x_1,...,x_N) \in \bigcup_{\Sigma\in\mathcal{F}} \Sigma^N$. Excluding configurations with $x_j = x_k$ for some $j \neq k$, this set is a strict subset of the space-like configurations $\mathscr{S}$. However, the (multi-time) Lorentz invariance of the wave equations can only be discussed if ψ

[5] Reasonable restrictions on the admissible wave functions may e.g. result from the existence and uniqueness theory of the law of motion of the HBD model.

is defined on the whole of $\mathscr{S}$. In any case, these facts further support the view of sec. 1.1.1.2 that it is unnecessary for the multi-time wave function to be defined on the whole of configuration space-time $\mathbb{R}^{4N}$.

2. The formula (2.2) for the crossing probability is only valid for hypersurfaces $\Sigma \in \mathcal{F}$. The HBD model so far does not make any statistical statements for hypersurfaces not belonging to the foliation. It was remarked in [42] that this fact need not stand in conflict with the usual quantum formalism. The main point is that the quantum formalism only refers to "measurements", i.e. special situations involving decoherence and registering devices consisting of many particles. In order to be able to speak of an experimental result, certain particle constellations (e.g. the ones comprising a computer memory) have to persist for sufficiently large times. Therefore, they will also be encoded in a particle configuration on one of the leaves Σ of the foliation. This leads to the conjecture that the measurement formalism implied by the HBD model is independent of the foliation.

2.2 On the description of subsystems in relativistic hypersurface Bohmian mechanics

The following section is taken from the article [50] by Detlef Dürr and the present author. It does not include those parts of [50] which are treated in other sections of the present thesis (this concerns secs. 2 and 3 (b) of [50]). Besides, minor changes concerning references to formulas and sections have been made.

2.2.1 Background and motivation

The possibility of describing a subsystem of a larger system in an autonomous way is basic to physics. In quantum physics entanglement prevails and an autonomous subsystem description is harder to justify than in classical physics: A quantum mechanical N-particle system possesses a wave function $\psi(\mathbf{x}_1, ...\mathbf{x}_N, t)$ where $\mathbf{x}_1, ..., \mathbf{x}_N$ are the particle coordinates. Suppose that a subsystem is formed by the particles with coordinates $q_1 = (\mathbf{x}_1, ..., \mathbf{x}_M)$. A simple way to associate a wave function with the subsystem exists if ψ has the form

$$\psi(q_1, q_2, t) \;=\; \varphi(q_1, t)\phi(q_2, t), \tag{2.3}$$

where $q_2 = (\mathbf{x}_{M+1}, ..., \mathbf{x}_N)$. Then φ can be regarded as the wave function of the subsystem. However, the presence of entanglement exactly means that ψ cannot be written in the form (2.3). This leads us to the question: which possibilities does quantum physics provide to describe the subsystem?

In open system quantum theory, a non-autonomous subsystem description is achieved by the reduced density matrix

$$W_{\mathrm{red}}(q_1, q_1', t) \;=\; \int dq_2\, \psi(q_1, q_2, t)\psi^*(q_1', q_2, t), \tag{2.4}$$

where the environment is "traced out", i.e. its actual state is ignored. The description is non-autonomous because the time evolution for W_{red} is not given by a closed equation. In

contrast, the association of the subsystem with a subsystem's wave function usually involves a preparation procedure like an ideal measurement which we informally describe by

$$\psi(q_1, q_2, t=0) \overset{\text{Schrödinger evolution}}{\longmapsto} \sum_\alpha \varphi_\alpha(q_1,t)\phi_\alpha(q_2,t) \overset{\text{collapse}}{\longmapsto} \varphi_{\alpha_0}(q_1,t)\phi_{\alpha_0}(q_2,t), \quad (2.5)$$

which leads to a projection onto a subsystem's wave function – a projection which depends on the wave function of the total system, the environment included. The projection procedure, also called the collapse of the wave function, is, however, not theoretically founded on the fundamental equations of quantum mechanics. This fact is also known as the measurement problem or objectification problem. In order to achieve a *justification* of the quantum formalism, a subsystem analysis is needed [48,71].

In Bohmian mechanics (BM; see e.g. [21,49,51,57]), the subsystem description is part of the theory and the objectification problem therefore does not even occur. For spin-less BM, one can define a *conditional wave function* ψ_{cond} by plugging into the wave function of the larger system $\Psi(q_1, q_2, t)$ the actual Bohmian environment configuration Q_2 [47]:

$$\psi_{\text{cond}}(q_1, t) \;=\; \frac{1}{\mathcal{N}}\Psi(q_1, Q_2(t), t), \quad (2.6)$$

where $\mathcal{N}$ is a normalization factor such that $\|\psi_{\text{cond}}\| = 1$.

The evolution of the subsystem configuration Q_1 then only depends on the environment configuration via the conditional wave function. Besides, it is also possible to express conditional probabilities for the subsystem via ψ_{cond} only. Therefore, the conditional wave function is basic to the statistical analysis of BM [47].

However, the description of a subsystem by the conditional wave function is typically not autonomous. Nevertheless, in certain situations autonomy can be attained and the conditional wave function then becomes effective [47], in the sense that even the implicit reference to the environment configuration (which is present in ψ_{cond}) is lost. More precisely, we say that a subsystem has the *effective wave function* ψ_{eff} (up to normalization) at time t if the wave function Ψ of the total system and the environment configuration $Q_2(t)$ satisfy

$$\Psi(q_1, q_2, t) \;=\; \psi_{\text{eff}}(q_1, t)\Phi(q_2, t) + \Psi^{\perp}(q_1, q_2, t), \quad (2.7)$$

where Φ and $\Psi^{\perp}$ have *macroscopically disjoint*[6] q_2-supports and $Q_2(t) \in \operatorname{supp}\Phi$.

The effective wave function reflects the idea of a quantum mechanical subsystem wave function that has been "prepared" by controlling the environment (e.g. experimental devices). ψ_{eff} persists as long as the Schrödinger evolution of the composite system does not destroy the effective product structure. If it exists, $\psi_{\text{eff}} = \psi_{\text{cond}}$.

For non-relativistic BM with spin, the conditional wave function does not exist anymore. Plugging the actual environment configuration into the wave function of the composite system yields a spinor-valued wave function which still contains all the spinor degrees of freedom of the environment. Denote the spin components of the wave function of the composite system by $\Psi^{s_1 s_2}$. Then naively generalizing eq. (2.6) would yield

$$\psi^{s_1 s_2}_{\text{cond}}(q_1, t) \;=\; \frac{1}{\mathcal{N}}\Psi^{s_1 s_2}(q_1, Q_2(t), t), \quad (2.8)$$

[6] See [47] for details.

which cannot be a wave function associated only with the subsystem, as the spin index of the environment is still present.

The substitute is a *conditional density matrix* [46] where the spinor degrees of freedom of the environment are traced out. Explicitly:

$$W^{\mathrm{nonrel}\, s_1}_{\mathrm{cond}\quad s_1'}(q_1, q_1', t) \;=\; \frac{1}{\mathcal{N}} \sum_{s_2} \Psi^{s_1 s_2}(q_1, Q_2(t), t) \Psi^{\dagger}_{s_1' s_2}(q_1, Q_2(t), t), \tag{2.9}$$

where $(\cdot)^\dagger$ denotes the conjugate transposed.

It turns out that the conditional density matrix functions in the same way as the conditional wave function in the spin-less case [46]. Of course, there are also situations in which an effective wave function exists [46]. It is clear that if this is the case the conditional density matrix is pure and given by the effective wave function. The converse is less obvious.

In the following, we extend the subsystem description to relativistic BM with spin, namely to the hypersurface Bohm-Dirac model [42]. It turns out that the description leads to a non-trivial generalization of the non-relativistic conditional density matrix. Having introduced the new conditional density matrix, we analyze its properties and find a lifting to a density operator on the Hilbert space $\mathcal{H}_\Sigma^{(N)}$ of eq. (1.73). Finally, we generalize the notion of the effective wave function and prove a lemma that clarifies its relation to the conditional density matrix.

2.2.2 Notation

For the rest of the section, we consider a HBD system S (i.e. a set of particles) composed of two parts: S_1, the subsystem of interest and S_2, the environment. Schematically, we write this as $S = S_1 \cup S_2$. The number of particles splits according to $N = N_1 + N_2$. The wave function of S is denoted by Ψ. We refer to its spin components as $\Psi^{s_1 s_2}$ and to the partial trace over these spin components as $\mathrm{tr}_{\mathbb{C}^{k_i}}$ where $k_i = 4^{N_i}$, $i = 1, 2$. If $\Psi = \psi_1 \otimes \psi_2$, then $\overline{\psi}_1 = \psi_1^\dagger \gamma_1^0 \cdots \gamma_{N_1}^0$ and $\overline{\psi}_2 = \psi_2^\dagger \gamma_{N_1+1}^0 \cdots \gamma_N^0$ where these gamma-matrices now act on $\mathbb{C}^{k_1}$ and $\mathbb{C}^{k_2}$, respectively. Consider a space-like hypersurface $\Sigma \in \mathcal{F}$ (we will use the symbol $\mathcal{S}$ for arbitrary space-like hypersurfaces). The configuration obtained by intersecting the world lines of the particles with Σ is denoted by $Q(\Sigma) = (Q_1(\Sigma), Q_2(\Sigma)) \in \Sigma^N$. A generic configuration space-time variable is called $q = (q_1, q_2) = (x_1, ..., x_{N_1}, x_{N_1+1}, ..., x_N) \in \mathbb{M}^N$. The complex conjugate of $z \in \mathbb{C}$ is denoted by z^*.

2.2.3 Results

In order to be able to make concrete calculations, we henceforth assume that the tensor current in the HBD law (2.1) is given by the Dirac current

$$j_\psi^{\mu_1 \cdots \mu_N} \;=\; \overline{\psi} \gamma_1^{\mu_1} \cdots \gamma_N^{\mu_N} \psi. \tag{2.10}$$

Note that a different tensor current would entail modifications of the results. However, the assumption of the Dirac current is plausible as it implies the usual $|\psi|^2$-density on flat hypersurfaces.

2.2.3.1 Conditional density matrix

We first aim at a subsystem description of S_1 by a conditional density matrix. To begin with, we rewrite eq. (2.1), observing (2.10), for the world lines of particles in S_1 by applying the identity

$$v^\dagger w = (v_1^*, ..., v_k^*) \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix} = v_1^* w_1 + ... + v_N^* w_N = \mathrm{tr}_{\mathbb{C}^k} \begin{pmatrix} v_1^* w_1 & & * \\ & \ddots & \\ * & & v_N^* w_N \end{pmatrix} = \mathrm{tr}_{\mathbb{C}^k}(w v^\dagger) \tag{2.11}$$

to the rhs. of eq. (2.1) with $v^\dagger = \overline{\Psi}(q)\gamma_1^{\mu_1}...\gamma_N^{\mu_N} \prod_{j\neq k} n_{\mu_j}(x_j)$ and $w = \Psi(q)$ for fixed q. This yields:

$$\frac{dX_k^{\mu_k}(s)}{ds} \;\propto\; \mathrm{tr}_{\mathbb{C}^k} \left[\Psi\overline{\Psi}\gamma_1^{\mu_1} \cdots \gamma_k^{\mu_k} \cdots \gamma_N^{\mu_N} \prod_{j\neq k} n_{\mu_j}(x_j) \right]_{q=Q(\Sigma)} . \tag{2.12}$$

Next, we split up the trace according to $\mathrm{tr}_{\mathbb{C}^k} \equiv \mathrm{tr}_{\mathbb{C}^{k_1}} \mathrm{tr}_{\mathbb{C}^{k_2}}$ and noting that the γ-matrices in eq. (2.12) commute, we obtain after rearranging:

$$\frac{dX_k^{\mu_k}(s)}{ds} \;\propto\; \mathrm{tr}_{\mathbb{C}^{k_1}} \left\{ \mathrm{tr}_{\mathbb{C}^{k_2}} \left[\Psi\Psi^\dagger \prod_{j\in S_2} \gamma_j^0 \gamma_j^{\mu_j} n_{\mu_j}(x_j) \right]_{q_2=Q_2(\Sigma)} \gamma_k^0 \gamma_k^{\mu_k} \prod_{j\neq k,\, j\in S_1} \gamma_j^0 \gamma_j \cdot n(x_j) \right\}_{q_1=Q_1(\Sigma).} \tag{2.13}$$

In slight abuse of notation, in eq. (2.13) we use the same symbols for the gamma matrices as before although now $\gamma_j^{\mu_j}$, $j = 1, ..., N_1$, act on $\mathbb{C}^{k_1}$ instead of $\mathbb{C}^k$. The separation of variables associated with S_1 and S_2 in eq. (2.13) leads to a rewriting of the HBD guidance law of the desired form:

$$\frac{dX_k^{\mu_k}(s)}{ds} \;\propto\; \mathrm{tr}_{\mathbb{C}^{k_1}} \left\{ W_{\mathrm{cond}}(q_1, q_1')\; \gamma_k^0 \gamma_k^{\mu_k} \prod_{j\neq k,\, j\in S_1} \gamma_j^0 \gamma_j \cdot n(x_j) \right\}_{q_1=q_1'=Q_1(\Sigma)} , \tag{2.14}$$

where

$$W_{\mathrm{cond}}{}_{s_1'}^{s_1}(q_1, q_1') \;:=\; \frac{1}{\mathcal{N}} \sum_{s_2} \Psi^{s_1 s_2}(q_1, Q_2(\Sigma)) \left[\Psi^\dagger(q_1', Q_2(\Sigma)) \prod_{j\in S_2} \gamma_j^0 \gamma_j \cdot n(X_j(\Sigma)) \right]_{s_1' s_2} \tag{2.15}$$

defines the components of the *conditional density matrix* for S_1 and

$$\mathcal{N} := \int_\Sigma d\sigma_1 \cdots \int_\Sigma d\sigma_{N_1} \mathrm{tr}_{\mathbb{C}^k} \left[\Psi(q_1, Q_2(\Sigma)) \Psi^\dagger(q_1, Q_2(\Sigma)) \prod_{j\in S_2} \gamma_j^0 \gamma_j \cdot n(X_j(\Sigma)) \prod_{j\in S_1} \gamma_j^0 \gamma_j \cdot n(x_j) \right] \tag{2.16}$$

is the appropriate normalization factor. Note that $\mathcal{N}$ is independent of the choice of the space-like hypersurface in the domain of integration. To see this, we write:

$$\begin{aligned}\mathcal{N} &= \int_\Sigma d\sigma_1 \cdots \int_\Sigma d\sigma_{N_1} \Psi^\dagger(q_1, Q_2(\Sigma)) \prod_{j\in S_2} \gamma_j^0 \gamma_j \cdot n(X_j(\Sigma)) \prod_{j\in S_1} \gamma_j^0 \gamma_j \cdot n(x_j) \Psi(q_1, Q_2(\Sigma)) \\ &= \int_\Sigma d\sigma_{1,\mu_1} \cdots \int_\Sigma d\sigma_{N_1,\mu_{N_1}} \; j^{\mu_1 \cdots \mu_{N_1} \mu_{N_1+1} \cdots \mu_N}[\Psi,\Psi]|_{q_2=Q_2(\Sigma)} \prod_{j\in S_2} n_{\mu_j}(X_j(\Sigma)), \quad (2.17)\end{aligned}$$

which can easily be shown to be independent of the hypersurface in the domain of integration assuming drop-off conditions for the wave function in the space-like directions in each particle coordinate and by using the divergence theorem.

Also note that for a flat foliation, i.e. $n(x) = (1,0,0,0)\ \forall x$ in a certain frame, our definition of W_{cond} coincides with the one for non-relativistic BM with spin in that frame (cf. eq. (2.9) and since $(\gamma_k^0)^2 = \mathbb{1}\,\forall k$).

The physical significance of W_{cond} is based on its dynamical role as well as its role in the statistical analysis. Eq. (2.14) establishes the dynamical role of W_{cond}. In order to analyze the statistical meaning of W_{cond}, we start from the crossing probability of $\Sigma \in \mathcal{F}$ of the HBD model (eq. (2.2)):

$$\text{Prob}\,(\text{particle } i \text{ crosses } \Sigma \text{ in } d\sigma_i,\ i = 1, ..., N) = \rho(x_1, ..., x_N) d\sigma_1 \cdots d\sigma_N. \quad (2.18)$$

Next, we condition:

$$\text{Prob}\,(\text{particle } i \text{ crosses } \Sigma \text{ in } d\sigma_i,\ i = 1, ..., N_1 | Q_2(\Sigma)) = \frac{\rho(x_1, ..., x_{N_1}, Q_2(\Sigma)) d\sigma_1 \cdots d\sigma_{N_1}}{\int_\Sigma d\sigma_1 \cdots \int_\Sigma d\sigma_{N_1} \rho(x_1, ..., x_{N_1}, Q_2(\Sigma))} \quad (2.19)$$

and comparing eq. (2.10) with

$$\rho(x_1, ..., x_N) = \overline{\Psi}(q) \gamma_1^{\mu_1} \cdots \gamma_N^{\mu_N} \Psi(q) \prod_{j=1}^N n_{\mu_j}(x_j) \quad \left(= j_\Psi^{\mu_1 \cdots \mu_N} \prod_{j=1}^N n_{\mu_j}(x_j) \right), \quad (2.20)$$

we repeat the same steps leading from eq. (2.1) to eqs. (2.12)-(2.14) to obtain:

$$\begin{aligned}&\text{Prob}\,(\text{particle } i \text{ crosses } \Sigma \text{ in } d\sigma_i,\ i = 1, ..., N_1 | Q_2(\Sigma)) \\ &= \text{tr}_{\mathbb{C}^{k_1}} \left[W_{\text{cond}}(q_1, q_1) \prod_{j\in S_1} \gamma_j^0 \gamma_j \cdot n(x_j) \right] d\sigma_1 \cdots d\sigma_{N_1}. \quad (2.21)\end{aligned}$$

Eqs. (2.19) and (2.21) also explain why the normalization (2.16) of W_{cond} is appropriate: $\mathcal{N} = \int_\Sigma d\sigma_1 \cdots \int_\Sigma d\sigma_{N_1} \rho(x_1, ..., x_{N_1}, Q_2(\Sigma))$.

Eq. (2.21) allows us to calculate expectation values, e.g. of the distribution of the subsystem configuration. Let $\hat{Q}_1$ denote the multiplication operator with q_1 on $\mathcal{H}_\Sigma^{(N_1)}$ for $\Sigma \in \mathcal{F}$; then its expectation value for a "state" characterized by W_{cond} is given by:

$$\langle \hat{Q}_1 \rangle_{W_{\text{cond}}} = \int_\Sigma d\sigma_1 \cdots \int_\Sigma d\sigma_{N_1} \; q_1 \, \text{tr}_{\mathbb{C}^{k_1}} \left[W_{\text{cond}}(q_1, q_1) \prod_{j\in S_1} \gamma_j^0 \gamma_j \cdot n(x_j) \right]. \quad (2.22)$$

2.2.3.2 Conditional density operator

We introduce an operator $\hat{W}_{\mathrm{cond}}$ on $\mathcal{H}_{\mathcal{S}}^{(N_1)}$ such that for $\mathcal{S} = \Sigma \in \mathcal{F}$, eq. (2.22) can be rewritten as the trace $\mathrm{tr}(\hat{W}_{\mathrm{cond}}\,\hat{Q}_1)$.

For this purpose, we define bras $\langle q_1, s_1|$ and kets $|q_1', s_1'\rangle$ with $\langle q_1, s_1|q_1', s_1'\rangle = \delta^{(3N_1)}(q_1 - q_1')\delta_{s_1 s_1'}$ and $\hat{Q}_1|q_1, s_1\rangle = q_1|q_1, s_1\rangle$ where $\langle\cdot,\cdot\rangle$ is the scalar product (1.77) on $\mathcal{H}_{\mathcal{S}}^{(N_1)}$ (cf. (1.73)).
Let

$$(\gamma n)(q_i) := \prod_{j\in S_i} \gamma_j^0 \gamma_j \cdot n(x_j), \quad \text{where} \quad (\gamma n)^\dagger(q_i) = (\gamma n)(q_i), \quad i = 1, 2, \tag{2.23}$$

since $(\gamma_k^{\mu_k})^\dagger = \gamma_k^0 \gamma_k^{\mu_k} \gamma_k^0$.

We now show that for every fixed q_i, $(\gamma n)(q_i)$, $i = 1, 2$, is a positive matrix. To see this, we consider the quadratic forms $\Psi^\dagger\,(\gamma n)(q_i)\,\Psi$, $i = 1, 2$ and rewrite them using eqs. (2.23) and (1.76):

$$\begin{aligned}
\Psi^\dagger\,(\gamma n)(q_1)\,\Psi &= j^{\mu_1\cdots\mu_{N_1}0\cdots0}[\Psi,\Psi] \prod_{j\in S_1} n_{\mu_j}(x_j), \\
\Psi^\dagger\,(\gamma n)(q_2)\,\Psi &= j^{0\cdots0\,\mu_{N_1+1}\cdots\mu_N}[\Psi,\Psi] \prod_{j\in S_2} n_{\mu_j}(x_j),
\end{aligned} \tag{2.24}$$

which both are greater or equal to zero as the tensor current $j^{\mu_1\cdots\mu_N}[\Psi,\Psi]$ is positive-definite.

As a consequence of the positivity and the self-adjointness of $(\gamma n)(q_i)$, it is possible to define $\sqrt{(\gamma n)(q_i)}$. The relation of $\langle q_1, s_1|\varphi\rangle$ to the components $\varphi^{s_1}(q_1)$ of a wave function $\varphi \in \mathcal{H}_{\mathcal{S}}^{(N_1)}$ then is:

$$\langle q_1, s_1|\varphi\rangle = \sum_{s_1'} \sqrt{(\gamma n)(q_1)}^{s_1}_{s_1'} \varphi^{s_1'}(q_1). \tag{2.25}$$

Using the abbreviation

$$\int_{\mathcal{S}^{N_1}} d^{3N_1}q_1 \;:=\; \int_{\mathcal{S}} d\sigma_1 \cdots \int_{\mathcal{S}} d\sigma_{N_1}, \tag{2.26}$$

we have

$$\begin{aligned}
\mathrm{tr}(\hat{W}_{\mathrm{cond}}\,\hat{Q}_1) &= \sum_{s_1} \int_{\mathcal{S}^{N_1}} d^{3N_1}q_1\, \langle q_1, s_1|\hat{W}_{\mathrm{cond}}\,\hat{Q}_1|q_1, s_1\rangle \\
&= \sum_{s_1} \int_{\mathcal{S}^{N_1}} d^{3N_1}q_1\; q_1\, \langle q_1, s_1|\hat{W}_{\mathrm{cond}}|q_1, s_1\rangle.
\end{aligned} \tag{2.27}$$

Comparing eq. (2.27) with eq. (2.22), we are led to define:

$$\langle q_1, s_1|\hat{W}_{\mathrm{cond}}|q_1', s_1'\rangle := \left(\sqrt{(\gamma n)(q_1)}\, W_{\mathrm{cond}}(q_1, q_1') \sqrt{(\gamma n)(q_1')}\right)^{s_1}_{s_1'}. \tag{2.28}$$

The action of $\hat{W}_{\mathrm{cond}}$ on a vector $\varphi \in \mathcal{H}_{\mathcal{S}}^{(N_1)}$, expressed in components, is given by:

$$(\hat{W}_{\mathrm{cond}}\,\varphi)^{s_1}(q_1) := \int_{\mathcal{S}^{N_1}} d^{3N_1}q_1' \sum_{s_1'} \left[W_{\mathrm{cond}}(q_1,q_1')(\gamma n)(q_1')\right]^{s_1}_{s_1'} \varphi^{s_1'}(q_1'). \tag{2.29}$$

Note that eqs. (2.25), (2.28) and (2.29) are chosen consistently, as one must have:

$$\begin{aligned}
&\left[\sqrt{(\gamma n)(q_1)}\,\hat{W}_{\mathrm{cond}}\,\varphi\right]^{s_1}(q_1) \overset{(2.25)}{=} \langle q_1,s_1|\hat{W}_{\mathrm{cond}}\,|\varphi\rangle \\
&\quad= \int_{\mathcal{S}^{N_1}} d^{3N_1}q_1' \sum_{s_1'} \langle q_1,s_1|\hat{W}_{\mathrm{cond}}\,|q_1',s_1'\rangle\langle q_1',s_1'|\varphi\rangle \\
&\overset{(2.28),(2.25)}{=} \int_{\mathcal{S}^{N_1}} d^{3N_1}q_1' \sum_{s_1'} \left(\sqrt{(\gamma n)(q_1)}\,W_{\mathrm{cond}}(q_1,q_1')\sqrt{(\gamma n)(q_1')}\right)^{s_1}_{s_1'} \left[\sqrt{(\gamma n)(q_1')}\varphi(q_1')\right]^{s_1'} \\
&\overset{(2.29)}{=} \left[\sqrt{(\gamma n)(q_1)}\,\hat{W}_{\mathrm{cond}}\,\varphi\right]^{s_1}(q_1).
\end{aligned} \tag{2.30}$$

In fact, as one expects, $\hat{W}_{\mathrm{cond}}$ (as an operator on $\mathcal{H}_{\Sigma}^{(N_1)}$)[7] can equivalently be derived from the projector $|\Psi\rangle\langle\Psi|$ on $\mathcal{H}_{\Sigma}^{(N)}$ by a partial trace. For this purpose, we straightforwardly generalize eq. (2.25) to $\mathcal{H}_{\Sigma}^{(N)}$:

$$\langle q_1,s_1,q_2,s_2|\Psi\rangle \;=\; \sum_{s_1's_2'} \sqrt{(\gamma n)(q_1)}^{s_1}_{s_1'} \sqrt{(\gamma n)(q_2)}^{s_2}_{s_2'}\, \Psi^{s_1's_2'}(q_1,q_2). \tag{2.31}$$

Then:

$$\begin{aligned}
&\sum_{s_2} \langle q_1,s_1,Q_2(\Sigma),s_2|\Psi\rangle\langle\Psi|q_1',s_1',Q_2(\Sigma),s_2\rangle \\
&\quad\overset{(2.31)}{=} \sum_{s_2}\sum_{\tilde{s}_1,\tilde{s}_2} \sqrt{(\gamma n)(q_1)}^{s_1}_{\tilde{s}_1} \sqrt{(\gamma n)(Q_2(\Sigma))}^{s_2}_{\tilde{s}_2} \Psi^{\tilde{s}_1\tilde{s}_2}(q_1,Q_2(\Sigma)) \\
&\qquad\times \sum_{\hat{s}_1,\hat{s}_2} \Psi^{\dagger}{}_{\hat{s}_1,\hat{s}_2}(q_1,Q_2(\Sigma)) \sqrt{(\gamma n)(q_1')}^{\hat{s}_1}_{s_1'} \sqrt{(\gamma n)(Q_2(\Sigma))}^{\hat{s}_2}_{s_2} \\
&\quad\overset{(2.15)}{=} \sum_{\tilde{s}_1,\hat{s}_1} \sqrt{(\gamma n)(q_1)}^{s_1}_{\tilde{s}_1}\, W_{\mathrm{cond}}{}^{\tilde{s}_1}_{\hat{s}_1}(q_1,q_1') \sqrt{(\gamma n)(q_1')}^{\hat{s}_1}_{s_1'} \\
&\quad\overset{(2.28)}{=} \langle q_1,s_1|\hat{W}_{\mathrm{cond}}|q_1',s_1'\rangle.
\end{aligned} \tag{2.32}$$

As expected, we have the following

Lemma 2.2.1 *$\hat{W}_{\mathrm{cond}}$ is a density operator on $\mathcal{H}_{\mathcal{S}}^{(N_1)}$.*

Proof: 1. Consider $\mathcal{H}_{\mathcal{S}}^{(N_1)}$ for a general space-like hypersurface $\mathcal{S}$. Then $\hat{W}_{\mathrm{cond}}$ is self-adjoint on $\mathcal{H}_{\mathcal{S}}^{(N_1)}$. In the proof we make use of the property $W_{\mathrm{cond}}^{\dagger}(q_1,q_1') = W_{\mathrm{cond}}(q_1',q_1)$. To see this, consider:

$$W_{\mathrm{cond}}^{\dagger}(q_1,q_1') \;=\; \left\{\mathrm{tr}_{\mathbb{C}^{k_2}}\left[\Psi(q_1,Q_2(\Sigma))\Psi^{\dagger}(q_1',Q_2(\Sigma))(\gamma n)(Q_2(\Sigma))\right]\right\}^{\dagger}. \tag{2.33}$$

[7] Note that the restriction to $\mathcal{H}_{\Sigma}^{(N_1)}$ instead of a general $\mathcal{H}_{\mathcal{S}}^{(N_1)}$ is necessary because of the use of $\langle q_1,s_1,Q_2(\Sigma),s_2|$ on $\mathcal{H}_{\Sigma}^{(N)}$ (cf. eq. (2.32)) which requires $(q_1,Q_2(\Sigma)) \in \Sigma^N$.

With the identity $(\mathrm{tr}_{\mathbb{C}^{k_2}} A)^\dagger = \mathrm{tr}_{\mathbb{C}^{k_2}}(A^\dagger)$ for the partial trace of matrices $A : \mathbb{C}^k \to \mathbb{C}^k$, eq. (2.33) becomes:

$$W^\dagger_{\mathrm{cond}}(q_1, q_1') \;=\; \mathrm{tr}_{\mathbb{C}^{k_2}}\Big[(\gamma n)^\dagger(Q_2(\Sigma))\Psi(q_1', Q_2(\Sigma))\Psi^\dagger(q_1, Q_2(\Sigma))\Big]. \tag{2.34}$$

Using the cyclic property of the partial trace, i.e. $\mathrm{tr}_{\mathbb{C}^{k_2}}(\mathbb{1}\otimes B\,A) = \mathrm{tr}_{\mathbb{C}^{k_2}}(A\,\mathbb{1}\otimes B)$ where $B : \mathbb{C}^{k_2} \to \mathbb{C}^{k_2}$, and observing (2.23), we finally obtain:

$$W^\dagger_{\mathrm{cond}}(q_1, q_1') \;=\; \mathrm{tr}_{\mathbb{C}^{k_2}}\Big[\Psi(q_1', Q_2(\Sigma))\Psi^\dagger(q_1, Q_2(\Sigma))(\gamma n)(Q_2(\Sigma))\Big] \;=\; W_{\mathrm{cond}}(q_1', q_1). \tag{2.35}$$

We proceed with the proof of the self-adjointness of $\hat{W}_{\mathrm{cond}}$:

$$\begin{aligned}
\langle\psi, \hat{W}_{\mathrm{cond}}\,\varphi\rangle^{(N_1)}_{\mathcal{S}} &\overset{(1.77),(2.29)}{=} \int_{\mathcal{S}^{N_1}} d^{3N_1}q_1 \int_{\mathcal{S}^{N_1}} d^{3N_1}q_1'\, \psi^\dagger(q_1)\,(\gamma n)(q_1)\, W_{\mathrm{cond}}(q_1, q_1')\,(\gamma n)(q_1')\,\varphi(q_1') \\
&\overset{(2.23),(2.35)}{=} \int_{\mathcal{S}^{N_1}} d^{3N_1}q_1 \int_{\mathcal{S}^{N_1}} d^{3N_1}q_1' \left[W_{\mathrm{cond}}(q_1', q_1)\,(\gamma n)(q_1)\,\psi(q_1)\right]^\dagger (\gamma n)(q_1')\,\varphi(q_1') \\
&\overset{(2.29)}{=} \int_{\mathcal{S}^{N_1}} d^{3N_1}q_1' \left[\hat{W}_{\mathrm{cond}}\,\psi\right]^\dagger (q_1')\,(\gamma n)(q_1')\,\varphi(q_1') \\
&\overset{(1.77)}{=} \langle \hat{W}_{\mathrm{cond}}\,\psi,\, \varphi\rangle^{(N_1)}_{\mathcal{S}}.
\end{aligned} \tag{2.36}$$

Thus, $\hat{W}_{\mathrm{cond}}$ is symmetric on $\mathcal{H}^{(N_1)}_{\mathcal{S}}$ and as a bounded operator it is therefore also self-adjoint.

2. To show the positivity of $\hat{W}_{\mathrm{cond}}$, we start with

$$\langle\varphi, \hat{W}_{\mathrm{cond}}\,\varphi\rangle^{(N_1)}_{\mathcal{S}} \;=\; \int_{\mathcal{S}^{N_1}} d^{3N_1}q_1 \int_{\mathcal{S}^{N_1}} d^{3N_1}q_1'\, \varphi^\dagger(q_1)\,(\gamma n)(q_1)\, W_{\mathrm{cond}}(q_1, q_1')\,(\gamma n)(q_1')\,\varphi(q_1') \tag{2.37}$$

and plug in the explicit form of ${W_{\mathrm{cond}}}^{s_1}_{s_1'}(q_1, q_1')$ from eq. (2.15). Next, we simplify the expression for ${W_{\mathrm{cond}}}^{s_1}_{s_1'}(q_1, q_1')$ by a multi-time Lorentz transformation L (see eqs. (1.14), (1.15)). As for fixed $Q(\Sigma)$ all $n(X_j(\Sigma))$ are constant time-like future-oriented unit vectors, we may choose L such that for the Lorentz-transformed normal covectors one has $n'(X_j(\Sigma)) \equiv (1,0,0,0),\ j = 1, ..., N$. Thus: $(\gamma n')(Q_2(\Sigma)) = \mathbb{1}$.

Continuing with the hereby simplified eq. (2.37) and dropping the Lorentz transformation primes for notational ease, we have:

$$\begin{aligned}
\langle\varphi, \hat{W}_{\mathrm{cond}}\,\varphi\rangle^{(N_1)}_{\mathcal{S}} \;&=\; \frac{1}{\mathcal{N}} \int_{\mathcal{S}^{N_1}} d^{3N_1}q_1 \int_{\mathcal{S}^{N_1}} d^{3N_1}q_1' \sum_{s_1, s_1'} \Bigg\{ \Big[\varphi^\dagger(q_1)\,(\gamma n)(q_1)\Big]_{s_1} \\
&\qquad \times \sum_{s_2} \Psi^{s_1 s_2}(q_1, Q_2(\Sigma))\Psi^\dagger_{s_1' s_2}(q_1', Q_2(\Sigma))\,\Big[(\gamma n)(q_1')\,\varphi(q_1')\Big]^{s_1'} \Bigg\} \\
&=\; \frac{1}{\mathcal{N}} \sum_{s_2} \left(\int_{\mathcal{S}^{N_1}} d^{3N_1}q_1 \sum_{s_1} \Big[\varphi^\dagger(q_1)\,(\gamma n)(q_1)\Big]_{s_1} \Psi^{s_1 s_2}(q_1, Q_2(\Sigma)) \right) \\
&\qquad \times \left(\int_{\mathcal{S}^{N_1}} d^{3N_1}q_1' \sum_{s_1'} \Psi^\dagger_{s_1' s_2}(q_1', Q_2(\Sigma))\,\Big[(\gamma n)(q_1')\,\varphi(q_1')\Big]^{s_1'} \right) \\
&\equiv\; \sum_{s_2} c^{s_2} c^*_{s_2} \;\geq 0,
\end{aligned} \tag{2.38}$$

where

$$c^{s_2} \;=\; \frac{1}{\sqrt{\mathcal{N}}}\int_{\mathcal{S}^{N_1}} d^{3N_1}q_1 \sum_{s_1}\Big[\varphi^\dagger(q_1)\,(\gamma n)(q_1)\Big]_{s_1}\,\Psi^{s_1 s_2}(q_1, Q_2(\Sigma)). \tag{2.39}$$

3. $\hat{W}_{\mathrm{cond}}$ is normalized. This straightforwardly follows from the fact that $\mathrm{tr}\,\hat{W}_{\mathrm{cond}}$ yields unity by eq. (2.21) when one applies the computational formula for the trace as used in eq. (2.27) and makes use of eq. (2.28). □

2.2.3.3 Effective wave function

Because of the dependence on $Q_2(\Sigma)$, W_{cond} typically does not evolve autonomously, i.e. according to its own multi-time system of von Neumann equations. We now turn to an autonomous subsystem description in terms of wave functions. For this purpose, we extend the definition of the effective wave function (cf. eq. (2.7)) to the HBD model. Assume that there exists a hypersurface $\Sigma \in \mathcal{F}$ such that $\Psi^{s_1 s_2}(q_1, q_2)$ and the actual configuration $Q(\Sigma) = (Q_1, Q_2)(\Sigma)$ on that hypersurface satisfy

$$\Psi^{s_1 s_2}(q_1, q_2) = \psi_1^{s_1}(q_1)\psi_2^{s_2}(q_2) + (\Psi^\perp)^{s_1 s_2}(q_1, q_2) \quad \forall q = (q_1, q_2) \in \Sigma^N, \tag{2.40}$$

with ψ_2 and $\Psi^\perp$ possessing *macroscopically disjoint* q_2-supports and $Q_2(\Sigma) \in \mathrm{supp}\,\psi_2$. Then for $q_1 \in \Sigma^{N_1}$ system S_1 is said to have the *effective wave function*

$$\psi_{\mathrm{eff}}^{s_1}(q_1) = \frac{\psi_1^{s_1}(q_1)}{\|\psi_1\|_\Sigma^{(N_1)}}. \tag{2.41}$$

It has the following properties:

1. If Σ is an equal-time hypersurface of a Lorentz frame, then our definition agrees with the non-relativistic one in that frame.

2. Inserting the product structure $\Psi(q_1, Q_2(\Sigma)) = \psi_1(q_1) \otimes \psi_2(Q_2(\Sigma))$ resulting from eq. (2.40) into the relativistic guidance equation (2.1) for a particle in S_1, we obtain:

$$\begin{aligned}\frac{dX_k^{\mu_k}(s)}{ds} &\propto \overline{\psi}_1(q_1) \otimes \overline{\psi}_2(Q_2(\Sigma))\,\gamma_k^{\mu_k} \prod_{j\neq k, j\in S_1} \gamma_j \cdot n(x_j)\;(\gamma n)(Q_2(\Sigma))\,\psi_1(q_1) \otimes \psi_2(Q_2(\Sigma))\Big|_{q_1=Q_1(\Sigma)} \\ &\propto \overline{\psi}_{\mathrm{eff}}(q_1)\gamma_1^{\mu_1}\cdots\gamma_k^{\mu_k}\cdots\gamma_{N_1}^{\mu_{N_1}}\psi_{\mathrm{eff}}(q_1) \prod_{j\neq k, j\in S_1} n_{\mu_j}(x_j)\Big|_{q_1=Q_1(\Sigma)}, \end{aligned} \tag{2.42}$$

where we absorbed the factor $\|\psi_1\|_\Sigma^{(N_1)}\,\overline{\psi}_2(Q_2(\Sigma))\,(\gamma n)(Q_2(\Sigma))\,\psi_2(Q_2(\Sigma)) \geq 0$ into the proportionality. One obtains the same result as in eq. (2.42) if one starts with a pure[8] W_{cond} in eq. (2.14).

3. We now come to the expression of conditional probabilities, starting from the conditional version of the crossing probability (cf. eq. (2.19)):

$$\mathrm{Prob}\,(\text{particle } i \text{ crosses } \Sigma \text{ in } d\sigma_i,\; i = 1, ..., N_1 | Q_2(\Sigma)) = \frac{\rho(q_1, Q_2(\Sigma))d\sigma_1\cdots d\sigma_{N_1}}{\int_{\Sigma^{N_1}} dq_1^{3N_1}\,\rho(q_1, Q_2(\Sigma))}. \tag{2.43}$$

[8] We apply the notion *pure* to the matrix W_{cond}, in the sense that it can be written as $W_{\mathrm{cond}}(q_1, q_1') = \psi(q_1)\psi^\dagger(q_1')$ for some wave function ψ. Otherwise, we call it *mixed*.

Into this equation we plug in the HBD density

$$\rho(q) = \overline{\Psi}(q)(\gamma n)(q_1)\,(\gamma n)(q_2)\Psi(q) \tag{2.44}$$

in the particular situation given by eq. (2.40), using that then $\Psi(q_1, Q_2(\Sigma)) = \psi_1(q_1) \otimes \psi_2(Q_2(\Sigma))$. This yields:

$$\begin{aligned}
&\text{Prob}\,(\text{particle } i \text{ crosses } \Sigma \text{ in } d\sigma_i,\ i = 1, ..., N_1 | Q_2(\Sigma)) \\
&= \frac{\overline{\psi}_1(q_1) \otimes \overline{\psi}_2(Q_2(\Sigma))\,(\gamma n)(q_1)\,(\gamma n)(Q_2(\Sigma))\,\psi_1(q_1) \otimes \psi_2(Q_2(\Sigma))\,d\sigma_1 \cdots d\sigma_{N_1}}{\int_{\Sigma^{N_1}} dq_1^{3N_1}\,\overline{\psi}_1(q_1) \otimes \overline{\psi}_2(Q_2(\Sigma))\,(\gamma n)(q_1)\,(\gamma n)((Q_2(\Sigma)))\,\psi_1(q_1) \otimes \psi_2(Q_2(\Sigma))} \\
&= \frac{\overline{\psi}_1(q_1)\,(\gamma n)(q_1)\,\psi_1(q_1)\,d\sigma_1 \cdots d\sigma_{N_1}}{\int_{\Sigma^{N_1}} dq_1^{3N_1}\,\overline{\psi}_1(q_1)\,(\gamma n)(q_1)\,\psi_1(q_1)} \\
&= \overline{\psi}_{\text{eff}}(q_1)\,(\gamma n)(q_1)\,\psi_{\text{eff}}(q_1)\,d\sigma_1 \cdots d\sigma_{N_1}.
\end{aligned} \tag{2.45}$$

Eq. (2.45) also explains the normalization in eq. (2.41).

4. The description of S_1 in terms of ψ_{eff} has the same form as the description of S in terms of Ψ (cf. eqs. (2.1), (2.42) and eqs. (2.2), (2.45)).

W_{cond} always exists; the effective wave function only exists in certain situations. If the effective wave function exists, W_{cond} is pure and given by the effective wave function. The converse is not as obvious and content of the following lemma.

Lemma 2.2.2 *$W_{\text{cond}}(q_1, q_1')$ is pure if and only if for $\Sigma \in \mathcal{F}$, $\Psi(q_1, Q_2(\Sigma))$ can be written as $\Psi^{s_1 s_2}(q_1, Q_2(\Sigma)) = \psi_1^{s_1}(q_1)\psi_2^{s_2}(Q_2(\Sigma))$.*

Proof: "$\Leftarrow$": Let $\Psi^{s_1 s_2}(q_1, Q_2(\Sigma)) = \psi_1^{s_1}(q_1)\psi_2^{s_2}(Q_2(\Sigma))$. Then, according to eqs. (2.15) and (2.23):

$$\begin{aligned}
{W_{\text{cond}}}_{s_1'}^{s_1}(q_1, q_1') &= \frac{1}{\mathcal{N}} \sum_{s_2} \psi_1^{s_1}(q_1)\psi_2^{s_2}(Q_2(\Sigma)) \left[\psi_1^\dagger(q_1')\psi_2^\dagger(Q_2(\Sigma))(\gamma n)(Q_2(\Sigma))\right]_{s_1' s_2} \\
&= \frac{1}{\mathcal{N}} \left\{ \sum_{s_2} \psi_2^{s_2}(Q_2(\Sigma)) \left[\psi_2^\dagger(Q_2(\Sigma))(\gamma n)(Q_2(\Sigma))\right]_{s_2} \right\} \psi_1^{s_1}(q_1){\psi_1}^\dagger_{s_1'}(q_1') \\
&\equiv \frac{1}{\widetilde{\mathcal{N}}} \psi_1^{s_1}(q_1){\psi_1}^\dagger_{s_1'}(q_1').
\end{aligned} \tag{2.46}$$

"$\Rightarrow$": We split the proof into two steps:

1. Simplification of the form of W_{cond}: Using the manifest Lorentz invariance of the HBD model, we simplify the form of W_{cond} employing the multi-time Lorentz transformation leading to $(\gamma n)(Q_2(\Sigma)) = \mathbb{1}$ (cf. eq. (1.15) and below). Note that by virtue of eq. (1.15) this transformation leaves a pure W_{cond} pure and a mixed W_{cond} mixed. We obtain:

$${W'}_{\text{cond}}{}_{s_1'}^{s_1}(q_1, q_1') = \frac{1}{\mathcal{N}} \sum_{s_2} \Psi'^{s_1 s_2}(L^{-1}(q_1, Q_2(\Sigma))) \left[(\Psi')^\dagger(L^{-1}(q_1', Q_2(\Sigma)))\right]_{s_1' s_2}. \tag{2.47}$$

For a flat foliation, this coincides with the definition of the conditional density matrix in the non-relativistic case (cf. eq. (2.9)).

2. Indirect proof of "⇒" via Schmidt decomposition: Assume that for $\Sigma \in \mathcal{F}$ and fixed q_1, $\Psi^{s_1 s_2}(q_1, Q_2(\Sigma))$ cannot be written as a tensor product of vectors in $\mathbb{C}^{k_1}$ and $\mathbb{C}^{k_2}$, respectively. Then according to the Schmidt decomposition there exist orthonormal bases $\{u_1, ..., u_{k_1}\}$ of $\mathbb{C}^{k_1}$ and $\{v_1, ..., v_{k_2}\}$ of $\mathbb{C}^{k_2}$ such that

$$\Psi(q_1, Q_2(\Sigma)) = \sum_{j=1}^{m} c_j \, u_j \otimes v_j, \quad \text{in components:} \quad \Psi^{s_1 s_2}(q_1, Q_2(\Sigma)) = c_{s_1} \, \delta^{s_1 s_2}, \tag{2.48}$$

where $m = \max\{k_1, k_2\}$ and the coefficients c_j are non-negative, uniquely determined by $\Psi(q_1, Q_2(\Sigma))$, and a number $l \geq 2$ of them is non-zero. Relabel such that these are the coefficients $c_1, ..., c_l$.

Similarly for fixed q_1',

$$\Psi(q_1', Q_2(\Sigma)) = \sum_{j=1}^{m} c_j' \, u_j' \otimes v_j', \quad \text{in components:} \quad \Psi^{s_1' s_2'}(q_1', Q_2(\Sigma)) = c_{s_1'}' \, \delta^{s_1' s_2'}, \tag{2.49}$$

where m is the same as before, the u_j' and v_j' define orthonormal bases of the respective spaces, and we can choose the first $l' \geq 2$ of the c_j' to be non-zero. Then, with the previous choice of the multi-time Lorentz transformation, the conditional density matrix takes the following form:

$$W_{\mathrm{cond}}{}_{s_1'}^{s_1}(q_1, q_1') = \frac{1}{\mathcal{N}} \sum_{s_2 = s_2'} c_{s_1} c_{s_1'}' \, \delta^{s_1 s_2} \delta_{s_1' s_2'} = \frac{1}{\mathcal{N}} c_{s_1} c_{s_1'}' \, \delta_{s_1'}^{s_1}. \tag{2.50}$$

Thus, as a diagonal matrix with $\min\{l, l'\} \geq 2$ non-zero entries, $W_{\mathrm{cond}}{}_{s_1'}^{s_1}(q_1, q_1')$ is not pure. □

The lemma shows that the mathematical structure in the definition of the effective wave function (cf. eq. (2.40)) follows from W_{cond} being pure. If in addition the "macroscopic disjointness", the key aspect to earn ψ_{eff} the attribute "effective", is given, the effective wave function is indeed the wave function that is uniquely determined by W_{cond}.

2.2.4 Indistinguishable particles

Particles with spin are usually thought of as being indistinguishable. Contrary to what one may think, this poses no problem for BM (cf. [44]). One only has to recognize that the appropriate configuration space in this case is the set of unordered configurations, i.e. the subsets of $\mathbb{R}^3$ with N elements: ${}^N\mathbb{R}^3 \equiv \{S \subset \mathbb{R}^3 : |S| = N\}$. This space is topologically nontrivial and the analysis of BM on this space leads to the familiar distinction between bosons and fermions (see e.g. [44]). The description of the spinor bundle on this configuration space becomes, however, a bit technical. Since such technicalities do not yield more insight into the autonomous subsystem description, we adopt a pragmatic point of view here: One may use an arbitrary labeling of the particles (and thus the ordered tensor product of spinors) and apply the corresponding (anti)symmetrization postulate. The crucial point is that the constructions in the definitions of W_{cond} and ψ_{eff} in fact commute with permutations of the particle labels. Thus, one may apply them without changes. In

particular, the distinction between environment particles and subsystem particles is not based on particle labels, but on the fact that some set of particles, e.g. particles belonging to a certain region $R \subset \Sigma$ of a hypersurface $\Sigma \in \mathcal{F}$, has the actual configuration $\{X(\Sigma), Y(\Sigma), ..., Z(\Sigma)\} \subset R$.

2.2.5 Outlook

In this section, we have derived a subsystem description for the HBD model. By construction, the quantities of the subsystem, the conditional density matrix and the effective wave function, have statistical meaning on the space-time structure given by the foliation $\mathcal{F}$.

This framework should be taken as the starting point for a further study of an effective relativistic "measurement formalism" which in turn should lead to a rigorous justification of the usual quantum formalism (as far as it exists for relativistic quantum theories). A thorough discussion of the measurement formalism as arising from non-relativistic BM has been achieved in [48, 49].

To appreciate the quest, note that a physical experiment with its state of motion defines an equal time hypersurface $\mathcal{S}$ in general not belonging to the foliation. That is, such a hypersurface $\mathcal{S}$ is not part of the space time structure defined by $\mathcal{F}$. Of course, the main interest lies in the statistics of "measurements" of the subsystems on such a "query hypersurface". In order to obtain these statistics, one has to relate the "$|\psi|^2$-probability" formula holding only on leaves of $\mathcal{F}$ to the usual formalism of operator-observables on query hypersurfaces. An analysis related in spirit was performed in [19] for a particular limiting case of Lorentz invariance and spin-less particles.

2.3 Bloch's considerations on the statistical meaning of multi-time wave functions in certain classes of "measurements"

In [20], Bloch showed how certain assumptions about the single-time measurement formalism together with the possibility to separately time-evolve a multi-time wave function in each time coordinate imply a statistical meaning of the multi-time wave function. A similar consideration was made by Berndl *et al.* in [19, sec. 3.1]. We present a synthesized version of both works here, clearly working out the assumptions and limits of the idea. We shall see that this analysis supports the conjecture that the effective measurement formalism resulting from the HBD model does not depend on the foliation (see the remark in sec. 2.1, point 2).

Assumptions:

1. The unitary part of the time-evolution of the multi-time wave function is given by[9] $\psi(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N) = e^{-iH_1(t_1-t_0)} \cdots e^{-iH_N(t_N-t_0)} \psi_0(t_0, \mathbf{x}_1, ..., t_0, \mathbf{x}_N)$ where ψ_0 is an initial wave function at some common time t_0.

[9] Bloch uses a different symbol for the partial time evolution operators, namely $D_j(t_j)$, instead of $e^{-iH_j t_j}$. His definitions of the operators D_j, however, imply $D_j(t_j) = e^{-iH_j t_j}$.

2. The orthodox single-time measurement formalism applies. In particular, $|\psi|^2(t, \mathbf{x}_1, ..., t, \mathbf{x}_N)$ is the probability density to find particle 1 at position $\mathbf{x}_1$, ... and particle N at position $\mathbf{x}_N$ at common time t.

3. At (common) time $t = \tau_1$, a "position measurement" is performed on particle 1 only, ... and at time $t = \tau_N$ on particle N only, where $\tau_1 < \cdots < \tau_N$. These "measurements" have definite results which are associated with projectors M_i acting only on the spin indices and particle coordinates of particle i. The M_i therefore commute with each other as well as with $e^{-iH_j t_j}$ for $j \neq i$.

Argument: Let the initial wave function be ψ_0 at common time $t = 0$. Then, by assumptions 2 and 3, at time $t = \tau_N$ the (effective) wave function after the sequence of "measurements" (and the associated collapses) is given by:

$$\psi(\tau_1, ..., \tau_N) \;=\; M_N e^{-i(\tau_N - \tau_{N-1})H} M_{N-1} e^{-i(\tau_{N-1} - \tau_{N-2})H} \cdots e^{-i(\tau_2 - \tau_1)H} M_1 e^{-i\tau_1 H} \psi_0, \tag{2.51}$$

where the space coordinates $\mathbf{x}_i$ are suppressed and H is the associated single-time Hamiltonian, $H = \sum_k H_k$.

Using assumption 1 and the fact that the M_i commute with $e^{-iH_j t_j}$ for $j \neq i$ as well as the projector properties $M_i^2 = M_i$ and $M_i^\dagger = M_i$, eq. (2.51) yields the following probability density for the "measurement results" associated with the operators M_i:

$$\begin{aligned} \psi^\dagger\psi(\tau_1, \mathbf{x}_1, ..., \tau_N, \mathbf{x}_N) \;&=\; \psi_0^\dagger(\mathbf{x}_1 \cdots \mathbf{x}_N)\, e^{i\tau_1 H_1} \cdots e^{iH_N \tau_N} M_N \cdots M_1\, e^{-i\tau_N H_N} e^{-i\tau_1 H_1} \psi_0(\mathbf{x}_1 \cdots \mathbf{x}_N) \\ &=\; \psi^\dagger(\tau_1, \mathbf{x}_1, ..., \tau_N, \mathbf{x}_N) M_N \cdots M_1 \psi(\tau_1, \mathbf{x}_1, ..., \tau_N, \mathbf{x}_N). \end{aligned} \tag{2.52}$$

In the case of "position measurements", the operators M_i are multiplication operators with the indicator functions of infinitesimal spatial regions in the $\mathbf{x}_i$ coordinates. We conclude that

$$\psi^\dagger\psi\,(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N) d^3\mathbf{x}_1 \cdots d^3\mathbf{x}_N \tag{2.53}$$

is the probability to find particle 1 at $\mathbf{x}_1$ in a "position measurement" at time τ_1, ... and particle N at $\mathbf{x}_N$ at time τ_N.

Lorentz invariance: We now show that even though the derivation of (2.53) involves a wave function collapse in a distinguished frame F, the result is Lorentz invariant.

To see this, rewrite (2.53) using the tensor current j_ψ and the oriented 3-volume elements $d\sigma_{i,\mu_i}(x_i) = d\sigma_i(x_i) n_{\mu_i}(x_i)$. Realizing $\psi^\dagger\psi = j_\psi^{0\cdots 0}$ and $j_\psi^{0\cdots 0} = j_\psi^{\mu_1 \cdots \mu_N} d\sigma_{1,\mu_1} \cdots d\sigma_{N,\mu_N}$ for $n(x_i) = (1, 0, 0, 0)$ and $d\sigma_i(x_i) = d^3\mathbf{x}_i$, $i = 1, ..., N$, eq. (2.53) becomes:

$$j_\psi^{\mu_1 \cdots \mu_N}(x_1, ..., x_N) d\sigma_{1,\mu_1} \cdots d\sigma_{N,\mu_N}, \tag{2.54}$$

which is manifestly Lorentz invariant and does not contain any reference to a distinguished frame. Thus, the results of Lorentz-transforming (2.53) and repeating the derivation for a collapse in a different frame coincide. The functional expression for the density furthermore agrees with the one for ρ_Σ in eq. (1.60), with the only difference that the space-time points in eq. (2.54) are arbitrary (i.e. do not have to lie on a space-like hypersurface Σ).

Discussion:

1. One expects that the HBD model with a flat foliation leads to a similar measurement formalism as used by Bloch. To obtain this measurement formalism, one would just have to copy the statistical analysis of non-relativistic BM in [47] and [48], replacing the conditional wave function with the conditional density matrix of sec. 2.2. Given the above-mentioned assumptions, one would thus indeed reach a measurement formalism for arbitrary query hypersurfaces, as suggested in sec. 2.2.5. Bloch's analysis then shows for a special case that the supposed measurement formalism is indeed independent of the foliation, as conjectured in [19,42].

2. Bloch's assumptions are quite special. As was shown in sec. 1.2.1.2, the separate existence of the partial time evolution operators $e^{-iH_j t_j}$ is only ensured in the interaction-free case. The overall dynamics for the effective wave function given by (2.51) then corresponds to non-interacting phases which are interrupted from time to time by external interventions. Moreover, the assumption that the M_i act only on the coordinates and the spin index of particle i further restricts the generality of Bloch's result. Consequently, the claim of Bloch (later taken up by Tomonaga [96]) that (2.53) establishes the physical meaning of the multi-time wave function appears to be too strong.

3. Bloch emphasizes in his paper that the result (2.53) were only valid for space-like arguments of the wave function. The likely reason for this is that the QED model of Dirac, Fock and Podolsky (on which Bloch's analysis is based) is only consistent on $\mathscr{S}$. However, the operators $e^{-iH_j t_j}$ do not respect the domain $\mathscr{S}$. Bloch's analysis thus does not apply to the QED model, and the restriction of (2.53) to space-like configurations is unnecessary.

We conclude that in view of these points, realistic relativistic models such as the HBD model may, due to their conceptual clarity, also prove particularly valuable for working out a more general relativistic measurement formalism.

2.4 Alternative realistic relativistic models

While the previous section has made plausible that the foliation of the HBD model does not appear in the respective measurement formalism, there remain some objectionable points with regard to a preferred foliation.

1. It is additional space-time structure, conflicting Maudlin's "relativistic constraint" (see the introduction). This fact holds even in the case that the foliation is extracted from the wave function.

2. There does not appear to be a convincing reason to choose a particular foliation.

3. It has not been shown (and is difficult to imagine) that no relativistic theory of particle motion exists which does not need a foliation and which also implies the quantum predictions in a suitable sense.

In view of this criticism, we regard it as helpful to briefly review the status of some existing realistic relativistic models which do not require a foliation: a Bohmian light cone model (sec. 2.4.1) and two different relativistic GRW models (sec. 2.4.2).

2.4.1 Bohmian light cone models

In order to formulate a non-local law of motion for relativistic particle trajectories, a foliation is not necessary. From a relativistic standpoint, the most natural generalization of the simultaneity surfaces in non-relativistic BM are light cones. In order to obtain a non-local law of motion (as necessary because of Bell's theorem [14]), one then has to use forward light cones, either exclusively or in addition to the retarded ones. A Bohmian law of this kind, based on a multi-time wave function, was suggested in [54]. It uses only advanced light cones[10]. Here, we present a time-symmetric version of the model. Let $X_k(s_k)$ denote the world lines of N particles, parametrized by an individual parameter s_k. Using a positive definite tensor current j_ψ similarly to before, the law reads:

$$\frac{dX_k^{\mu_k}}{ds_k} \;\Big\|\; \frac{1}{2}\left[j_\psi^{\mu_1 \dots \mu_N} \prod_{j \neq k} \frac{dX_j^{\nu_j}}{ds_j}(s_{j,\mathrm{ret}}(p_k))\eta_{\mu_j \nu_j} + j_\psi^{\mu_1 \dots \mu_N} \prod_{j \neq k} \frac{dX_j^{\nu_j}}{ds_j}(s_{j,\mathrm{adv}}(p_k))\eta_{\mu_j \nu_j} \right], \tag{2.55}$$

where $p_k = X_k(s_k)$, $s_{j,\mathrm{ret}}(p_k)$ ($s_{j,\mathrm{adv}}(p_k)$) are the values of the world line parameter s_j where the retarded (advanced) light cone at p_k intersects the trajectory of the j-th particle and the $\|$-sign denotes parallelity of two vectors. The tensor current j_ψ is to be evaluated at $(X_1(s_{1,\mathrm{ret}}(p_k)), ..., X_N(s_{N,\mathrm{ret}}(p_k))$ in the first summand and at $(X_1(s_{1,\mathrm{adv}}(p_k)), ..., X_N(s_{N,\mathrm{adv}}, (p_k))$ in the second.

Further heuristic reasons why to choose this particular law were given in [54]. However, the problem of the model is that it does apparently not possess an equivariant measure. It is therefore unknown whether (or how) it can be analyzed statistically. Because of this lack of "statistical transparency", to use the terminology of [19], theories of this kind are often discarded (see e.g. [98, sec. 3.3.2]), leaving the HBD model as the best state of the art of relativistic Bohmian theories.

Discussion of "statistical transparency": It becomes obvious that the main reason to use a preferred foliation in relativistic Bohmian mechanics is "statistical transparency". In [54, 98], this term is taken as the requirement that there exists an equivariant density. However, this requirement is not really a physical one, as it is dictated only by the particular way scientists are accostumed to perform a statistical analysis. While *some* way to perform a statistical analysis is indeed required to obtain a useful theory, one may doubt that the requirement of an *equivariant density* should hold in the light of relativity. The reason is that the notion of equivariance presupposes a velocity vector field, i.e. a structure which already requires a notion of simultaneity. Thus, one cannot make the argument that a foliation – which mainly provides a notion of simultaneity – is necessary for "statistical transparency", because the argument is circular.

[10] Assuming interplaying macroscopic and microscopic arrows of time, they are relabeled as retarded light cones.

This line of thought rather motivates to search for alternative ways of performing a statistical analysis. What makes this task difficult is that the law of motion and the statistical analysis have to harmonize in some way – and neither of the two is fixed. Note, however, that the need for alternative ways of a statistical analysis is not limited to Bohmian quantum theory. An analogous situation is encountered for Wheeler-Feynman electrodynamics[11]. The same holds true for other laws of motion which do not involve a structure similar to a velocity vector field.

Remark concerning the requirements on the wave function level of the theory: Note that the question of an adequate statistical analysis of relativistic laws of motion may also have consequences for the wave function level of the theory. To see this more clearly, recall that the main role of the multi-time wave equations is to ensure the continuity equations of the tensor current j_ψ. However, the purpose of these continuity equations in turn is to guarantee the equivariance property for the HBD law of motion (eq. (2.1)) which may not be appropriate for a relativistic law of motion which is not based on a velocity vector field. For example, there is no apparent reason why the tensor current in eq. (2.55) should be divergence-free. Thus, an alternative way for the statistical analysis may indeed have far-reaching consequences for the construction principles of multi-time wave equations and these, in turn, for the entailed mathematical structures such as the Hilbert space picture.

2.4.2 Relativistic GRW models

Apart from the HBD model, multi-time wave functions are also required to formulate relativistic GRW (Ghirardi-Rimini-Weber) models. We do not attempt to give a complete overview of these models here. Instead, we refer to the respective sources and introduce only those parts of the theories which contribute to the question of the requirements on multi-time wave function and its conceptual role in the theory.

GRW models can be distributed into two main classes: those with a *mass density ontology* (GRWm [10]) and those with a *flash ontology* (GRWf [97]), i.e. where discrete events in space-time are supposed to represent the physical world. Their common structure is a modification of the wave equations by stochastic terms (and corresponding new constants of nature). This results in an *objective collapse* of the wave functions, so that objects in physical space are effectively localized at a rate depending on the size (e.g. number of particles or mass) of the object in question.

2.4.2.1 GRWm

The GRWm model [10] uses a unitary wave function dynamics between Hilbert spaces $\mathcal{H}_\Sigma$ for arbitrary space-like hypersurfaces Σ. The elements of the $\mathcal{H}_\Sigma$ are, of course, multi-time wave functions. For N "particles" (i.e. N arguments of the multi-time wave function), these Hilbert spaces correspond to the $\mathcal{H}_\Sigma^{(N)}$ of eq. (1.73). According to lemma 1.3.3, unitary wave function dynamics between these Hilbert spaces exist if there is a *conserved and positive-definite tensor current* $j_\psi^{\mu_1 \cdots \mu_N}$. The unitary time evolution is then interrupted from time

[11] Note that for WF theory, the task seems clearer than for Bohmian light cone models because at least the law of motion is canonical.

to time by jumps defined by a stochastic law the exact nature of which is not important here.

Given $j_\psi^{\mu_N \cdots \mu_N}$, one defines the law for the mass density, a 4-vector m_μ in the Dirac case, as follows. Compared to [10, eq. (4)] we use a slight generalization by allowing for arbitrary positive-definite and conserved tensor currents, not just the free Dirac current (1.59).

$$m_\mu(x) \;=\; \sum_{i=1}^{N} m_i \delta_\mu^{\mu_i} \int_{PLC(x)^{N-1}} \left(\prod_{j \neq i} d\sigma_j^{\mu_j}(y_i) \right) j_{\psi_{PLC(x)}, \mu_1 \cdots \mu_i \cdots \mu_N}, \tag{2.56}$$

where $j_\psi^{\mu_1 \cdots \mu_N}(x_1, ..., x_N)$ is evaluated at $(y_1, ..., y_{i-1}, x, y_{i+1}, ..., y_N)$ and $PLC(x)$ denotes the past light cone at x.

Note that the multi-time wave function carries an additional hypersurface label Σ ($\Sigma = PLC(x)$ in eq. (2.56) which is a limiting case of space-like hypersurfaces). The reason for this is that a collapse at $p \in \mathbb{R}^4$ globally modifies all wave functions ψ_Σ with p in the past of Σ. Thus, ψ_Σ and $\psi_{\Sigma'}$, where $\Sigma \neq \Sigma'$, may differ even in points $q \in \Sigma \cap \Sigma'$.

Given an appropriate law for the collapses, the statistics of the resulting model can be shown to agree (approximately) with any quantum-mechanical model with probability density given by ρ_Σ of eq. (1.60).

2.4.2.2 GRWf

The GRWf model, as presented in [97], requires a unitary evolution map between Hilbert spaces of the form $\mathcal{H}_{\Sigma_1}^{(1)} \otimes \cdots \otimes \mathcal{H}_{\Sigma_N}^{(1)}$. It was originally constructed for free multi-time Dirac equations with external potential. However, given a positive-definite and divergence-free tensor current $j_\psi^{\mu_1 \cdots \mu_N}$ on $\mathbb{R}^{4N}$ it can be formulated without essential changes as we sketch now (otherwise following [97, sec. 3.3]).

The model assumes N initial flashes $X_1, ..., X_N \in \mathbb{R}^4$ to be given. Then N time differences $\Delta t_i > 0,\ i = 1, ..., N$ are randomly determined and N hyperboloids Σ_i with time-like distance Δt_i to X_i are constructed. These are the surfaces on which the next generation of flashes is determined, according to the probability distribution

$$\mathrm{Prob}(Y_1 \in d\sigma_1, ..., Y_N \in d\sigma_N) \;=\; \varrho(y_1, ..., y_N) d\sigma_1 \cdots d\sigma_N, \tag{2.57}$$

where the notation "$d\sigma_i$" is the same as in eq. (2.2).

The density ϱ is given by

$$\varrho(y_1, ..., y_N) = \int_{\prod_i \Sigma_i} d\sigma_1(z_1) \cdots d\sigma_N(z_N) |f_{\Sigma_1}(y_1, z_1)|^2 \cdots |f_{\Sigma_N}(y_N, z_N)|^2 j_\psi^{\mu_1 \cdots \mu_N} n_{\mu_1}(z_1) \cdots n_{\mu_N}(z_N), \tag{2.58}$$

where the f_{Σ_i} are certain "jump factors" (e.g. Gaussians on Σ_i) by which the wave function becomes modified as follows. One randomly determines N new flashes $Y_1, ..., Y_N$ according to (2.57). Then one replaces ψ by a new (collapsed) ψ' defined by

$$\psi'(z_1, ..., z_N) \;:=\; \frac{f_{\Sigma_1}(Y_1, z_1) \cdots f_{\Sigma_N}(Y_N, z_N) \psi(z_1, ..., z_N)}{\varrho^{1/2}(Y_1, \cdots, Y_N)} \tag{2.59}$$

on $\prod_i \Sigma_i$ and on $\mathbb{R}^{4N}$ by extension via the multi-time equations.

Iterating the whole procedure, one obtains a set of flashes which represents a discrete space-time world, the statistics of which is (approximately) in agreement with the quantum predictions based on ρ_Σ of eq. (1.60).

We thus see that both GRWm and GRWf models (approximately) recover the quantum predictions without the use of a foliation. However, this is only possible using (unexplained) stochastic terms.

2.5 Conclusion

Both the HBD model as well as the GRWm/f models – as different as their pictures of the world may be – suggest that the connection of the multi-time wave function ψ with physics derives from the tensor current j_ψ. The tensor current is used to formulate covariant laws for the primitive ontology (eqs. (2.1), (2.56) and (2.57), (2.58)). Besides this main role, it also has a secondary role which follows from said laws: its density component yields the statistical distribution on space-like hypersurfaces (in "measurements").

Even though the discussed models all have their individual open issues (the preferred foliation in the case of the HBD model and the unexplained stochastic terms in the case of the GRW models), we have now reached an exemplary conceptual framework of the role of multi-time wave functions in relativistic quantum theory. In the following chapters, we proceed to the search for interacting evolution equations which are compatible with this framework.

Chapter 3

A relativistically interacting exactly solvable multi-time model for two mass-less Dirac particles in $1+1$ dimensions

This chapter has been published in similar form as the article [67] by the present author. In the version at hand, several sections have been adapted to avoid overlap with other chapters. This concerns sections I, II, IV A and B, VI B as well as VIII of [67].

3.1 Background and motivation

The implausibility of potentials in multi-time wave equations (see sec. 1.2.1.2) motivates the search for alternative mechanisms of interactions. Here we explore such a different approach, inspired by the field of zero-range physics (see [1] for an overview). The main idea is that boundary conditions for the wave function may yield physically interesting effects and even interaction while the formal differential operator in the wave equation is the free one. This clearly avoids the use of potentials. However, the direct application of the methods used in zero-range physics, developed for the single-time formalism, is not possible for multi-time equations.

To illustrate this claim, recall the standard functional-analytic treatment of single-time wave equations (see e.g. [82,83] and [51, chap. 14])

$$i\frac{\partial}{\partial t}\varphi \;=\; H\varphi, \tag{3.1}$$

where H is a self-adjoint operator on a Hilbert space $\mathcal{H}$, most often $\mathcal{H} = L^2(\mathcal{Q}) \otimes \mathbb{C}^k$, where $\mathcal{Q} \subset \mathbb{R}^{Nd}$ is the physically accessible part of configuration space. Usually, H is an unbounded operator with domain $\mathcal{D}(H) \subsetneq \mathcal{H}$. The specification of $\mathcal{D}(H)$ is important for physics, as it includes potential boundary conditions which influence time evolution and spectrum. H is the generator of a strongly continuous unitary one-parameter group $U(t) = \exp(-iHt)$. If $\varphi_0 \in \mathcal{D}(H)$, then $U(t)\varphi_0 \in \mathcal{D}(H)$ and $U(t)\varphi_0$ satisfies eq. (3.1). The

unitarity of U ensures conservation of the norm of the wave function which is essential for the statistical meaning of the wave function.

It is crucial to note that within this approach the allowed boundary conditions are timeless, as the Hilbert space $L^2(\mathcal{Q}) \otimes \mathbb{C}^k$ does not include time. Spatial boundary conditions prescribed in this way are automatically extended for all times t. An example for two particles is:

$$\varphi(\mathbf{x}_1, \mathbf{x}_2, t) \;=\; 0 \quad \text{for } \mathbf{x}_1 = \mathbf{x}_2, \; \forall t. \tag{3.2}$$

For multi-time wave functions, the method can be generalized straightforwardly using theorem 1.2.1 on strongly continuous unitary N-parameter groups $U(t_1, ..., t_N)$ on the same Hilbert space $\mathcal{H}$ as above. We recall that such a group can be constructed if and only if the generators H_j of the one-parameter subgroups $U(0, ..., 0, t_j, 0, ..., 0)$ are self-adjoint, commute pairwise and have a common domain $\mathcal{D}(H_j) \equiv \mathcal{D}$, independent of j. This implies: if $\psi_0 \in \mathcal{D}$, then $U(t_1, ..., t_N)\psi_0 \in \mathcal{D}$ and $U(t_1, ..., t_N)\psi_0$ obeys the multi-time equations (1.16).

However, one crucial aspect changes: boundary conditions are still supposed to be expressed via the domain $\mathcal{D}$ which makes no reference to time. Consequently, the boundary conditions are automatically extended in *all* time coordinates, e.g.:

$$\psi(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2) \;=\; 0 \quad \text{for } \mathbf{x}_1 = \mathbf{x}_2, \; \forall\, t_1, t_2. \tag{3.3}$$

Using the connection between single-time and multi-time equations (eq. (1.9)), one can see that condition (3.3) in fact differs from the corresponding one in the single-time formalism (3.2), although one might have $\mathcal{D}(H) = \mathcal{D}$. Namely, eq. (3.2) translated into the multi-time formalism via (1.9) reads:

$$\psi(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2) = 0 \quad \text{for } \mathbf{x}_1 = \mathbf{x}_2, \; t_1 = t_2 \tag{3.4}$$

with the condition "for $t_1 = t_2$" instead of "$\forall\, t_1, t_2$". However, boundary conditions like (3.3) for spatio-temporal configurations which may be time-like do not have a clear meaning. It thus seems that the functional-analytic approach is not adequate for multi-time equations on domains with boundaries, since it automatically implements too many[1] and physically unreasonable boundary conditions. Therefore, a different method is required to implement the idea that boundary conditions could lead to relativistically invariant interaction for multi-time wave functions. In order for the boundary conditions to be Lorentz invariant, time should also be admitted in their formulation. We suggest to take a step back and view the multi-time equations (1.16) as a general overdetermined system of PDEs on a subset of configuration space-time $\mathbb{R}^{N(1+d)}$, treating space and time on equal footing.

Of course, such a change in methods raises important questions, such as:

1. How does one prove existence and uniqueness of solutions?

2. How is probability conservation guaranteed and which notion thereof is adequate in the relativistic regime?

[1] It may well be that the only common domain $\mathcal{D}$ of self-adjointness of the H_j's is the one corresponding to the free operators, i.e. one where no boundary condition such as (3.4) is prescribed.

(In the functional-analytic treatment, the two points are conveniently answered by the notion of self-adjointness.)

In this chapter, we provide a model for which both questions can be answered definitely and precisely, bearing in mind also the physical aspects of interaction and Lorentz invariance. For this purpose, we consider a two-time system of mass-less Dirac equations in one spatial dimension ($d = 1$) on the domain of space-like configurations.

The choice of the model is explained as follows: The dimensionality both allows for an explicit solution in the mass-less case as well as leads to the situation that a certain natural Lorentz-invariant boundary in configuration space-time, the set of coincidence points, has the right dimensionality to have impact on the time evolution[2]. Moreover, the Dirac equation is Lorentz invariant, reflects the expected dispersion relation, and possesses a conserved tensor current with a positive component that can play the role of a probability density. The choice of domain is explained by the considerations about the necessity of multi-time wave functions in sec. 1.1.1.2. Interestingly, this immediately raises the question of boundary conditions since the domain of space-like configurations has a non-empty boundary: the light-like configurations. This provides a natural reason to study the idea of relativistic interaction by boundary conditions.

The chapter is structured as follows: We begin with introducing the model, as defined by its multi-time equations, domain, and initial conditions as well as boundary conditions at the space-time points of coincidence. Next, the general solution is found (lemma 3.3.1) and the existence and uniqueness of C^k-solutions[3] is studied by a generalized method of characteristics (theorem 3.3.3). Employing the relativistic notion of probability conservation of sec. 1.3.2 combined with Stokes' theorem, we then determine a general class of boundary conditions which guarantees it (theorem 3.4.2). We proceed with proving the Lorentz invariance of the model, and particularly of the boundary conditions (lemma 3.5.1). Moreover, a criterion for what constitutes interaction is suggested and applied to the model, showing that it is indeed interacting in this sense (theorem 3.6.2). The time evolution and effect of the interaction are explicitly illustrated using the example of initially localized wave packets for each of the two particles. Finally, the implications of antisymmetry for the boundary conditions in the case of indistinguishable particles are analyzed (lemma 3.7.1).

3.2 The model

Our model is based on a two-time wave function for two Dirac (spin-$\frac{1}{2}$) particles in $(1+1)$-dimensional space-time:

$$\psi : \ \Omega \subset \mathbb{R}^2 \times \mathbb{R}^2 \quad \longrightarrow \quad \mathbb{C}^2 \otimes \mathbb{C}^2, \qquad (t_1, z_1, t_2, z_2) \quad \longmapsto \quad \psi(t_1, z_1, t_2, z_2). \tag{3.5}$$

As mentioned before, the physically natural choice of the domain Ω is the set $\mathscr{S}$ of space-like configurations, given by[4]:

$$\mathscr{S} \ := \ \{(t_1, z_1, t_2, z_2) \in \mathbb{R}^2 \times \mathbb{R}^2 : (t_1 - t_2)^2 - (z_1 - z_2)^2 < 0\}. \tag{3.6}$$

[2] In a functional-analytic setting, the dimensionality of the boundary to allow for zero-range interactions is known to depend sensitively on the order of the differential operator and the dimension of configuration space [94].

[3] The differentiability referes to both space and time.

[4] We denote vectors in $1+1$-dimensional space-time $\mathbb{R}^2$ by $x = (t, z)$.

Initial data should be prescribed on a surface $\mathcal{I}$ of the form $\mathcal{I} = (\Sigma_0 \times \Sigma_0) \cap \Omega$ where Σ_0 is a space-like hypersurface. We choose:

$$\mathcal{I} \;:=\; \{(t_1, z_1, t_2, z_2) \in \mathscr{S} : t_1 = t_2 = 0\}, \tag{3.7}$$

i.e. a Σ_0 corresponding to $t = 0$.

In order to obtain a fully Lorentz invariant model, boundary conditions have to be prescribed on a Lorentz invariant subset of $\partial\Omega$. The first natural choice is the whole of $\partial\mathscr{S}$, i.e. the set

$$\mathscr{L} \;:=\; \{(t_1, z_1, t_2, z_2) \in \mathscr{S} : (t_1 - t_2)^2 - (z_1 - z_2)^2 = 0\} \tag{3.8}$$

of light-like configurations. However, this set has dimension three, as compared to dimension two of $\mathcal{I}$, so one expects it to lead to an overdetermined initial boundary value problem (IBVP)[5]. The second natural choice – and the one we shall make – is the set $\mathscr{C}$ of *coincidence points* in space-time, given by:

$$\mathscr{C} := \{(t_1, z_1, t_2, z_2) \in \mathbb{R}^2 \times \mathbb{R}^2 : t_1 = t_2, z_1 = z_2\}. \tag{3.9}$$

As two-time wave equations we use the free $(1+1)$-dimensional mass-less Dirac equations acting on the spin indices of the first and second particle, respectively. In this case, the Dirac γ-matrices are 2×2-matrices. Choosing the representation

$$\gamma^0 = \sigma_1, \quad \gamma^1 = \sigma_1\sigma_3, \tag{3.10}$$

the multi-time Dirac equations (1.18) can be written as

$$\begin{aligned} i\frac{\partial}{\partial t_1}\psi(t_1, z_1, t_2, z_2) \;&=\; -i\,\sigma_3 \otimes \mathbb{1}_2 \frac{\partial}{\partial z_1}\,\psi(t_1, z_1, t_2, z_2), \\ i\frac{\partial}{\partial t_2}\psi(t_1, z_1, t_2, z_2) \;&=\; -i\,\mathbb{1}_2 \otimes \sigma_3 \frac{\partial}{\partial z_2}\,\psi(t_1, z_1, t_2, z_2). \end{aligned} \tag{3.11}$$

In the case with mass, additional terms $m_1\,\sigma_1 \otimes \mathbb{1}_2$ and $m_2\,\mathbb{1}_2 \otimes \sigma_1$ appear in front of ψ on the right hand side.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.12}$$

are the Pauli matrices.

Note that the compatibility conditions (1.55) are satisfied, as the matrices appearing in the first and second equation of (3.11) are constant and commute. Furthermore, the free Dirac current $j_\psi^{\mu\nu} = \overline{\psi}\gamma_1^\mu\gamma_2'\psi$ is divergence-free in all of its indices as a consequence of eqs. (3.11).

To summarize, the model is given by:

$$\begin{cases} \text{Eqs. (3.11) on } \mathscr{S}, \\ \psi_i = g_i \text{ on } \mathcal{I},\ i = 1, 2, 3, 4, \\ \text{boundary conditions on } \mathscr{C}. \end{cases} \tag{3.13}$$

[5]This claim will be given further evidence in sec. 4.3.

Here, $\psi_i,\ i=1,2,3,4$ denote the components of ψ with respect to the (ordered) basis

$$\mathcal{B} = (e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2), \tag{3.14}$$

where e_i are the canonical basis vectors of $\mathbb{C}^2$. $g_i,\ i=1,2,3,4$ are arbitrary complex-valued C^k-functions on $\overline{\mathcal{I}}$. The form of admissible boundary conditions will be explored in the next section.

For future convenience, note the following explicit representation for arbitrary complex-valued 2×2 matrices $A=(a_{ij}),\ B=(b_{ij})$ with respect to $\mathcal{B}$:

$$A\otimes \mathbb{1}_2 = \begin{pmatrix} a_{11} & 0 & a_{12} & 0 \\ 0 & a_{11} & 0 & a_{12} \\ a_{21} & 0 & a_{22} & 0 \\ 0 & a_{21} & 0 & a_{22} \end{pmatrix}, \quad \mathbb{1}_2 \otimes B = \begin{pmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{pmatrix}. \tag{3.15}$$

3.3 Existence and uniqueness

In this section, it is shown which type of boundary conditions ensures existence and uniqueness of a C^k-solution (in space and time and for any $k\in\mathbb{N}$) of the two-time equations. This is achieved using a generalized version of the method of characteristics.

Lemma 3.3.1 *On any open and connected domain $D\subset\mathbb{R}^2\times\mathbb{R}^2$, the general solution of the two-time system* (3.11) *is given by:*

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}(t_1,z_1,t_2,z_2) \;=\; \begin{pmatrix} f_1(z_1-t_1, z_2-t_2) \\ f_2(z_1-t_1, z_2+t_2) \\ f_3(z_1+t_1, z_2-t_2) \\ f_4(z_1+t_1, z_2+t_2) \end{pmatrix}, \tag{3.16}$$

where $f_j:\mathbb{R}^2\to\mathbb{C},\ j=1,2,3,4$ are C^1-functions.

Proof: Using eq. (3.15), we explicitly write out eq. (3.11):

$$\begin{aligned} i\frac{\partial}{\partial t_1}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} &= -i\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\frac{\partial}{\partial z_1}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \\ i\frac{\partial}{\partial t_2}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} &= -i\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}\frac{\partial}{\partial z_2}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \end{aligned} \tag{3.17}$$

We see that our choice of basis in spin space makes all occurring matrices diagonal. The structure of the equations becomes very simple. For example, for ψ_1 we have:

$$\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial z_1}\right)\psi_1 = 0, \quad \left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial z_2}\right)\psi_1 = 0 \quad \Rightarrow\ \psi_1(t_1,z_1,t_2,z_2) = f_1(z_1-t_1, z_2-t_2), \tag{3.18}$$

where f_1 is C^1. The claim for the other components follows analogously. □

It is instructive to understand this result in geometrical terms. Eq. (3.16) implies that the components of the solution are constant along certain two-dimensional surfaces in $\mathbb{R}^2 \times \mathbb{R}^2$ (for some $c_1, c_2 \in \mathbb{R}$):

$$\begin{aligned} S_1(c_1,c_2) &:= \{(t_1,z_1,t_2,z_2) \in \mathbb{R}^2 \times \mathbb{R}^2 : z_1 - t_1 = c_1,\ z_2 - t_2 = c_2\}, \\ S_2(c_1,c_2) &:= \{(t_1,z_1,t_2,z_2) \in \mathbb{R}^2 \times \mathbb{R}^2 : z_1 - t_1 = c_1,\ z_2 + t_2 = c_2\}, \\ S_3(c_1,c_2) &:= \{(t_1,z_1,t_2,z_2) \in \mathbb{R}^2 \times \mathbb{R}^2 : z_1 + t_1 = c_1,\ z_2 - t_2 = c_2\}, \\ S_4(c_1,c_2) &:= \{(t_1,z_1,t_2,z_2) \in \mathbb{R}^2 \times \mathbb{R}^2 : z_1 + t_1 = c_1,\ z_2 + t_2 = c_2\}, \end{aligned} \tag{3.19}$$

where the index i in S_i refers to the component ψ_i that is constant along S_i. This behavior closely resembles the method of characteristics (see e.g. [24, 56]). We therefore call the surfaces S_i *multi-time characteristics*. They allow for a simple and powerful method to study the IBVP. Note that for the unbounded domain $\mathbb{R}^2 \times \mathbb{R}^2$, lemma 3.3.1 already yields existence and uniqueness of solutions for the initial value problem (3.13), with $f_i(x,y)$ from eq. (3.16) given by $g_i(x,y)$. For more complex domains such as $\Omega = \mathscr{S}$, one has to know more about the topological structure (in particular the connectedness).

Lemma 3.3.2 *The domain $\mathscr{S}$ is the disjoint union of the sets $\mathscr{S}_1$ and $\mathscr{S}_2$ where*

$$\begin{aligned} \mathscr{S}_1 &:= \{(t_1,z_1,t_2,z_2) \in \mathbb{R}^2 \times \mathbb{R}^2 : (t_1-t_2)^2 - (z_1-z_2)^2 < 0,\ z_1 < z_2\}, \\ \mathscr{S}_2 &:= \{(t_1,z_1,t_2,z_2) \in \mathbb{R}^2 \times \mathbb{R}^2 : (t_1-t_2)^2 - (z_1-z_2)^2 < 0,\ z_1 > z_2\}. \end{aligned} \tag{3.20}$$

Furthermore, $\mathscr{S}_1$ and $\mathscr{S}_2$ cannot be connected by a curve lying entirely in $\mathscr{S}$.

Proof: The first statement is obvious from the definition $\mathscr{S}$ (eq. (3.6)). The second statement follows because $\mathscr{S}_1, \mathscr{S}_2$ are disjoint and open (as can be seen from eq. (3.20)). □

This splitting of $\mathscr{S}$ into path-wise disjoint parts implies that one should formulate the IBVP separately for $\mathscr{S}_1, \mathscr{S}_2$. In particular, this allows for more subtle boundary conditions as limits within either $\mathscr{S}_1$ or $\mathscr{S}_2$. To identify these limits would mean to reduce the number of possibilities to prescribing that ψ has to be continuous across the boundary. It may, however, be physically reasonable to admit singularities (including jumps) of ψ at the boundary. In fact, this situation is generic in the field of zero-range physics [1] where similar singularities appear for δ-interactions.

Now we come to the main result of this section: the formulation of the initial boundary value problem and the corresponding proof of the existence and uniqueness of solutions.

Theorem 3.3.3 *Let $k \in \mathbb{N}$. Given complex-valued C^k-functions $h_j^{\pm}$ as well as $g_i^{(j)}$ ($i = 1,2,3,4$; $j = 1,2$) such that (3.23) holds, there exists a unique solution ψ which is C^k on $\mathscr{S}_1$ and $\mathscr{S}_2$ for the initial boundary value problem defined by:*

1. *For $\mathscr{S}_1$:*

$$\begin{aligned} \psi_i(0,z_1,0,z_2) &= g_i^{(1)}(z_1,z_2), \quad i = 1,2,3,4 \text{ for } z_1 < z_2, \text{ i.e. on } \mathscr{I}_1 := \mathscr{I} \cap \mathscr{S}_1, \\ \psi_3(t,z-0,t,z+0) &= h_1^+(t,z) \text{ for } t \geq 0, \text{ i.e. on } \mathscr{C}, \\ \psi_2(t,z-0,t,z+0) &= h_1^-(t,z) \text{ for } t < 0, \text{ i.e. on } \mathscr{C}. \end{aligned} \tag{3.21}$$

2. For $\mathscr{S}_2$:

$$\begin{aligned}
\psi_i(0,z_1,0,z_2) &= g_i^{(2)}(z_1,z_2),\ i=1,2,3,4 \text{ for } z_1>z_2, \text{ i.e. on } \mathcal{I}_2 := \mathcal{I}\cap\mathscr{S}_2,\\
\psi_2(t,z+0,t,z-0) &= h_2^+(t,z) \text{ for } t\geq 0, \text{ i.e. on } \mathscr{C},\\
\psi_3(t,z+0,t,z-0) &= h_2^-(t,z) \text{ for } t<0, \text{ i.e. on } \mathscr{C}.
\end{aligned} \tag{3.22}$$

Here,"± 0" denotes the corresponding limits, e.g. $\psi(z-0,z+0) := \lim_{\varepsilon\to 0}\psi(z-\varepsilon,z+\varepsilon)$.

Furthermore, it is required that the initial conditions satisfy the boundary conditions, i.e.:

$$\begin{aligned}
g_3^{(1)}(z,z) &= h_1^+(0,z)\ \forall z\in\mathbb{R},\\
g_2^{(1)}(z,z) &= h_1^-(0,z)\ \forall z\in\mathbb{R},\\
g_2^{(2)}(z,z) &= h_2^+(0,z)\ \forall z\in\mathbb{R},\\
g_3^{(2)}(z,z) &= h_2^-(0,z)\ \forall z\in\mathbb{R},
\end{aligned} \tag{3.23}$$

and also that these transitions between initial and boundary values be of regularity C^k.

Proof: We only show the statement for $\mathscr{S}_1$; the one for $\mathscr{S}_2$ follows analogously. The proof is structured as follows. First, we identify the part of $\mathscr{S}_1$ where each component of ψ is formally determined by initial data, i.e. their domain of dependence. Next, we check if there are also parts of $\mathscr{S}_1$ where the ψ_i are not specified by initial data. We continue by demonstrating that the above-mentioned boundary conditions formally yield the missing values of the ψ_i. Subsequently, we make sure that the constructions actually work by explicitly demonstrating that there exist curves within the characteristic surfaces connecting each point in $\mathscr{S}$ with exactly one initial or boundary value. Finally, we explicitly write down the solution in terms of initial data and show that it is indeed C^k.

1. *Domain of dependence of the initial data:* Consider the initial conditions in (3.21). Using the general solution (eq. (3.16)), we find: $\psi_i(0,z_1,0,z_2) = f_i(z_1,z_2) \overset{!}{=} g_i^{(1)}(z_1,z_2)$, $z_1<z_2$. Formally, this equation determines $f_i = f_i(x,y)$ as a function on $\{(x,y)\in\mathbb{R}^2 : x<y\}$. Geometrically, this means that the characteristics $S_i(x,y)$ intersect $\mathcal{I}$ in a single point $(0,x,0,y)_i$ for all i. Then ψ_i is constant along $S_i(x,y)$. This consideration demonstrates uniqueness. However, existence is only guaranteed if one can connect the initial values with a continuous curve within S_i that also remains in $\mathscr{S}_1$. This is shown under point 4.

2. *Complement of the domain of dependence of the initial data:*

   (a) $\psi_1(t_1,z_1,t_2,z_2) = f_1(z_1-t_1,z_2-t_2)$: We know from 1. that there exist points $(t_1,z_1,t_2,z_2)\in\mathscr{S}_1$ such that $z_1-t_1<z_2-t_2$. However, is $z_1-t_2>z_2-t_2$ also possible in $\mathscr{S}_1$? To answer this question, consider: $z_1-t_2>z_2-t_2 \Leftrightarrow z_1-z_2\geq t_1-t_2$. In $\mathscr{S}_1$, $z_1-z_2<0$ which implies $t_1-t_2<0$ and therefore $|z_1-z_2|<|t_1-t_2|$. This inequality states that the configuration (t_1,z_1,t_2,z_2) has to be time-like, in contradiction to $\mathscr{S}_1\subset\mathscr{S}$. So there are no points in $\mathscr{S}_1$ which require the function $f_1(x,y)$ to be defined for $x>y$. We proceed similarly for the other components.

(b) $\psi_4(t_1,z_1,t_2,z_2) = f_4(z_1+t_1, z_2+t_2)$: $z_1+t_1 > z_2+t_2 \Leftrightarrow t_1-t_2 > z_2-z_1$. Since $z_1 < z_2$ in $\mathscr{S}_1$, we obtain: $|t_1-t_2| > |z_1-z_2|$, so also $f_4(x,y)$ is only required for $x<y$.

(c) $\psi_2(t_1,z_1,t_2,z_2) = f_2(z_1-t_1, z_2+t_2)$: $z_1-t_1 > z_2+t_2 \Leftrightarrow -t_1-t_2 > z_2-z_1$. This time, the inequality can always be satisfied, e.g by choosing $z_1 < z_2$ arbitrarily and $t_1 = t_2 \equiv t$ with $t < (z_1-z_2)/2$. Thus, $f_2(x,y)$ is not yet determined fully by initial values. Note that this case appears only for $t_1+t_2<0$.

(d) $\psi_3(t_1,z_1,t_2,z_2) = f_3(z_1+t_1, z_2-t_2)$: $z_1+t_1 > z_2-t_2 \Leftrightarrow t_1+t_2 > z_2-z_1$. Again, this can happen for all values of z_1+t_2, z_2-t_2, e.g. for $t_1=t_2\equiv t$ with $t > (z_2-z_1)/2$. Note that this case requires $t_1+t_2>0$.

3. *Domain of dependence of the boundary values:*

(a) The condition $\psi_2(t,z-0,t,z+0) \stackrel{!}{=} h_1^-(t,z),\ t<0$ yields (leaving away the limit "± 0" for notational ease): $f_2(z-t,z+t) = h_1^-(t,z)$. Indeed, this determines the missing values $f_2(x,y),\ x\geq y$ exactly once, as the map $\Phi : \{(t,z)\in\mathbb{R}^2 : t<0\} \to \{(x,y)\in\mathbb{R}^2 : x>y\},\ (t,z)\mapsto(z-t,z+t)$ is bijective.

(b) Similarly, the condition $\psi_3(t,z-0,t,z+0) \stackrel{!}{=} h_1^+(t,z),\ t\geq 0$ determines $f_3(x,y),\ x\geq y$ exactly once as the map $\tilde{\Phi} : \{(t,z)\in\mathbb{R}^2 : t\geq 0\} \to \{(x,y)\in\mathbb{R}^2 : x\geq y\},\ (t,z)\mapsto(z+t,z-t)$ is bijective.

4. *Proof of existence:* We have to make sure that both initial values as well as boundary values can be transported along a mult-time characteristic while staying in $\mathscr{S}_1$. Then the aforesaid considerations show that the functions f_i are determined uniquely.

(a) For ψ_1: We have to show that there exists a continuous curve connecting (t_1,z_1,t_2,z_2) to $(0,z_1-t_1,0,z_2-t_2)$ while staying within a multi-time characteristic S_1 defined by $z_1-t_1=c_1,\ z_2-t_2=c_2$ and also in $\mathscr{S}_1$. In fact, such a path is given by:

$$\gamma_1 : [0,1] \to S_1\cap\mathscr{S}_1, \quad \gamma_1(\tau) := (\tau t_1, z_1-t_1+\tau t_1, \tau t_2, z_2-t_2+\tau t_2). \quad (3.24)$$

Obviously: $\gamma_1(0) = (0,z_1-t_1,0,z_2-t_2)$, $\gamma_1(1) = (t_1,z_1,t_2,z_2)$. Besides, the z_1-component of $\gamma_2(\tau)$ has to be smaller than the z_2-component: $z_1-t_1+\tau t_1 < z_2-t_2+\tau t_2 \Leftrightarrow z_2-z_1 > (t_2-t_1)(1-\tau)$. This inequality is satisfied because in $\mathscr{S}_1$, we have $z_1<z_2$ and $|z_1-z_2|>|t_1-t_2|$.

Furthermore, one has to ensure that $\gamma_1(\tau)$ always yields a space-like configuration. To see this, consider:

$$\begin{aligned} \tau^2(t_1-t_2)^2 &< (z_1-t_1+\tau t_1-z_2+t_2-\tau t_2)^2 \\ \Leftrightarrow \quad 0 &< [(z_1-z_2)+(t_2-t_1)]^2 - 2\tau(z_1-z_2)(t_2-t_1) - 2\tau(t_2-t_1)^2. \end{aligned}$$

Now we use $-(t_2-t_1)^2 > -(z_2-z_1)^2$ for the last summand which yields:

$$\begin{aligned} &[(z_1-z_2)+(t_2-t_1)]^2 - 2\tau(z_1-z_2)(t_2-t_1) - 2\tau(t_2-t_1)^2 \\ &> [(z_1-z_2)+(t_2-t_1)]^2 - \tau[(z_1-z_2)^2 + 2(z_1-z_2)(t_2-t_1) - (t_2-t_1)^2] \\ &= [(z_1-z_2)+(t_2-t_1)^2](1-\tau). \end{aligned}$$

For $\tau \in (0,1)$ this is indeed greater than zero and for $\tau = 0, 1$ the claim is evident, anyway.

For the other components we only state the corresponding curves. The proof that they stay within $S_i \cap \mathscr{S}_1$ is analogous to the one above. In case of ψ_2, ψ_3 the curves start at boundary values, i.e. stay only within $S_i \cap \overline{\mathscr{S}}_1$.

(b) For ψ_4:

$$\gamma_4 : [0,1] \to S_4 \cap \mathscr{S}_1, \quad \gamma_4(\tau) := (\tau t_1, z_1 + t_1 - \tau t_1, \tau t_2, z_2 + t_2 - \tau t_2). \quad (3.25)$$

(c) For ψ_2:

$$\gamma_2 : [0,1] \to S_4 \cap \overline{\mathscr{S}}_1,$$
$$\gamma_2(\tau) := \begin{cases} (\tau t_1, z_1 - t_1 + \tau t_1, \tau t_2, z_2 + t_2 - \tau t_2) & \text{for } z_1 - t_1 < z_2 + t_2; \\ ((-z_1 + z_2 + t_1 + t_2)/2 + \tau(z_1 - z_2 + t_1 - t_2), (z_1 + z_2 - t_1 + t_2)/2 + \tau(z_1 - z_2 + t_1 - t_2), & \\ (-z_1 + z_2 + t_1 + t_2)/2 + \tau(z_1 - z_2 - t_1 + t_2), (z_1 + z_2 - t_1 + t_2)/2 + \tau(-z_1 + z_2 + t_1 - t_2)) & \\ \text{for } z_1 - t_1 > z_2 + t_2. \end{cases} \quad (3.26)$$

The rather lengthy formula in the second case arises from a simple consideration. As before, $p = (t_1, z_1, t_2, z_2)$ is the point where we want to show the solution to be determined. Next, one determines the point (t, z, t, z) of intersection of $S_2(z_1 - t_1, z_2 + t_2)$ with $\mathscr{C}$, obtaining $t = (-z_1 + z_2 + t_1 + t_2)/2$ and $z = (z_1 + z_2 - t_1 + t_2)/2$. Then: $\gamma_2(\tau) = (t + \tau(t_1 - t), z + \tau(z_1 - z), t + \tau(t_2 - t), z - \tau(z_2 - z))$.

(d) For ψ_3:

$$\gamma_3 : [0,1] \to S_3 \cap \overline{\mathscr{S}}_1,$$
$$\gamma_3(\tau) := \begin{cases} (\tau t_1, z_1 + t_1 - \tau t_1, \tau t_2, z_2 - t_2 + \tau t_2) & \text{for } z_1 + t_1 < z_2 - t_2; \\ ((z_1 - z_2 + t_1 + t_2)/2 + \tau(-z_1 + z_2 + t_1 - t_2), (z_1 + z_2 + t_1 - t_2)/2 + \tau(z_1 - z_2 - t_1 + t_2), & \\ (z_1 - z_2 + t_1 + t_2)/2 + \tau(-z_1 + z_2 - t_1 + t_2), (z_1 + z_2 + t_1 - t_2)/2 + \tau(-z_1 + z_2 - t_1 + t_2)) & \\ \text{for } z_1 + t_1 > z_2 - t_2. \end{cases} \quad (3.27)$$

The expression in the second case results from an analogous consideration as for ψ_2, the only change being the use of the point (t, z, t, z) of intersection of $S_3(z_1 + t_1, z_2 - t_2)$ with $\mathscr{C}$.

5. *Explicit solution and C^k property:* Collecting the results from the previous points, we obtain on $\mathscr{S}_1$:

$$\begin{aligned} \psi_1(t_1, z_1, t_2, z_2) &= g_1^{(1)}(z_1 - t_1, z_2 - t_2), \\ \psi_2(t_1, z_1, t_2, z_2) &= \begin{cases} g_2^{(1)}(z_1 - t_1, z_2 + t_2) & \text{for } z_1 - t_1 < z_2 + t_2 \\ h_1^-((-z_1 + z_2 + t_1 + t_2)/2, (z_1 + z_2 - t_1 + t_2)/2) & \text{for } z_1 - t_1 \geq z_2 + t_2 \end{cases} \\ \psi_3(t_1, z_1, t_2, z_2) &= \begin{cases} g_3^{(1)}(z_1 + t_1, z_2 - t_2) & \text{for } z_1 + t_1 < z_2 - t_2 \\ h_1^+((z_1 - z_2 + t_1 + t_2)/2, (z_1 + z_2 + t_1 - t_2)/2) & \text{for } z_1 + t_1 \geq z_2 - t_2 \end{cases} \\ \psi_4(t_1, z_1, t_2, z_2) &= g_4^{(1)}(z_1 + t_1, z_2 + t_2). \end{aligned} \quad (3.28)$$

From this formula, we immediately see that ψ_1, ψ_4 are C^k on $\mathscr{S}_1$ as $g_1^{(1)}, g_2^{(1)}$ are C^k. For the other two components a similar argument holds true if additionally the transition between the two cases is C^k.

(a) For ψ_2: The critical points are at $u := z_1 - t_1 = z_2 + t_2$. We obtain as a condition that

$$g_2^{(1)}(u,u) \stackrel{!}{=} h_1^-(0,u) \; \forall u \in \mathbb{R} \tag{3.29}$$

and that this transition be C^k. We recognize this as one of the conditions in (3.23) in the statement of the theorem.

(b) For ψ_3: The critical points are at $v := z_1 + t_1 = z_2 - t_2$. We obtain as a condition that

$$g_3^{(1)}(v,v) \stackrel{!}{=} h_1^+(0,v) \; \forall v \in \mathbb{R} \tag{3.30}$$

and that this transition be C^k. This is again one of the conditions in (3.23). □

Remark: Note that for the definition of the functions $h_j^{\pm}$ one can make use of those components of ψ that are already determined by initial values at the boundary point (t,z,t,z) in question.

3.4 Boundary conditions derived from probability conservation

In this section, we combine the relativistic notion of probability conservation (see sec. 1.3.2) and especially its formulation via the current form ω_j (see sec. 1.3.3) with Stokes' theorem in order to extract a general class of probability-conserving boundary conditions. This formulation is applicable to our model as the free Dirac equations conserve the tensor current $j_\psi^{\mu\nu} = \overline{\psi}\gamma_1^\mu\gamma_2^\nu\psi$. Before stating the main result, we formulate a lemma which allows us to control the spreading of the wave function.

Lemma 3.4.1 *Consider the IBVP defined by* (3.21), (3.22) *and let* Σ *denote a space-like hypersurface. Then, if the initial data are compactly supported on* $\mathcal{I}$*, they are compactly supported on all sets of the form* $(\Sigma \times \Sigma) \cap \mathscr{S}$*.*

Proof: This can be seen immediately from the explicit solution (3.28). (Influences propagate with finite speed along the multi-time characteristics.) □

Theorem 3.4.2 *Let* $\varepsilon_{\mu\nu}$ *denote the Levi-Civita symbol. Assume that the initial data are of regularity* C^k, $k \in \mathbb{N}$, *and compactly supported on* $\mathcal{I}$*. Then the following conditions for the tensor current guarantee probability conservation in the sense of criterion* (1.71)*:*

$$\begin{aligned} \varepsilon_{\mu\nu} j_\psi^{\mu\nu}(t, z-0, t, z+0) &\stackrel{!}{=} 0, \quad t,z \in \mathbb{R}, \\ \varepsilon_{\mu\nu} j_\psi^{\mu\nu}(t, z+0, t, z-0) &\stackrel{!}{=} 0, \quad t,z \in \mathbb{R}. \end{aligned} \tag{3.31}$$

Expressed in terms of the components of ψ*, these conditions are equivalent to:*

$$\begin{aligned} \psi_2(t, z-0, t, z+0) &\stackrel{!}{=} e^{-i\theta_1(t,z)} \psi_3(t, z-0, t, z+0), \quad t,z \in \mathbb{R}, \\ \psi_2(t, z+0, t, z-0) &\stackrel{!}{=} e^{-i\theta_2(t,z)} \psi_3(t, z+0, t, z-0), \quad t,z \in \mathbb{R} \end{aligned} \tag{3.32}$$

for arbitrary functions $\theta_1, \theta_2 : \mathbb{R}^2 \to [-\pi, \pi)$. *(In order for* ψ *to be* C^k*, they have to be* C^k*-functions, too.)*

Remark:

1. Conditions (3.31) have the physical meaning that the probability flux from $\mathscr{S}_1$ into $\mathscr{C}$ and from $\mathscr{S}_2$ into $\mathscr{C}$ has to vanish, separately. They are therefore a subclass of all conditions on j that lead to probability conservation. Boundary conditions with a similar meaning are widely used to express confinement of particles in certain spatial regions (see e.g. [51, chaps. 12,14] for a physically motivated discussion). The crucial difference here is that the boundary set is determined by internal relations between the particles, not by external geometry.

2. Note that the boundary conditions (3.32) are of the form (3.21), (3.22) with the property (3.23). Thus, theorem 3.3.3 ensures the existence and uniqueness of a C^k-solution on $\mathscr{S}_1$ and $\mathscr{S}_2$ of the corresponding IBVP.

Proof: The idea is to use Stokes' theorem for a closed surface S of the form $S = [(\Sigma_1 \times \Sigma_1) \cap \mathscr{S}] \cup [(\Sigma_2 \times \Sigma_2) \cap \mathscr{S}] \cup M$ where Σ_1, Σ_2 are space-like hypersurfaces and M is the rest of the closed surface. Then, because of $d\omega_j = 0$, one obtains equality of the normalization integrals (1.66) if the contribution of M vanishes. Parts of the contribution of M vanish because ψ is compactly supported on sets of the form $(\Sigma \times \Sigma) \cap \mathscr{S}$ according to lemma 3.4.1. Demanding that the remaining parts also vanish leads to conditions on the tensor current.

We split the proof into two parts: the first one to establish the conditions on the current such that the normalization integral of the wave function is equal for all space-like hypersurfaces Σ and the second one to derive the equivalent conditions for the components of ψ.

1. We first show that S can be understood as a closed surface in an appropriate sense. To this end, we define finite versions of Σ_1, Σ_2. Pick $p_1 \in \Sigma_1$, $p_2 \in \Sigma_2$. Then let

$$\Sigma_i^R \;:=\; \{p \in \Sigma_i : -(p^0 - p_i^0)^2 + (\mathbf{p} - \mathbf{p}_i)^2 < R^2\}, \quad i = 1, 2. \tag{3.33}$$

   For R large enough and $q \in [(\Sigma_i \backslash \Sigma_i^R) \times (\Sigma_i \backslash \Sigma_i^R)] \cap \mathscr{S}$ we have $\psi(q) = 0$ as ψ is compactly supported on sets of the form $(\Sigma \times \Sigma) \cap \mathscr{S}$. Consequently, one obtains

$$\int_{(\Sigma_i \times \Sigma_i) \cap \mathscr{S}} \omega_j \;=\; \int_{(\Sigma_i^R \times \Sigma_i^R) \cap \mathscr{S}} \omega_j, \quad i = 1, 2. \tag{3.34}$$

   It is therefore permitted to replace Σ_i with Σ_i^R for the purpose of the argument.

   Now we construct a closed surface S_R as follows: Let $V_{\Sigma_1, \Sigma_2} \subset \mathbb{R}^{1+d}$ be the volume between Σ_1, Σ_2, i.e. if $t_\Sigma(\mathbf{x})$ denotes the time coordinate of the unique point $p \in \Sigma$ with spatial coordinates $\mathbf{x}$, then

$$V_{\Sigma_1, \Sigma_2} \;:=\; \{(\tau, \mathbf{y}) \in \mathbb{R}^{1+d} : t_{\Sigma_1}(\mathbf{y}) < \tau < t_{\Sigma_2}(\mathbf{y}) \vee t_{\Sigma_2}(\mathbf{y}) < \tau < t_{\Sigma_1}(\mathbf{y})\}. \tag{3.35}$$

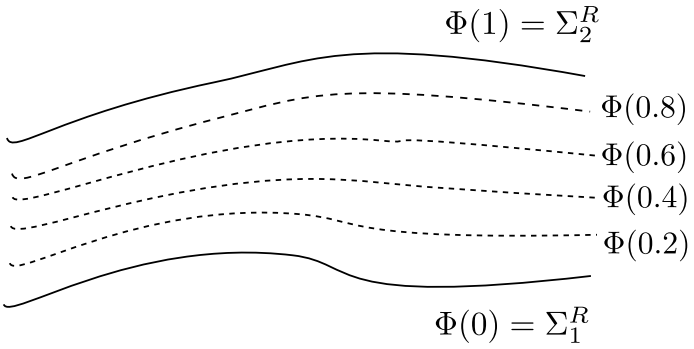

Figure 3.1: A continuous deformation of Σ_1^R into Σ_2^R.

Next, consider a continuous deformation of Σ_1^R into Σ_2^R (see fig. 3.1), i.e. a smooth map

$$\Phi : [0,1] \to \{\Sigma \subset \overline{V}_{\Sigma_1,\Sigma_2} : \Sigma \text{ space-like surface}\}, \quad \text{with } \Phi(0) = \Sigma_1^R,\ \Phi(1) = \Sigma_2^R. \tag{3.36}$$

Now let

$$S_R \ := \ \partial \left(\bigcup_{s\in[0,1]} [\Phi(s) \times \Phi(s)] \cap \mathscr{S} \right). \tag{3.37}$$

By construction, S_R is a closed surface. It has the form

$$S_R \ = \ [(\Sigma_1^R \times \Sigma_1^R) \cap \mathscr{S}] \cup [(\Sigma_2^R \times \Sigma_2^R) \cap \mathscr{S}] \cup M_1 \cup M_2, \tag{3.38}$$

where $M_1 = M \cap \operatorname{supp}\psi$ and $M_2 = M \backslash M_1$ (so $\psi \equiv 0$ on M_2). From eq. (3.37) it can then be seen that M_1 consists of those points $p \in \Phi(s) \times \Phi(s)$ which do not lie in $\mathscr{S}$. As $\Phi(s)$ is a space-like surface, all points $p = (x,y),\ x,y \in \Phi(s)$ with $x \neq y$ are contained in $\mathscr{S}$. The remaining ones therefore belong to the set $\mathscr{C}$ of coincidence points and it follows that $M_1 \subset \mathscr{C}$.

At this point, a subtlety appears: Recall that the values of ψ are not defined on $\mathscr{C}$ (as $\mathscr{C} \nsubseteq \mathscr{S}$). Rather, one has to consider the corresponding limits in $\mathscr{S}_1$ and $\mathscr{S}_2$. Instead of S_R, one should consider the union of $S_R^{(1)}$ with $S_R^{(2)}$ where $S_R^{(i)},\ i = 1,2$ are defined by eq. (3.37) using $\mathscr{S}_i,\ i = 1,2$ instead of $\mathscr{S}$. They have the form

$$S_R^{(i)} \ = \ [(\Sigma_1^R \times \Sigma_1^R) \cap \mathscr{S}_i] \cup [(\Sigma_2^R \times \Sigma_2^R) \cap \mathscr{S}_i] \cup M_1 \cup M_2^{(i)}, \tag{3.39}$$

where M_1 is the same as above and $\psi \equiv 0$ on $M_2^{(i)},\ i = 1,2$.

Let V_R denote the volume enclosed by S_R. Using Stokes' theorem for S_R, we obtain:

$$\begin{aligned} \int_{S_R} \omega_j \ &= \ \int_{V_R} d\omega_j \ \overset{\text{eq.}(1.72)}{=} \ 0 \\ \Rightarrow \quad \int_{(\Sigma_1^R \times \Sigma_1^R)\cap\mathscr{S}} \omega_j \ &= \ \int_{(\Sigma_2^R \times \Sigma_2^R)\cap\mathscr{S}} \omega_j \ - \int_{M_1} \omega_j^{(1)} \ + \int_{M_1} \omega_j^{(2)}, \end{aligned} \tag{3.40}$$

where $\omega_j^{(i)}$ is shorthand for taking the limit $\varepsilon \to 0$ for $\psi(t, z + (-1)^i\varepsilon, t, z - (-1)^i\varepsilon)$, $i = 1, 2$ in the expression for ω_j. Orientation conventions have to be considered to obtain the correct signs in front of the integrals.

Thus, we obtain independence of the normalization integrals from Σ if

$$\int_{M_1} \omega_j^{(1)} \;=\; \int_{M_1} \omega_j^{(2)}. \tag{3.41}$$

We specialize[6] to the case

$$\int_{M_1} \omega_j^{(1)} \;=\; \int_{M_1} \omega_j^{(2)} \;=\; 0. \tag{3.42}$$

This condition will be satisfied if the current form obtained from the corresponding limit vanishes on M_1, or more generally on $\mathscr{C}$. The latter condition is reasonable to demand because the construction has to work for any Σ_1, Σ_2.

So far, the construction works for any dimension d. We now specialize to $d = 1$. In order to obtain an appropriate condition on j, we express ω_j using relative coordinates

$$\begin{aligned} & z = z_1 - z_2, \quad Z = z_1 + z_2, \quad \tau = t_1 - t_2, \quad T = t_1 + t_2 \\ \Leftrightarrow \quad & z_1 = \tfrac{1}{2}(Z + z), \quad z_2 = \tfrac{1}{2}(Z - z), \quad t_1 = \tfrac{1}{2}(T + \tau), \quad t_2 = \tfrac{1}{2}(T - \tau). \end{aligned} \tag{3.43}$$

This yields:

$$\begin{aligned} \omega_j \;=\; & \tfrac{1}{2} j_\psi^{00} dz \wedge dZ - \tfrac{1}{4}(j_\psi^{10} + j_\psi^{01}) d\tau \wedge dZ + \tfrac{1}{4}(j_\psi^{10} - j_\psi^{01}) d\tau \wedge dz \\ & - \tfrac{1}{4}(j_\psi^{10} - j_\psi^{01}) dT \wedge dZ - \tfrac{1}{4}(j_\psi^{10} + j_\psi^{01}) dz \wedge dT + \tfrac{1}{2} j_\psi^{11} d\tau \wedge dT. \end{aligned} \tag{3.44}$$

Now, on $\mathscr{C}$ we have $\tau = 0, z = 0$. Thus, we find:

$$\omega_j(t, z+(-1)^i 0, t, z-(-1)^i 0) \;=\; \tfrac{1}{4}(j_\psi^{01} - j_\psi^{10})(t, z+(-1)^i 0, t, z-(-1)^i 0)\, dT \wedge dZ \tag{3.45}$$

which leads to the following condition for the tensor current:

$$(j_\psi^{01} - j_\psi^{10})(t, z + (-1)^i 0, t, z - (-1)^i 0) \;\stackrel{!}{=}\; 0, \quad i = 1, 2. \tag{3.46}$$

Recalling $\varepsilon_{\mu\nu} j_\psi^{\mu\nu} = j_\psi^{01} - j_\psi^{10}$, one easily verifies that these are exactly the conditions stated in eq. (3.31).

2. Next, we show that the conditions for the current (which are bilinear in ψ) are actually equivalent to the linear relations between the components of ψ stated in eq. (3.32). For this purpose, consider

$$\begin{aligned} j_\psi^{01} - j_\psi^{10} \;=\; & \psi^\dagger(\gamma_1^0\gamma_2^0\gamma_1^0\gamma_2^1 - \gamma_1^0\gamma_2^0\gamma_1^1\gamma_2^0)\psi \;=\; \psi^\dagger(\mathbb{1}_2 \otimes \sigma_3 - \sigma_3 \otimes \mathbb{1}_2)\psi \\ & \stackrel{\text{eqs.(3.12),(3.15)}}{=} (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*) \begin{pmatrix} 0 & & & \\ & -2 & & \\ & & 2 & \\ & & & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \\ \Rightarrow \quad j_\psi^{01} - j_\psi^{10} \;=\; 0 \quad \Leftrightarrow \quad & |\psi_2|^2 \;=\; |\psi_3|^2. \end{aligned} \tag{3.47}$$

[6] The general case would lead to compensating currents from $\mathscr{S}_1$ to $\mathscr{S}_2$ and the other way around. This would mean that the particles could pass each other – which seems somewhat odd in $d = 1$. This case will be analyzed in more detail in chap. 4.

This relation is satisfied if and only if there exists a phase function θ such that $\psi_2 = e^{-i\theta}\psi_3$. Applied to eq. (3.46), this yields the claim (3.32). □

Remark: Note that the strategy used in the proof can immediately be generalized to arbitrary particle numbers and dimensions and domains. Furthermore, it is purely geometrical and therefore leads to Lorentz invariant conditions for the tensor current, if the domain Ω is Lorentz invariant[7].

3.5 Lorentz invariance

In this section, we address the issue of Lorentz invariance of the constructions used in the model. First, we state clearly our understanding of Lorentz invariance. Then we briefly review some basic representation theory of the one-dimensional proper Lorentz group and discuss the invariance of the model. The main result is the proof that the probability-conserving boundary conditions (3.32) are indeed Lorentz invariant under certain conditions on the phase functions. We also point out a subclass of conditions for which the invariance is manifest.

3.5.1 The meaning of Lorentz invariance for the model

For the model to be Lorentz invariant, we require the following points:

1. If a function ψ solves the multi-time wave equations in one frame, it also solves the equations in every other frame. Furthermore, the equations have the same functional form in all frames.
2. Probability conservation holds in all frames.
3. In any frame, initial data can be given on $(\Sigma_t \times \Sigma_t) \cap \mathscr{S}$ where Σ_t is an equal-time hypersurface.
4. If a function ψ satisfies the boundary conditions in one frame, it satisfies the Lorentz-transformed boundary conditions in every other frame. These boundary conditions have the same functional form in all frames.

Before commenting on these points, we state the transformation properties in question.

3.5.2 Representation of the one-dimensional proper Lorentz group

In $1+1$ dimensions, the proper Lorentz group $\mathcal{L}_+^\uparrow$ has only one generator, the boost generator in z-direction ($x_i = (t_i, z_i)$). For the spinor representation acting on the spin index of the i-th particle, this is:

$$S_i^{01} \;=\; \tfrac{1}{4}[\gamma_i^0, \gamma_i^1]. \tag{3.48}$$

Under the action of an element $\Lambda \in \mathcal{L}_+^\uparrow$, a two-time wave function transforms as

$$\psi(x_1, x_2) \;\overset{\Lambda}{\longmapsto}\; \psi'(x_1, x_2) \;\equiv\; S_1[\Lambda]\, S_2[\Lambda]\, \psi(\Lambda^{-1}x_1, \Lambda^{-1}x_2), \tag{3.49}$$

[7] We call a set A Lorentz invariant if and only if for each point $p \in A$ the Lorentz transformed point p' is also contained within the set.

where

$$S_i[\Lambda] \;=\; \exp(\beta S_i^{01}). \tag{3.50}$$

Here, β is a real parameter determined by Λ. Finally, for later use, note the following relation:

$$\gamma_i^\mu S_i[\Lambda] \;=\; S_i[\Lambda]\, \Lambda^\mu_\nu \gamma_i^\nu. \tag{3.51}$$

The afore-noted information is already sufficient to discuss the requirements mentioned in the previous subsection:

1. By the standard arguments about the Lorentz invariance of the Dirac equation (see eg. [95]), one can show that the multi-time Dirac equations (3.11) indeed transform covariantly. Recalling the argument in sec. 1.1.1.2, we note that in order to discuss the Lorentz invariance of the wave equations, it is crucial to consider a multi-time wave function. Moreover, this consideration also requires the domain Ω to be Lorentz invariant. The space-like configurations $\mathscr{S}$ are of course such a Lorentz invariant set.

   Furthermore, under $\Lambda \in \mathcal{L}^\uparrow_+$, one obtains

   $$j_\psi^{\mu\nu}(x_1, x_2) \quad \overset{\Lambda}{\longmapsto} \quad \Lambda^\mu_\rho\, \Lambda^\nu_\sigma\, j_\psi^{\rho\sigma}(\Lambda^{-1}x_1, \Lambda^{-1}x_2), \tag{3.52}$$

   i.e. $j_\psi^{\mu\nu}$ transforms similarly to a tensor, the only difference being that the argument of j is an element of configuration space-time instead of just space-time.

2. As shown in theorem 3.4.2, if probability conservation holds on one space-like hypersurface, it holds on all space-like hypersurfaces. This, of course, includes the equal-time hypersurfaces for all frames.

3. So far, we assumed the initial data to be given in one particular frame. However, as the choice of this frame is not fixed by any circumstance, one can simply choose the coordinates such that the initial data surface is actually of the desired form.

4. The Lorentz invariance of the boundary conditions is the most subtle point. Because of the transformation properties of j, the conditions on the tensor current are easily seen to be Lorentz invariant (see eq. (3.31)). However, for the conditions (3.32) on the components of ψ, Lorentz invariance is not manifest and the transformation behavior has to be checked explicitly.

3.5.3 Lorentz invariance of the boundary conditions

Lemma 3.5.1 *The current-conserving boundary conditions*

$$\begin{aligned}
\psi_2(t, z-0, t, z+0) \;&\overset{!}{=}\; e^{-i\theta_1}\, \psi_3(t, z-0, t, z+0), \quad t, z \in \mathbb{R}, \\
\psi_2(t, z+0, t, z-0) \;&\overset{!}{=}\; e^{-i\theta_2}\, \psi_3(t, z+0, t, z-0), \quad t, z \in \mathbb{R}
\end{aligned} \tag{3.53}$$

are Lorentz invariant if the functions θ_1, θ_2 transform as Lorentz scalars, i.e. if

$$\theta_i(t, z) \;\overset{\Lambda}{\longmapsto}\; \theta_i(\Lambda^{-1}(t, z))\ \forall \Lambda \in \mathcal{L}^\uparrow_+,\ i = 1, 2. \tag{3.54}$$

Proof: We explicitly determine the transformation properties of the components ψ_i. According to eq. (3.49), we need to calculate the matrices $S_1[\Lambda], S_2[\Lambda]$ via formula (3.50). We have:

$$\begin{aligned} S_1[\Lambda] &= \exp(\beta\tfrac{1}{2}\gamma_1^0\gamma_1^1) \overset{\text{eq.(3.10)}}{=} \exp(\beta\tfrac{1}{2}\sigma_3 \otimes \mathbb{1}_2) \\ &\overset{\text{eq.(3.15)}}{=} \sum_{k=0}^{\infty} \frac{(\beta/2)^k}{k!} \begin{pmatrix} \mathbb{1}_2 & \\ & (-\mathbb{1}_2)^k \end{pmatrix} \\ &= \cosh\beta \cdot \mathbb{1}_4 + \sinh\beta \begin{pmatrix} \mathbb{1}_2 & \\ & -\mathbb{1}_2 \end{pmatrix}, \end{aligned} \tag{3.55}$$

$$\begin{aligned} S_2[\Lambda] &= \exp(\beta\tfrac{1}{2}\gamma_2^0\gamma_2^0) \overset{\text{eq.(3.10)}}{=} \exp(\beta\tfrac{1}{2}\mathbb{1}_2 \otimes \sigma_3) \\ &\overset{\text{eq.(3.15)}}{=} \sum_{k=0}^{\infty} \frac{(\beta/2)^k}{k!} \begin{pmatrix} 1 & & & \\ & (-1)^k & & \\ & & (-1)^k & \\ & & & 1 \end{pmatrix} \\ &= \cosh\beta \cdot \mathbb{1}_4 + \sinh\beta \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}. \end{aligned} \tag{3.56}$$

It follows that:

$$S_1[\Lambda]S_2[\Lambda] = \cosh^2\beta\,\mathbb{1}_4 + 2\cosh\beta\sinh\beta \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix} + \sinh^2\beta \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}. \tag{3.57}$$

Thus

$$\psi_i(x_1, x_2) \overset{\Lambda}{\longmapsto} \psi_i(\Lambda^{-1}x_1, \Lambda^{-1}x_2) \quad \text{for } i = 2, 3. \tag{3.58}$$

Using this transformation property in eq. (3.53) together with eq. (3.54) immediately yields the claim. □

Lemma 3.5.2 *In the case of $e^{-i\theta_k(t,z)} \equiv \pm i$, the boundary conditions* (3.53) *can be rewritten in the following manifestly Lorentz invariant form:*

$$\begin{aligned} \varepsilon_{\mu\nu}\gamma_1^\mu\gamma_2^\nu\psi(t, z-0, t, z+0) &\overset{!}{=} \pm i(\mathbb{1}_4 + \gamma_1^5\gamma_2^5)\psi(t, z-0, t, z+0), \quad t, z \in \mathbb{R}, \\ \varepsilon_{\mu\nu}\gamma_1^\mu\gamma_2^\nu\psi(t, z+0, t, z-0) &\overset{!}{=} \pm i(\mathbb{1}_4 + \gamma_1^5\gamma_2^5)\psi(t, z+0, t, z-0), \quad t, z \in \mathbb{R}, \end{aligned} \tag{3.59}$$

where

$$\gamma_k^5 := i\gamma_k^0\gamma_k^1, \quad k = 1, 2. \tag{3.60}$$

Proof:

$$\varepsilon_{\mu\nu}\gamma_1^\mu\gamma_2^\nu = \gamma_1^0\gamma_2^1 - \gamma_1^1\gamma_2^0 = \sigma_1\otimes\mathbb{1}_2\cdot\mathbb{1}_2\otimes(\sigma_1\sigma_3) - (\sigma_1\sigma_3)\otimes\mathbb{1}_2\cdot\mathbb{1}_2\otimes\sigma_1 \;=\; \begin{pmatrix} & & 0 & 0 \\ & & 2 & 0 \\ 0 & -2 & & \\ 0 & 0 & & \end{pmatrix},$$

$$\mathbb{1}_4 + \gamma_1^5\gamma_2^5 = \mathbb{1}_4 + i\sigma_3\otimes\mathbb{1}_2\cdot i\mathbb{1}_2\otimes\sigma_3 = \mathbb{1}_4 - \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 0 & & & \\ & 2 & & \\ & & 2 & \\ & & & 0 \end{pmatrix}. \tag{3.61}$$

Thus, we obtain:

$$\begin{aligned} \varepsilon_{\mu\nu}\gamma_1^\mu\gamma_2^\nu\psi \;&\stackrel{!}{=}\; \pm i(\mathbb{1}_4+\gamma_1^5\gamma_2^5)\psi \\ \Leftrightarrow\quad \begin{pmatrix} & & 0 & 0 \\ & & 2 & 0 \\ 0 & -2 & & \\ 0 & 0 & & \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} \;&\stackrel{!}{=}\; \pm i\begin{pmatrix} 0 & & & \\ & 2 & & \\ & & 2 & \\ & & & 0 \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}. \end{aligned} \tag{3.62}$$

This is in turn equivalent to the following conditions

$$\begin{aligned} 0 \;&=\; 0, \\ \psi_3 \;&=\; \pm i\psi_2, \\ -\psi_2 \;&=\; \pm i\psi_3, \\ 0 \;&=\; 0. \end{aligned} \tag{3.63}$$

The claim follows. □

Remark: For the manifestly Lorentz invariant boundary conditions (3.59), one can use the usual representation-independent strategy to prove Lorentz invariance.

Assume that the conditions are fulfilled in one frame F. Now consider the same conditions in another frame F' that is connected with the former one by a Lorentz transformation Λ. We have to show that (3.59) is satisfied as a consequence of the transformation law (3.49) for ψ as well as (3.59) for F. Consider eq. (3.59) for F':

$$\begin{aligned} \varepsilon_{\mu\nu}\gamma_1^\mu\gamma_2^\nu\psi'(x_1,x_2) \;&\stackrel{!}{=}\; \pm i(\mathbb{1}_4+\gamma_1^5\gamma_2^5)\psi'(x_1,x_2) \\ \Leftrightarrow\quad \varepsilon_{\mu\nu}\gamma_1^\mu\gamma_2^\nu S_1[\Lambda]S_2[\Lambda]\,\psi(\Lambda^{-1}x_1,\Lambda^{-1}x_2) \;&\stackrel{!}{=}\; \pm i(\mathbb{1}_4+\gamma_1^5\gamma_2^5)S_1[\Lambda]S_2[\Lambda]\,\psi(\Lambda^{-1}x_1,\Lambda^{-1}x_2), \end{aligned} \tag{3.64}$$

where $x_1 = (t, z\pm 0)$ and $x_2 = (t, x_2 \mp 0)$. As $S_1[\Lambda]$ and $S_2[\Lambda]$ are invertible and because $(\Lambda^{-1}x_1, \Lambda^{-1}x_2)$ again has the form $(t', z'\pm 0, t', z'\mp 0)$, it is sufficient to prove that $S_1[\Lambda]S_2[\Lambda]$ commutes with both $\varepsilon_{\mu\nu}\gamma_1^\mu\gamma_2^\nu$ as well as $\mathbb{1}_4+\gamma_1^5\gamma_2^5$. Consider first

$$\varepsilon_{\mu\nu}\,\gamma_1^\mu\gamma_2^\nu\,S_1[\Lambda]S_2[\Lambda] \stackrel{\text{eq.}(3.51)}{=} S_1[\Lambda]S_2[\Lambda]\,\varepsilon_{\mu\nu}\,\Lambda_\rho^\mu\Lambda_\sigma^\nu\,\gamma_1^\rho\gamma_2^\sigma = S_1[\Lambda]S_2[\Lambda]\,\varepsilon_{\rho\sigma}\,\gamma_1^\rho\gamma_2^\sigma, \tag{3.65}$$

where in the equality we used $\det(\Lambda)\varepsilon_{\rho\sigma} = \varepsilon_{\mu\nu}\,\Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\sigma}$ as well as $\det(\Lambda) = 1$ for $\Lambda \in \mathcal{L}^{\uparrow}_{+}$.

In order to show that $S_1[\Lambda]S_2[\Lambda]$ commutes with $\mathbb{1}_4 + \gamma^5_1\gamma^5_2$, it is sufficient that the generators $S^{01}_k,\ k = 1,2$ commute with $\gamma^5_1\gamma^5_2$. We have: $S^{01}_k\gamma^5_j = \gamma^5_j S^{01}_k,\ j,k = 1,2$. For $j \neq k$, this is obvious and in case $j = k$ the equation easily follows from $S^{01}_k = \frac{1}{2}\gamma^0_k\gamma^1_k$ (see eq. (3.48)) as well as $\gamma^5_k = i\gamma^0_k\gamma^1_k$ (see eq. (3.60)).

3.6 Interaction

In this section, we analyze the physical meaning of the boundary conditions (3.53) and in particular the question of whether they lead to interaction. In order to address this question appropriately, we suggest a simple and clear-cut notion of interaction. Then we use the explicit solution of our model to determine the time evolution initial product wave functions. From this result, we gain physical insight into the detailed nature of the time evolution implied by our model. Moreover, we can use it to conclude that the model indeed leads to interaction.

A criterion for interaction: Most often, "interaction" in quantum mechanics is simply defined by the presence of an interaction potential in the Hamiltonian. This notion of interaction is obviously not adequate for models such as ours where one aims at implementing interaction effects via boundary conditions. A more general criterion is needed.
A quantum-mechanical model is called *free* if every initial product wave function (in the particle coordinates and spin indices) remains a product wave function during time evolution. It is called *interacting* if there exist initial product wave functions that do not stay product wave functions during time evolution.

Evolution of an initial product wave function: We first explicitly determine the time evolution of an arbitrary initial wave function on $\mathscr{S}_1$.

Lemma 3.6.1 *For our model defined with boundary conditions* (3.32)*, the time evolution of an initial wave function on $\mathscr{S}_1$, i.e.*

$$\psi_i(0,z_1,0,z_2) \;=\; g^{(1)}_i(z_1,z_2) \quad \text{for } z_1 < z_2, \quad i = 1,2,3,4, \tag{3.66}$$

where these initial data are assumed to satisfy the boundary conditions, is given by

$$\begin{aligned}
\psi_1(t_1,z_1,t_2,z_2) &= g^{(1)}_1(z_1-t_1,z_2-t_2),\\
\psi_2(t_1,z_1,t_2,z_2) &= \begin{cases} g^{(1)}_2(z_1-t_1,z_2+t_2) & \text{for } z_1-t_1<z_2+t_2\\ e^{-i\theta_1((-z_1+z_2+t_1+t_2)/2,(z_1+z_2-t_1+t_2)/2)}g^{(1)}_3(z_2+t_2,z_1-t_1) & \text{for } z_1-t_1\geq z_2+t_2 \end{cases}\\
\psi_3(t_1,z_1,t_2,z_2) &= \begin{cases} g^{(1)}_3(z_1+t_1,z_2-t_2) & \text{for } z_1+t_1<z_2-t_2\\ e^{i\theta_1((z_1-z_2+t_1+t_2)/2,(z_1+z_2+t_1-t_2)/2)}g^{(1)}_2(z_2-t_2,z_1+t_1) & \text{for } z_1+t_1\geq z_2-t_2 \end{cases}\\
\psi_4(t_1,z_1,t_2,z_2) &= g^{(1)}_4(z_1+t_1,z_2+t_2).
\end{aligned} \tag{3.67}$$

Proof: We make use of the explicit solution (3.28) where this time the functions $h^{\pm}_1$ are given by ψ_2,ψ_3 via the boundary conditions (3.32). As noted before, we can solve the model on $\mathscr{S}_1$ autonomously.
For ψ_1,ψ_4 the claim is directly given by eq. (3.28). For ψ_2,ψ_3 there are two different cases:

1. For $z_1 + t_1 \geq z_2 - t_2$, ψ_2 is determined by initial data (see eq. (3.28)) and we have:

$$\begin{aligned}\psi_3(t_1, z_2, t_2) &= h_1^+(\underbrace{\tfrac{z_1 - z_2 + t_1 + t_2}{2}}_{t}, \underbrace{\tfrac{z_1 + z_2 + t_1 - t_2}{2}}_{z}) \\ &= e^{i\theta_1(t,z)}\psi_2(t, z-0, t, z+0) = e^{i\theta_1(t,z)} g_2^{(1)}(z-t, t+z) \\ &= e^{i\theta_1((z_1 - z_2 + t_1 + t_2)/2, (z_1 + z_2 + t_1 - t_2)/2)} g_2^{(1)}(z_2 - t_2, z_1 + t_1). \end{aligned} \tag{3.68}$$

2. Similarly, for $z_1 - t_1 \geq z_2 + t_2$, ψ_3 is determined by initial data and we obtain:

$$\begin{aligned}\psi_2(t_1, z_2, t_2) &= h_1^-(\underbrace{\tfrac{-z_1 + z_2 + t_1 + t_2}{2}}_{t'}, \underbrace{\tfrac{z_1 + z_2 - t_1 + t_2}{2}}_{z'}) \\ &= e^{i\theta_1(t',z')}\psi_2(t', z'-0, t', z'+0) = e^{i\theta_1(t',z')} g_2^{(1)}(z'-t', t'+z') \\ &= e^{-i\theta_1((-z_1 + z_2 + t_1 + t_2)/2, (z_1 + z_2 - t_1 + t_2)/2)} g_3^{(1)}(z_2 + t_2, z_1 - t_1). \end{aligned} \tag{3.69}$$

Noting that the two cases are exclusive on $\mathscr{S}_1$, we have determined ψ uniquely (see eq. (3.28)) and the claim follows. □

We can use the lemma to determine the time evolution of an initial product wave function on $\mathscr{S}_1$, given by $\psi(0, z_1, 0, z_2) = \phi(z_1) \otimes \chi(z_2)$ for $z_1 < z_2$ where ϕ, χ are two-component spinors. ψ has the components

$$\psi_1 = \phi_1\chi_1, \quad \psi_2 = \phi_1\chi_2, \quad \psi_3 = \phi_2\chi_1, \quad \psi_4 = \phi_2\chi_2. \tag{3.70}$$

Theorem 3.6.2 *Our model, defined by eqs.* (3.21), (3.22) *and* (3.53), *is interacting in the sense of the criterion presented above.*

Proof: Applying lemma 3.6.1 to the initial product wave function of eq. (3.70), we obtain:

$$\begin{aligned}\psi_1(t_1, z_1, t_2, z_2) &= \phi_1(z_1 - t_1)\chi_1(z_2 - t_2), \\ \psi_2(t_1, z_1, t_2, z_2) &= \begin{cases} \phi_1(z_1 - t_1)\chi_2(z_2 + t_2) & \text{for } z_1 - t_1 < z_2 + t_2 \\ e^{-i\theta_1((-z_1 + z_2 + t_1 + t_2)/2, (z_1 + z_2 - t_1 + t_2)/2)}\phi_2(z_2 + t_2)\chi_1(z_1 - t_1) & \text{for } z_1 - t_1 \geq z_2 + t_2 \end{cases} \\ \psi_3(t_1, z_1, t_2, z_2) &= \begin{cases} \phi_2(z_1 + t_1)\chi_1(z_2 - t_2) & \text{for } z_1 + t_1 < z_2 - t_2 \\ e^{i\theta_1((z_1 - z_2 + t_1 + t_2)/2, (z_1 + z_2 + t_1 - t_2)/2)}\phi_1(z_2 - t_2)\chi_2(z_1 + t_1) & \text{for } z_1 + t_1 \geq z_2 - t_2 \end{cases} \\ \psi_4(t_1, z_1, t_2, z_2) &= \phi_2(z_1 + t_1)\chi_2(z_2 + t_2) \end{aligned} \tag{3.71}$$

for $(t_1, z_1, t_2, z_2) \in \mathscr{S}_1$.

For $t_1, t_2 \neq 0$, the wave function ψ in eq. (3.71) is in general not a product wave function (3.70). This can clearly be seen from the case differentiation for the components ψ_2, ψ_3. (The corresponding non-interacting model would merely lead to translated initial data in all components.) We have therefore found an example for an initial product wave function which becomes entangled with the time evolution and the claim follows. □

Details of the interaction: As thm. 3.6.2 is an important result, we shall analyze the interaction effect in more detail. To this end, we consider a particularly simple class of initial product wave functions (nonzero only on $\mathscr{S}_1$):

$$\psi(0,z_1,0,z_2) = \phi(z_1)\otimes\chi(z_2) \quad \text{where} \quad \phi_2 \equiv \chi_1 \equiv 0, \quad \phi_1 = \tilde{\phi}\,1_{[a,b]}, \quad \chi_2 = \tilde{\chi}\,1_{[c,d]}. \tag{3.72}$$

Here, $\tilde{\phi},\tilde{\chi}:\mathbb{R}\to\mathbb{C}$ are smooth functions with support in $[a,b]$ and $[c,d]$, respectively. We choose $a<b<c<d$. The symbol $1_{[x,y]}$ denotes the characteristic function of the interval $[x,y]$, i.e. $1_{[x,y]}(z)=1$ if $z\in[x,y]$ and 0 otherwise. We have multiplied $\tilde{\phi},\tilde{\chi}$ with the characteristic functions of their support to make more explicit when they vanish.

As a consequence of (3.72), we have:

$$\begin{aligned}
\psi_1(0,z_1,0,z_2) &= \psi_3(0,z_1,0,z_2) = \psi_4(0,z_1,0,z_2) = 0,\\
\psi_2(0,z_1,0,z_2) &= \tilde{\phi}(z_1)\,\tilde{\chi}(z_2)\,1_{[a,b]}(z_1)\,1_{[c,d]}(z_2).
\end{aligned} \tag{3.73}$$

Specifically, we note that $\psi_2(0,z_1,0,z_2)=0$ for $z_2\geq z_1$. Therefore, ψ satisfies the boundary conditions (3.53) in the form $0=0$. Eq. (3.72) represents a wave function which is initially well-localized and associated with definite spin state $e_1\otimes e_2$ (corresponding to ψ_2).

According to lemma 3.6.1, the explicit time evolution of ψ on $\mathscr{S}_1$ is given by

$$\begin{aligned}
\psi_1 &\equiv \psi_4 \equiv 0,\\
\psi_2(t_1,z_1,t_2,z_2) &= \tilde{\phi}(z_1-t_1)\tilde{\chi}(z_2+t_2)\,1_{[a,b]+t_1}(z_1)\,1_{[c,d]-t_2}(z_2)\\
&\quad\times\Theta(-z_1+t_1+z_2+t_2),\\
\psi_3(t_1,z_1,t_2,z_2) &= e^{i\theta_1((z_1-z_2+t_1+t_2)/2,(z_1+z_2+t_1-t_2)/2)}\tilde{\phi}(z_2-t_2)\tilde{\chi}(z_1+t_1)\\
&\quad\times 1_{[a,b]+t_2}(z_2)\,1_{[c,d]-t_1}(z_1)\,\Theta(z_1+t_1-z_2+t_2),
\end{aligned} \tag{3.74}$$

where Θ denotes the Heaviside function. Note that in eq. (3.74) one can leave away the Θ functions as they only set the function to zero where it vanishes anyway.

Eq. (3.74) allows for a simple graphical representation (see fig. 3.2). Focusing on times $t_1=t_2=t$, we see that at $t=0$ each particle has an associated wave packet localized in a certain region. These regions do not overlap. For $t>0$, the wave packets are translated towards each other with speed $c=1$, so that the gap between them shrinks with speed 2. There is no dispersion in the mass-less case. Of course, these wave packets are not actually functions on physical space but on different copies thereof, as factors of configuration space. As soon as they meet, a scattering process happens. The wave packets swap place, i.e. the one associated with particle 1 becomes associated with particle 2 and the other way around. This process is associated with a phase. The particles move in opposite directions as before – and the corresponding contribution to the wave function is one associated with different spin.

In summary, the interaction has range zero, respects retardation and produces a change from one spin component into another. Due to this behavior, the model only describes scattering processes. "Bound states" or "resonances" do not appear. (To define these concepts rigorously, one can use the single-time model obtained by restricting the multi-time wave function to a common time.)

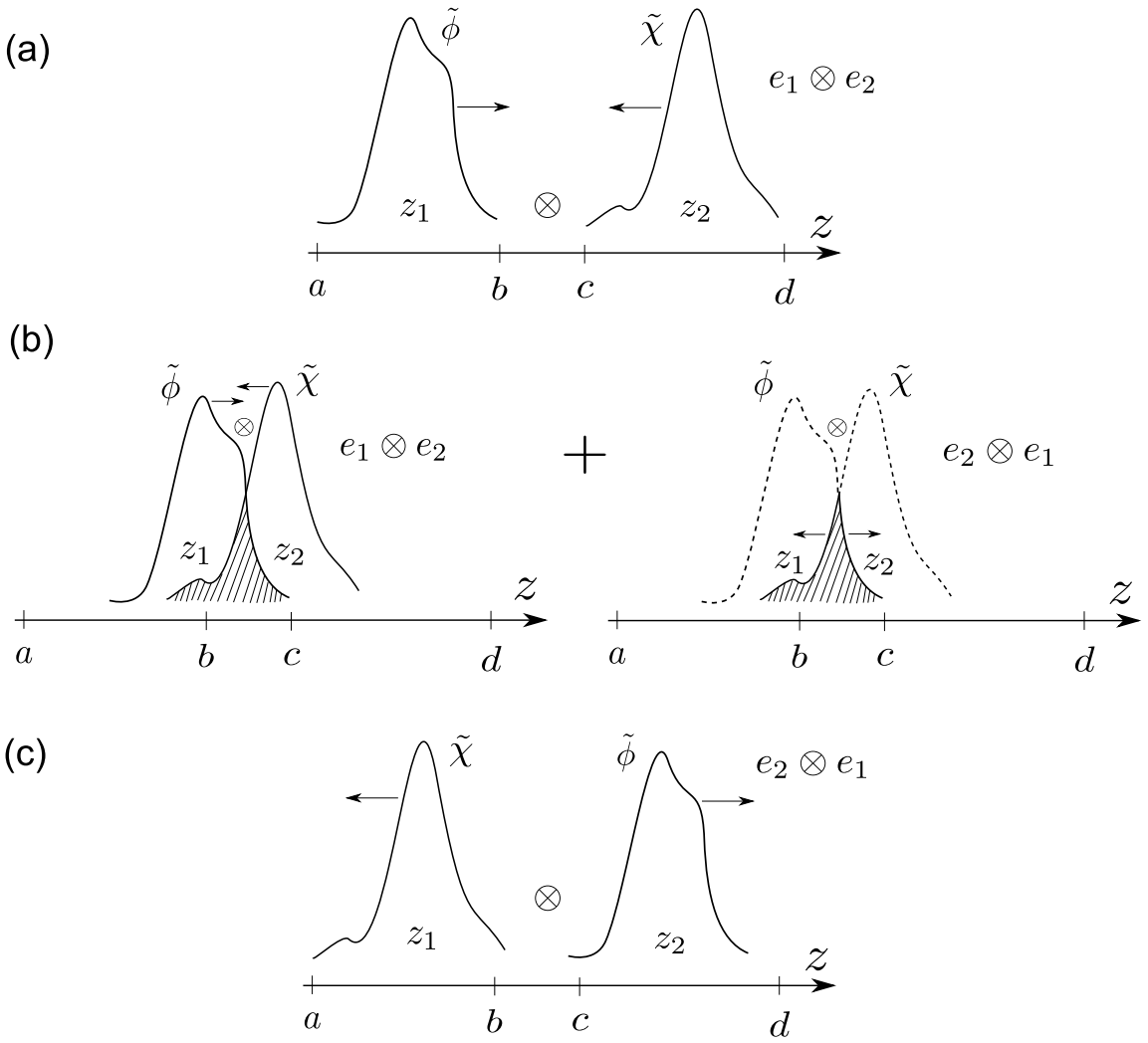

Figure 3.2: Schematic illustration of the interaction. (a) $t_1 = t_2 = 0$: The wave packets move towards each other (in different parts of configuration space) with speed $c = 1$ and without dispersion. $\tilde{\phi}$ is associated with particle 1 and $\tilde{\chi}$ with particle 2. The only non-zero component of the total wave function is ψ_2 (associated with $e_1 \otimes e_2$). (b) $(d-a)/2 > t_1 = t_2 > (c-b)/2$: The wave packets overlap (if plotted in the same space). The hatched area for $e_1 \otimes e_2$ has left $\mathscr{S}_1$. It reappears with a phase in the component for $e_2 \otimes e_1$. The wave packets have swapped place. The hatched parts of $\tilde{\phi}$ are now associated with particle 2 and the hatched parts of $\tilde{\chi}$ with particle 1. (c) $t_1 = t_2 > (d-a)/2$: The wave packets have passed each other. The only non-zero component is ψ_3 (associated with $e_2 \otimes e_1$).

3.7 Indistinguishable particles

If we describe indistinguishable fermions, the correct transformation properties[8] of the wave function are

$$\psi^{s_1 s_2}(x_1, x_2) \;=\; -\psi^{s_2 s_1}(x_2, x_1). \tag{3.75}$$

Here, the double index $s_1 s_2$ where $s_1, s_2 = -1, 1$ is a different way of denoting the spin components. We have:

$$\psi^{-1-1} \;=\; \psi_1, \quad \psi^{-11} \;=\; \psi_2, \quad \psi^{1-1} \;=\; \psi_3, \quad \psi^{11} \;=\; \psi_4. \tag{3.76}$$

One may ask: is our model compatible with these transformation rules? In order to answer this question, we note that given a solution of the IBVP with boundary conditions (3.32) on $\mathscr{S}$, we may use eq. (3.75) to extend it antisymmetrically to $\mathscr{S}_2$. It is easy to see that as a consequence of the two-time Dirac equation (3.11) on $\mathscr{S}_1$ it also solves the two-time Dirac equation on $\mathscr{S}_2$, with initial data that are the antisymmetric extension of those on $\mathscr{S}_1$. However, under which circumstances are the boundary conditions on $\mathscr{S}_2$ satisfied?

To answer this question, we consider the transformation behavior of the boundary conditions (3.32) on $\mathscr{S}_1$:

$$\begin{aligned}
& \psi_2(t, z-0, t, z+0) \;\stackrel{!}{=}\; e^{-i\theta_1(t,z)} \psi_3(t, z-0, t, z+0), \quad t, z \in \mathbb{R} \\
\overset{\text{antisym.}}{\longmapsto} \quad & -\psi_3(t, z+0, t, z-0) \;\stackrel{!}{=}\; -e^{-i\theta_1(t,z)} \psi_2(t, z+0, t, z-0), \quad t, z \in \mathbb{R} \\
\Leftrightarrow \quad & \psi_2(t, z+0, t, z-0) \;\stackrel{!}{=}\; e^{+i\theta_1(t,z)} \psi_2(t, z+0, t, z-0), \quad t, z \in \mathbb{R},
\end{aligned}$$

which is a boundary condition on $\mathscr{S}_2$. Comparison with eq. (3.32) yields the following result:

Lemma 3.7.1 *The IBVP defined by eqs.* (3.21), (3.22) *and* (3.32) *is compatible with antisymmetry under particle exchange* (3.75) *if the initial data are antisymmetric and if* $\theta_2 = -\theta_1$.

3.8 Discussion and outlook

In this chapter, we have provided an explicit construction of an interacting quantum-mechanical model which conforms to the strictest requirements of relativistic invariance, as embodied by the multi-time formalism. In particular, the free Dirac current $j^{\mu\nu}_\psi = \overline{\psi}\gamma_1^\mu \gamma_2^\nu \psi$ is conserved which makes the model compatible with the HBD model (compare sec. 2.1) as well as the GRWm theory (sec. 2.4.2.1). Note that the same does not hold true for the GRWf model (sec. 2.4.2.2) because the latter requires the multi-time dynamics to be defined on $\mathbb{R}^{N(1+d)}$ instead of $\mathscr{S}$. Together with one of these realistic models, our one-dimensional model thus constitutes an interacting theory free of the measurement problem, achieving a solution of both first and second-class difficulties (see the introduction). The interaction described by the model corresponds to a scattering process with zero range which is associated with a spin flip.

[8] Eq. (3.75) is a straightforward generalization of the well-known antisymmetry properties of a single-time wave function, i.e. $\varphi^{s_1 s_2}(\mathbf{x}_1, \mathbf{x}_2, t) = -\varphi^{s_2 s_1}(\mathbf{x}_2, \mathbf{x}_1, t)$.

Recall the notion of a single-time reducible theory from sec. 1.2.3. Noting that the boundary conditions (3.32) of our model are formulated at only one time, we see that it is indeed single-time reducible[9]. A single-time formulation can be obtained by restricting the multi-time wave function to a single global time via eq. (1.9). Then the multi-time model (3.13) yields a single-time model (using the chain rule to obtain a single wave equation from the multi-time wave equations).

The thus obtained single-time model allows for a comparison with the more familiar functional-analytic approach. It can be analyzed on its own terms using the methods of zero-range physics. This changes the notion of a solution from "classical" to "weak". Nevertheless, we expect such an approach to lead to results for the subclass of classical solutions which are similar to the results for the solutions of our model when evaluated at equal times. For the functional-analytic approach, one expects that the phase functions in eq. (3.32) must not depend on t – and therefore, by Lorentz invariance, not on z, either. Unsurprisingly, our approach is slightly more general in this regard[10], as it is specifically designed for an equal treatment of space and time variables (see the introduction).

In view of the success of the methods for this very simple model, it is natural to ask for possible generalizations with respect to several aspects:

- **Different boundary conditions:** The class of boundary conditions (3.32) was deduced by a top-down approach, starting from the requirements of existence and uniqueness and then successively adding the physical requirements of probability conservation, Lorentz-invariance and antisymmetry. We did not attempt to show that it is the most general class compatible with these requirements. In particular, as noted in footnote 6, there do also exist further probability-conserving conditions on the tensor current, corresponding to a non-vanishing transfer of total probability form $\mathscr{S}_1$ to $\mathscr{S}_2$ and vice versa. A study of alternative boundary conditions and of the generality of the present ones will be presented in sec. 4.1.

- $N > 2$ **particles:** A generalization to N particles is straightforward and will be given in sec. 4.2. The main difficulty is to find a class of boundary conditions which can be formulated in a simple way for every N.

- **Non-zero masses:** The study of the mass-less case is mainly a technical simplification. The conservation properties of the tensor current as well as the derivation of the class of probability-conserving boundary conditions are independent of the presence of mass terms in the multi-time equations. However, the strategy of the existence and uniqueness proof was to make use of the fact that the solution has to be constant along the multi-time characteristics. This is not true anymore for the case with mass. Preliminary investigations have led us to the idea that it might be possible to reformulate the simultaneous differential systems of multi-time equations into a single system of multi-time integral equations. For these integral equations, fixed point arguments could be used to prove the existence and uniqueness of a solution.

[9] Of course, single-time reducibility does not mean that the multi-time formulation is dispensable. On the contrary, it is essential, for otherwise Lorentz invariance of the wave equations cannot be discussed (see sec. 1.1.1.2)

[10] Note, however, that this greater generality of the method may not be needed because translationally invariant phase functions do not depend on t and z.

A different (albeit more indirect) way to approach the case $m \neq 0$ goes via the single-time reduced model[11]. Assuming that the self-adjointness of the single-time Hamiltonian can be shown on a domain implementing the boundary conditions, one can always add a bounded term without changing the situation. Then one could use the inductive method of Petrat and Tumulka mentioned in sec. 1.2.3 for single-time reducible theories with finite propagation speed to prove existence and uniqueness of solutions for the multi-time model.

- **Higher dimensions:** An immediate generalization of the model to $d > 1$ is not feasible. One can see this from the following consideration. The boundary conditions as the mechanism of interaction are derived from the fact that the integral $\int_{\mathscr{C}} \omega_j$ has to vanish to ensure probability conservation. However, ω_j is in general a $2d$-form and $\mathscr{C} = \{(x_1, x_2) \in \mathbb{R}^{1+d} \times \mathbb{R}^{1+d} : x_1 = x_2\}$ is $(1+d)$-dimensional. Thus, $\mathscr{C}$ is a zero-measure set for $d > 1$ and the integral vanishes without boundary conditions. As uniqueness of solutions follows from probability conservation according to thm. 1.3.3, no boundary conditions are required from a mathematical perspective. Prescribing boundary conditions in spite of this would either influence the wave function only on a low-dimensional set or lead to (possibly complicated) restrictions on the initial data. Without a clear physical reason for conditions of this kind, this option does not seem sensible. We therefore conclude that our construction has to be modified in order to produce interaction effects for $d > 1$.

- **Different domains:** Motivated by the fact that $\int_{\mathscr{C}} \omega_j$, the flux through the boundary, vanishes for $d > 1$, one can try to find a different Lorentz invariant domain which yields a non-vanishing flux through the boundary, i.e. $\int_{\mathscr{B}} \omega_j$ where $\mathscr{B}$ has a dimension of at least $2d$. Such a domain is for example given by the space-like configurations with a minimum space-like distance α:

$$\mathscr{S}_\alpha \;:=\; \{(x_1, x_2) \in \mathbb{R}^{1+d} \times \mathbb{R}^{1+d} : (x_1^0 - x_2^0)^2 - (\mathbf{x}_1 - \mathbf{x}_2)^2 < -\alpha^2\}. \tag{3.77}$$

  The question of whether consistent, Lorentz invariant and probability-conserving dynamics exist on $\mathscr{S}_\alpha$ will be answered in sec. 4.3 for the one-dimensional case. Note, however, that this idea is not fully satisfactory from the physical point of view as one introduces an additional constant α without an apparent deeper reason. Thus, if consistent dynamics should exist on $\mathscr{S}_\alpha$, the limit $\alpha \to 0$ would be particularly interesting. In this limit, the boundary conditions are prescribed on the light-like configurations $\mathscr{L}$, a situation which faintly resembles the mechanism of interaction in Wheeler-Feynman electrodynamics.

[11] The single-time reduced model will be considered in more detail in sec. 4.2.6

Chapter 4

Generalizations of the $1+1$-dimensional model

This section deals with several generalizations of the model in chap. 3. Firstly, we treat the questions of different boundary conditions as well as of the generality of the previous ones (sec. 4.1). Secondly, the model is generalized to the N-particle case (sec. 4.2). Finally, the issue of the existence of dynamics on a domain of configurations with a minimum space-like distance α is addressed (sec. 4.3).

4.1 Different boundary conditions

4.1.1 Probability conservation

Recall from the proof of thm. 3.4.2 that the following condition on the tensor current is sufficient to ensure probability conservation in the sense of condition (1.71):

$$\varepsilon_{\mu\nu}\left[j_\psi^{\mu\nu}(t,z-0,t,z+0)-j_\psi^{\mu\nu}(t,z+0,t,z-0)\right] \;=\; 0 \quad \forall t,z\in\mathbb{R}. \tag{4.1}$$

We now show that as compared to the previous class (3.32) there exists a new type of boundary conditions implying (4.1) without rendering the individual summands in eq. (4.1) and therefore the flux from $\mathscr{S}_1$ to $\mathscr{S}_2$ (and vice versa) zero.

Lemma 4.1.1 *Let $w_1=\psi_2+\psi_3$ and $w_2=\psi_2-\psi_3$. Then the following boundary conditions ensure condition* (4.1)*:*

$$w_1(t,z-0,t,z+0) \;=\; w_1(t,z+0,t,z-0) \quad \forall t,z\in\mathbb{R}, \tag{4.2}$$

$$w_2(t,z+0,t,z-0)-w_2(t,z-0,t,z+0) \;=\; -i\beta w_1(t,z-0,t,z+0) \quad \forall t,z\in\mathbb{R}, \tag{4.3}$$

where β is a real number.

Remarks:

1. Similar boundary conditions were used by Albeverio *et al.* to express certain types of zero-range interactions in the case of a single Dirac particle [1, appendix J, eqs. (J.6), (J.7)]. Analogously, our boundary conditions do not exhaust the class of boundary conditions which lead to compensating but non-vanishing currents between $\mathscr{S}_1$ and $\mathscr{S}_2$. However, a complete classification of all boundary conditions of this kind is tedious and does not lead to more physical insight. Therefore, we only analyze the class given by (4.2), (4.3) as an example.

2. The boundary conditions (4.2), (4.3) enforce a jump in the combination of the components of ψ which is expressed by w_2. Its strength depends on w_1 and is, in addition, controlled by the real parameter β. The case $\beta = 0$ corresponds to continuity of both w_1 and w_2 (and therefore ψ_1 and ψ_2) between $\mathscr{S}_1$ and $\mathscr{S}_2$. Continuity of w_1 between the different parts of the domain is always enforced by condition (4.2). Because of this situation, we henceforth refer to eqs. (4.2), (4.3) as *jump boundary conditions*.

Proof: We begin by rewriting condition (4.1) in components, denoting $(t, z-0, t, z+0)$ by 0^- and $(t, z+0, t, z-0)$ by 0^+.

$$\begin{aligned} & \varepsilon_{\mu\nu}\left[j_\psi^{\mu\nu}(t,z-0,t,z+0) - j_\psi^{\mu\nu}(t,z+0,t,z-0)\right] \;=\; 0 \\ \Leftrightarrow\quad & j_\psi^{01}(0^-) - j_\psi^{10}(0^-) \;=\; j_\psi^{01}(0^+) - j_\psi^{10}(0^+) \\ \Leftrightarrow\quad & |\psi_3|^2(0^-) - |\psi_2|^2(0^-) \;=\; |\psi_3|^2(0^+) - |\psi_2|^2(0^+). \end{aligned} \tag{4.4}$$

Next, we show that conditions (4.2) and (4.3) imply (4.4). One has:

$$\begin{aligned} |\psi_2|^2 &= \tfrac{1}{4}(|w_1|^2 + |w_2|^2 + w_1^* w_2 + w_2^* w_1), \\ |\psi_3|^2 &= \tfrac{1}{4}(|w_1|^2 + |w_2|^2 - w_1^* w_2 - w_2^* w_1). \end{aligned} \tag{4.5}$$

Using these relations in eq. (4.4), we obtain:

$$\begin{aligned} (4.4) \quad \Leftrightarrow\quad & (w_1^* w_2)(0^-) + (w_2^* w_1)(0^-) \;=\; (w_1^* w_2)(0^+) + (w_2^* w_1)(0^+) \\ \overset{(4.2)}{\Leftrightarrow}\quad & w_1^*(0^-)w_2(0^-) + w_2^*(0^-)w_1(0^-) \;=\; w_1^*(0)w_2(0^+) + w_1(0)w_2^*(0^+) \\ \Leftrightarrow\quad & w_1^*(0^-)[w_2(0^-) - w_2(0^+)] + w_1(0^-)[w_2(0^-) - w_2(0^+)]^* \;=\; 0 \\ \overset{(4.3)}{\Leftrightarrow}\quad & w_1^*(0^-)\, i\beta w_1(0^-) + w_1(0^-)[i\beta w_1(0^-)]^* \;=\; 0, \end{aligned} \tag{4.6}$$

which is true. □

4.1.2 Existence and uniqueness

Next, we show that the jump boundary conditions do indeed lead to the existence and uniqueness of solutions. This is achieved using the general existence and uniqueness theorem 3.3.3.

Lemma 4.1.2 *The IBVP given by eqs.* (3.21), (3.22) *and* (3.23), *where the functions* $h_i^\pm$ *are defined by eqs.* (4.2) *and* (4.3), *possesses a unique solution.*

Proof: We only have to show that the functions $h_i^{\pm}$, $i = 1, 2$ are well-defined by the jump boundary conditions. Then existence and uniqueness follow from thm. 3.3.3.
From the proof of thm. 3.3.3, we know the following:

- For $t > 0$, $\psi_2(0^-)$ and $\psi_3(0^+)$ are determined by the initial data. $\psi_3(0^-)$ and $\psi_2(0^+)$ are not determined solely by the initial data.
- For $t < 0$, $\psi_3(0^-)$ and $\psi_2(0^+)$ are determined by the initial data. $\psi_2(0^-)$ and $\psi_3(0^+)$ are not determined by the initial data alone.

Next, we use the jump boundary conditions and the above-mentioned limits of the wave function at the boundary to define the functions $h_i^{\pm}$, $i = 1, 2$ in the general form of the boundary conditions (see eqs. (3.21) and (3.22)). To this end, we write down the jump boundary conditions (4.2) and (4.3) in terms of the components $\psi_2(0^{\pm})$ and $\psi_3(0^{\pm})$ and solve for the undetermined quantities.
Let $t > 0$. Then:

$$(3.21) \quad \Leftrightarrow \quad \underbrace{\psi_3(0^-) - \psi_2(0^+)}_{\text{undetermined}} = \underbrace{\psi_3(0^+) - \psi_2(0^-)}_{\text{determined}}, \tag{4.7}$$

$$(3.21) \quad \Leftrightarrow \quad \underbrace{\psi_2(0^+) + \psi_3(0^-)}_{\text{undetermined}} - \underbrace{\psi_3(0^+) - \psi_2(0^-)}_{\text{determined}} = -i\beta[\underbrace{\psi_2(0^-)}_{\text{det.}} + \underbrace{\psi_3(0^-)}_{\text{undet.}}] \tag{4.8}$$

Now we solve (4.7) for $\psi_3(0^-)$ and plug in the result into (4.8). This way, we obtain an equation for $\psi_2(0^+)$ in terms of quantities which are all determined by initial data only:

$$\psi_2(0^+) = \frac{2\psi_2(0^-) - i\beta\psi_3(0^+)}{2 + i\beta} \equiv h_2^+. \tag{4.9}$$

Recalling the meaning of the abbreviations $0^{\pm}$, this defines the function h_2^+. Inserting (4.9) back into eq. (4.7) yields:

$$\psi_3(0^-) = \frac{2\psi_3(0^+) - i\beta\psi_2(0^-)}{2 + i\beta} \equiv h_1^+. \tag{4.10}$$

Let $t < 0$. An analogous procedure as above results in:

$$\psi_3(0^+) = \frac{2\psi_3(0^-) + i\beta\psi_2(0^+)}{2 - i\beta} \equiv h_2^-, \tag{4.11}$$

$$\psi_2(0^-) = \frac{2\psi_2(0^+) + i\beta\psi_3(0^-)}{2 - i\beta} \equiv h_1^-. \tag{4.12}$$

This completes the proof. □

4.1.3 Lorentz invariance and antisymmetry

In order to discuss the physical meaning of the jump boundary conditions, we now analyze their compatibility with the further requirements of Lorentz invariance and antisymmetry of the wave function.

Lemma 4.1.3 *The jump boundary conditions* (4.2), (4.3) *are Lorentz invariant.*

Proof: Recall from eq. (3.58) that under a Lorentz transformation Λ, $\psi_i(x_1,x_2) \overset{\Lambda}{\longmapsto} \psi_i(\Lambda^{-1}x_1,\Lambda^{-1}x_2)$ for $i=2,3$. Since only ψ_2 and ψ_3 appear in eqs. (4.2), (4.3) and the boundary conditions are prescribed on the Lorentz invariant set $\mathscr{C}$, this immediately yields the claim. □

Lemma 4.1.4 *In the case of an antisymmetric solution, the jump boundary conditions* (4.2), (4.3) *imply that ψ_2 and ψ_3 are continuous across the boundary, i.e.*

$$\psi_2(0^-) = \psi_2(0^+), \tag{4.13}$$
$$\psi_3(0^-) = \psi_3(0^+). \tag{4.14}$$

Proof: Antisymmetry of the wave function implies: $\psi_2(0^-) = -\psi_3(0^+)$, $\psi_2(0^+) = -\psi_3(0^-)$. Inserting these relations into eq. (4.7) and eliminating $\psi_2(0^\pm)$ yields eq. (4.13). Re-inserting (4.13) into eq. (4.7) then gives eq. (4.14). □

Remark: First of all, note that the most important physical properties besides interaction have now been established for the jump boundary conditions. Recall, however, from the proof of thm. 3.3.3 that the interaction effects take place in the components ψ_2 and ψ_3 only. They furthermore require some kind of discontinuity of the wave function across the boundary. Therefore, for indistinguishable particles, the continuity of ψ_2 and ψ_3 shown in lemma 4.1.4 implies that the jump boundary conditions lead to a non-interacting model, i.e. one agreeing with the free multi-time Dirac equations on $\mathbb{R}^2 \times \mathbb{R}^2$. Interesting effects appear only for distinguishable particles.

4.1.4 Interaction effects for distinguishable particles

In order to analyze the character of the supposed interaction in more detail and to prove that the model is indeed interacting for distinguishable particles, we now determine the explicit solution for the general IBVP with jump boundary conditions.

Lemma 4.1.5 *The explicit solution of the IBVP defined by eqs.* (3.21), (3.22) *and* (3.23), *where the functions $h_i^\pm$ are given by eqs.* (4.9)-(4.12), *reads:*

1. *On $\mathscr{S}_1$:*

$$\begin{aligned}
\psi_1(t_1,z_1,t_2,z_2) &= g_1^{(1)}(z_1-t_1,z_2-t_2),\\
\psi_2(t_1,z_1,t_2,z_2) &= \begin{cases} g_2^{(1)}(z_1-t_1,z_2+t_2) & \text{for } z_1-t_1<z_2+t_2 \\ \frac{2g_2^{(2)}(z_1-t_1,z_2+t_2)+i\beta g_3^{(1)}(z_2+t_2,z_1-t_1)}{2-i\beta} & \text{for } z_1-t_1\geq z_2+t_2 \end{cases}\\
\psi_3(t_1,z_1,t_2,z_2) &= \begin{cases} g_3^{(1)}(z_1+t_1,z_2-t_2) & \text{for } z_1+t_1<z_2-t_2 \\ \frac{2g_3^{(2)}(z_1+t_1,z_2-t_2)-i\beta g_2^{(1)}(z_2-t_2,z_1+t_1)}{2+i\beta} & \text{for } z_1+t_1\geq z_2-t_2 \end{cases}\\
\psi_4(t_1,z_1,t_2,z_2) &= g_4^{(1)}(z_1+t_1,z_2+t_2).
\end{aligned} \tag{4.15}$$

2. On $\mathscr{S}_2$:

$$\begin{aligned}
\psi_1(t_1,z_1,t_2,z_2) &= g_1^{(2)}(z_1-t_1,z_2-t_2),\\
\psi_2(t_1,z_1,t_2,z_2) &= \begin{cases} g_2^{(2)}(z_1-t_1,z_2+t_2) & \text{for } z_1-t_1>z_2+t_2\\ \frac{2g_2^{(1)}(z_1-t_1,z_2+t_2)-i\beta g_3^{(2)}(z_2+t_2,z_1-t_1)}{2+i\beta} & \text{for } z_1-t_1\leq z_2+t_2 \end{cases}\\
\psi_3(t_1,z_1,t_2,z_2) &= \begin{cases} g_3^{(2)}(z_1+t_1,z_2-t_2) & \text{for } z_1+t_1>z_2-t_2\\ \frac{2g_3^{(1)}(z_1+t_1,z_2-t_2)+i\beta g_2^{(2)}(z_2-t_2,z_1+t_1)}{2-i\beta} & \text{for } z_1+t_1\leq z_2-t_2 \end{cases}\\
\psi_4(t_1,z_1,t_2,z_2) &= g_4^{(2)}(z_1+t_1,z_2+t_2). \qquad (4.16)
\end{aligned}$$

Proof: The idea is to make use of the formula for the explicit solution of the IBVP, as given in eq. (3.28) for $\mathscr{S}_1$. As the jump boundary conditions mix the parts of the solution on $\mathscr{S}_1$ and $\mathscr{S}_2$, we have to solve the IBVP on these two parts of the domain simultaneously. In a completely analogous way as in thm. 3.3.3, one obtains a formula for the explicit solution on $\mathscr{S}_2$:

$$\begin{aligned}
\psi_1(t_1,z_1,t_2,z_2) &= g_1^{(2)}(z_1-t_1,z_2-t_2),\\
\psi_2(t_1,z_1,t_2,z_2) &= \begin{cases} g_2^{(2)}(z_1-t_1,z_2+t_2) & \text{for } z_1-t_1>z_2+t_2\\ h_2^{+}((-z_1+z_2+t_1+t_2)/2,(z_1+z_2-t_1+t_2)/2) & \text{for } z_1-t_1\leq z_2+t_2 \end{cases}\\
\psi_3(t_1,z_1,t_2,z_2) &= \begin{cases} g_3^{(2)}(z_1+t_1,z_2-t_2) & \text{for } z_1+t_1>z_2-t_2\\ h_2^{-}((z_1-z_2+t_1+t_2)/2,(z_1+z_2+t_1-t_2)/2) & \text{for } z_1+t_1\leq z_2-t_2 \end{cases}\\
\psi_4(t_1,z_1,t_2,z_2) &= g_4^{(2)}(z_1+t_1,z_2+t_2). \qquad (4.17)
\end{aligned}$$

Next, we have to determine the functions $h_i^{\pm}$ in terms of the initial data $g_j^{(i)}$. Collecting the results of eqs. (4.9)-(4.12) and using eq. (4.17) to express $\psi_i(0^{\pm})$ via $g_i^{(j)}$ (i.e. $\psi_i(0^-) = g_i^{(1)}(t,z,t,z)$ and $\psi_i(0^+) = g_i^{(2)}(t,z,t,z)$, $i=1,2$), we find:

$$\begin{aligned}
h_1^+(t,z) &= \frac{2g_3^{(2)}(z+t,z-t)-i\beta g_2^{(1)}(z-t,z+t)}{2+i\beta},\\
h_1^-(t,z) &= \frac{2g_2^{(2)}(z-t,z+t)+i\beta g_3^{(1)}(z+t,z-t)}{2-i\beta},\\
h_2^+(t,z) &= \frac{2g_2^{(1)}(z-t,z+t)-i\beta g_3^{(2)}(z+t,z-t)}{2+i\beta},\\
h_2^-(t,z) &= \frac{2g_3^{(1)}(z+t,z-t)+i\beta g_2^{(2)}(z-t,z+t)}{2-i\beta}. \qquad (4.18)
\end{aligned}$$

Inserting the $h_i^{\pm}$ back into the second cases for ψ_2 and ψ_3 in eqs. (3.28) and (4.17) yields the claim. □

From the form of the explicit solution (eqs. (4.15) and (4.16)), especially from the non-trivial superpositions in ψ_2, ψ_3, we can read off that generically entanglement is created in the time evolution and therefore obtain:

Corollary 4.1.6 *In the case of distinguishable particles, the model with jump boundary conditions is interacting in the sense of the criterion in sec. 3.6.*

Details of the interaction: In order to understand the interaction effect in more detail, we analyze the time evolution of the same initial product wave function as considered previously in sec. 3.6, eq. (3.72):

$$\begin{aligned} g_1^{(j)} &\equiv g_3^{(j)} \equiv g_4^{(j)} \equiv 0, \quad j=1,2, \\ g_2^{(1)}(z_1,z_2) &= \tilde{\phi}(z_1)\tilde{\chi}(z_2)1_{[a,b]}(z_1)1_{[c,d]}(z_2), \\ g_2^{(2)} &\equiv 0, \end{aligned} \tag{4.19}$$

where $a<b<c<d$. Note that the jump boundary conditions are satisfied in the form $0=0$ as $\psi_i(0^{\pm})=0$ for $t=0$ and all i.
The time evolution yields:

$$\psi_1 \equiv \psi_4 \equiv 0 \text{ on } \mathscr{S}. \tag{4.20}$$

Furthermore, on $\mathscr{S}_1$:

$$\begin{aligned} \psi_2(t_1,z_1,t_2,z_2) &= \begin{cases} g_2^{(1)}(z_1-t_1,z_2+t_2) & \text{for } z_1-t_1>z_2+t_2 \\ 0 & \text{else} \end{cases} \\ \psi_3(t_1,z_1,t_2,z_2) &= \begin{cases} 0 & \text{for } z_1+t_1>z_2-t_2 \\ \frac{-i\beta g_2^{(1)}(z_2-t_2,z_1+t_1)}{2+i\beta} & \text{else} \end{cases} \end{aligned} \tag{4.21}$$

And on $\mathscr{S}_2$:

$$\begin{aligned} \psi_2(t_1,z_1,t_2,z_2) &= \begin{cases} 0 & \text{for } z_1-t_1>z_2+t_2 \\ \frac{2g_2^{(1)}(z_1-t_1,z_2+t_2)}{2+i\beta} & \text{else} \end{cases} \\ \psi_3 &\equiv 0. \end{aligned} \tag{4.22}$$

The result of the calculation is represented schematically in figure 4.1. The initial product wave function is localized in $\mathscr{S}_1$ and associated with spin component ψ_2 (corresponding to $e_1 \otimes e_2$). If plotted in the same space, the product factors move towards each other with the time evolution until they start overlapping. When this happens, the overlapping part is split into one part which is transmitted into $\mathscr{S}_2$ and another part which is scattered back into $\mathscr{S}_1$. The distribution of these parts is controlled by the parameter β. The transmitted part remains in the same spin component while the reflected part changes its spin component to ψ_3 (corresponding to $e_2 \otimes e_1$). For large times, the initial part in $\mathscr{S}_1$ for ψ_2 vanishes and only the transmitted and reflected parts remain.

Compared to the previous type of interactions (see sec. 3.6), both the existence of a transmitted part and the detailed factors of this part and the reflected one represent new features. The presence of a transmitted part illustrates that there does indeed exist a non-zero flux between $\mathscr{S}_1$ to $\mathscr{S}_2$[1]. This shows in detail that the jump boundary conditions are different from the previous ones.

[1] This fact can be seen more generally also from eqs. (4.15) and (4.16).

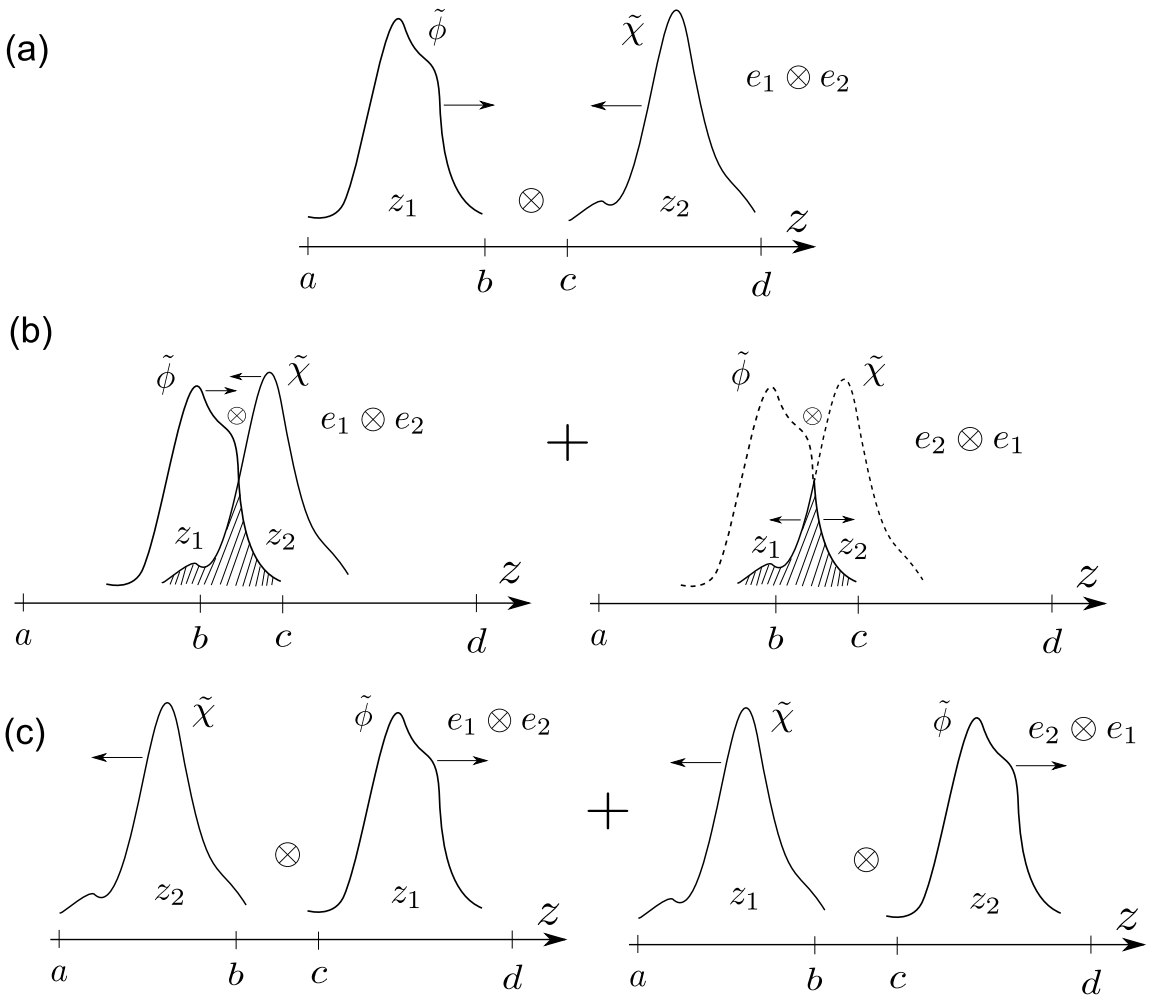

Figure 4.1: Schematic illustration of the interaction for the jump boundary conditions. (a) $t_1 = t_2 = 0$: The wave packets move towards each other. $\tilde{\phi}$ is associated with particle 1 and $\tilde{\chi}$ with particle 2. The only non-zero component of the total wave function is ψ_2 (associated with $e_1 \otimes e_2$). (b) $(d-a)/2 > t_1 = t_2 > (c-b)/2$: The wave packets overlap if plotted in the same space. The hatched area for $e_1 \otimes e_2$ has left $\mathscr{S}_1$. One part of it (associated with a factor $2/(2+i\beta)$) is transmitted to $\mathscr{S}_2$. The other part reappears in the component for $e_2 \otimes e_1$ with a factor $-i/(2+i\beta)$. The wave packets have swapped place. The hatched parts of $\tilde{\phi}$ are now associated with particle 2 and the hatched parts of $\tilde{\chi}$ with particle 1. (c) $t_1 = t_2 > (d-a)/2$: The wave packets in each component have passed each other. Contrary to the previous type of boundary conditions (see fig. 3.2) there exists both a transmitted part (associated with $e_1 \otimes e_2$, in $\mathscr{S}_2$) and a reflected part (associated with $e_2 \otimes e_1$, in $\mathscr{S}_1$).

Limiting cases:

1. $\beta \to 0$: From eqs. (4.15), (4.15) one can see that then only the transmitted parts exist. ψ_2 and ψ_3 are continuous across the boundary $\mathscr{C}$ between $\mathscr{S}_1$ and $\mathscr{S}_2$, because the boundary conditions (4.2), (4.3) reduce to continuity for $\beta \to 0$. The time evolution of these components agrees with the non-interacting one, i.e. the one obtained by solving the free multi-time Dirac equations on $\mathbb{R}^2 \times \mathbb{R}^2$.

2. $\beta \to \infty$: In this case, the transmitted parts vanish and eqs. (4.15), (4.15) show that the reflected parts are all associated with the phase factor -1. At first glance, the situation seems similar to the one for the previous type of boundary conditions (3.32). However, it crucially differs from the latter in that the initial data have to satisfy the jump conditions (4.2) and (4.3) which enforce an infinite jump in one of the components of the wave function (ψ_3). Strictly speaking, this is no valid initial condition.

4.1.5 Conclusion

The class of jump boundary conditions leads to a mathematically well-defined and Lorentz invariant dynamics which conserves probability. However, it is only interacting in the case of distinguishable particles. One obtains two different kinds of processes: (i) scattering off each other, and (ii) passing through each other. Process (ii) may seem unphysical for a theory of point particles such as an HBD model, as at one instance of time both particles then occupy the same space-point which, strictly speaking, contradicts the notion of a two-particle configuration[2]. All in all, the jump boundary conditions constitute an interesting generalization of possible dynamics for the $1+1$-dimensional model but the previous class of boundary conditions is more universally applicable and harmonizes better with a Bohmian theory of point particles.

4.2 N particles

This section is taken from the paper [69] by Lukas Nickel and the present autor with only minor changes.

The main difficulty of an N-particle generalization of the $1+1$-dimensional model is to find a concise approach which allows to deal with every $N \in \mathbb{N},\ N \geq 2$. While in chap. 3 we started out with proving the existence and uniqueness for a very general class of boundary conditions and restricted this class according to the various physical requirements, we shall employ a bottom-up approach here. We directly specialize on the case of indistinguishable particles, extract a class of physically reasonable boundary conditions and only then prove existence and uniqueness for the resulting model.

The section is structured as follows. In sec. 4.2.1 we introduce the model as defined by its multi-time equations, domain, initial values and boundary conditions. In sec. 4.2.2,

[2] For the GRWm model one reaches a different conclusion, as it describes a different microscopic reality. The GRWf model, on the other hand, is not compatible with the $1+1$-dimensional model, anyway, as it requires the multi-time wave function to be defined not only on $\mathscr{S}$ but on $\mathbb{R}^2 \times \mathbb{R}^2$, precluding the possibility of interaction via boundary conditions.

the implications of antisymmetry are discussed and the general solution of the multi-time equations is found using a generalized version of the method of characteristics. Sec. 4.2.3 deals with extracting a general class of probability-conserving boundary conditions. In sec. 4.2.4 the requirements of Lorentz invariance are checked and the before-mentioned class of boundary conditions is shown to satisfy them. Our main result is an existence and uniqueness theorem (sec. 4.2.5). In sec. 4.2.6 we finally give a general argument that the model is interacting and show that one can regard the interaction as given by an effective δ-potential at equal times.

4.2.1 Definition of the model

Analogously to the two-particle case (see sec. 3.2), our model is based on a multi-time wave function for N mass-less Dirac particles on the set of space-like configurations

$$\mathscr{S} \;:=\; \{(t_1, z_1, ..., t_N, z_N) \in \mathbb{R}^{2N} : (t_j - t_k)^2 - (z_j - z_k)^2 < 0 \;\forall j \neq k\} \tag{4.23}$$

in $1+1$-dimensional space-time. The appropriate spin space is $\mathcal{S} = (\mathbb{C}^2)^{\otimes N}$. Thus, ψ has 2^N spin components $\psi_i,\ i = 1, ..., 2^N$.

As multi-time evolution equations we prescribe the system of N mass-less Dirac equations

$$i\gamma_k^\mu \partial_{k,\mu}\, \psi(x_1, ..., x_N) \;=\; 0, \quad k = 1, ..., N, \tag{4.24}$$

where $x_k = (t_k, z_k)$, $\partial_{k,\mu} = \frac{\partial}{\partial x_k^\mu}$ and γ_k^μ is the μ-th Dirac gamma matrix acting on the spin index of the k-th particle. Using the representation (3.10), the multi-time Dirac equations (4.24) take a diagonal form (which can be seen by multiplying eq. (4.24) with γ_k^0 from the left). This results in

$$\left(\frac{\partial}{\partial t_k} + \sigma_{3,k}\frac{\partial}{\partial z_k}\right) \psi(t_1, z_1, ..., t_N, z_N) \;=\; 0, \quad k = 1, ..., N, \tag{4.25}$$

where $\sigma_{3,k}$ is the third Pauli matrix, $\sigma_3 = \mathrm{diag}(1, -1)$, acting on the spin index of the k-th particle.

Initial data are prescribed on the set

$$\mathscr{I} \;:=\; \{(t_1, z_1, ..., t_N, z_N) \in \overline{\Omega} : t_1 = \cdots = t_N = 0\}. \tag{4.26}$$

Since $\mathscr{S}$ has a non-empty boundary $\partial\mathscr{S}$, one should expect that boundary conditions are needed to ensure the uniqueness of solutions. At this point, we leave open the exact nature of the boundary conditions. It will be clarified by further considerations about Lorentz invariance and probability conservation.
The structure of the model can be summarized as

$$\begin{cases} \text{the system of equations (4.25) on } \Omega = \mathscr{S}, \\ \text{initial conditions on } \mathscr{I}, \\ \text{boundary conditions on } \partial\Omega. \end{cases} \tag{4.27}$$

4.2.2 Antisymmetry, general solution and multi-time characteristics

In this section, we first show how the antisymmetry of the wave function for indistinguishable particles allows to reduce the domain from $N!$ disconnected parts to a single connected one. Using a notation for the spin components which is tailor-made for the multi-time equations (4.25) we show how this facilitates the explicit determination of their general solution. This leads to the notion of multi-time characteristics.

4.2.2.1 Antisymmetry and reduction of the domain

Following the spirit outlined at the beginning of the section, we make simplifications wherever physically reasonable in order to achieve a concise model for which existence and uniqueness can be proved elegantly. The first simplification is the assumption of indistinguishable particles. This is natural, considering that the particles are not dynamically distinguished by eqs. (4.24) alone. Denote the spin components of ψ by $\psi_{s_1 \dots s_N}$ where each s_i can take the values ± 1. We write

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \vdots \\ \psi_{2^N} \end{pmatrix} \equiv \begin{pmatrix} \psi_{--\cdots--} \\ \psi_{--\cdots-+} \\ \psi_{--\cdots+-} \\ \vdots \\ \psi_{++\cdots++} \end{pmatrix}. \tag{4.28}$$

Indistinguishability implies the following antisymmetry condition for the wave function. Let $\pi \in S^N$ be a permutation. Then

$$\psi_{s_{\pi(1)} \dots s_{\pi(N)}}(x_{\pi(1)}, ..., x_{\pi(N)}) \stackrel{!}{=} (-1)^{\mathrm{sgn}(\pi)} \psi_{s_1 \dots s_N}(x_1, ..., x_N). \tag{4.29}$$

We now use this condition to relate a solution of (4.27) on the different parts of the domain $\Omega = \mathscr{S}$ (see eq. (4.23)). Note that in one spatial dimension $\mathscr{S}$ separates into $N!$ disjoint parts which can be classified according to the relation of the spatial coordinates z_k, e.g. $z_2 < z_1 < z_5 < z_3 < \dots$ Using the permutation group S^N, we write $\mathscr{S}$ as the disjoint union of open sets as follows:

$$\begin{aligned} \mathscr{S} &= \bigsqcup_{\pi \in S^N} \mathscr{S}_\pi, \\ \text{where} \quad \mathscr{S}_\pi &:= \left\{ (t_1, z_1, ..., t_N, z_N) \in \mathscr{S} : z_{\pi(1)} < \dots < z_{\pi(N)} \right\}. \end{aligned} \tag{4.30}$$

The crucial point is the following: given a solution of the model, as defined by (4.27) on $\mathscr{S}_1$ (corresponding to $z_1 < \dots < z_N$), the antisymmetric extension via eq. (4.29) yields a solution on $\mathscr{S}_\pi$, provided the boundary and initial conditions are chosen to be compatible with antisymmetry. Note that this reduces the possible classes of initial boundary value problems (IBVPs) (4.27) to an autonomous IBVP on $\mathscr{S}_1$. We shall employ this strategy in the following. Our new model may be summarized according to (4.27) with $\mathscr{S}$ replaced by $\mathscr{S}_1$.

4.2.2.2 Multi-time characteristics and general solution

Using the notation (4.28), we express the diagonalized multi-time Dirac equations (4.25) for a fixed component $\psi_{s_1 \dots s_N}$ as

$$\left(\frac{\partial}{\partial t_k} - s_k \frac{\partial}{\partial z_k} \right) \psi_{s_1 \dots s_k \dots s_N} \;=\; 0, \quad k = 1, ..., N. \tag{4.31}$$

Note that (4.31) imposes N equations for each of the 2^N spin components $\psi_{s_1 \dots s_N}$. This simple form of the equations allows to find the general solution.

Lemma 4.2.1 *The general solution of eqs.* (4.31) *is given by*

$$\psi_{s_1 \dots s_N}(t_1, z_1, ..., t_N, z_N) \;=\; f_{s_1 \dots s_N}\left(z_1 + s_1 t_1, ..., z_N + s_N t_N\right), \tag{4.32}$$

where $f_{s_1 \dots s_N} \colon \mathbb{R}^N \to \mathbb{C}$ *are* C^1*-functions,* $s_1 = \pm 1, ..., s_N = \pm 1$.

Proof: The result is obvious when one is familiar with the notation. Simply write out eq. (4.31) for $\psi_{s_1 \dots s_N}$ separately: $\left(\frac{\partial}{\partial t_1} - s_1 \frac{\partial}{\partial z_1} \right) \psi_{s_1 \dots s_N} = 0$, ..., $\left(\frac{\partial}{\partial t_N} - s_N \frac{\partial}{\partial z_N} \right) \psi_{s_1 \dots s_N} = 0$. This implies the form (4.32). □

The form of the general solution motivates the following definition, analogous to the one in sec. 3.3.

Definition: Let $p = (t_1, z_1, \dots t_N, z_N) \in \mathbb{R}^{2N}$. Then we call

$$c_k \;:=\; z_k + s_k t_k \tag{4.33}$$

the *characteristic values* at p associated with the component $\psi_{s_1 \dots s_N}$.
Furthermore, we define the *multi-time characteristics* of the components $\psi_{s_1 \dots s_N}$ by

$$S_{s_1 \dots s_N}(c_1, ..., c_N) \;:=\; \{(t_1, z_1, ..., t_N, z_N) \in \mathbb{R}^{2N} : z_k + s_k t_k = c_k\}. \tag{4.34}$$

With these definitions, one can reformulate lemma 4.2.1 as follows: the components $\psi_{s_1 \dots s_N}$ of solutions of (4.31) are constant on the respective multi-time characteristics (4.34). Note that this implies existence and uniqueness on the domain $\mathbb{R}^{2N}$ for an initial value problem at $t_1 = ... = t_N = 0$, the functions $f_{s_1 \dots s_N}$ being given by the initial values. However, as known from chap. 3, this does not in general hold true for a domain with a boundary such as $\mathscr{S}_1$.

4.2.3 Derivation of boundary conditions from probability conservation

In this section, we use the current form ω_j in an analogous way to sec. 3.4 to extract a sufficient condition for probability conservation on the components of the wave-function. Because of the reduction of the domain in the previous section, we may focus on $\mathscr{S}_1$ only.

Lemma 4.2.2 *Probability conservation on* $\mathscr{S}_1$ *in the sense of*

$$\int_{\Sigma^N \cap \mathscr{S}_1} \omega_j \;=\; \int_{(\Sigma')^N \cap \mathscr{S}_1} \omega_j \tag{4.35}$$

for all space-like hypersurfaces Σ, Σ' holds if the wave function ψ is compactly supported on all sets of the form $\Sigma^N \cap \mathscr{S}_1$ and if

$$\omega_j|_{\mathscr{C}_1} \;=\; 0, \tag{4.36}$$

where

$$\mathscr{C}_1 := \{(t_1, z_1, \ldots t_N, z_N) \in \partial\mathscr{S}_1 \,|\, \exists k : t_k = t_{k+1} \wedge z_k = z_{k+1}\}\,. \tag{4.37}$$

Remark:

1. The assumption of compact support of the wave function (or, alternatively, of suitable drop-off conditions) with respect to spatial directions is needed as a technical assumption in the proof. It is reasonable because the multi-time Dirac equations have finite propagation speed (see eq. (4.32)). Consequently, compactly supported initial data imply the desired property.

2. Note that the wave function is, strictly speaking, not defined on $\partial\mathscr{S}_1$. When using values of the wave function at the boundary (such as in eq. (4.36)), we assume that the wave function is continuous[3] and refer to the corresponding limit in $\mathscr{S}_1$. In this way, jumps of the wave function across the boundaries of different $\mathscr{S}_\pi$ are admitted. In fact, singularities of this kind are typical for zero-range interactions [1, appendix J].

Proof: We adopt the idea of the proof of thm. 3.4.2 and generalize it to N particles. Let Σ, Σ' be space-like hypersurfaces. We construct a suitable submanifold with boundary in order to be able to use Stokes' theorem.

Let $t_\Sigma(z)$ denote the time coordinate of the unique point $p = (t_\Sigma(z), z) \in \Sigma$. Let $R > 0$ and consider the following set:

$$V_R \;:=\; \left\{(t_1, z_1, \ldots, t_N, z_N) \in \overline{\mathscr{S}}_1 \;\middle|\; \begin{array}{c} \exists\tau \in [0,1] : \forall k : t_k = t_\Sigma(z_k) + \tau\,(t_{\Sigma'}(z_k) - t_\Sigma(z_k)) \\ \text{and } |z_k| \leq R \end{array} \right\} \tag{4.38}$$

V_R is a bounded and closed, thus compact, $(N+1)$-dimensional submanifold of $\mathbb{R}^{2N}$ with boundary

$$\partial V_R \;=\; (\Sigma^N \cap \mathscr{S}_1) \cup ((\Sigma')^N \cap \mathscr{S}_1) \cup M_1 \cup M_2, \tag{4.39}$$

where M_2 is the subset of V_R with $|z_k| = R$ for some k and

$$M_1 \;=\; V_R \cap \partial\mathscr{S}_1. \tag{4.40}$$

Because of the first condition in the definition of V_R, a configuration in V_R always is an element of $\mathcal{S}^N$ for some space-like hypersurface $\mathcal{S}$. Therefore, it can only be an element of $M_1 \subset \partial\mathscr{S}_1$ (i.e. light-like) if $\exists k : t_k = t_{k+1}$ and $z_k = z_{k+1}$. This implies $M_1 \subset \mathscr{C}_1$.

In the limit $R \to \infty$, the integral $\int_{M_2} \omega_j$ vanishes because of the compact support of the wave function. Thus, it follows from Stokes' theorem, together with $d\omega_j = 0$, that

$$0 \;=\; \lim_{R\to\infty} \int_{V_R} d\omega_j = \lim_{R\to\infty} \int_{\partial V_R} \omega_j = -\int_{\Sigma^N \cap \mathscr{S}_1} \omega_j + \int_{(\Sigma')^N \cap \mathscr{S}_1} \omega_j + \int_{M_1} \omega_j. \tag{4.41}$$

[3] The assumption of continuity will be justified in sec. 4.2.5 where it is shown that a unique C^k solution exists for an appropriate IBVP.

The minus sign in front of the first integral on the rhs. is due to orientation conventions. Thus, probability conservation in the sense of eq. (4.35) holds iff $\int_{M_1} \omega_j = 0$. In order to make this integral vanish for all possible choices of Σ, Σ', the condition

$$\omega_j|_{\mathscr{C}_1} \stackrel{!}{=} 0 \tag{4.42}$$

has to be satisfied. □

Next, we study the implications of condition (4.42) for the components of the wave function. For simplicity, we first focus on the special case of equal-time hypersurfaces in a fixed but otherwise arbitrary Lorentz frame.

Lemma 4.2.3 *Let $\mathscr{C}_{1,t} := \{(t_1, z_1, ..., t_N, z_N) \in \mathscr{C}_1 : t_1 = \cdots = t_N\}$. Then the condition for probability conservation on equal-time hypersurfaces Σ_t in a particular Lorentz frame, i.e.* (4.42) *with $\mathscr{C}_1$ replaced by $\mathscr{C}_{1,t}$, holds if and only if the following condition is satisfied:*

$$\psi^\dagger(p)\,(\sigma_{3,k} - \sigma_{3,k+1})\,\psi(p) \;=\; 0 \quad \forall p \in \mathscr{C}_{1,t}^{(k)} \quad \forall k = 1, ..., N-1, \tag{4.43}$$

where $\mathscr{C}_{1,t}^{(k)} := \{(t_1, z_1, ..., t_N, z_N) \in \mathscr{C}_{1,t} : z_k = z_{k+1}\}$.
Furthermore, eq. (4.43) *can be rewritten as*

$$\sum_{\substack{(s_1,...,s_N)\in\{-,+\}^N \\ s_k \neq s_{k+1}}} s_{k+1} |\psi_{s_1...s_N}|^2(p) \;=\; 0 \quad \forall p \in \mathscr{C}_{1,t}^{(k)}. \tag{4.44}$$

Proof: We have to evaluate the condition $\omega_j|_{\mathscr{C}_{1,t}} = 0$. Note that for $p \in \mathscr{C}_{1,t}$ there exists $k \in \{1, ..., N-1\}$ such that $p = (t, z_1, ..., t, z_k = z, t, z_{k+1} = z, ..., t, z_N)$.

Next, we calculate $\omega_j|_{\mathscr{C}_{1,t}}$ according to eq. (1.70), recalling that in this case $x_k^0 = t$ and $x_j^1 = z_j$ as well as $z_k = z_{k+1} = z$. All terms with more than one index $\mu_l = 1$ in $j^{\mu_1 \cdots \mu_l \cdots \mu_N}$ vanish because they contain $dt \wedge dt = 0$. Moreover, the terms with $\mu_k = \mu_{k+1} = 0$ do not contribute, either, as they contain $dz \wedge dz = 0$. We are left with terms for which all indices μ_j are equal to zero apart from the k-th or the $(k+1)$-th:

$$\begin{aligned}
\omega_j(p) \;&=\; -j^{0\ldots(\mu_k=0)(\mu_{k+1}=1)\ldots 0}(p)\, dz_1 \wedge \cdots \wedge dz_{k-1} \wedge dz \wedge dt \wedge dz_{k+2} \wedge \cdots \wedge dz_N \\
&\quad - j^{0\ldots(\mu_k=1)(\mu_{k+1}=0)\ldots 0}(p)\, dz_1 \wedge \cdots \wedge dz_{k-1} \wedge dt \wedge dz \wedge dz_{k+2} \wedge \cdots \wedge dz_N \\
&=\; \left(j^{0\ldots 10\ldots 0} - j^{0\ldots 01\ldots 0}\right)(p)\, dz_1 \wedge \cdots \wedge dz_{k-1} \wedge dz \wedge dt \wedge dz_{k+2} \wedge \cdots \wedge dz_N
\end{aligned} \tag{4.45}$$

This expression vanishes if and only if the bracket is zero. This yields condition (4.43):

$$0 \;=\; (j^{0\ldots 10\ldots 0} - j^{0\ldots 01\ldots 0})(p) \;=\; \psi^\dagger(p)\,(\sigma_{3,k} - \sigma_{3,k+1})\,\psi(p). \tag{4.46}$$

Written out in components, eq. (4.46) reads:

$$0 \;=\; \sum_{(s_1,...,s_N)\in\{-,+\}^N} \left(-s_k |\psi_{s_1...s_N}|^2(p) + s_{k+1} |\psi_{s_1...s_N}|^2(p)\right), \tag{4.47}$$

where k was defined above.

Summands with $s_k = s_{k+1}$ cancel out. We are left with

$$0 \;=\; \sum_{\substack{(s_1,\ldots,s_N)\in\{-,+\}^N \\ s_k\neq s_{k+1}}} 2s_{k+1}|\psi_{s_1\ldots s_N}|^2(p). \tag{4.48}$$

which yields (4.44). □

We now take the following approach in order to find adequate boundary conditions that lead to probability conservation on general space-like hypersurfaces. First, we choose a subclass of (4.44) which turns out to be Lorentz invariant (see the next section) and to ensure the existence of a solution (see sec. 4.2.5). Then we prescribe the condition on the whole set $\mathscr{C}_1$ and show that it is indeed sufficient to ensure condition (4.36) and therefore probability conservation on general hypersurfaces.

It is useful to define the sets

$$\mathscr{C}_{k,k+1} \;:=\; \{(t_1,z_1,\ldots,t_N,z_N)\in\partial\mathscr{S}_1 |\, t_k = t_{k+1} \wedge z_k = z_{k+1}\}\,. \tag{4.49}$$

One can then write $\mathscr{C}_1 = \bigcup_{k=1}^{N-1}\mathscr{C}_{k,k+1}$ (see eq. (4.37)).

Lemma 4.2.4 *Let $\varphi^{(k)}\in(-\pi,\pi]$ for $k=1,\ldots,N-1$. Then the boundary conditions*

$$\psi_{s_1\ldots s_{k-1}+-s_{k+2}\ldots s_N} \;\stackrel{!}{=}\; e^{i\varphi^{(k)}}\psi_{s_1\ldots s_{k-1}-+s_{k+2}\ldots s_N} \;\text{ on }\; \mathscr{C}_{k,k+1},\quad k=1,\ldots,N-1 \tag{4.50}$$

imply probability conservation on all space-like hypersurfaces in the sense of eq. (4.35).

Proof: It was shown in lemma 4.2.2 that eq. (4.35) follows if

$$\omega_j|_{\mathscr{C}_1} \;=\; 0. \tag{4.51}$$

We now demonstrate that this equation indeed holds. Pick a point $p\in\mathscr{C}_1$. Then $\exists k : p\in\mathscr{C}_{k,k+1}$. Condition (4.50) at this point yields:

$$|\psi_{s_1\ldots s_{k-1}+-s_{k+2}\ldots s_N}|^2(p) \;=\; |\psi_{s_1\ldots s_{k-1}-+s_{k+2}\ldots s_N}|^2(p). \tag{4.52}$$

It follows that

$$j^{\mu_1\mu_2\ldots01\ldots\mu_N}(p) \;=\; j^{\mu_1\mu_2\ldots10\ldots\mu_N}(p) \tag{4.53}$$

because the expression for the current is diagonal in the components. In the formula for $\omega_j(p)$ (eq. (1.70)), we first perform the sum over the indices μ_k,μ_{k+1} and afterwards over the rest. Then there are four possibilities in the summands:

- $(\mu_k,\mu_{k+1}) = (0,0)$ or $(1,1)$: These do not contribute because the coordinates of the k-th and $(k+1)$-th particles are equal, say to (t,z), so either $dz\wedge dz=0$ or $dt\wedge dt=0$ appears as a factor in the wedge product.

- $(\mu_k,\mu_{k+1}) = (0,1)$ or $(1,0)$. One can see that these two factors cancel each other because (abbreviating the other factors in the wedge product by A and B)

$$\begin{aligned} &j^{\mu_1\ldots(\mu_k=0)(\mu_{k+1}=1)\ldots\mu_N} A\wedge dt\wedge dz\wedge B + j^{\mu_1\ldots(\mu_k=1)(\mu_{k+1}=0)\ldots\mu_N} A\wedge dz\wedge dt\wedge B \\ &\quad= \left(j^{\mu_1\ldots01\ldots\mu_N}(p) - j^{\mu_1\ldots10\ldots\mu_N}(p)\right) A\wedge dt\wedge dz\wedge B \;\stackrel{(4.53)}{=}\; 0. \end{aligned} \tag{4.54}$$

Therefore, the probability-conserving property $\omega_j(p)=0$ holds. □

4.2.4 Lorentz invariance

As in sec. 3.5, the Lorentz invariance of the N-particle model requires the invariance of the domain, the multi-time wave equations and the boundary conditions. Apart from the last point, the invariance is already manifest. In this section, we show that also the class (4.50) of probability-conserving boundary conditions is Lorentz invariant, meaning that the Lorentz-transformed boundary conditions are satisfied as a consequence of the old ones.

Recall that the transformation behavior of multi-time wave functions under a Lorentz transformation $\Lambda : x \mapsto x'$ in the proper Lorentz group $\mathcal{L}_+^\uparrow$ is given by

$$\psi'(x_1, ..., x_N) \;=\; S[\Lambda] \otimes \cdots \otimes S[\Lambda] \psi(\Lambda^{-1} x_1, ..., \Lambda^{-1} x_N), \tag{4.55}$$

where

$$S[\Lambda] \;=\; \exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right), \quad \sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]. \tag{4.56}$$

Here, ω is an antisymmetric $(1+d) \times (1+d)$ matrix which characterizes Λ.

For $d = 1$, there exists only one independent generator σ^{01} corresponding to a boost in z-direction. We denote the corresponding boost parameter by $\beta \in \mathbb{R}$. One obtains

$$S[\Lambda] \;=\; \begin{pmatrix} \cosh\beta + \sinh\beta & 0 \\ 0 & \cosh\beta - \sinh\beta \end{pmatrix}. \tag{4.57}$$

As the matrix is diagonal due to our choice of γ-matrices, it is easy to calculate the N-fold tensor product in eq. (4.55). In this way, we find that the components of ψ transform as

$$\psi'_{s_1 ... s_N}(x_1, ..., x_N) \;=\; \prod_{k=1}^{N} (\cosh\beta - s_k \sinh\beta)\, \psi_{s_1 ... s_N}(\Lambda^{-1} x_1, ..., \Lambda^{-1} x_N). \tag{4.58}$$

In other words: one obtains a factor of $(\cosh\beta - \sinh\beta)$ for every plus and a factor of $(\cosh\beta + \sinh\beta)$ for every minus in the index $(s_1 ... s_N)$. Hence, components with an equal number of plus and minus signs transform in the same way.

Example: We focus on the case $N = 3$ in order to motivate the general form (4.50) of the boundary conditions. Consider a boundary point $p = (t, z_1, t, z, t, z) \in \mathscr{C}_{1,t}$. We use eq. (4.44) to explicitly compute what the condition of probability conservation amounts to:

$$\omega_j(p) \;=\; 0 \quad \Leftrightarrow \quad |\psi_{--+}|^2(p) - |\psi_{-+-}|^2(p) + |\psi_{+-+}|^2(p) - |\psi_{++-}|^2(p) \;=\; 0. \tag{4.59}$$

Now we Lorentz transform this condition according to eq. (4.58) using the identity $(\cosh\beta - \sinh\beta)(\cosh\beta + \sinh\beta) = 1$:

$$0 = (\cosh\beta - \sinh\beta)\left(|\psi_{--+}|^2 - |\psi_{-+-}|^2\right)(p') + (\cosh\beta + \sinh\beta)\left(|\psi_{+-+}|^2 - |\psi_{++-}|^2\right)(p'), \tag{4.60}$$

where $p' = (\Lambda^{-1}(t, z_1), \Lambda^{-1}(t, z), \Lambda^{-1}(t, z))$. This equation cannot possibly be Lorentz invariant as a whole. Thus, demanding Lorentz invariance, we have to split up eq. (4.59) into two separate conditions relating only components which have the same number of plus and minus signs in their indices, i.e.

$$|\psi_{-+-}|^2(p) - |\psi_{--+}|^2(p) \;=\; 0, \quad |\psi_{+-+}|^2(p) - |\psi_{++-}|^2(p) \;=\; 0. \tag{4.61}$$

These equations are equivalent to

$$\psi_{--+}(p) \;=\; e^{i\varphi_-}\psi_{-+-}(p), \qquad \psi_{++-}(p) \;=\; e^{i\varphi_+}\psi_{+-+}(p). \tag{4.62}$$

A priori, the phases $\varphi_\pm$ could be functions of all particle coordinates. Invariance under Poincaré transformations leads to the requirement that the phases $\varphi_\pm$ should only depend on the Minkowski distances of pairs of particles, $(x_i - x_j)^\mu (x_i - x_j)_\mu$. In the 3-particle case, there is only one such variable, $s^2 := (t_1 - t)^2 - (z_1 - z)^2$. However, the Minkowski distance s^2 changes along a multi-time characteristic. This would lead to a contradiction because the solution has to be constant along the characteristic (see lemma 4.2.1). Thus, $\varphi_\pm$ must be constant in order for solutions to exist. A further investigation of the existence and uniqueness problem shows that even $\varphi_+ = \varphi_-$ is necessary.

We arrive at the following general picture. Exchanging $s_k \leftrightarrow s_{k+1}$ in $\psi_{s_1 \ldots s_k s_{k+1} \ldots s_N}$ on $\mathscr{C}_{k,k+1}$ only yields a phase factor which must not depend on the other spin indices (but may depend on k). This motivates the choice of boundary conditions (4.50). The following lemma shows that they are indeed Lorentz invariant.

Lemma 4.2.5 *The probability conserving boundary conditions of lemma 4.2.4, i.e.*

$$\psi_{s_1 \ldots s_{k-1} + - s_{k+2} \ldots s_N} \;\stackrel{!}{=}\; e^{i\varphi^{(k)}} \psi_{s_1 \ldots s_{k-1} - + s_{k+2} \ldots s_N} \quad \text{on} \quad \mathscr{C}_{k,k+1}, \quad k = 1, \ldots, N-1 \tag{4.63}$$

are Lorentz invariant.

Proof: According to eq. (4.58), eq. (4.63) has the same form in every Lorentz frame. Besides, the sets $\mathscr{C}_{k,k+1}$ on which the condition is prescribed are Lorentz invariant. □

Remark: One may ask if there are other possible choices of boundary conditions which lead to $\omega_j = 0$ at the boundary and are Lorentz invariant. The example shows that for $N = 3$ we have already found the only one. For $N \geq 4$ there may exist more complicated boundary conditions with the desired properties. However, aiming at a model valid for all $N \geq 2$, we do not further pursue this question here.

4.2.5 Existence and uniqueness of solutions

We now come to the main result of the N-particle section: the theorem on the existence and uniqueness of solutions for the boundary conditions discussed so far (thm. 4.2.6). Furthermore, we find an explicit formula for the unique solution of the respective IBVP.

We start by providing some intuition about the behavior of the solution for $N = 3$. The main idea is to make use of the fact that the components of the solution have to be constant along the multi-time characteristics (see sec. 3.3).

Example: For $N = 3$ the wave function has $2^3 = 8$ components. According to lemma 4.2.1, these are constant along their respective multi-time characteristics. We visualize the multi-time characteristics as follows (see fig. 4.2). One can see from eq. (4.34) that the multi-time characteristics are the Cartesian product of $N = 3$ lines. These lines are plotted in the same space-time diagram. Any combination of three points on the different lines constitutes an element of the respective multi-time characteristic. The slopes of the various

lines characterize the associated component $\psi_{s_1 \ldots s_N}$. More precisely, a line with negative (positive) slope for particle k is associated with the appearance of $s_k = +1$ ($s_k = -1$) in the index of ψ.

Fig. 4.2 shows a multi-time characteristic $S_{+++}(c_1, c_2, c_3)$ for the component ψ_{+++} where the c_k are defined by a certain point $p = (A, B, C) \in \mathscr{S}_1$ (see eq. (4.33)). ψ_{+++} is determined at p by initial data because the whole characteristic $S_{+++}(c_1, c_2, c_3)$ is contained in $\mathscr{S}_1$. This can be seen from the fact that every three points on different lines are space-like related. Besides, the value $\psi(p)$ is determined *uniquely* by initial data as there exists a unique intersection point of $S_{+++}(c_1, c_2, c_3)$ with the surface $t_1 = t_2 = t_3 = 0$, given by $(0, c_1, 0, c_2, 0, c_3)$. Thus, we obtain

$$\psi_{+++}(t_1, z_1, t_2, z_2, t_3, z_3) \;=\; psi_{+++}(0, c_1, 0, c_2, 0, c_3) \;=\; g_{+++}(c_1, c_2, c_3), \tag{4.64}$$

where g_{+++} are the initial values for ψ_{+++}.

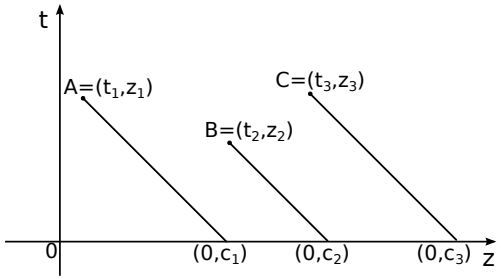

Figure 4.2: A multi-time characteristic $S_{+++}(c_1, c_2, c_3)$ for the component ψ_{+++}. $S_{+++}(c_1, c_2, c_3)$ is the Cartesian product of three lines which are plotted in the same space-time diagram. Every triple of points on different lines, e.g. (A, B, C), is contained in $\mathscr{S}_1$.

For a component $\psi_{s_1 s_2 s_3}$ containing plus as well as minus signs in the index, for example ψ_{+-+}, the situation is different (see fig. 4.3). One can see that intersections of the lines defining a characteristic S_{+-+} do occur in the diagram.

When an intersection occurs, the multi-time characteristic leaves $\mathscr{S}_1$. Therefore, a situation as in fig. 4.3 can happen: tracing back the multi-time characteristic to the initial data surface, one leaves the domain. Thus, $\psi_{+-+}(A, B, C)$ is not defined solely by initial values. To obtain the value of $\psi_{+-+}(A, B, C)$, we first realize $\psi_{+-+}(A, B, C) = \psi_{+-+}(P, P, C)$. Then we employ the boundary conditions to obtain $\psi_{+-+}(A, B, C) = \psi_{+-+}(P, P, C) = e^{i\varphi^{(1)}} \psi_{-++}(P, P, C)$. The component ψ_{-++} is now determined at (P, P, C) by initial data in a similar way as before, i.e. $\psi_{-++}(P, P, C) = g_{-++}(c_2, c_1, c_3)$. Summarizing the relations, we obtain:

$$\psi_{+-+}(t_1, z_1, t_2, z_2, t_3, z_3) \;=\; e^{i\varphi^{(1)}} g_{-++}(c_2, c_1, c_3). \tag{4.65}$$

Figuratively speaking, this amounts to exchanging particle labels and picking up a phase while leaving the domain on the way back to the initial data surface.

The considerations above hint at a general idea: it is possible to obtain an explicit formula for the solutions of the IBVP (see eq. (4.68)) by a process of successively tracing back components to collisions, using the boundary conditions to switch the component, tracing back to the next collision and finally arriving at the initial data. In this way, one

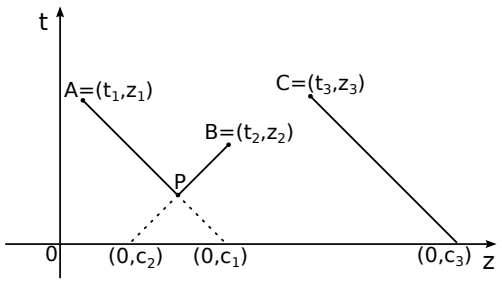

Figure 4.3: A multi-time characteristic S_{+-+} for ψ_{+-+}, depicted for the same configuration as in fig. 4.2. One cannot trace back the lines to the initial data surface $\mathcal{I}$ because one leaves $\mathscr{S}_1$ at the point P. Instead one has to first make use of the boundary conditions and can then trace back the multi-time characteristic for ψ_{-++} which corresponds to the same lines but with particle labels 1 and 2 exchanged.

can also determine the values of components of ψ with multiple intersections of the lines constituting the multi-time characteristic, as in fig. 4.4. This motivates the theorem below.

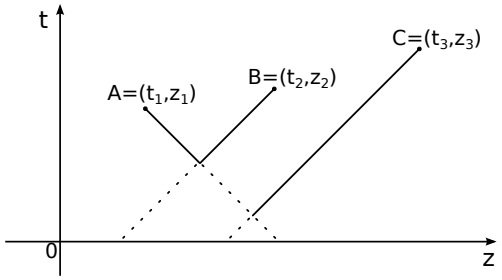

Figure 4.4: A multi-time characteristic with several intersections for the component ψ_{+--}.

Theorem 4.2.6 *Let $m \in \mathbb{N}$ and choose initial data $g_j \in C^m\left(\mathcal{I} \cap \overline{\mathscr{S}}_1, \mathbb{C}\right)$, $j = 1, ..., 2^N$ which also satisfy the boundary conditions, i.e.*

$$g_{s_1 ... s_{k-1} + - s_{k+2} ... s_N} \;=\; e^{i\varphi^{(k)}} g_{s_1 ... s_{k-1} - + s_{k+2} ... s_N} \quad \text{on } \mathcal{I} \cap \mathscr{C}_{k,k+1}, \quad k = 1, ..., N-1 \tag{4.66}$$

and let this transition be C^m.
Then there exists a unique solution $\psi \in C^m\left(\overline{\mathscr{S}}_1, (\mathbb{C}^2)^{\otimes N}\right)$ of the IBVP (4.27) with $\Omega = \mathscr{S}_1$, boundary conditions (4.50) and initial data

$$\psi_{s_1 ... s_N}(0, z_1, ..., 0, z_N) \;=\; g_{s_1 ... s_N}(z_1, ..., z_N), \quad z_1 \leq \cdots \leq z_N. \tag{4.67}$$

If all characteristic values $c_k = z_k + s_k t_k$ are different, the components of the solution are explicitly given by

$$\psi_{s_1 ... s_N}(t_1, z_1, ..., t_N, z_N) \;=\; e^{i\phi^{\pi}_{s_1 ... s_N}} g_{s_{\pi(1)} ... s_{\pi(N)}}\left(c_{\pi(1)}, ..., c_{\pi(N)}\right), \tag{4.68}$$

where π is the permutation with

$$c_{\pi(1)} < \cdots < c_{\pi(N)}. \tag{4.69}$$

$\phi^{\pi}_{s_1\ldots s_N}$ is the phase which is uniquely determined by the $\varphi^{(k)}$ of eq. (4.50) via the definition below.
If some of the c_k are equal, $\psi_{s_1\ldots s_N}$ is given by the continuation of eq. (4.68). In addition, the model ensures probability conservation on general space-like hypersurfaces and is Lorentz invariant.

Definition: A pair $(k,l) \in \{1,...,N\}^2$ with $k<l$ is said to be a *collision* of a transposition $\pi \in S_N$ iff $\pi(k) > \pi(l)$.

Definition: The phases $\phi^{\pi}_{s_1\ldots s_N}$ appearing in thm. 4.2.6 are given by the conditions

1. $\phi^{\mathrm{id}}_{s_1\ldots s_N} = 0$,
2. Let τ_k be the transposition of k and $k+1$. If π can be decomposed as $\pi = \tau_k \circ \sigma$ where σ is a permutation with fewer collisions than π, then

$$\phi^{\pi}_{s_1\ldots s_N} \;=\; \phi^{\sigma}_{s_1\ldots s_{k+1}s_k\ldots s_N} + s_k\varphi^{(k)}. \tag{4.70}$$

Lemma 4.2.7 *The phases $\phi^{\pi}_{s_1\ldots s_N}$ exist and are uniquely determined.*

Proof: We proceed via induction over the number of collisions.
Induction start: If π has no collision, $\pi = \mathrm{id}$. Thus, the phase $\phi^{\mathrm{id}}_{s_1\ldots s_N}$ is determined uniquely by the first condition in the definition. If π has exactly one collision, then it is just a transposition of neighboring elements, so there exists some k with $\pi = \tau_k = \tau_k \circ \mathrm{id}$ and the phase is uniquely determined by (4.70) as $\phi^{\pi}_{s_1\ldots s_N} = s_k\varphi^{(k)}$.
Induction step: Assume that all phases $\phi^{\pi'}_{s_1\ldots s_N}$ for permutations with $n \geq 1$ collisions are uniquely determined and let π have $n+1$ collisions. It is known from the general theory of permutations that there exists at least one permutation σ with n collisions and a neighboring transposition τ_s such that $\pi = \tau_s \circ \sigma$. However, it may be possible to decompose π in two different ways:

$$\pi \;=\; \tau_s \circ \sigma \;=\; \tau_k \circ \kappa, \tag{4.71}$$

where $s,k \in \{1,\ldots,N\}$, $s \neq k$ and σ,κ are permutations with at least n collisions. In order for these two permutations to have one collision less than π, we see that $(k,k+1)$ and $(s,s+1)$ must be collisions of π.

To show that despite the different ways of decomposition, the corresponding phases are uniquely defined, we make use of the fact that the phases need only be defined for a certain type of permutation. To characterize them, we prove an auxiliary claim: in the above situation, τ_s commutes with τ_k because $|s-k| \neq 1$.

Claim: Let $\psi_{s_1\ldots s_N}$ and $(t_1,z_1,\ldots,t_N,z_N) \in \mathscr{S}_1$ such that there is a collision, i.e. a pair (a,b) with $a<b$ and $c_a > c_b$. Then one of the following two possibilities holds:

$$\left\{ \begin{array}{l} \text{either } s_a = +1 \;\wedge\; s_b = -1 \;\wedge\; t_a > 0 \;\wedge\; t_b > 0 \\ \quad \text{or} \quad s_a = -1 \;\wedge\; s_b = +1 \;\wedge\; t_a < 0 \;\wedge\; t_b < 0 \end{array} \right. . \tag{4.72}$$

Proof of the Claim: We know that $a<b$, $c_a > c_b$ and $z_a < z_b$. We show that $s_a = +1$ implies $s_b = -1 \;\wedge\; t_a > 0 \;\wedge\; t_b > 0$; the second case follows analogously.

If $s_a = +1$, then

$$\begin{aligned} & c_a > c_b \\ \Leftrightarrow \quad & z_a + t_a > z_b + s_b t_b \\ \Leftrightarrow \quad & t_a - (s_b t_b) > z_b - z_a = |z_b - z_a|. \end{aligned} \tag{4.73}$$

If now $s_b = +1$, this would be a contradiction to the points (t_a, z_a) and (t_b, z_b) being space-like separated. Hence, $s_b = -1$, so we have:

$$|z_b - z_a| < t_a + t_b = |t_a + t_b|. \tag{4.74}$$

This implies that t_a and t_b cannot both be negative. So assume that one of them is negative, w.l.o.g. $t_a > 0, t_b < 0$. But then $|t_a - t_b| > |t_a + t_b|$ and

$$|z_k - z_a| < |t_a + t_b| < |t_a - t_b|, \tag{4.75}$$

which also is a contradiction to the points being space-like. Thus, one must have $t_a > 0, t_b > 0$ which proves the claim.

Because of the specific sign combinations that allow for collisions, the claim shows that if $(s, s+1)$ is a collision, neither $(s-1, s)$ nor $(s+1, s+2)$ can be a collision. Therefore, $|k - s| \geq 2$ and τ_k, τ_s commute.

We use the commutability of τ_k and τ_s to define a third permutation

$$\rho := \tau_k \circ \tau_s \circ \pi = \tau_k \circ \sigma = \tau_s \circ \kappa, \tag{4.76}$$

which by construction has $n-1$ collisions, i.e. one less than σ and κ. This means that the seemingly different representations of $\phi^\pi_{s_1 \dots s_N}$,

$$\begin{aligned} \phi^\pi_{s_1 \dots s_N} &= \phi^\sigma_{s_1 \dots s_{s+1} s_s \dots s_N} + s_s \varphi^{(s)} \quad \text{and} \\ \phi^\pi_{s_1 \dots s_N} &= \phi^\kappa_{s_1 \dots s_{k+1} s_k \dots s_N} + s_k \varphi^{(k)}, \end{aligned} \tag{4.77}$$

are in fact equal. This can be seen from the fact that the different ways of decomposing π via eq. (4.71) yield (using (4.70)):

$$\begin{aligned} \phi^\pi_{s_1 \dots s_N} = \phi^\sigma_{s_1 \dots s_{s+1} s_s \dots s_N} + s_s \varphi^{(s)} &= \phi^\rho_{s_1 \dots s_{k+1} s_k \dots s_{s+1} s_s \dots s_N} + s_k \varphi^{(k)} + s_s \varphi^{(s)} \\ &= \phi^\kappa_{s_1 \dots s_{k+1} s_k \dots s_N} + s_k \varphi^{(k)} = \phi^\pi_{s_1 \dots s_N}. \end{aligned} \tag{4.78}$$

This finishes the proof of uniqueness of the phases because by the induction hypothesis, the phases associated with ρ, σ and κ exist and are uniquely determined. □

Proof of the theorem: The points of Lorentz invariance and probability conservation are clear from lemma 4.2.5 and lemma 4.2.4, respectively. Furthermore, we already know that the uniqueness of solutions in a weak sense follows from probability conservation by virtue of thm. 1.3.3. If the function defined by eq. (4.68) is indeed m times continuously differentiable, it follows from continuity that it is also unique as a C^m-function

Thus, it only remains to show that the function given by (4.68) is indeed a classical solution of the IBVP. In order to prove this, the following four points have to be verified:

1. *Differentiability:* We need to prove that $\psi \in C^m\left(\mathscr{S}_1, (\mathbb{C}^2)^{\otimes N}\right)$. As the initial values satisfy $g_j \in C^m(\mathcal{I}, \mathbb{C})\ \forall j = 1, ..., 2^N$, this property is inherited by ψ_j via the characteristics. To see this, note that eq. (4.68) just makes use of a translation of the initial values along straight lines in the (t_k, z_k) spaces.
   However, we need to consider those points separately where the permutation π changes. This exactly happens when at least two of the characteristic values c_j are equal. But then the C^m-property of ψ is ensured by the requirement that the initial values must satisfy the boundary conditions (eq. (4.66)) and that the transition shall be C^m.

2. The function defined by eq. (4.68) *solves the system of Dirac equations in* $\mathscr{S}_1$. This follows from lemma 4.2.1 because the components of the solution are indeed constant along the respective multi-time characteristics and only depend on the characteristic values c_k.

3. *The initial conditions* (4.67) *are satisfied:* At a point $(0, z_1, 0, z_2, ..., 0, z_N) \in \mathcal{I} \cap \overline{\mathscr{S}}_1$, we have $c_k = z_k\ \forall k$ and thus $c_{\pi(1)} \leq c_{\pi(2)} \leq \cdots \leq c_{\pi(N)}$ is fulfilled for $\pi = \mathrm{id}$. Therefore, formula (4.68) reduces to

$$\psi_{s_1 \ldots s_N}(0, z_1, ..., 0, z_N) \;=\; g_{s_1 \ldots s_N}(c_1, ..., c_N) \tag{4.79}$$

   which is equivalent to (4.67).

4. *The boundary conditions* (4.50) *are satisfied:* Let $k \in \{1, ..., N-1\}$ and $(t_1, z_1, ..., t_k = t, z_k = z, t_{k+1} = t, z_{k+1} = z, ..., t_N, z_N) \in \mathscr{C}_{k,k+1}$. We consider two components of ψ where only the k-th and $(k+1)$-th sign is exchanged, or more formally: let $(s_1, \ldots s_N)$, $(\tilde{s}_1, \ldots \tilde{s}_N) \in \{-1, +1\}^N$ with $s_l = \tilde{s}_l\ \forall l \notin \{k, k+1\}$ and $(s_k, s_{k+1}) = (+, -) = (\tilde{s}_{k+1}, \tilde{s}_k)$. The respective characteristic values are given by $c_k = z_k + s_k t_k$ and $\tilde{c}_k = z_k + \tilde{s}_k t_k$.
   Now observe the property $c_l = \tilde{c}_l\ \forall l \notin \{k, k+1\}$ and $c_k = \tilde{c}_{k+1}$, $\tilde{c}_{k+1} = c_k$. Let π be the permutation that leads to $c_{\pi(1)} \leq \cdots \leq c_{\pi(N)}$. The permutation σ needed to achieve $\tilde{c}_{\sigma(1)} \leq \cdots \leq \tilde{c}_{\sigma(N)}$ is given by $\sigma = \tau_k \circ \pi$ and it has one collision less than π with respect to the indices $\tilde{s}_k$. Inserting this into eq. (4.68) yields

$$\begin{aligned}
\psi_{\tilde{s}_1 \ldots \tilde{s}_N} &\overset{(4.68)}{=} e^{i\phi^{\sigma}_{\tilde{s}_1 \ldots \tilde{s}_N}} g_{\tilde{s}_{\sigma(1)} \ldots \tilde{s}_{\sigma(N)}}\left(\tilde{c}_{\sigma(1)}, ..., \tilde{c}_{\sigma(N)}\right) \\
&\overset{(4.70)}{=} e^{i\phi^{\pi}_{s_1 \ldots s_N}} e^{-i\varphi^{(k)}} g_{s_{\pi(1)} \ldots s_{\pi(N)}}\left(c_{\pi(1)}, ..., c_{\pi(N)}\right) \\
&\overset{(4.68)}{=} e^{-i\varphi^{(k)}} \psi_{s_1 \ldots s_N}.
\end{aligned} \tag{4.80}$$

   This shows that (4.50) is valid.

These four points establish existence; the function given by (4.68) is indeed the solution of the IBVP. □

Remark:

1. Uniqueness of solutions can also be proven differently than by invoking thm. 1.3.3, namely by directly showing that every solution of the IBVP has to fulfil eq. (4.68). A possible proof goes via induction over the number of collisions.

2. Note that on purely dimensional grounds it is remarkable that solutions for the IBVP with boundary conditions (4.50) do exist. Because the dimension of $\mathscr{C}_{k,k+1}$ is $(N-1)(1+d)$ which for $N > 1+d$ is greater than Nd, the dimension of the initial data surface $\mathcal{I} \cap \mathscr{S}_1$, one might have suspected this not to be the case.

4.2.6 Interaction and effective potential

In addition to the mathematical and physical features already established, we now prove that our model is interacting. Moreover, we outline how the interaction can be described effectively at equal times using self-adjoint extensions of the free two-particle Dirac Hamiltonian.

In sec. 3.6, a general criterion for interaction was given. We call a physical model *interacting* iff it generates entanglement, i.e. iff there exist wave functions that are initially product states and become entangled during time evolution. Note that for the antisymmetrized wave functions we are considering, a product means "wedge product". We now present a simple argument for why our model is interacting in this sense.

Lemma 4.2.8 *The model defined by* (4.27) *with* $\Omega = \mathscr{S}_1$ *and boundary conditions* (4.50) *is interacting if there exists* $k \in \{1, \dots, N-1\}$ *with* $\varphi^{(k)} \neq \pi$.

Proof: W.l.o.g. $k = 1$. Let the initial conditions be such that $\psi|_{\mathcal{I}}$ is a product wave function. In particular, this means that there exist functions $\alpha, \beta, \gamma, \delta \in C^m(\mathbb{R}, \mathbb{C})$ and $\zeta \in C^m(\mathbb{R}^{N-2}, \mathbb{C})$ with

$$\begin{aligned} g_{+-+\cdots+}(z_1, ..., z_N) &= \alpha(z_1)\beta(z_2)\zeta(z_3, ..., z_N), \\ g_{-++\cdots+}(z_1, ..., z_N) &= \gamma(z_1)\delta(z_2)\zeta(z_3, ..., z_N) \end{aligned} \tag{4.81}$$

for $z_1 \leq \cdots \leq z_N$.
Antisymmetry (4.29) implies:

$$\alpha(z_1)\beta(z_2) = -\gamma(z_2)\delta(z_1). \tag{4.82}$$

Consider the solution at a point $p = (t, z_1, ..., t, z_N) \in \mathscr{S}_1$ with common time $t > 0$. The auxiliary claim in the proof of lemma 4.2.7 implies that the characteristic values at p with respect to the component $\psi_{+-+\cdots+}$ are in ascending order iff $z_1 + t \leq z_2 - t$. Thus we can use formula (4.68) to obtain $\psi_{+-+\cdots+}(p)$, with the permutation π being the identity if $z_1 \leq z_2 - 2t$ and the transposition τ_1 if $z_1 > z_2 - 2t$. Written via the Heaviside function Θ, this yields

$$\begin{aligned} \psi_{+-+\cdots+}(p) &= g_{+-+\cdots+}(c_1, ...c_N)\,\Theta(z_2 - z_1 - 2t) \\ &\quad + e^{i\varphi^{(1)}} g_{-++\cdots+}(c_2, c_1, c_3, ..., c_N)\,\Theta(2t + z_1 - z_2) \\ &= \alpha(c_1)\beta(c_2)\zeta(c_3, ..., c_N)\,\Theta(z_2 - z_1 - 2t) \\ &\quad + e^{i\varphi^{(1)}} \gamma(c_2)\delta(c_1)\zeta(c_3, ..., c_N)\,\Theta(2t + z_1 - z_2)\,. \end{aligned} \tag{4.83}$$

Using (4.82), the expression simplifies to

$$\psi_{+-+\cdots+}(p) = \alpha(c_1)\beta(c_2)\zeta(c_3, \dots, c_N)\left(\Theta(c_2 - c_1) - e^{i\varphi^{(1)}}\Theta(c_1 - c_2)\right). \tag{4.84}$$

This function contains the Heaviside function of a combination of t, z_1 and z_2 in a non-factorizable way. The Θ-function cannot be left away for general initial values (as might be the case if they were zero in some regions). Furthermore, because of the prefactor $e^{i\varphi^{(1)}}$ of the second summand, we cannot write it as a product as long as $\varphi^{(1)} \neq \pi$. □

Effective single-time model: In the following, we show how an effective single-time model can be obtained from our model when considered at equal times $t_1 = t_2 = t$. Even though manifest Lorentz invariance is lost for a single-time model, we consider it instructive to connect with this familiar setting.

For simplicity, let $N = 2$. We denote the single-time wave function by $\chi(z_1, z_2, t) := \psi(t, z_1, t, z_2)$. Then the single-time model is given by the domain $\{(z_1, z_2, t) \in \mathbb{R}^3 : z_1 \neq z_2\}$, initial data at $t = 0$, boundary conditions (4.50) (with t_1, t_2 replaced by t in all the constructions) and the wave equation

$$i\frac{\partial \chi}{\partial t} \;=\; -i\left(\sigma_3 \otimes \mathbb{1}\frac{\partial}{\partial z_1} + \mathbb{1} \otimes \sigma_3 \frac{\partial}{\partial z_2}\right)\chi \;\equiv\; \hat{H}\chi. \tag{4.85}$$

Note that eq. (4.85) is obtained from the multi-time equations (4.24) by the chain rule.

We introduce new coordinates $u = \frac{1}{2}(z_1 - z_2)$ and $v = \frac{1}{2}(z_1 + z_2)$. The Hamiltonian becomes

$$\hat{H} \;=\; -i\,\mathrm{diag}(\partial_v, \partial_u, -\partial_u, -\partial_v). \tag{4.86}$$

The boundary condition (4.50) can be reformulated using antisymmetry, i.e. $\chi_2(u, v, t) = -\chi_3(-u, v, t)$, with the result

$$\begin{aligned} \lim_{u \nearrow 0} \chi_2(u, v, t) \;&=\; \lim_{u \searrow 0} -e^{-i\varphi}\chi_2(u, v, t), \quad t \in \mathbb{R}, \\ \lim_{u \nearrow 0} \chi_3(u, v, t) \;&=\; \lim_{u \searrow 0} -e^{i\varphi}\chi_3(u, v, t), \quad t \in \mathbb{R}. \end{aligned} \tag{4.87}$$

The components χ_1 and χ_4 evolve freely and have to be continuous and zero at $u = 0$ because of antisymmetry. For $\varphi = \pi$, eq. (4.87) also reduces to the condition of continuity. In that case, the model becomes free – in agreement with lemma 4.2.8.

The boundary conditions (4.87) can be implemented in the functional-analytic setting by a family of self-adjoint extensions of $\hat{H}$ parametrized by φ [78]. Moreover, the unitary groups generated by these self-adjoint extensions can also be obtained as the limits of unitary groups generated by Hamiltonians $\hat{H}_n = \hat{H} + V_n$ with

$$V_n(u) \;=\; \mathrm{diag}\,(0, V_n(u), -V_n(u), 0)\,, \tag{4.88}$$

where the potentials converge to the δ-function, $V_n(u) \to (\pi - \varphi)\delta(u)$, in the distributional sense. In essence, this is what the physics literature shows [23,74][4]. Recalling $u = \frac{1}{2}(z_1 - z_2)$, our model can therefore be considered a relativistic version of the $1+1$-dimensional multi-particle Dirac equation with a spin-dependent $\delta(z_1 - z_2)$-potential for $\varphi \neq \pi$.

[4] Note that [23, 74] consider a single-particle Dirac equation. A comparison with these papers is nevertheless possible since, as evident from eqs. (4.86), (4.88) the single-time equation (4.85) with additional potential (4.88) decouples and $\tilde{\chi} := (\chi_2, \chi_3)$ satisfies a one-particle Dirac equation in u, i.e. $i\partial_t \tilde{\chi}(u, v) = [-i\sigma_3\partial_u + \sigma_3 V_n(u)]\,\tilde{\chi}(u, v)$.

4.2.7 Conclusion

In this section, we have achieved an N-particle generalization of the model in chap. 3 which, analogously to the latter, combines the most important qualitative physical properties for a multi-time theory: probability conservation, manifest Lorentz invariance and antisymmetry. The existence of the conserved tensor current again ensures the compatibility with the HBD and GRWm models.

The main results are (a) the extraction of the class (4.50) of boundary conditions which guarantees the desired physical properties and (b) the proof of the existence of dynamics and the explicit formula for solutions. Concerning (a), we believe that this class is the only one compatible with the physical requirements which can be formulated for general N (see the remark at the end of sec. 4.2.4). Concerning (b), the possibility to explicitly determine the solution of an interacting relativistic N-particle model is remarkable and may serve as both a pedagogical example as well as a testing ground for general claims about relativistic quantum mechanics.

4.3 Non-existence of dynamics for configurations with a minimal space-like distance α

This section is taken from the paper [69, sec. 8] by Lukas Nickel and the present author with only minor changes.

We return to the question of whether there exist consistent Lorentz-invariant and probability-conserving dynamics on the domain $\mathscr{S}_\alpha$ of space-like configurations with a minimum space-like distance α (defined in eq. (3.77)). As emphasized in sec. 3.8, this is relevant for the question of whether an extension of the idea of interaction by boundary conditions for multi-time wave equations to higher dimensions is possible. However, for higher dimensions there are, at present, no methods available to treat the existence and uniqueness theory on spatio-temporally nontrivial domains. Therefore, we focus on the simplest case $d = 1$ and $N = 2$ here, for which the method of multi-time characteristics is applicable, and show that the answer to the question is negative.

First, we demonstrate that there can only be one kind of boundary conditions with the desired properties. In a second step we then prove that the corresponding IBVP on $\mathscr{S}_\alpha$ does not possess non-trivial solutions. We make use of the following definitions:

$$\begin{aligned} \mathscr{S}_\alpha^+ &:= \{(t_1, z_1, t_2, z_2) \in \mathscr{S}_\alpha : z_1 - z_2 > 0\}, \\ \mathscr{S}_\alpha^- &:= \{(t_1, z_1, t_2, z_2) \in \mathscr{S}_\alpha : z_1 - z_2 < 0\}. \end{aligned} \tag{4.89}$$

Then: $\mathscr{S}_\alpha = \mathscr{S}_\alpha^+ \cup \mathscr{S}_\alpha^-$.

Lemma 4.3.1 *Let $\alpha > 0$ and $N = 2$. For the multi-time Dirac equations (4.24) on the domain $\mathscr{S}_\alpha$, there exist no other Poincaré invariant boundary conditions which lead to probability conservation on every space-like hypersurface and which are compatible with antisymmetry, besides the ones given by:*

$$\psi_{+-}(p) \;=\; e^{\pm i\varphi}\psi_{-+}(p) \quad \forall p \in \partial\mathscr{S}_\alpha^\pm \tag{4.90}$$

with a fixed $\varphi \in (-\pi, \pi]$.

Proof: Let $p = (t_1, z_1, t_2, z_2) \in \partial\mathscr{S}_\alpha$. Because the two points (t_1, z_1) and (t_2, z_2) are space-like separated, there exists a Lorentz frame with $t_1 = t_2$. Working in this frame, we can either write $p = (t_p, z, t_p, z + \alpha)$ or $p = (t_p, z, t_p, z - \alpha)$. The idea is to use Stokes' theorem in a similar way as in the proof of lemma 4.2.2 in order to obtain a condition for probability conservation on equal-time hypersurfaces $\Sigma_{\tau_1}, \Sigma_{\tau_2}$ in the considered Lorentz frame. Here, w.l.o.g. $\tau_1 < \tau_2$. Let

$$V \;:=\; \left\{(t, z_1, t, z_2) \in \overline{\mathscr{S}}_\alpha : \tau_1 \leq t \leq \tau_2\right\}. \tag{4.91}$$

V plays the same role as V_R in eq. (4.38) for R sufficiently large. Following the strategy of the proof of lemma 4.2.2, one deduces:

$$0 \;=\; \int_V d\omega_j \;=\; \int_{\partial V} \omega_j. \tag{4.92}$$

Note that in contrast to the proof of lemma 4.2.2 but similar to the proof of thm. 3.4.2, there now exist two connected components of the domain $\mathscr{S}_\alpha$. Therefore, probability conservation in the form

$$\int_{(\Sigma_{\tau_1} \times \Sigma_{\tau_1}) \cap \mathscr{S}_\alpha} \omega_j \;=\; \int_{(\Sigma_{\tau_2} \times \Sigma_{\tau_2}) \cap \mathscr{S}_\alpha} \omega_j \tag{4.93}$$

is equivalent to

$$\int_{M^{(1)}} \omega_j \;=\; \int_{M^{(2)}} \omega_j, \tag{4.94}$$

where $M^{(j)} = \{(t, z_1, t, z_2) \in \partial\mathscr{S}_\alpha : \max\{z_1, z_2\} = z_j \;\wedge \tau_1 < t < \tau_2\}$ for $j = 1, 2$. Observe that from $(t, z_1, t, z_2) \in M^{(j)}$ it follows that $z_j = z_{3-j} + \alpha$. Furthermore, antisymmetry implies:

$$\omega_j(t, z, t, z + \alpha) \;=\; -\omega_j(t, z + \alpha, t, z). \tag{4.95}$$

This can be seen from the fact that on $\mathscr{C}_1$, $\omega_j = (|\psi_{-+}|^2 - |\psi_{+-}|^2)\, dt \wedge dz$ (see the proof of lemma 4.2.3).
Inserting (4.95) into eq. (4.94) allows us to conclude:

$$\int_{M^{(1)}} \omega_j \;=\; -\int_{M^{(1)}} \omega_j \;=\; 0. \tag{4.96}$$

As this relation must hold for every τ_1, τ_2, we must have $\omega_j(p) = 0$. In components:

$$|\psi_{+-}(p)|^2 - |\psi_{-+}(p)|^2 \;=\; 0 \quad\Leftrightarrow\quad \psi_{+-}(p) \;=\; e^{i\varphi(p)}\psi_{-+}(p), \tag{4.97}$$

where $\varphi : \partial\mathscr{S}_\alpha \to (-\pi, \pi]$ could in principle be a function which is not constant. Because p is an arbitrary boundary point, this equation must hold on the whole of $\partial\mathscr{S}_\alpha$. Moreover, the requirement of Poincaré invariance has the consequence that $\varphi(p)$ has to be locally constant (see the example preceding lemma (4.2.4)). The domain $\mathscr{S}_\alpha$ has the two connected components $\mathscr{S}_\alpha^\pm$ and by antisymmetry one obtains:

$$\varphi|_{\mathscr{S}_\alpha^+} \;=\; -\,\varphi|_{\mathscr{S}_\alpha^-}\,. \tag{4.98}$$

Thus, indeed no other boundary conditions than (4.90) are permitted. □

Remark:

1. A similar proof for distinguishable particles shows that in this case another possibility appears. Similarly to sec. 4.1, the two contributions in eq. (4.94) can cancel instead of vanishing individually. However, this canceling is not physically sensible here because it would imply a non-vanishing current from $\mathscr{S}_\alpha^+$ to $\mathscr{S}_\alpha^-$ and vice versa. Provided the Born rule holds, the particles could then swap place instantaneously *over a distance*, which seems unphysical. We therefore take the class (4.90) to be the right one also for distinguishable particles.

2. The boundary conditions (4.90) do indeed imply Poincaré invariance and probability conservation. However, this will not be shown explicitly as they do not lead to the existence of dynamics (see the following lemma).

Lemma 4.3.2 *Let $\alpha > 0$ and consider the IBVP given by*

$$\begin{cases} i\gamma_k^\mu \partial_{k,\mu}\psi(t_1,z_1,t_2,z_2) &= 0 \text{ for } k=1,2, \\ \psi(0,z_1,0,z_2) &= g(z_1,z_2), \\ \psi_{+-} &= e^{i\varphi}\,\psi_{-+} \text{ on } \partial\mathscr{S}_\alpha \end{cases} \tag{4.99}$$

on the domain $\mathscr{S}_\alpha$. Here, $\varphi \in (-\pi,\pi]$ and $g : \{(z_1,z_2)\in\mathbb{R}^2 : |z_1-z_2|>\alpha\} \to \mathbb{C}^4$ is supposed to be a C^1-function.
Then, if there exist real numbers $a_1 < b_1 < a_2 < b_2$ with $g_{+-}(a_1,a_2) \neq g_{+-}(b_1,b_2)$ or $g_{-+}(a_1,a_2) \neq g_{-+}(b_1,b_2)$ the IBVP (4.99) does not have any C^1-solution.

Proof: Assume that there exist real numbers $a_1 < b_1 < a_2 < b_2$ with $g_{-+}(a_1,a_2) \neq g_{-+}(b_1,b_2)$. The case of g_{+-} is similar and will not be shown explicitly. Suppose that ψ is a solution of (4.99). We obtain a contradiction by constructing points (t_1,y_1,t_2,y_2) and $(s_1,x_1,s_2,x_2) \in \mathscr{S}_\alpha$ which lie on the same multi-time characteristic with respect to the component ψ_{+-} (see fig. 4.5).
The construction proceeds as follows:

1. Choose a point (t_1,y_1,t_2,y_2) on the same multi-time characteristic of ψ_{+-} as $(0,a_1,0,a_2)$ and on the boundary of $\mathscr{S}_\alpha$, i.e.

$$\begin{cases} a_1 &= y_1 - t_1, \\ a_2 &= y_2 + t_2, \\ (t_1-t_2)^2 &= (y_1-y_2)^2 - \alpha^2. \end{cases} \tag{4.100}$$

   This in particular implies:

$$\psi_{-+}(t_1,y_1,t_2,y_2) \;=\; g_{-+}(a_1,a_2). \tag{4.101}$$

2. Consider the set of points (s_1,x_1,s_2,x_2) on the same multi-time characteristic as $(0,b_1,0,b_2)$ and on the boundary of $\mathscr{S}_\alpha$, i.e.

$$\begin{cases} b_1 &= x_1 - s_1, \\ b_2 &= x_2 + s_2, \\ (s_1-s_2)^2 &= (x_1-x_2)^2 - \alpha^2. \end{cases} \tag{4.102}$$

   This means

$$\psi_{-+}(s_1,x_1,s_2,x_2) \;=\; g_{-+}(b_1,b_2). \tag{4.103}$$

3. Now select a point (s_1, x_1, s_2, x_2) on the same multi-time characteristic as (t_1, y_1, t_2, y_2) with respect to the component ψ_{+-}, i.e.

$$\begin{cases} x_1 + s_1 &= y_1 + t_1, \\ x_2 - s_2 &= y_2 - t_2. \end{cases} \tag{4.104}$$

This implies that the value at (t_1, y_1, t_2, y_2) can be obtained in two different ways: firstly by using the boundary condition at that point and secondly by going along the characteristic surface[5] to (s_1, x_1, s_2, x_2) and using the value from there. In formulas:

$$\begin{aligned} \psi_{+-}(t_1, y_1, t_2, y_2) &\overset{\text{b.c.}}{=} e^{i\varphi}\, \psi_{-+}(t_1, y_1, t_2, y_2) \\ &\overset{(4.101)}{=} e^{i\varphi}\, g_{-+}(a_1, a_2). \end{aligned} \tag{4.105}$$

$$\begin{aligned} \psi_{+-}(t_1, y_1, t_2, y_2) &\overset{\text{char.}}{=} \psi_{+-}(s_1, x_1, s_2, x_2) \\ &\overset{\text{b.c.}}{=} e^{i\varphi}\, \psi_{-+}(s_1, x_1, s_2, x_2) \\ &\overset{(4.103)}{=} e^{i\varphi}\, g_{-+}(b_1, b_2). \end{aligned} \tag{4.106}$$

Thus:

$$g_{-+}(b_1, b_2) = g_{-+}(a_1, a_2), \tag{4.107}$$

in contradiction to the assumption.

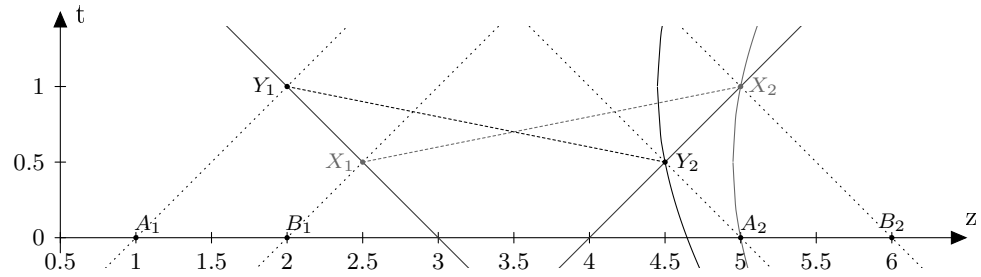

Figure 4.5: Construction in the proof for values $a_1 = 1$, $b_1 = 2$, $a_2 = 5$, $b_2 = 6$ and $\alpha = \sqrt{6}$. The points are $A_j = (a_j, 0)$, $B_j = (b_j, 0)$, and $Y_j = (y_j, t_j)$, $X_j = (x_j, s_j)$ for $j = 1, 2$. The black hyperbola consists of points with space-like distance α to Y_1 and the grey one of those with space-like distance of α to X_1. The configurations (X_1, X_2) and (Y_1, Y_2) lie on the same multi-time-characteristic, comprised of the Cartesian product of the two solid black lines.

This proves the claim, provided the points we use do exist. Indeed, the combination of the eight equations (4.100), (4.102) and (4.104) with eight unknowns leads to rather lengthy

[5] One may wonder how to obtain a path connecting the two points which neither leaves the characteristic nor the domain. This is achieved as follows. Concatenate the two linear paths from (t_1, y_1, t_2, y_2) to (t_1, y_1, s_2, x_2) and from (t_1, y_1, s_2, x_2) to (s_1, x_1, s_2, x_2), i.e. first move the right point from Y_2 to X_2 and afterwards the left from Y_1 to X_1. One can see from the hyperbolas in figure 4.5 that this path only leaves $\mathscr{S}_\alpha$ at its endpoints.

quadratic equations the general solution of which can be found below. One explicit solution is given in the figure. □

Remark: The lemma shows that the most general Lorentz invariant and probability-conserving IBVP (4.99) on $\mathscr{S}_\alpha$ is over-determined. From eq. (4.107) we conclude that the only admissible initial data are those for which g_{-+} is constant (and thus also g_{+-}). Due to normalization, this constant has to be zero. The two other components are exactly those which are not affected by the boundary conditions. Moreover, it becomes clear from the proof that the problem originates from the fact that the dimension of $\partial\mathscr{S}_\alpha$ is too high which implies (regardless of initial conditions) that certain components of the wave function have to be constant on sets as the initial data surface. This problem cannot simply be avoided by prescribing boundary conditions only on a part of the boundary due to the requirement of Lorentz invariance.

Explicit formulas for the points used in the proof of lemma 4.3.2: In the following we state the solutions of the eight equations (4.100), (4.102) and (4.104) which are used in the proof of lemma 4.3.2.

$$\begin{aligned}
y_1 &= a_1 + \frac{1}{2}\left(-a_1 + b_1 + \frac{1}{2}(a_2 - 2b_1 + b_2) - \frac{1}{2}\xi\right),\\
2t_1 &= -a_1 + b_1 + \frac{1}{2}(a_2 - 2b_1 + b_2) - \frac{1}{2}\xi,\\
y_2 &= a_1 + \frac{1}{2}\left(-a_1 + b_1 + \frac{1}{2}(a_2 - 2b_1 + b_2) + \frac{1}{2}\xi\right),\\
t_2 &= \frac{a_2 - b_2 + 2\left(\alpha^2 - b_1^2 + 2b_1b_2 - b_2^2 + (b_2 - b_1)\left(\frac{1}{2}(a_2 - 2b_1 + b_2) - \frac{1}{2}\xi\right)\right)}{(4b_1 - 4b_2 + 2(a_2 - 2b_1 + b_2) - 2\xi)},\\
x_1 &= b_1 + \frac{1}{4}(a_2 - 2b_1 + b_2) + \frac{1}{4}\xi,\\
s_1 &= \frac{1}{4}(a_2 - 2b_1 + b_2) + \frac{1}{4}\xi,\\
x_2 &= \frac{b_2 - \alpha^2 - b_1^2 + 2b_1b_2 - b_2^2 + (b_2 - b_1)\left(\frac{1}{2}(a_2 - 2b_1 + b_2) - \frac{1}{2}\xi\right)}{(2b_1 - 2b_2 + (a_2 - 2b_1 + b_2) + \xi)},\\
s_2 &= \frac{\alpha^2 - b_1^2 + 2b_1b_2 - b_2^2 + (b_2 - b_1)\left(\frac{1}{2}(a_2 - 2b_1 + b_2) - \frac{1}{2}\xi\right)}{(2b_1 - 2b_2 + (a_2 - 2b_1 + b_2) + \xi)},
\end{aligned} \tag{4.108}$$

where

$$\xi = \sqrt{\frac{(b_2 - a_2)^2(b_1 - a_1) + 4\alpha^2(b_2 - a_2)}{b_1 - a_1}}. \tag{4.109}$$

The radicand is positive since $a_1 < b_1$ and $a_2 < b_2$.

4.4 Discussion and outlook

In this chapter we have seen that various generalizations of our model, such as different boundary conditions and the N-particle case, can be achieved with mathematical rigor. The

possibility of the N-particle generalization may be taken to further motivate the question of whether also a QFT model for a multi-time wave function on the set $\mathscr{S}_{\mathrm{var}}$ of space-like configurations with arbitrary particle number (see eq. (1.23)) can be formulated and shown consistent in a similar way. Such a model could then be combined with a Bell-type (Bohmian) quantum field theory [45]. A good starting point might be the case of a finite number of N-particle sectors.

However, a generalization of the model to the domain $\mathscr{S}_\alpha$ with the desired properties was shown to be impossible for $d = 1$. While not strictly excluded, this result motivates us to believe that physically sensible dynamics do not exist on $\mathscr{S}_\alpha$ for $d > 1$, either. On this basis, the limit $\alpha \to 0$ (which is attractive because of its parallels with Wheeler-Feynman theory) can also be excluded.

These aspects show that, while valuable to demonstrate that the most important qualitative physical features can be achieved for a manifestly Lorentz invariant quantum theory, the idea of interaction by boundary conditions is too limited to describe more realistic physical systems. This motivates us to analyze a different class of multi-time wave equations with a different mechanism for relativistic interactions which can be formulated for $1 + 3$ space-time dimensions: the Two-Body Dirac equations.

Chapter 5

The Two-Body Dirac equations of constraint theory

This chapter is based on the paper *On the question of current conservation for the Two-Body Dirac equations of constraint theory* by the present author [68]. Introduction and discussion have been modified in order to fit with the other chapters of the thesis. Overlapping parts, such as sec. 2 of [68], have been left away or merged into the general chapters on the multi-time formalism, especially into chap. 1, sec. 1.3.

5.1 Background and motivation

In the 1980s, a formulation of classical mechanics using Dirac's formalism of constraints [39,41,64,65,86] led, via canonical quantization, to the formulation of equations for a multi-time wave function of two directly interacting particles in $1+3$ space-time dimensions (see e.g. [27, 70, 88] and references therein)[1]. For our purpose, the most important outcome of this development were the *Two-Body Dirac (2BD) equations* [26, 27, 29, 31, 33, 75, 88]. These are simultaneous relativistic equations for a 16-component multi-time wave function $\psi(x_1, x_2)$ which relates the space-time coordinates x_i of two spin-$\frac{1}{2}$ particles $i = 1, 2$. They take the following form:

$$\begin{aligned} D_1\psi(x_1, x_2) &= 0, \\ D_2\psi(x_1, x_2) &= 0, \end{aligned} \tag{5.1}$$

where $D_i = D_{i,0} + \tilde{V}_i(x, \hat{P})$. Here, $D_{i,0}$ is the operator in the free Dirac equation $D_{i,0}\psi = 0$ for the i-th particle in manifestly covariant form (see eq. (1.18)) and $\tilde{V}_i(x, \hat{P})$ are functions of $x = x_1 - x_2$ as well as the total momentum operator $\hat{P}$.

Compared to the previously analyzed types of multi-time equations, the structure of the 2BD equations differs precisely by the inclusion of these momentum-dependent interaction terms. This clearly avoids the no-go theorem 1.2.5 for potentials in Hamiltonian multi-time wave equations (which are assumed to be momentum-independent). However, the structural change in the equations implies that one leaves the known territory of the existence and

[1] Apparently, the development was largely independent (and unaware of) the work of Dirac [38], Tomonaga [96] and Schwinger [93].

uniqueness theories of quantum-mechanical wave equations, including the generalizations made in the previous chapters. This raises difficult and important questions, such as:

1. Are the Two-Body Dirac equations mathematically consistent or is it at least plausible that solutions exist?

2. Are they compatible with a probabilistic meaning of the wave function? More specifically, does there exist a conserved tensor current $j_\psi^{\mu\nu}$ with a positive density component as in sec. 1.3?

These questions do, of course, also concern the general validity of applications of the 2BD equations to bound state problems, such as phenomenological calculations of mesonic spectra (see e.g. [26,28,30,61]).

In view of this situation, the present chapter aims at a clarification of whether the 2BD equations fit within the general conceptual framework of chaps. 1 and 2. First, we introduce the mathematical idea underlying the 2BD equations which allows them to satisfy a certain necessary compatibility condition (sec. 5.2.2). We then discuss its physical consequences, in particular the necessity of the appearance of a certain covariantization of the spatial distance in the center of momentum frame and the compatibility of this fact with the theory of relativity. Subsequently, we introduce and focus on a concrete and important class of the 2BD equations for particle-antiparticle pairs which was first suggested by Sazdjian [88] (sec. 5.2.3). This class is related to other forms of the 2BD equations like the one used by Crater and Van Alstine (see [31,75]) so that our analysis is sufficiently general. We point out open mathematical issues (sec. 5.2.4) and propose a preliminary understanding (sec. 5.2.5).

After this, we come to the main part of this chapter: the question of whether an appropriate tensor current exists. To approach this question, we show that the free Dirac current is not conserved (sec. 5.3.1). Possible replacements can, however, be found. Nevertheless, there does not seem to exist a general argument that any of them is positive definite. We therefore continue the analysis by posing the question of whether further conditions can render the currents or, equivalently, the associated scalar product positive definite. A related analysis of Sazdjian [90] is discussed and found incomplete. The main open points are the questions of 1. how one should regard the further assumptions, (a) as restrictions on the admitted space of functions (sec. 5.3.2) or (b) as restrictions on the admissible potentials 5.3.3, as well as 2. whether the two options (a), (b) or a combination thereof are physically reasonable. These questions are answered and the consequences are drawn. Furthermore, we briefly point out that the class of gauge transformations for the 2BD equations is restricted as compared to the free case (sec. 5.4). The chapter ends with a discussion of the results and their implications.

5.2 The Two-Body Dirac equations as multi-time evolution equations

5.2.1 Notation

We use the following abbreviations:

$$\begin{aligned} & x = x_1 - x_2, \quad X = (x_1 + x_2)/2, \\ & \hat{p}_{k,\mu} = i\frac{\partial}{\partial x_k^\mu}, \quad k = 1, 2, \\ & \hat{p} = (\hat{p}_1 - \hat{p}_2)/2, \quad \hat{P} = \hat{p}_1 + \hat{p}_2. \end{aligned} \tag{5.2}$$

5.2.2 Implications of the consistency condition for the general form of the equations

Our starting point is the following general form of multi-time wave equations:

$$D_i\psi(x_1, x_2) \;=\; 0, \quad i = 1, 2. \tag{5.3}$$

The crucial point is that in contrast to the form of the Hamiltonian multi-time system (eq. (1.47)), which led to the no-go theorem 1.2.5, we now allow the operators D_i to be differential operators of *any order*, including infinity.

Recall from sec. 1.2.1.1 that certain compatibility conditions have to be satisfied in order for the wave equations (5.3) to have solutions. However, because the D_i are not necessarily first-order differential operators, eqs. (5.3) cannot be cast into the form (1.47). Therefore, the consistency condition (1.55) is not appropriate. For general operators D_i, one cannot expect to find a simple replacement of eq. (1.55) which is also both necessary and sufficient. However, a necessary condition reads

$$[D_1, D_2] \;=\; \lambda_1 D_1 + \lambda_2 D_2, \tag{5.4}$$

where at least formally the λ_i can be operators.

To see that (5.4) is a necessary condition, consider a solution ψ of (5.3). Assume that one cannot write $[D_1, D_2]$ in the form (5.4). In general, we then have $[D_1, D_2] = \lambda_1 D_1 + \lambda_2 D_2 + R$ where $R\psi \neq 0$. Thus: $[D_1, D_2]\psi = R\psi \neq 0$ in contradiction to the fact that $[D_1, D_2]\psi = 0$ trivially holds for any solution of (5.3) on which the action of $D_1 D_2$ and $D_2 D_1$ is well-defined.

Remark: Note that in the case that the operators D_i are of first order, condition (5.4) bears some similarity with the consistency condition of Frobenius' theorem from differential topology. However, it was discussed in [80, sec. 2.4] that the conditions for multi-time wave functions are, even for first-order operators, different from the statement of Frobenius' theorem, one of the reasons being the number of components of ψ.

Furthermore, note that (5.4) applied to the first-order multi-time equations (1.47) seems to lead to a weaker condition than the previous condition (1.55). For (5.4) it is sufficient for the right hand side to be a linear combination of the operators $\left(i\frac{\partial}{\partial t_k} - H_k\right)$ instead of having to vanish. At first glance, the result in [80], according to which (1.55) is a necessary and

sufficient condition, thus seems to deem (5.4) too weak a condition. However, [80] is tied to the case that the operators H_k are operators on Hilbert space (or more generally operator-valued functions of the time variables). As the operators D_i contain time derivatives and therefore cannot be regarded as operators on Hilbert space, it does not follow from [80] that the right hand side of eq. (5.4) has to vanish. Still, the question of a necessary *and* sufficient condition for the system of equations (5.3) remains open.

The important question now is: *which operators D_i satisfy condition* (5.4)*?* In order to reduce this question to a tractable form, we assume with [88] that operators D_i take the following form:

$$D_i \;=\; D_{i,0} + D_{3-i,0}\hat{V}, \quad i = 1, 2, \tag{5.5}$$

where $D_{i,0}$ are the operators of the corresponding free equations and $\hat{V}$ is an operator the structure of which is yet to be determined. Generally, it may depend on x, $\hat{p}$ and $\hat{P}$ as well as the gamma matrices $\gamma_1^\mu, \gamma_2^\nu$. A dependence on X is excluded as one aims at a Poincaré invariant theory. Note that (5.5) introduces a relation between the interaction terms in the two wave equations (5.3). We remark that the form (5.5) may imply that the wave function of eq. (5.3) cannot be directly identified with the wave function of, say, the Breit equation, or the class of 2BD equations of Crater and Van Alstine. A wave function transformation may be necessary to relate the two types of wave functions (see [31, 75]). Notwithstanding, we analyze the theory resulting from (5.5) on its own terms.

Using the form (5.5), we obtain:

$$\begin{aligned}[D_1, D_2] \;=\;& D_{1,0}D_{2,0} + D_{1,0}^2\hat{V} + D_{2,0}\hat{V}D_{2,0} + D_{2,0}\hat{V}D_{1,0}\hat{V} \\ & - D_{2,0}D_{1,0} - D_{2,0}^2\hat{V} - D_{1,0}\hat{V}D_{1,0} - D_{1,0}\hat{V}D_{2,0}\hat{V}.\end{aligned} \tag{5.6}$$

As the $D_{i,0}$ are supposed to correspond to the operators in a free wave equation (acting only on the coordinates and spin indices of the i-th particle), we have $[D_{1,0}, D_{2,0}] = 0$ and the first summands in the lines of eq. (5.6) cancel.

Aiming to bring (5.6) into the form of the right hand side of eq. (5.4), we calculate the expression

$$\begin{aligned}-[D_{1,0}, \hat{V}]D_1 + [D_{2,0}, V]D_2 \;=\;& -D_{1,0}\hat{V}D_{1,0} - D_{1,0}\hat{V}D_{2,0}\hat{V} + \hat{V}D_{1,0}^2 \\ & + D_{2,0}\hat{V}D_{2,0} + D_{2,0}\hat{V}D_{1,0}\hat{V} - \hat{V}D_{2,0}^2.\end{aligned} \tag{5.7}$$

Comparing eqs. (5.6) and (5.7), condition (5.4) is satisfied if

$$[D_{1,0}^2 - D_{2,0}^2, \hat{V}] \;=\; 0. \tag{5.8}$$

Specializing to the Dirac case, we evaluate (5.8) for

$$D_{i,0} \;=\; \gamma_i \cdot \hat{p}_i - m_i. \tag{5.9}$$

Then: $D_{i,0}^2 = \hat{p}_i^2 + m_i^2$ and eq. (5.8) reduces to:

$$[\hat{p}_1^2 - \hat{p}_2^2, \hat{V}] \;=\; 0. \tag{5.10}$$

Rewriting this equation using total and relative momentum operators yields:

$$[\hat{P} \cdot \hat{p}, \hat{V}] \;=\; 0. \tag{5.11}$$

Now, because of Poincaré invariance, $\hat{V}$ must not depend on X. Thus, we arrive at the condition

$$\hat{P}^\mu \frac{\partial \hat{V}}{\partial x^\mu} \stackrel{!}{=} 0. \tag{5.12}$$

To further evaluate eq. (5.12), note that the only Poincaré-invariant quantities involving x are functions of $x \cdot q$ where q is a quantity transforming as a 4-vector (e.g. $x, \hat{P}, \hat{p}, \gamma_1, \gamma_2$). Thus, $\frac{\partial \hat{V}}{\partial x^\mu} \propto q_\mu$ and (5.12) requires $q \perp \hat{P}$ in the Minkowski sense. This can in general only be achieved if q_μ has the form

$$q_\mu = \hat{\pi}_\mu^\nu \tilde{q}_\nu, \tag{5.13}$$

where $\hat{\pi}_\mu^\nu := \left(1 - \frac{\hat{P}_\mu \hat{P}^\nu}{\hat{P} \cdot \hat{P}}\right)$ is (an operator version of) the projection operator on the subspace "orthogonal to $\hat{P}$" and $\tilde{q}$ again transforms as a vector.
The most important consequence of this can be seen by considering the case that $\tilde{q} = x$. Then eq. (5.13) implies that the only dependence of $\hat{V}$ on x may be via

$$\hat{x}_\perp^\mu := \hat{\pi}_\nu^\mu x^\nu = x^\mu - \frac{\hat{P} \cdot x \, \hat{P}^\mu}{\hat{P} \cdot \hat{P}}. \tag{5.14}$$

To see this in detail, note that $x \cdot q = x \cdot \hat{x}_\perp = x \cdot (\hat{\pi} x) = x \cdot (\hat{\pi}^2 x) = \hat{x}_\perp \cdot \hat{x}_\perp$. We emphasize that $\hat{x}_\perp$ involves the total momentum operator and therefore is an operator itself.

On the meaning of $\hat{x}_\perp$: In classical mechanics, the analog $x_\perp$ of the operator $\hat{x}_\perp$, i.e. where $\hat{P}$ in eq. (5.2) is replaced by a time-like 4-vector P, has the following meaning. Consider the relative spatial coordinate $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$ in the center of momentum (c.m.) frame, i.e. the frame where the total momentum 4-vector takes the form $(P^0, 0, 0, 0)$. Then $x_\perp = (0, \mathbf{x})$. In this way, one can see that $x_\perp$ is the covariantization of $(0, \mathbf{x})$.

Note that generalizing a non-relativistic law by the replacement $(0, \mathbf{x}) \to x_\perp$ would arouse suspicion. If P were the total momentum of the total system, it would be inacceptable because in this context total quantities do not have any physical meaning but are only used to define coordinate systems. However, the use of $x_\perp$ is restricted to autonomous two-particle systems which are to be thought of as subsystems of a larger system. Then the total momentum P is meaningful. Yet one might object against the use of any preferred frame, even if it is dynamically preferred, such as the c.m. frame. However, this criticism is alleviated by the fact that the replacement $(0, \mathbf{x}) \to x_\perp$ has never been used in the derivation of the necessity of $\hat{x}_\perp$. Rather, the crucial point is the form (5.5) of the operators D_i – which is far from directly assuming a covariantization of a non-relativistic law of motion[2].

Further remarks:

1. One can also motivate the necessity of $\hat{x}_\perp$ very concisely in the context of two-body Klein-Gordon equations of the form $(\hat{p}_i^2 - m_i^2 - \hat{V})\psi = 0, \; i = 1, 2$ where $\hat{V}$ is a scalar and Poincaré invariant potential (operator) [29, sec. II]. However, this derivation is of limited significance for the approach to the 2BD equations taken here because the

[2] A related subtlety of the notion of Lorentz invariance was critically discussed by Bell in [13].

square of the operators (5.5) does not in general yield $\hat{p}_i^2 - m_i^2 - \hat{V}$ with a scalar potential $\hat{V}$[3].

2. The necessity of $\hat{x}_\perp$ does not follow from the connection of the 2BD equations with the Bethe-Salpeter equation (see [89]). Rather, the insight that $\hat{x}_\perp$ is necessary to formulate differential equations of the type (5.3) is itself used to make the so-called "relativistic instantaneous approximation" which creates a manifest $\hat{x}_\perp$-dependence of the potential terms.

3. The occurrence of $\hat{x}_\perp$ constitutes a non-local feature of the 2BD equations, in the sense that $\hat{V}$ involves non-trivial functions of differential operators. It therefore cannot be represented as a simple multiplication or differential operator. We shall see in sec. 5.3 that this kind of "non-locality" (i.e. the momentum-dependence of $\hat{V}$) causes difficulties with the current conservation.

5.2.3 Two-Body Dirac equations for fermion-antifermion systems

For the rest of the chapter, we now specialize to an important class of 2BD equations which is almost identical to one discussed above: the case of spin-$\frac{1}{2}$ particle-antiparticle pairs. This case is particularly relevant as one aims at a theoretical description of mesons and their spectra [26,28,30,61]. Equations for fermions with the same charge are, on the other hand, not believed to describe bound states (as are the equations below) and therefore not to lead to particularly interesting subsystem dynamics.

The class of 2BD equations for particle-antiparticle pairs (first introduced by Sazdjian in [88, sec. VI]) is given by

$$\begin{aligned} D_1\psi(x_1,x_2) &\equiv \left[\gamma_1\cdot\hat{p}_1 - m_1 - (-\gamma_2\cdot\hat{p}_2 + m_2)\hat{V}\right]\psi(x_1,x_2) = 0, \\ D_2\psi(x_1,x_2) &\equiv \left[\gamma_2\cdot\hat{p}_2 + m_2 + (\gamma_1\cdot\hat{p}_1 + m_1)\hat{V}\right]\psi(x_1,x_2) = 0. \end{aligned} \tag{5.15}$$

Here, ψ is a 16-component wave function. According to standard sign conventions, particle 2 is the anti-particle.

The form of the equations is motivated similarly as the approach via eqs. (5.3) and (5.5), the only difference being that one has to account for the symmetries of the fermion-antifermion system. More precisely, the first of the equations (5.15) has to be obtained from the second via charge conjugation and mass exchange [88, p. 3411]. This changes some signs as compared to the form of the operators D_i in (5.5).

$\hat{V}$ is an operator which may depend on $\hat{P}, \hat{p}, \hat{x}_\perp$ and the γ-matrices in a Poincaré invariant manner. The symmetry of the fermion-antifermion system demands [88, p. 3411]:

$$\hat{V}(1,2;\gamma_1,\gamma_2) = \hat{V}(2,1;-\gamma_2,-\gamma_1), \tag{5.16}$$

where "$1 \leftrightarrow 2$" indicates the exchange of particle labels in quantities such as $\hat{p}_i, x_i$ (but not the γ-matrices). We remark that here and in the following the notation $\hat{V}(...)$ is only meant to emphasize possible dependencies on certain variables. $\hat{V}$ is always the same operator.

[3] Note that starting from a different point, Crater and Van Alstine were in fact able to derive 2BD equations with scalar interactions as "square roots" of corresponding scalar interacting two-body Klein-Gordon equations [27].

Then the 2BD equations (5.15) also satisfy the compatibility condition (5.4) because the following relation holds [88, p. 3412]:

$$[D_1, D_2] \;=\; -[\gamma_1 \cdot \hat{p}_1, \hat{V}]D_1 + [\gamma_2 \cdot \hat{p}_2, \hat{V}]D_2. \tag{5.17}$$

Special choices of $\hat{V}$ may yield $[D_1, D_2] = 0$ [88, sec. VII].

Single-time irreducibility: Recall the notion of a single-time reducible theory from sec. 1.2.3. If the 2BD equations (5.15) were single-time reducible, we would have to obtain a single-time equation by considering $i\partial_t \varphi(\mathbf{x}_1, \mathbf{x}_2; t)$ for $\varphi(\mathbf{x}_1, \mathbf{x}_2; t) = \psi(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2, t)|_{t_1=t_2=t}$. However, contrary to the Hamiltonian multi-time equations (1.16), such a reduction is not possible here as one cannot solve the 2BD equations (5.15) for $i\partial_{t_1}\psi$ and $i\partial_{t_2}\psi$. Therefore, the 2BD equations are *single-time irreducible*.

5.2.4 Basic mathematical questions

Due to the dependence of $\hat{V}$ on $\hat{P}$, the 2BD equations (5.15) are of infinite order both in space and time coordinates. This immediately raises difficult mathematical questions which are generally not addressed in the literature (compare e.g. [27,88]). Besides the question of the compatibility of the 2BD equations which was already discussed in sec. 5.2.2, one may ask:

1. What are appropriate initial data?
2. What is an adequate space of solutions?

Concerning 1., note that in contrast to a first-order multi-time system (1.47), one does not expect initial data to consist only of prescribing the wave function for configurations on a space-like hypersurface as in (1.49). For wave equations of n-th order one would rather expect that also $(n-1)$-th time derivatives have to be prescribed. If this analogy extended to infinite order, this understanding of time evolution for the 2BD equations could not make sense since prescribing all derivatives of an (analytic) function on a Cauchy surface is equivalent to writing down the solution on the whole of $\mathbb{R}^8$ (for two particles). One may, however, hope that the fact that the infinite order only arises from the dependence of $\hat{V}$ on the total momentum operator may help to identify sensible initial data (see section 5.2.5).

With respect to point 2, note that the possible spaces of initial data are usually a good starting point for defining Hilbert spaces on which at least the non-relativistic existence and uniqueness theory is usually based. Due to the occurrence of powers of $\hat{P}$ to infinite order, and therefore of time derivatives, it is clear that this setting cannot be used without major changes. Moreover, the considerations in section 1.3 show that the natural Hilbert spaces depend, via the scalar product (1.77) (and corresponding statements about self-adjointness etc.), on the form of the conserved tensor current of the theory (1.58). The question of conserved currents for the 2BD theory will be addressed in section 5.3.

5.2.5 A preliminary mathematical understanding

In this subsection, we propose a way how one can understand the 2BD equations in a preliminary way for superpositions of eigenfunctions of the total momentum operator.

Assume that the only momentum dependence of $\hat{V}$ is via $\hat{P}$ (explicitly via P^2 or implicitly via $\hat{x}_\perp$). We write: $\hat{V} = \hat{V}(\hat{x}_\perp, \hat{P})$. Let ψ_P be an eigenfunction of $\hat{P}$, i.e. a function of the form[4]

$$\psi_P(x_1, x_2) \;=\; \tilde{\psi}(x) e^{-iP\cdot X}. \tag{5.18}$$

Then

$$\hat{V}(\hat{x}_\perp, \hat{P})\psi_P \;\equiv\; V(x_\perp, P)\psi_P, \tag{5.19}$$

where $V(x_\perp, P)$ is the matrix-valued function which is obtained by replacing $\hat{P}$ in $\hat{x}_\perp$ with its eigenvalue P. In this way, we can regard $\hat{V}$ as a multiplication operator.

For ψ_P, eqs. (5.15) constitute a first order system of differential equations, analogous to (1.47). The analogy with (1.49) suggests that adequate initial data are of the form of prescribing $\psi_P(x_1, x_2)$ on a space-like hypersurface, e.g. for $x_1^0 = x_2^0 = 0$:

$$\psi_P(0, \mathbf{x}_1, 0, \mathbf{x}_2) \;\stackrel{!}{=}\; \tilde{\psi}_0(\mathbf{x}) e^{i\mathbf{P}\cdot\mathbf{X}}, \quad \mathbf{x} \in \mathbb{R}^3. \tag{5.20}$$

The role of the two eqs. (5.15) then is to (a) time-evolve $\tilde{\psi}$ in x^0 and (b) determine P^0.

More generally, one should consider superpositions of eigenfunctions of $\hat{P}$. These functions are necessary to describe localized wave packets[5] on the configuration space of two particles. Let Z denote further quantities, e.g. the relative momentum eigenvalues, which classify a suitable space of "relative coordinate wave functions" $\tilde{\psi}(x)$. Then:

$$\psi(x_1, x_2) \;=\; \int dZ \int d^3\mathbf{P}\, c(\mathbf{P}, Z) \tilde{\psi}_Z(x) e^{-iP\cdot X}, \tag{5.21}$$

where it is understood that each P^0 is determined by eqs. (5.15) by demanding that $\tilde{\psi}_Z(x)e^{-iP\cdot X}$ be a solution for every $Z, \mathbf{P}$.

The further strategy in this chapter is the following: Setting aside the question of the solution theory[6], we assume that solutions of the form (5.21) exist, at least for superpositions of finitely many eigenvalues of $\hat{P}$. This permits us to analyze the central physical question of whether there exist adequate conserved currents for the 2BD equations.

5.3 The question of current conservation

Recall the central place of the tensor current $j^{\mu\nu}[\psi_1, \psi_2]$ and especially $j^{\mu\nu}[\psi, \psi]$ in the general structure of a multi-time theory (sec. 1.3). The importance of j has also been recognized by various authors in the context of the Two-Body Dirac equations, in particular for the question of how to construct scalar products and corresponding Hilbert spaces, see [70, 84] for the spin-less Klein-Gordon case and [88, 90] for the Dirac case with spin.

[4] Note that an HBD law (2.1) then does *not* imply that the Bohmian center of mass coordinate $X(s) = (X_1(s) + X_2(s))/2$ (for $m_1 = m_2$) moves on a straight line.

[5] Note that the issue of localization is not as problematic for the 2BD equations as e.g. in relativistic quantum field theory where the Hamiltonian is assumed to be bounded from below. The reason is that, as with the single-particle Dirac equation, for the 2BD equations negative eigenvalues of the energy-momentum operators are possible.

[6] A good starting point for the question of existence and uniqueness might be to first specialize on the case of a total momentum eigenfunction (5.18). One may then hope that via (5.21) a suitable space of solutions can be constructed.

Here, we first review previous results for the 2BD equations, adding details and clearly stating critical assumptions (sec. 5.3.1). It turns out that the free Dirac current is not conserved. While possible replacements can be found, they are neither unique nor simple. We follow Sazdjian [90] to pick a particular one in order to further analyze the resulting theory. The expression for the scalar product is in general not positive definite and a comprehensive analysis of the associated problems has apparently not been performed yet. Therefore, we discuss whether the "scalar product" can be rendered positive definite by (a) restrictions on the function space (sec. 5.3.2) or (b) restrictions on the potential terms (sec. 5.3.3).

5.3.1 Previous results

In the following we assume that $\hat{V}$ satisfies the following hermiticity condition:

$$V^\dagger(x_\perp, P) \;=\; \gamma_1^0\gamma_2^0 V(x_\perp, P)\gamma_1^0\gamma_2^0, \tag{5.22}$$

where V (without the hat) was introduced in eq. (5.19).

Claim 1: *Let $\psi_P, \psi_{P'}$ be eigenfunctions of $\hat{P}$. Then the free Dirac tensor current $j^{\mu\nu}_{\rm free}[\psi_P, \psi_{P'}] = \overline{\psi}_P\gamma_1^\mu\gamma_2^\nu\psi_{P'}$ is conserved if and only if no interaction terms $-(-\gamma_2\cdot\hat{p}_2 + m_2)\hat{V}$ and $(\gamma_1\cdot\hat{p}_1 + m_1)\hat{V}$, respectively, are present in the 2BD equations.*

Proof: Consider

$$i\partial_{1,\mu}\left(\overline{\psi}_P\gamma_1^\mu\gamma_2^\nu\psi_{P'}\right) \;=\; -(\gamma_1^\mu\hat{p}_{1,\mu}\psi_P)^\dagger\gamma_1^0\gamma_2^0\gamma_2^\nu\psi_{P'} \;+\; \overline{\psi}_P\gamma_2^\nu(\gamma_1^\mu\hat{p}_{1,\mu}\psi_{P'}). \tag{5.23}$$

Denote[7] $\hat{V}\psi_P$ by $V_P\psi_P$ – which still contains a $x_\perp$-dependence. The first of the 2BD equations (5.15) yields:

$$\gamma_1\cdot\hat{p}_1\,\psi_{P'} \;=\; [m_1 + (-\gamma_2\cdot\hat{p}_2 + m_2)V_{P'}]\,\psi_{P'}. \tag{5.24}$$

Using the relations (5.22) as well as $(\gamma_k^\mu)^\dagger = \gamma_k^0\gamma_k^\mu\gamma_k^0$ it follows that

$$(\gamma_1\cdot\hat{p}_1\,\psi_P)^\dagger \;=\; \overline{\psi}_P\left[V_P(m_2 + \gamma_2\cdot\overleftarrow{\hat{p}_2}) + m_1\right]\gamma_1^0\gamma_2^0, \tag{5.25}$$

where the arrow indicates the direction in which the derivative acts. Combining eqs. (5.23), (5.24) and (5.25), we obtain:

$$i\partial_{1,\mu}(\overline{\psi}_P\gamma_1^\mu\gamma_2^\nu\psi_{P'}) \;=\; \overline{\psi}_P\left[-V_P(m_2 + \gamma_2\cdot\overleftarrow{\hat{p}_2})\gamma_2^\nu + \gamma_2^\nu(-\gamma_2\cdot\hat{p}_2 + m_2)V_{P'}\right]\psi_{P'}. \tag{5.26}$$

We note the following points:

1. The term with m_1 has dropped out.
2. The term with m_2, i.e. $-V_P m_2\gamma_2^\nu + \gamma_2^\nu m_2 V_{P'}$ yields zero only in the case that V does not contain γ_2-matrices and for $P = P'$.

[7] This replaces the previous notation $V(x_\perp, P)$ to fit in the equations.

3. Even in the latter case, the remaining term $-V_P\gamma_2\cdot \overleftarrow{\hat{p}_2}\ \gamma_2^\nu - \gamma_2^\nu\gamma_2\cdot\hat{p}_2 V_{P'}$ does not vanish because $[\gamma_2^\nu, \gamma_2\cdot\hat{p}_2]\neq 0$.

4. If $P\neq P'$ and V is not constant, not even special choices for V make the rhs. of (5.26) vanish. The appearance of both V_P and $V_{P'}$ is unavoidable because the basic mechanism which allows the 2BD equations to circumvent the no-go theorems is the use of these momentum-dependent terms.

5. The only case in which the free Dirac current is conserved is that the 2BD equations do not contain the interaction terms from the very beginning.

An analogous reasoning for $-i\partial_{2,\nu}\left(\overline{\psi}_P\gamma_1^\mu\gamma_2^\nu\psi_{P'}\right)$ establishes the claim. □

Claim 2 (see [90, p. 1625]): *There exist currents $j^{\mu\nu}_{\rm int}[\psi_P,\psi_{P'}]$ which are conserved by the 2BD equations.*

Proof: One looks for a term $j^{\mu\nu}_{\rm add}[\psi_P,\psi_{P'}]$ such that

$$j^{\mu\nu}_{\rm int}[\psi_P,\psi_{P'}] \ :=\ j^{\mu\nu}_{\rm free}[\psi_P,\psi_{P'}] + j^{\mu\nu}_{\rm add}[\psi_P,\psi_{P'}] \tag{5.27}$$

is conserved. We leave away the square brackets $[\psi_P,\psi_{P'}]$ in the following for notational ease but emphasize that in order to be able to treat V as a matrix this understanding is crucial. Defining

$$F_1^\nu := \partial_{1,\mu} j^{\mu\nu}_{\rm free},\quad F_2^\mu := \partial_{2,\nu} j^{\mu\nu}_{\rm free},\quad F := \partial_{1,\mu}\partial_{2,\nu} j^{\mu\nu}_{\rm free}, \tag{5.28}$$

we see that $j^{\mu\nu}_{\rm add}$ has to be a solution of the equations

$$\partial_{1,\mu} j^{\mu\nu}_{\rm add} = -F_1^\nu,\quad \partial_{2,\nu} j^{\mu\nu}_{\rm add} = -F_2^\mu. \tag{5.29}$$

Such a solution is easy to find [90, eq. (3.19)]. Let $G(x-x')$ be a Green's function of the four-dimensional wave equation, i.e.

$$\Box_x G(x-x') \ =\ \delta^{(4)}(x-x'). \tag{5.30}$$

Then for any pair of Green's functions $G_i,\ i=1,2$, a solution of (5.29) is given by:

$$\begin{aligned} j^{\mu\nu}_{\rm add}(x_1,x_2) \ :=\ & -\partial_1^\mu \int d^4x_1'\, G_1(x_1-x_1')F_1^\nu(x_1',x_2) - \partial_2^\nu \int d^4x_2'\, G_2(x_2-x_2')F_2^\mu(x_1,x_2') \\ & + \partial_1^\mu\partial_2^\nu \int d^4x_1' d^4x_2'\, G_1(x_1-x_1')G_2(x_2-x_2')F(x_1',x_2'). \qquad \Box \end{aligned} \tag{5.31}$$

Remark:

1. Note that the approach of finding divergence-free tensor currents via eq. (5.27) is motivated by relativistic invariance. It is quite different from the situation for a non-local potential in the Schrödinger equation (see [66]). There, the free spatial current and density do not satisfy a continuity equation, either. In oder to obtain a continuity equation, one modifies only the spatial current, retaining the density component $|\psi|^2$. However, in the relativistic case, an analogous approach is not possible because modifications of only the spatial components of the tensor current would destroy the transformation properties of the latter.

2. From eq. (5.31) it is obvious that $j^{\mu\nu}_{\text{add}}$ *is not defined uniquely.* One has to make a choice of the Green's functions $G_i,\ i = 1, 2$. Sazdjian's choice is[8] $G_i \equiv G_A,\ i = 1, 2$ where G_A is the advanced Green's function, with the reason that this would be "the only solution of (5.29) which vanishes when the interaction is switched off" [90, p. 1625].

3. The construction of j_{int} is very general (one might even say too general) since it would have worked for any F_1^ν, F_2^μ defined by eq. (5.28). So, what is the significance of the existence of conserved j_{int}'s? A good answer would be to point out, for example, a unique current with the required properties such as a positive component (see sec. 1.3). However, a *general argument* why any of the possible definitions j_{int} should yield a positive definite current simply does not exist.

4. Nevertheless, one may ask the question of whether given *further assumptions* the currents are positive definite. Further assumptions might even be plausible, for example if they concern special potentials $\hat{V}$. In the end, it is only important that the currents are positive definite for realistic choices of $\hat{V}$. An approach involving further assumptions was chosen by Sazdjian [90] which we shall critically review next.

Sazdjian's paper does not directly address the question of whether the currents j_{int} are positive definite but rather with the one of whether the associated scalar product (see eqs. (1.77), (1.79)) is. However, these two questions are equivalent as long as the wave functions admitted in the construction of the tensor currents (and in the scalar product) are not subject to restrictions which forbid localized wave packets.

Sazdjian's results for the scalar product derived from $j^{\mu\nu}_{\text{int}}$ according to (1.77) are as follows. The above-mentioned choice of Green's functions leads to the following expression[9] for a scalar product, for two eigenfunctions[10] $\psi_P, \psi_{P'}$ of $\hat{P}$ and the special case of $\Sigma = \Sigma_t$, i.e. an equal-time hypersurface with normal covector field $n \equiv (1, 0, 0, 0)$ [90, eq. (5.11)]:

$$\begin{aligned}\langle \psi_P, \psi_{P'} \rangle_{\Sigma_t} &:= \int_{\Sigma_t \times \Sigma_t} d\sigma(x_1) d\sigma(x_2)\, j^{\mu\nu}_{\text{int}}[\psi_P, \psi_{P'}](x_1, x_2) n_\mu(x_1) n_\nu(x_2) \\ &= \lim_{\varepsilon \to 0} \int d^3\mathbf{X}\, d^3\mathbf{x}\ \overline{\psi}_P \left[\gamma_1^0 \gamma_2^0 - V^*_{P'} \gamma_1^0 \gamma_2^0 V_P \right. \\ &\qquad \left. + (P^{0\prime} + P^0) \frac{V_{P'+i\varepsilon n} - V_{P - i\varepsilon n}}{P^{0\prime} - P^0 + 2i\varepsilon} \right] \psi_{P'}, \end{aligned} \tag{5.32}$$

where $(\cdot)^*$ denotes complex conjugation (without transposition). The limit $\varepsilon \to 0^+$ comes from the definition of $G_A(x)$ by its Fourier transform

$$G_A(x) = \lim_{\varepsilon \to 0^+} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{k^2 - 2ik^0\varepsilon}. \tag{5.33}$$

[8] Sazdjian only uses eq. (5.31) with $G_1 \equiv G_2$.

[9] We have adopted our notation conventions. Besides, the range of integration is corrected according to the remark in sec. 1.3 below eq. (1.79) so that the integration over x_1 and x_2 is over the same equal-time hypersurface instead of two different ones.

[10] For more general wave functions such as in (5.21), the definition of $\langle \cdot, \cdot \rangle_{\Sigma_t}$ can be extended by linearity.

Let $\psi_P = e^{-iP\cdot X}\phi_1(x)$, $\psi_{P'} = e^{-iP'\cdot X}\phi_2(x)$. Then for $P^2 = (P')^2$, Sazdjian obtains from eq. (5.32) [90, eq. (5.12)]:

$$\langle \psi_P, \psi_{P'} \rangle_{\Sigma_t} \;=\; (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{p}') \int d^3\mathbf{x}\, \overline{\phi}_1(x) \left[\gamma_1^0\gamma_2^0 - V_P \gamma_1^0\gamma_2^0 V_P + 4(P^0)^2 \frac{\partial V_P}{\partial (P^2)} \right] \phi_2(x), \tag{5.34}$$

where $x = (x^0 = 0, \mathbf{x})$. We note that because of the delta function one can use $P = P'$ everywhere inside the integral.

Remark: Ignoring the delta function, for $P = P'$, eq. (5.34) should yield the square of a norm. It therefore has to be positive. However, in general (i.e. independently of V_P) only the first term in the square brackets of eq. (5.34), which corresponds to the usual expression $\int \psi^\dagger\psi = \int \overline{\psi}\gamma_1^0\gamma_2^0\psi$ in the Dirac case, yields a positive contribution. Thus, as recognized by Sazdjian [90, p. 1631][11]:

> "If the potential V is explicitly independent of P^2 in the c.m. frame, the expression of the norm (5.34) shows that its kernel still depends on V. This implies that V must satisfy some inequality conditions to guarantee the positivity of the norm. This question was examined in more detail in Ref. [88, sec. VII A]."

In Ref. [88, sec. VII A], it is suggested to make a wave function transformation [88, eq. (7.1)] which would map the norm given by eq. (5.34) to the free $\int |\psi|^2$-norm. However, this is only possible for operators $\hat{V}$ for which V_P does not depend on P^2 in the c.m. frame [88, p. 3423] and if in addition the following condition [88, eq. (7.6)] is satisfied:

$$\tfrac{1}{4}\,\mathrm{Tr}\,(\gamma_1 \cdot \frac{\hat{P}}{\sqrt{\hat{P}^2}}\;\gamma_2 \cdot \frac{\hat{P}}{\sqrt{\hat{P}^2}}\hat{V}) \;<\; 1. \tag{5.35}$$

Presumably, the trace is to be taken over the spin components of $\hat{V}$. The question of whether the independence of V_P of P^2 in the c.m. frame is a reasonable assumption is not clarified in [88]. The more recent article [75, eqs. (A4), (A9)] even seems to show the contrary.

However, this confusing point set aside, a much more basic question is left open. Because $V = V_P$ and because P is a property of ψ_P, i.e. of the wave function, it is unclear how one should regard conditions that lead to the positivity of (5.34):

1. as conditions on the space of admissible wave functions, or

2. as conditions on the operators $\hat{V}$, given their domain?

These questions are discussed in none of the references [75, 88, 90] for the 2BD case.

In the following subsections we analyze the consequences of these two possibilities (see sec. 5.3.2 for possibility 1 and sec. 5.3.3 for possibility 2).

[11] For clarity, notation and references in the quote have been adapted to our conventions, without changes in content.

5.3.2 Can the positivity of norm and currents be guaranteed by restriction of the function space?

Recall from section 1.3.4 that the hope is to be able to regard the 2BD equations as defining an evolution map[12] for any pair of space-like hypersurfaces Σ, Σ', i.e.

$$U_{\Sigma\to\Sigma'} : \mathcal{H}^{(2)}_{\Sigma} \to \mathcal{H}^{(2)}_{\Sigma'}, \quad \psi_\Sigma \mapsto \psi_{\Sigma'} \tag{5.36}$$

with $\psi_\Sigma(q) = \psi_{\Sigma'}(q)$ if $q \in \Sigma \cap \Sigma'$. $U_{\Sigma\to\Sigma'}$ should be unitary in the scalar product defined by j_{int} according to eq. (1.77).

However, as we saw above, this construction does not yield a scalar product on $\mathcal{H}^{(2)}_\Sigma$ because it is in general not positive definite. Thus, we define

$$\mathcal{H}^{\text{pos}}_\Sigma := \{\phi \in \mathcal{H}^{(2)}_\Sigma : \langle\phi,\phi\rangle_\Sigma < \infty \wedge \langle\phi,\phi\rangle_\Sigma > 0\} \cup \{0\} \tag{5.37}$$

as the subspaces of $\mathcal{H}^{(2)}_\Sigma$ for which $\langle\cdot,\cdot\rangle_\Sigma$ is actually positive definite. The question is: *does* $\mathcal{H}^{\text{pos}}_\Sigma$ *define an acceptable space of functions?*

To decide on this question, consider the following points:

1. It is not clear anymore that $\mathcal{H}^{\text{pos}}_\Sigma$ contains all physically relevant functions. One can see this e.g. from (5.21) and (5.34). Any reasonable quantum mechanical matter theory should be able to describe localized wave packets. To construct these wave packets, one in general requires all Fourier modes ψ_P as in (5.21). However, for a general $\hat{V}$, the requirement of positivity of (5.34) implies conditions on the P's such that some are not admitted in the construction of wave packets. Furthermore, these conditions are mathematically quite involved and do not serve a clear physical purpose.

2. The $\mathcal{H}^{\text{pos}}_\Sigma$ do not, in general, define Hilbert spaces. Completeness may be violated. Even worse, the $\mathcal{H}^{\text{pos}}_\Sigma$ may not even be vector spaces. Consequently, the mathematical structures, on which the usual quantum formalism is built up, break down, including the self-adjoint operator observables as well as the standard approach to define the time evolution. Of course, one may consider the option of further reducing the admissible functions by replacing $\mathcal{H}^{\text{pos}}$ with some Hilbert space $\mathcal{H}^*$ contained in it. In fact, a similar route was suggested by Sazdjian [90, p. 1624]. This, however, further strengthens the criticism of point 1 and still leaves open the question of whether the usual mathematical structures can be defined.

A simple analogy: To appreciate the problems that accompany the restricted function spaces $\mathcal{H}^{\text{pos}}_\Sigma$ and $\mathcal{H}^*$, consider the following example. Let the Hilbert space of our theory be given by $\mathbb{C}^2$ with "scalar product"

$$\langle v, w\rangle_A := v^\dagger A w \quad \text{where} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.38}$$

[12] Note that this question is independent of the fact that the operators D_i appearing in the 2BD equations are not operators on Hilbert space.

Of course, $\langle\cdot,\cdot\rangle_A$ does not define a scalar product on $\mathbb{C}^2$. So in analogy to (5.37) we define:

$$\mathcal{H}^{\text{pos}} \;:=\; \{v \in \mathbb{C}^2 : v^\dagger A v > 0\} \cup \{0\} \tag{5.39}$$

as the subset on which $\langle\cdot,\cdot\rangle_A$ actually is a scalar product. We note that e.g. $(0,1)$, which may be a physically relevant vector to represent a spin state, is not contained in $\mathcal{H}^{\text{pos}}$ (cf. point 1).

Moreover, $\mathcal{H}^{\text{pos}}$ is not a vector space, because for $v_1 = (1, \frac{1}{2}) \in \mathcal{H}^{\text{pos}}$ and $v_2 = (-1, \frac{1}{2}) \in \mathcal{H}^{\text{pos}}$, the sum $v_1 + v_2 = (0,1)$ is not an element of $\mathcal{H}^{\text{pos}}$. Furthermore, it is also not complete, as the following example illustrates. Consider the sequence given by $v_n = (1, 1 - 1/n)$. We have: $\langle v_n, v_n\rangle_A = 1 - (1 - 1/n)^2 > 0$ and thus $v_n \in \mathcal{H}^{\text{pos}}$. However, the limit $v = (1,1)$ has norm zero, i.e. $v \notin \mathcal{H}^{\text{pos}}$. These problems are analogous to point 2 above. Note that they can be overcome by defining even smaller Hilbert spaces $\mathcal{H}^*$ as the span of $(1,0)$. ($\mathcal{H}^*$ is a complete vector space for which $\langle\cdot,\cdot\rangle_A$ defines a scalar product.) However, even more physically interesting vectors get lost this way.

To extend the analogy, suppose that the "wave equation" of our theory is given by

$$i\frac{d}{dt}u \;=\; Bu, \tag{5.40}$$

where

$$B \;=\; \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{5.41}$$

B is self-adjoint with respect to the canonical scalar product on $\mathbb{C}^2$ but not with respect to $\langle\cdot,\cdot\rangle_A$. Thus, it defines a time evolution on $\mathbb{C}^2$ but not necessarily on $\mathcal{H}^{\text{pos}}$. Let us analyze the consequences. Given $u(t=0) \equiv u_0$, we have

$$u(t) \;=\; \exp(-iBt)u_0 \;=\; (\mathbb{1}_2 \cos t \; - iB \sin t)u_0. \tag{5.42}$$

Let $u_0 = (1,0) \in \mathcal{H}^{\text{pos}} \cap \mathcal{H}^*$. Then $u(t) = (\cos t, -\sin t)$, which is in general not an element of $\mathcal{H}^{\text{pos}}$ (neither of $\mathcal{H}^*$).

One may try admitting only initial data $u_0 = (a,b) \in \mathcal{H}^{\text{pos}}$ for which also $u(t) \in \mathcal{H}^{\text{pos}}\ \forall t$. Then $|a| > |b|$. We have $u(t) = (a\cos t + b\sin t, -a\sin t + b\cos t)$ and therefore

$$u^\dagger(t)Au(t) \;=\; (|a|^2 - |b|^2)(\cos^2 t - \sin^2 t) + 4\,\mathrm{Re}\,(a^*b)\cos t \sin t. \tag{5.43}$$

We ask: do $a, b \in \mathbb{C}$ with $|a| > |b|$ exist which make this expression positive for every t? For an answer, consider (5.43) for (i) $t = \pi/4$ and (ii) $t = 3\pi/4$. In the case (i), we have $\sin t = \cos t = 1/\sqrt{2}$ and obtain as a condition that $\mathrm{Re}\,(a^*b) > 0$. In the case (ii), $\sin t = -\cos t = 1/\sqrt{2}$ and we obtain the condition $\mathrm{Re}\,(a^*b) < 0$, in contradiction to (i). We conclude that the restriction to $\mathcal{H}^{\text{pos}}$ is not in any way consistent with the given time evolution (5.40) (neither is the restriction to $\mathcal{H}^*$). This illustrates the problem of defining the time evolution of point 2 above.

Comparison with Klein-Gordon theory: If the above analogy extends to the case of the 2BD equations, the logical consequence is to reject restrictions on the function space. However, in view of previous claims about the consistency of a Hilbert space picture for

interacting two-body Klein-Gordon (KG) equations in [70,84,90], one may wonder if points 1 and 2 are actually as severe as they seem to be.

These sources (especially [90, sec. III]) show the following. In the KG case, one can identify potentials such that the scalar product given by the conserved tensor currents of the theory according to (1.77) is positive definite on a subspace $\mathcal{H}^*$ of the Hilbert space $\mathcal{H}_\Sigma$ of the theory. Then $\mathcal{H}^*$ is again a Hilbert space, corresponding to one of four possible choices of the sign of eigenvalues of the operators $p_1 \cdot \hat{P}/\hat{P}^2$ and $p_2 \cdot \hat{P}/\hat{P}^2$. One may thus hope that the problems of point 2 do not appear.

However, this approach disregards problem 1: the subspace $\mathcal{H}^*$ does not contain all physically relevant functions. One cannot, for example, represent localized wave packets by wave functions in $\mathcal{H}^*$. To do so would require basis vectors from all of $\mathcal{H}_\Sigma$. Furthermore, problems with the self-adjointness of operator observables may occur (see also [84, p. 66]). A completely analogous situation is encountered in free one-particle KG theory [92, chap. 3]. In this case, one draws the logical consequence that the KG equation theory cannot be considered a self-contained one-particle theory. By the same arguments, one also has to reject the approach via $\mathcal{H}^*$ towards interacting two-body KG theory.

Conclusion: One may wonder whether or not the situation for the KG theory has any significance for the 2BD theory. As remarked after the discussion of the meaning of $\hat{x}_\perp$ in sec. 5.2.2, the square of the 2BD equations does in general not yield interacting scalar KG equations. Moreover, recalling the quote at the end of section 5.3.1, the implications of the two-body KG theory on the 2BD theory are limited. For the 2BD case, the "scalar product" is not positive definite even if the norm is independent of P^2 in the c.m. frame (see also [88, p. 1627-28, 1631]). Furthermore, we note that the approach in the KG case involves both restrictions on the potential operator $\hat{V}$ as well as restrictions on the function space. The restrictions on the function space turned out to be inacceptable whereas there is no reason to reject restrictions on the potentials as long as they include the ones used in applications. In view of this situation, together with points 1-2 (as illustrated by the analogy), we conclude that restrictions on the function space are also inacceptable for the 2BD theory[13]. The question of whether, on the other hand, there exist sensible restrictions on the potentials such that the scalar product is positive definite will be the subject of the next section.

5.3.3 Do special operators $\hat{V}$ exist for which scalar product and currents are always positive definite?

Consider eq. (5.34) "in the c.m. frame", i.e. for $P = (P^0, 0, 0, 0)$. Then $\hat{x}_\perp$ acts as the multiplication operator with the spatial part $\mathbf{x}$ of the relative coordinate. Demanding that $\langle \psi_P, \psi_P \rangle_{\Sigma_t}$ be positive for all eigenfunctions ψ_P of $\hat{P}$, we obtain the following condition for V_P:

$$\overline{\phi}(\mathbf{x}) \left[\gamma_1^0 \gamma_2^0 - V_P \gamma_1^0 \gamma_2^0 V_P + 4(P^0)^2 \frac{\partial V_P}{\partial (P^2)} \right] \phi(\mathbf{x}) \ \geq 0. \tag{5.44}$$

This condition should be satisfied for a reasonably general class of functions ϕ, e.g. for all $\phi \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^{16}$. We note the following points:

[13] One may even hope that the situation is better in the 2BD theory, in the sense that the free Dirac current is positive definite, as opposed to the free KG current.

1. V_P, which depends on $\hat{P}$ via the quantities $x_\perp = (0, \mathbf{x})$ and $P^2 = (P^0)^2$ in the c.m. frame, has to be bounded[14] with respect to (a) P^2 and (b) $x_\perp^2 = -\mathbf{x}^2$. (a) is easy to achieve, e.g. in the case that V does not depend on $(P^0)^2$ in the c.m. frame. (b) is a real restriction. We shall see the consequences below.

2. Do solutions V_P of eq. (5.44) exist? To answer this question, consider the class of scalar functions $V_P \equiv f(-x_\perp^2)$. In the c.m. frame they take the form $f(\mathbf{x}^2)$ which is independent of $(P^0)^2$. Thus, the term $4(P^0)^2 \frac{\partial V_P}{\partial (P^2)}$ in eq. (5.44) vanishes. Making use of the fact that $f(-\mathbf{x}^2)$ is real-valued as a consequence of eq. (5.22) in the scalar case, condition (5.44) reduces to

$$\begin{aligned} \phi^\dagger(\mathbf{x}) \left[\mathbb{1} - |f(\mathbf{x})|^2\right] \phi(\mathbf{x}) \;&\geq\; 0 \quad \forall \phi \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^{16},\ \forall \mathbf{x} \in \mathbb{R}^3 \\ \Leftrightarrow \quad |f(\mathbf{x}^2)| \;&\leq\; 1 \ \forall \mathbf{x} \in \mathbb{R}^3. \end{aligned} \tag{5.45}$$

Thus, we conclude that *there do indeed exist special operators,* e.g. $\hat{V} \equiv f(-\hat{x}_\perp^2)$ with $|f(y)| < 1\ \forall y \in \mathbb{R}$, *for which the scalar product is positive definite on a general function space,* e.g. for superpositions of eigenfunctions of $\hat{P}$ (5.21) with suitable drop-off conditions[15] for $|x_\perp^2| \to \infty$. Given any smooth and real-valued function $g(y)$, such a function f can be constructed as

$$f(y) \;:=\; \tanh g(y). \tag{5.46}$$

One may, however, ask: *are these restrictions on $\hat{V}$ physically reasonable*[16]*?*

We try to answer this question by using realistic potentials derived from quantum field theory in [75, appendix A]. One such possibility for scalar interactions in lowest order perturbation theory is [75, eqs. (A4), (2.17), (2.20)]:

$$\hat{V}_1 \;:=\; \tanh\left[-\frac{1}{2\sqrt{\hat{P}^2}} \frac{g_1 g_2}{4\pi} \frac{\exp\left(-\mu\sqrt{-\hat{x}_\perp^2}\right)}{\sqrt{-\hat{x}_\perp^2}}\right], \tag{5.47}$$

where $g_1, g_2 \in \mathbb{R}$, $\mu > 0$. The question is: *does $\hat{V}_1$ satisfy the positivity condition* (5.44)*?*

We first note that $V_{1,P}$ (i.e. $\hat{V}_1$ where $\hat{P}$ is replaced by an eigenvalue P) does indeed explicitly depend on P^2 even for $P = (P^0, 0, 0, 0)$. This feature is shared with other possible potentials derived from QFT (see[17] [75, appendix A]). Thus, we can neither use the simplified condition (5.45) nor the before-mentioned condition (5.35) of Sazdjian.

[14] "Bounded" in this context means that the absolute values of the eigenvalues of $V_P = V(x_\perp, P)$ are bounded.

[15] Note that because of the form of the kernel of the scalar product (5.34) the drop-off conditions may become modified as compared to the case $\hat{V} \equiv 0$. If this turns out problematic, one could easily avoid the situation by demanding that V_P goes to zero for $|x_\perp^2| \to \infty$ sufficiently fast.

[16] Note that when comparing such a bounded $\hat{V}$ with, say, a Coulomb potential (which is unbounded), one may have to take into account a wave function transformation (see [75] and the remark below eq. (5.5)).

[17] Note that also for Crater's and Van Alstine's form of the equations the potentials explicitly depend on $\sqrt{P^2}$, called w in the references (see e.g. [26, appendix A]).

Let us evaluate condition (5.44) for $V_{1,P}$ for eigenfunctions of $\hat{P}$ and in the c.m. frame. Then: $P^2 = (P^0)^2$ and $\sqrt{-x_\perp^2} = |\mathbf{x}|$. We have:

$$\frac{\partial V_{1,P}}{\partial (P^0)^2} \;=\; \frac{1}{4}|P^0|^{-3}\frac{g_1 g_2}{4\pi}\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|}\frac{1}{\cosh^2\left[-\frac{1}{2|P^0|}\frac{g_1 g_2}{4\pi}\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|}\right]}. \tag{5.48}$$

Let

$$y \;:=\; \frac{1}{2|P^0|}\frac{g_1 g_2}{4\pi}\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|}. \tag{5.49}$$

Evidently, $y > 0$. Then eq. (5.44) becomes

$$\phi^\dagger\left[\mathbb{1}(1-\tanh^2(-y)) + 2\gamma_1^0\gamma_2^0\,\frac{y}{\cosh^2(-y)}\right]\phi \;\geq 0 \quad \forall y > 0,\ \forall \phi \in \mathbb{C}^{16}. \tag{5.50}$$

Note that γ_j^0 has the eigenvalues ± 1 for each j. Thus, eq. (5.50) yields the two conditions

$$1 - \tanh^2(-y) \pm \frac{2y}{\cosh^2(-y)} \;\geq 0 \quad \forall y > 0. \tag{5.51}$$

However, for "$-$", the function $h(y) := 1 - \tanh^2(-y) - \frac{2y}{\cosh^2(-y)}$ is negative for $y > \frac{1}{2}$, corresponding to $|\mathbf{x}|e^{\mu|\mathbf{x}|} < \frac{1}{|P^0|}\frac{g_1 g_2}{4\pi}$. Consequently, wave functions with internal part $\phi(\mathbf{x})$ with support concentrated around $|\mathbf{x}| = 0$ have negative "norm" and "probability density".

Comparison with the norm used by Crater and Van Alstine: Building on Sazdjian's work [90], Crater and Van Alstine also considered the question of an adequate norm [31, p. 9]. Their derivation of the norm is based on the following wave function transformation between the wave function ψ of eq. (5.15) and the wave function $\tilde{\psi}$ appearing in the so-called "hyperbolic form" of their equations [31, eqs. (52), (53)]:

$$\psi \;=\; \cosh(\Delta)\tilde{\psi}, \tag{5.52}$$

where

$$\Delta \;=\; \tanh^{-1}(\hat{V}). \tag{5.53}$$

We note that this transformation is not a simple mathematical object because it evidently depends on the operator $\hat{V}$. We continue assuming that it does indeed exist (which may yield further conditions on the potentials or on the function space) and analyze the consequences for the norm.

Employing the transformation (5.52) for eq. (5.34), Crater and Van Alstine obtain (see [31] and [34, appendix B]; the result is adapted to our notation):

$$\langle\tilde{\psi}_P, \tilde{\psi}_{P'}\rangle_{\Sigma_t} \;=\; (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')\int d^3\mathbf{x}\,\tilde{\phi}_1^\dagger(x)\left[\mathbb{1} - 4P^2\gamma_1^0\gamma_2^0\frac{\partial\Delta_P}{\partial(P^2)}\right]\tilde{\phi}_2(x), \tag{5.54}$$

where the $\tilde{\phi}_i$ are defined analogously to the ϕ_i in eq. (5.34) and Δ_P is the operator Δ with $\hat{P}$ replaced by its eigenvalue P. The symbol $\tilde{(\cdot)}$ indicates that the wave function transformation (5.52) has been made.

Considering eq. (5.54), we note that the expression for the norm (i.e. for $P = P'$) reduces to the usual $\int |\psi|^2$-expression *for energy-independent potentials* and is then positive without further restriction on the potentials. However, as evident from both [75, appendix A] and [26, appendix A], *realistic choices of the potentials explicitly depend on the energy* $\sqrt{P^2}$. Thus, equivalent restrictions on the potentials as given by condition (5.44) also appear following Crater's and Van Alstine's approach. This, of course, has to be the case if the wave function transformation is to yield an equivalence between the 2BD equations of Sazdjian (5.15) and the 2BD equations of Crater and Van Alstine.

More precisely, one can see from eq. (5.54) that the following condition has to be satisfied by Δ:

$$\tilde{\phi}^\dagger(\mathbf{x}) \left[\mathbb{1} - 4P^2 \gamma_1^0 \gamma_2^0 \frac{\partial \Delta_P}{\partial (P^2)} \right] \tilde{\phi}(\mathbf{x}) \ \geq 0 \tag{5.55}$$

for all $\tilde{\phi} \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^{16}$.

We now evaluate this condition for the choice of Δ corresponding to $\hat{V}_1$ from above (see eq. (5.47), [75, eq. (A4)]). Then:

$$\Delta_1 \ = \ \tanh^{-1}(\hat{V}_1) \ = \ -\frac{1}{2\sqrt{\hat{P}^2}} \frac{g_1 g_2}{4\pi} \frac{\exp\left(-\mu\sqrt{-\hat{x}_\perp^2}\right)}{\sqrt{-\hat{x}_\perp^2}}. \tag{5.56}$$

After a short and elementary calculation similar to the one leading from eq. (5.44) to eq. (5.51), condition (5.55) reduces to

$$|\mathbf{x}| e^{\mu|\mathbf{x}|} \ < \ \frac{g_1 g_2}{4\pi}, \tag{5.57}$$

which is the same condition as before, with the same consequences.

Conclusion: It is in principle possible to guarantee the positive definiteness of the scalar product by special choices of the potential operator $\hat{V}$. This is particularly easy to achieve in the scalar case and for $\hat{V}$'s which are independent of $\hat{P}^2$. Realistic choices for $\hat{V}$ such as $\hat{V}_1$ from eq. (5.47), however, are not independent of $\hat{P}^2$. This has the consequence that $\hat{V}_1$ does not satisfy the condition (5.44) for positive definiteness of the scalar product and of the probability density. One may suspect that other realistic choices for $\hat{V}$ might suffer from the same problem. Therefore, they might not lead to a self-contained quantum mechanical two-particle theory which can possibly make statistical predictions in its own right.

5.4 Gauge invariance

In this section, we briefly comment on how the unusual mathematical structure of the 2BD equations influences the notion of gauge invariance.

According to the view put forward in sec. 1.3, one should regard the tensor current $j^{\mu\nu}[\psi, \psi]$, not the wave function ψ, as the physical object. Transformations ψ which leave j invariant are considered pure gauge. In the case of free multi-time Dirac equations, one has $j^{\mu\nu}[\psi, \psi] = \overline{\psi} \gamma_1^\mu \gamma_2'^\nu \psi$ and the gauge transformations are given by (see also [80]):

$$\psi(x_1, x_2) \ \mapsto \ e^{-i\theta(x_1, x_2)} \psi(x_1, x_2). \tag{5.58}$$

In the 2BD case, however, the tensor currents have to be modified (see sec. 5.3.1). The possible replacements are momentum-dependent, i.e. their form depends on the wave function itself. Consequently, the class of gauge transformations changes. In particular, the standard transformations (5.58) cannot in general be considered gauge transformations anymore, because e.g. $\psi \mapsto e^{-iP\cdot X}\psi$ may change the eigenvalue P of $\hat{P}$ and P in turn is crucial for the form of j_{int}. The class of gauge transformations is thus reduced to transformations

$$\theta(x_1, x_2) \;\equiv\; \tilde{\theta}(x) \tag{5.59}$$

which do not involve the coordinate X on which $\hat{P}$ acts. Note, however, that the general gauge transformations (5.58) may introduce terms into the multi-time equations (5.3) which are not Poincaré invariant. This is not possible using only the restricted class of gauge transformations (5.59).

5.5 Discussion

In this chapter, we critically reviewed the 2BD equations, placing them into the context of the multi-time formalism for the first time. As compared to the class of Hamiltonian multi-time equations of sec. 1.2.1, the 2BD questions achieve a mechanism of interaction by making use of interaction terms including arbitrary powers of the total momentum operator $\hat{P}$. These in turn are required to define the variable $\hat{x}_\perp$, a certain covariantization of the spatial relative coordinate in the center of momentum frame, which is needed to satisfy a necessary compatibility condition.

However, the use of $\hat{P}$ entails that the 2BD equations are of infinite order (and single-time irreducible). Consequently, the question arose whether they fit within the general framework for multi-time wave functions of chaps. 1 and 2. The main concern was whether there still exist conserved tensor currents, as required for probability conservation (see sec. 1.3). It turned out that the free Dirac current is not conserved. There do, however, exist possible replacements. These replacements are not unique and there is no general argument why for any of them the currents should be positive definite for arbitrary potentials and on a general function space. This led to the question of whether the currents can be rendered positive definite by restricting the function space or the admitted class of potentials. Our analysis started out from a previous one by Sazdjian which was, however, shown incomplete. In particular, Sazdjian did not discuss the question if further restrictions to render the currents positive definite are to be regarded as restrictions of the function space or of the potentials, so that further work was necessary.

First, we showed in detail that restrictions of the function space are not acceptable. The reason is that, roughly speaking, Fourier modes are excluded which are necessary for the representation of localized wave packets, for the self-adjointness of operator observables and for the usual way of defining the time evolution.

Second, we analyzed the implications of the requirement of positivity of the currents on the allowed form of the potentials, given a sufficiently general function space. The results were twofold: on the one hand, we found that it is indeed possible to identify a general class of potentials with the desired property. On the other hand, potentials which were suggested as physically accurate in the literature may in fact violate the requirements for positive definite currents. It should be emphasized that in any case the form of the

probability density changes as a consequence of the fact that the form of the currents depends on the chosen potential. This is of relevance also for applications, e.g. concerning transition rates. For spectra, in contrast, the form of the probability density is unimportant.

As there are several possible motivations to study the 2BD equations, the implications of the results on restrictions for the admissible potentials can be regarded in different ways.

On the one hand, for applications where realistic potentials are required, doubts are raised that phenomenological calculations of meson spectra based on the 2BD equations such as in [26,30,34,75,100] do have a theoretical justification. To resolve the doubts would require to check the positivity condition (5.44) (or, equivalently, (5.55)) for the potential used. This, however, has not been done in the literature, nor does an awareness of the problem seem to exist. In fact, we found that for a simple choice of the potential which was derived by Sazdjian from the Bethe-Salpeter equation, the positivity condition is violated. The fact that this topic has not received attention before is somewhat surprising, considering that one of Crater's and Van Alstine's main reasons to introduce the first version of the 2BD equations was that the Bethe-Salpeter equation possesses negative-norm states – and therefore does not have a clear physical interpretation [27].

Furthermore, assuming that physically realistic potentials could be found which also satisfy the positivity condition, the modified probability density as compared to the $|\psi|^2$-density seems unusual. It would therefore be interesting to subject the modified density to experimental tests, for example by determining transition rates. In this respect, Crater and coworkers investigated decay rates of quarkonium and positronium into two photons[18], considering the effects of the modified norm (5.54) [34]. The theoretical results obtained compare well with other phenomenological approaches, but still lie outside of the error bars of the experimental data in all cases. These differences between theory and experiment are particularly interesting, considering that they appear at a place which is critical from a purely theoretical point of view. As we stressed in sec. 5.3.1, the tensor current is not unique, requiring the choice of two Green's functions in eq. (5.31). Note that the theoretical results for the mesonic spectra given in [34], which are independent of the exact form of the tensor current, fit much better with the experimental data.
One could take these findings as a motivation to study the question of whether modifying the potentials or (as is particularly interesting) making a different choice of the tensor current could improve the theoretical results. However, before immediately drawing the consequence that such modifications are required, one should not forget that further (possibly critical) assumptions are involved in the process of calculating decay rates via the 2BD equations. This is obvious from the fact that the 2BD equations as a strict two-particle theory do not, by themselves, accommodate processes with variable particle numbers. A theoretical justification to nevertheless calculate decay rates using solutions of the 2BD equations therefore cannot be contained in the framework of the 2BD equations alone but has to come e.g. from quantum field theory.

On the other hand, for foundational aspects in relativistic quantum theory, it seems remarkable that there do exist interaction terms for multi-time equations at all which satisfy the minimal requirements of Lorentz invariance and compatibility with a probabilistic meaning of the wave function. It is interesting to note that the compatibility of these as-

[18] I am grateful to H. W. Crater for pointing this out to me.

pects in $1+3$ space-time dimensions was achieved for a single-time irreducible theory (see sec. 5.2.3). One class of these equations is given by[19]:

$$\begin{aligned}\left\{\gamma_1\cdot\hat{p}_1 - m_1 - (-\gamma_2\cdot\hat{p}_2 + m_2)\tanh\left[g(-\hat{x}_\perp^2)\right]\right\}\psi(x_1,x_2) &= 0,\\ \left\{\gamma_2\cdot\hat{p}_2 + m_2 + (\gamma_1\cdot\hat{p}_1 + m_1)\tanh\left[g(-\hat{x}_\perp^2)\right]\right\}\psi(x_1,x_2) &= 0,\end{aligned} \tag{5.60}$$

where $g(y)$ is an arbitrary smooth and real-valued function. The expression for the associated positive tensor current is rather lengthy and can be calculated via eqs. (5.28), (5.31). The corresponding scalar product, evaluated on equal-time hypersurfaces of a special frame, is given by (5.34). As stressed above, the tensor current involved in this construction is not unique. Such a non-uniqueness of the currents is, however, not an uncommon situation in quantum physics. One can for example always add a term which is divergence-free – and sometimes this is even appropriate. Moreover, the additional freedom in choosing a Green's function in eq. (5.31) might (in more general situations than (5.60)) help to reconcile experimental and theoretical results for decay rates.

Finally, one may wonder whether a similar approach as for the 2BD equations can be taken also for $N>2$ particles. However, appropriate wave equations in a closed form have never been found. This may be due to the fact that there does not exist a generalization of the variable $x_\perp$ for N particles which allows to satisfy the necessary compatibility condition of the wave equations in a similar way as for two particles [32, 91].

[19] Note that a similarly looking class of 2BD equations was suggested by Crater and Van Alstine [31, eqs. (52), (53)]. Eq. (5.60) is a subclass of these equations for which the positivity of the scalar product and currents has been checked in sec. 5.3.3. For the general class in [31], positivity may be violated.

Perspective

In this thesis, we presented – and in parts developed – a comprehensive physical and mathematical framework for multi-time wave functions. The combinations of the interacting wave equations of chaps. 3-5 with a hypersurface Bohm-Dirac guidance law or a GRWm theory of chap. 2 may provide the first examples for interacting realistic relativistic quantum theories. They show possible ways how the two main problems of relativistic quantum theory, the problem of mathematically consistent relativistic interactions and the measurement problem, can be avoided, achieving both physical clarity and mathematical precision. Thus, the main goal, as formulated in the introduction, could be achieved. Furthermore, the developed models provide explicit counterexamples to various general impossibility claims in the "folklore" of relativistic quantum physics, which seem to often result from an interpretational over-generalization of certain no-go theorems. Examples include the impossibility of localized particles *per se* in relativistic quantum physics (sometimes said to follow from Malament's theorem [72]) as well as the necessity of particle creation and annihilation for quantum-mechanical relativistic interactions (often stated in textbooks on QFT, e.g. [79], and sometimes said to follow from the "no interaction theorem" [35]).

Our work offers several concrete starting points for future research.

1. The subsystem description for the HBD model in sec. 2.2 provides the basis for a derivation of an effective measurement formalism. As shown in sec. 2.3, this has so far only been achieved in strongly restricted situations.

2. The $1+1$-dimensional model of chaps. 3 and 4 most likely allows for further generalizations, such as non-zero masses and the QFT case.

3. The findings about the restrictions on the potentials in the Two-Body Dirac equations advise a study of whether the potentials used in applications satisfy them and are therefore conceptually adequate. Furthermore, one may speculate that the insight that there are further possible choices of conserved tensor currents may help to reconcile experimental data and theoretical calculations for transition and decay rates for mesons.

Finally, in view of the fact that the discussed interacting models are exemplary and (in one or the other way) too restricted to describe realistic (actual) physical systems, we now speculate about how further progress for interacting relativistic multi-time theories may be achieved. Considering the various no-go theorems on Hamiltonian theories and the fact that none of our interacting models is Hamiltonian, we do not regard further efforts in this direction fruitful. Even though the $1+1$-dimensional model is in fact single-time reducible

and nevertheless interacting, it would be highly interesting to focus future research on single-time irreducible theories. These theories are of particular conceptual value in view of the fact that then a multi-time formulation is not only necessary to achieve manifest Lorentz invariance but also provides new possibilities for the formulation of interacting dynamics.

A particularly interesting possibility for single-time irreducible multi-time wave equations is given by multi-time integral equations. There are several further clues pointing in this direction: the supposed possibility of reformulation of the $1+1$-dimensional model in terms of an integral equation, the analogy with Wheeler-Feynman electrodynamics as well as the Bethe-Salpeter equation (from which the Two-Body Dirac equations can in some sense be derived). It will furthermore be sketched in the appendix that appropriate multi-time integral equations imply potential equations for the single-time wave function when retardation effects are neglected.

The main reason why we did not attempt a more detailed study of multi-time integral equations here is that for these it is in general an open question how the wave function attains statistical meaning. In addition to the usual problems with UV-divergencies, the Bethe-Salpeter equation is, for example, known to possess states with negative "norm". This, of course, simply means that the respective density is not adequate for the theory. While it has to our best knowledge not been excluded that different conserved and positive definite tensor currents can be found (similarly to the situation for the Two-Body Dirac equations), it may also be that the framework of conserved tensor currents of sec. 1.3 is too restricted to encompass integral equations.

One may thus raise the question of why to remain within this framework. From the perspective of Bohmian mechanics, the reason is the possibility of a statistical analysis via an equivariant density. However, as discussed in sec. 2.4.1, one would not expect the concept of an equivariant density to be adequate for Bohmian light cone models or, more generally, other models which do not make use of additional space-time structure besides the space-time metric. It becomes clear that one faces a complex of difficult and interrelated questions, which, in our opinion, seems exciting and full of novel possibilities at the same time. Perhaps an approach via simplified, exactly solvable toy models for the Bethe-Salpeter equation, such as the Wick-Cutkosky model [77], could help to make progress.

In conclusion, we hope that the insights from this thesis provide a fresh and unconventional perspective on the foundations of relativistic quantum physics.

Appendix: Derivation of a potential equation from a multi-time integral equation

In the following, we further support the claim that multi-time integral equations may provide a promising mechanism for relativistic interactions by showing (in a non-rigorous way) that a multi-time integral equation inspired by the Bethe-Salpeter equation yields a single-time wave equation with potential when time delay effects are neglected.

Consider the integral equation

$$\psi(x_1, x_2) \;=\; e_1 e_2 \int d^4x_1' \int d^4x_2'\, G_1(x_1 - x_1') G_2(x_2 - x_2') \delta((x_1' - x_2')^2) \psi(x_1', x_2'), \qquad (61)$$

where the $G_j,\ j = 1, 2$ are the respective Green's functions of the free wave equation (e.g. the free KG equation) for the j-th particle and the e_j are charges. Eq. (61) is a version of the BS equation for scalar particles in the ladder approximation (compare [55, chap. 6]). We now perform the non-relativistic limit of eq. (61) employing the following assumptions:

1. One can replace the G_j's by the Green's functions of the respective free Schrödinger equations, i.e.

$$i\frac{\partial}{\partial t_j} G_j(t_j, \mathbf{x}_j) \;=\; -\frac{1}{2m_j} \Delta_j G_j(t_j, \mathbf{x}_j) + \delta^{(4)}(t_j, x_j). \qquad (62)$$

2. The retardation is negligible, i.e. one is allowed to make the replacement

$$\begin{aligned} \delta((t_1 - t_2)^2 - (\mathbf{x}_1 - \mathbf{x_2})^2) \;&=\; \frac{1}{2}\left[\frac{\delta(t_1 - t_2 - |\mathbf{x}_1 - \mathbf{x}_2|)}{|\mathbf{x}_1 - \mathbf{x}_2|} + \frac{\delta(t_1 - t_2 + |\mathbf{x}_1 - \mathbf{x}_2|)}{|\mathbf{x}_1 - \mathbf{x}_2|}\right] \\ &\longmapsto\; \frac{\delta(t_1 - t_2)}{|\mathbf{x}_1 - \mathbf{x}_2|}. \end{aligned} \qquad (63)$$

Assumption 1 aims at obtaining an equation of first order in time. It is motivated by an analogous assumption of Dirac in his derivation of a potential equation from a QED model [38]. Heuristically, the assumption asserts that one may treat the matter as moving with low speed in relation to the reference frame. Assumption 2 amounts to requiring that the wave function does not change much when one replaces $t_1 - t_2 \pm |\mathbf{x}_1 - \mathbf{x}_2|/c$ with $t_1 - t_2$. Its plausibility is immediately seen for SI units as then $c \gg 1$.

Next, consider

$$\begin{aligned} i\frac{\partial}{\partial t}\psi(t,\mathbf{x}_1,t,\mathbf{x}_2) &= i\frac{\partial}{\partial t_1}\psi(t_1,\mathbf{x}_1,t_2,\mathbf{x}_2)|_{t_1=t_2=t} + i\frac{\partial}{\partial t_2}\psi(t_1,\mathbf{x}_1,t_2,\mathbf{x}_2)|_{t_1=t_2=t} \\ &\overset{(61),(62)}{=} \left(-\frac{1}{2m_1}\Delta_1 - \frac{1}{2m_2}\Delta_2\right)\psi(t,\mathbf{x}_1,t,\mathbf{x}_2) + I_1 + I_2, \end{aligned} \tag{64}$$

where

$$\begin{aligned} I_1 &:= e_1e_2\int d^4x_2'\, G_2(t-x_2^{0\prime},\mathbf{x}_2-\mathbf{x}_2')\delta((t-t_2')^2-(\mathbf{x}_1-\mathbf{x}_2')^2)\psi(t,\mathbf{x}_1,t_2',\mathbf{x}_2'), \\ I_2 &:= e_1e_2\int d^4x_1'\, G_1(t-x_1^{0\prime},\mathbf{x}_1-\mathbf{x}_1')\delta((t_1'-t)^2-(\mathbf{x}_1'-\mathbf{x}_2)^2)\psi(t_1',\mathbf{x}_1',t,\mathbf{x}_2). \end{aligned} \tag{65}$$

We focus on I_1 and make use of assumption 2. This results in

$$I_1 \;\mapsto\; e_1e_2\int d^3x_2'\, G_2(0,\mathbf{x}_2-\mathbf{x}_2')\frac{1}{|\mathbf{x}_1-\mathbf{x}_2'|}\psi(t,\mathbf{x}_1,t,\mathbf{x}_2'). \tag{66}$$

Now the Green's functions of the Schrödinger equation have the property

$$G_j(0,\mathbf{x}) \;=\; \delta^{(3)}(\mathbf{x}), \tag{67}$$

so we obtain:

$$I_1 \;\mapsto\; \frac{e_1e_2}{|\mathbf{x}_1-\mathbf{x}_2|}\psi(t,\mathbf{x}_1,t,\mathbf{x}_2). \tag{68}$$

Proceeding analogously for I_2, eq. (64) becomes

$$i\frac{\partial}{\partial t}\psi(t,\mathbf{x}_1,t,\mathbf{x}_2) \;=\; \left[-\frac{1}{2m_1}\Delta_1 - \frac{1}{2m_2}\Delta_2 + \frac{2e_1e_2}{|\mathbf{x}_1-\mathbf{x}_2|}\right]\psi(t,\mathbf{x}_1,t,\mathbf{x}_2). \tag{69}$$

This is the two-particle Schrödinger equation with Coulomb potential[20] for the single-time wave function (1.9).

Conclusion: Contrary to the Hamiltonian multi-time wave equations of sec. 1.2.1, multi-time integral equations are capable of implying single-time wave equations with a potential. The latter arises as an effective description from neglecting the retardation. It is noteworthy that for multi-time integral equations the problems associated with the consistency condition (1.55), as required for systems of multi-time differential equations, do not occur. The reason is that one prescribes only a single equation and not many which may contradict each other. (Systems of integro-differential multi-time equations are obtained from eq. (61) by letting the free operators, i.e. $\left(i\frac{\partial}{\partial t_j}+\frac{1}{2m_j}\Delta_j\right)$, $j=1,2$, act on it. To see their consistency without reducing it to the question of the existence of solutions of the integral equation (61) seems difficult.)

[20] We are not concerned with prefactors here.

Bibliography

[1] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden. *Solvable Models in Quantum Mechanics*. AMS Chelsea Publishing, 2005.

[2] V. Allori, S. Goldstein, R. Tumulka, and N. Zanghì. On the Common Structure of Bohmian Mechanics and the Ghirardi-Rimini-Weber Theory. *Brit. J. Phil. Sci.*, 59:353–389, 2008. arXiv:quant-ph/0603027v4.

[3] V. Allori, S. Goldstein, R. Tumulka, and N. Zanghì. Predictions and Primitive Ontology in Quantum Foundations: A Study of Examples. *Brit. J. Phil. Sci.*, 65:323–352, 2008. arXiv:1206.0019v2.

[4] A. O. Barut. The covariant many-body problem in quantumelectrodynamics. *J. Math. Phys.*, 32:1091–1095, 1991.

[5] A. Bassi and G.C. Ghirardi. Dynamical Reduction Models. *Phys. Rept.*, 379:257–426, 2003. arXiv:quant-ph/0302164.

[6] A. Bassi and G.C. Ghirardi. The Trieste Lecture of John Stewart Bell. *J. Phys. A*, 40(12):2919, 2007. A video of the lecture is also available.

[7] G. Bauer. *Ein Existenzsatz für die Wheeler-Feynman-Elektrodynamik*. Herbert-Ute-Verlag Wissenschaft, 1997. In German.

[8] G. Bauer, D. A. Deckert, and D. Dürr. On the existence of dynamics in Wheeler-Feynman electromagnetism. *Zeitschrift für angewandte Mathematik und Physik*, 64:1087–1124, 2013.

[9] G. Bauer, D. A. Deckert, D. Dürr, and G. Hinrichs. On Irreversibility and Radiation in Classical Electrodynamics of Point Particles. *J. Stat. Phys.*, 154:610–622, 2014. arXiv:arXiv:1306.3756v1.

[10] D. Bedingham, D. Dürr, G.C. Ghirardi, S. Goldstein, R. Tumulka, and N. Zanghì. Matter Density and Relativistic Models of Wave Function Collapse. *J. Stat. Phys.*, 154:623–631, 2014. arXiv:1111.1425v4.

[11] J. S. Bell. Against 'measurement'. In *Speakable and Unspeakable in Quantum Mechanics*, pages 213–231. Cambridge University Press, 1987.

[12] J. S. Bell. Are there quantum jumps? In *Speakable and Unspeakable in Quantum Mechanics*, pages 201–212. Cambridge University Press, 1987.

[13] J. S. Bell. Beables for quantum field theory. In *Speakable and Unspeakable in Quantum Mechanics*, pages 173–180. Cambridge University Press, 1987.

[14] J. S. Bell. Bertlmann's socks and the nature of reality. In *Speakable and Unspeakable in Quantum Mechanics*, pages 139–158. Cambridge University Press, 1987.

[15] J. S. Bell. How to teach special relativity. In *Speakable and Unspeakable in Quantum Mechanics*, pages 67–80. Cambridge University Press, 1987.

[16] J. S. Bell. La nouvelle cuisine. In *Speakable and Unspeakable in Quantum Mechanics*, pages 233–248. Cambridge University Press, 1987.

[17] J. S. Bell. On the impossible pilot wave. In *Speakable and Unspeakable in Quantum Mechanics*, pages 159–168. Cambridge University Press, 1987.

[18] J. S. Bell. The theory of local beables. In *Speakable and Unspeakable in Quantum Mechanics*, pages 52–62. Cambridge University Press, 1987.

[19] K. Berndl, D. Dürr, S. Goldstein, and N. Zanghì. EPR-Bell Nonlocality, Lorentz Invariance, and Bohmian Quantum Theory. *Phys. Lett. A*, 53:2062–2073, 1996. arXiv:quant-ph/9510027v1.

[20] F. Bloch. Die physikalische Bedeutung mehrerer Zeiten in der Quantenelektrodynamik. *Phys. Z. d. Sowjetunion*, 5:301–315, 1934. In German.

[21] D. Bohm and B. J. Hiley. *The undivided universe.* Routledge, 1993.

[22] G. Breit. Dirac's Equation and the Spin-Spin Interactions of Two Electrons. *Phys. Rev.*, 39:616–624, 1932.

[23] M. G. Calkin, D. Kiang, and Y. Nogami. Proper treatment of the delta function potential in the one dimensional Dirac equation. *Am. J. Phys.*, 55(8):737–739, 1987.

[24] R. Courant and D. Hilbert. *Methods of Mathematical Physics II.* Interscience Publishers, 1962.

[25] W. L. Craig and Q. Smith. *Einstein, Relativity and Absolute Simultaneity.* Routledge, 2008.

[26] H. W. Crater and J. Schiermeyer. Applications of two-body Dirac equations to the meson spectrum with three versus two covariant interactions, SU(3) mixing, and comparison to a quasipotential approach. *Phys. Rev. D*, 82:094020, 2010.

[27] H. W. Crater and P. Van Alstine. Two-body Dirac equations. *Ann. Phys.*, 148:57–94, 1983.

[28] H. W. Crater and P. Van Alstine. Relativistic Naive Quark Model for Spinning Quarks in Mesons. *Phys. Rev. Lett.*, 53:1527–1530, 1984.

[29] H. W. Crater and P. Van Alstine. Two-body Dirac equations for particles interacting through world scalar and vector potentials. *Phys. Rev. D*, 36:3007–3036, 1987.

[30] H. W. Crater and P. Van Alstine. Two-body Dirac equations for meson spectroscopy. *Phys. Rev. D*, 37:1982–2000, 1988.

[31] H. W. Crater and P. Van Alstine. Extension of two-body Dirac equations to general covariant interactions through a hyperbolic transformation. *J. Math. Phys.*, 31:1998, 1990.

[32] H. W. Crater and P. Van Alstine. Structure of quantum-mechanical relativistic two-body interactions for spinning particles. *Found. Phys.*, 24:297–328, 1994.

[33] H. W. Crater and P. Van Alstine. A tale of three equations: Breit, Eddington-Gaunt, and Two-Body Dirac. *Found. Phys.*, 27:67–79, 1997.

[34] H. W. Crater, C.-Y. Wong, and P. Van Alstine. Tests of two-body Dirac equation wave functions in the decays of quarkonium and positronium into two photons. *Phys. Rev. D*, 74:054028, 2006.

[35] D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan. Relativistic Invariance and Hamiltonian Theories of Interacting Particles. *Rev. Mod. Phys.*, 35:350–375, 1963.

[36] P. C. W. Davies. Extension of Wheeler-Feynman quantum theory to the relativistic domain. I. Scattering processes. *J. Phys. A*, 4:836–845, 1971.

[37] D.-A. Deckert. *Electrodynamic Absorber Theory - A Mathematical Study*. PhD thesis, Mathematisches Institut, Ludwig-Maximilians-Universität München, 2009. `http://edoc.ub.uni-muenchen.de/11421/`.

[38] P. A. M. Dirac. Relativistic Quantum Mechanics. *Proc. R. Soc. Lond. A*, 136:453–464, 1932.

[39] P. A. M. Dirac. *Lectures on Quantum Mechanics*. Dover, 2001.

[40] P. A. M. Dirac, V. A. Fock, and B. Podolsky. On Quantum Electrodynamics. In J. Schwinger, editor, *Selected Papers on Quantum Electrodynamics*, pages 29–40. Dover, 1958.

[41] P. Droz-Vincent. Hamiltonian Systems of Relativistic Particles. *Reports on Mathematical Physics*, 8:79–101, 1975.

[42] D. Dürr, S. Goldstein, K. Münch-Berndl, and N. Zanghì. Hypersurface Bohm-Dirac models. *Phys. Rev. A*, 60:2729–2736, 1999. arXiv:quant-ph/9801070v2.

[43] D. Dürr, S. Goldstein, T. Norsen, W. Struyve, and N. Zanghì. Can Bohmian mechanics be made relativistic? *Proc. R. Soc. A*, 470(2162):20130699, 2014. arXiv:1307.1714v2.

[44] D. Dürr, S. Goldstein, J. Taylor, R. Tumulka, and N. Zanghì. Topological Factors Derived from Bohmian Mechanics. *Ann. Henri Poincaré*, 7:791–807, 2006. arXiv:quant-ph/0601076v1.

[45] D. Dürr, S. Goldstein, R. Tumulka, and N. Zanghì. Bell-type quantum field theories. *J. Phys. A*, 38:R1, 2005. arXiv:quant-ph/0407116v1.

[46] D. Dürr, S. Goldstein, R. Tumulka, and N. Zanghì. On the Role of Density Matrices in Bohmian Mechanics. *Found. Phys.*, 35(3):449–467, 2005. arXiv:quant-ph/0311127v1.

[47] D. Dürr, S. Goldstein, and N. Zanghì. Quantum Equilibrium and the Origin of Absolute Uncertainty. *J. Stat. Phys.*, 67:843–907, 1992. arXiv:quant-ph/0308039v1.

[48] D. Dürr, S. Goldstein, and N. Zanghì. Quantum Equilibrium and the Role of Operators as Observables in Quantum Theory. *J. Stat. Phys.*, 116:959–1055, 2004. arXiv:quant-ph/0308038v2.

[49] D. Dürr, S. Goldstein, and N. Zanghì. *Quantum Physics without Quantum Philosophy.* Springer, 2013.

[50] D. Dürr and M. Lienert. On the description of subsystems in relativistic hypersurface Bohmian mechanics. *Proc. R. Soc. A*, 470(2169):20140181, 2014. arXiv:1403.1464v2.

[51] D. Dürr and S. Teufel. *Bohmian Mechanics.* Springer, 2009.

[52] R. P. Feynman. The Development of the Space-Time View of Quantum Electrodynamics. 1965. Nobel Lecture. `http://www.nobelprize.org/nobel_prizes/physics/laureates/1965/feynman-lecture.html`.

[53] J. Glimm and A. Jaffe. *Quantum Physics – A Functional Integral Point of View.* Springer, 1987.

[54] S. Goldstein and R. Tumulka. Opposite arrows of time can reconcile relativity and nonlocality. *Classical and Quantum Gravity*, 20(3):557, 2003. arXiv:quant-ph/0105040v5.

[55] W. Greiner and J. Reinhardt. *Quantum Electrodynamics.* Springer, 2009. 4th Edition.

[56] B. Gustafsson, H.-O. Kreiss, and J. Oliger. *Time Dependent Problems and Difference Methods.* Wiley Interscience, 1995.

[57] P. R. Holland. *The Quantum Theory of Motion.* Cambridge University Press, 1993.

[58] L. P. Horwitz and F. Rohrlich. Constraint relativistic quantum dynamics. *Phys. Rev. D*, 24:1528–1542, 1981.

[59] D. Howard. 'Nicht Sein Kann Was Nicht Sein Darf', or the Prehistory of EPR, 1909-1935: Einstein's Early Worries about the Quantum Mechanics of Composite Systems. In A.I. Miller, editor, *Sixty-Two Years of Uncertainty.* Plenum Press, 1990.

[60] F. Hoyle and J. V. Narlikar. Cosmology and action-at-a-distance electrodynamics. *Rev. Mod. Phys.*, 67:113–155, 1995.

[61] H. Jallouli and H. Sazdjian. Relativistic effects in the pionium lifetime. *Phys. Rev. D*, 58:014011, 1998.

[62] H. Kleinert. *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics and Financial Markets.* World Scientific, 2009. 5th Edition.

[63] K. Königsberger. *Analysis 2.* Springer, 2004.

[64] A. Komar. Constraint formalism of classical mechanics. *Phys. Rev. D*, 18:1881–1886, 1978.

[65] A. Komar. Interacting relativistic particles. *Phys. Rev. D*, 18:1887–1893, 1978.

[66] C. Li, L. Wan, Y. Wei, and J. Wang. Definition of current density in the presence of a non-local potential. *Nanotechnology*, 19:155401, 2008.

[67] M. Lienert. A relativistically interacting exactly solvable multi-time model for two massless Dirac particles in 1+1 dimensions. *J. Math. Phys.*, 56(4):042301, 2015. arXiv:1411.2833v2.

[68] M. Lienert. On the question of current conservation for the Two-Body Dirac equations of constraint theory. *J. Phys. A*, 48(32):325302, 2015.

[69] M. Lienert and L. Nickel. A simple explicitly solvable interacting relativistic N-particle model. *J. Phys. A*, 48(32):325301, 2015.

[70] G. Longhi and L. Lusanna. Bound-state solutions, invariant scalar products, and conserved currents for a class of two-body relativistic systems. *Phys. Rev. D*, 34:3707–3731, 1986.

[71] G. Ludwig. *Foundations of Quantum Mechanics I.* Springer, 1983.

[72] D. Malament. In Defense of Dogma: Why There Cannot be a Relativistic Quantum Mechanics of (Localizable) Particles. In R. Clifton, editor, *Perspectives on Quantum Reality*, volume 57 of *The University of Western Ontario Series in Philosophy of Science*, pages 1–10. Springer, 1996.

[73] T. Maudlin. Space-time in the quantum world. In J.T. Cushing, editor, *Bohmian Mechanics and Quantum Theory: An Appraisal*, pages 285–307. Kluwer Academic Publishers, 1996.

[74] B. H. J. McKellar and G. J. Stephenson Jr. Klein Paradox and the Dirac-Kronig-Penney model. *Phys. Rev. A*, 36(6):2566–2569, 1987.

[75] J. Mourad and H. Sazdjian. The two-fermion relativistic wave equations of constraint theory in the Pauli-Schrödinger form. *J. Math. Phys.*, 35(12):6379–6406, 1994.

[76] N. Nakanishi. A General Survey of the Theory of the Bethe-Salpeter Equation. *Prog. Theor. Phys. Supplement*, 43:1–81, 1969.

[77] N. Nakanishi. Review of the Wick-Cutkosky Model. *Prog. Theor. Phys. Supplement*, 95:1–24, 1988.

[78] L. Nickel. Master's thesis. On relativistic interactions in quantum theories, 2015. Mathematical Institute, Ludwig-Maximilians-Universität, München.

[79] M. E. Peskin and D. V. Schroeder. *An Introduction to Quantum Field Theory.* Westview Press, 1995.

[80] S. Petrat and R. Tumulka. Multi-Time Schrödinger Equations Cannot Contain Interaction Potentials. *J. Math. Phys.*, 55(032302), 2014. arXiv:1308.1065v2.

[81] S. Petrat and R. Tumulka. Multi-Time Wave Functions for Quantum Field Theory. *Ann. Phys.*, 345:17–54, 2014. arXiv:1309.0802v2.

[82] M. Reed and B. Simon. *Methods of Modern Mathematical Physics I: Functional Analysis.* Academic Press, 1980.

[83] M. Reed and B. Simon. *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness.* Academic Press, 2005.

[84] V. A. Rizov, H. Sazdjian, and I. T. Todorov. On the relativistic quantum mechanics of two interacting spinless particles. *Ann. Phys.*, 165(1):59–97, 1985.

[85] G. Roepstorff. *Path Integral Approach to Quantum Physics.* Springer, 1994.

[86] F. Rohrlich. Evolution and covariance in constraint dynamics. *Phys. Rev. D*, 25:2576–2583, 1982.

[87] E. E. Salpeter and H. A. Bethe. A Relativistic Equation for Bound-State Problems. *Phys. Rev.*, 84:1232–1242, 1951.

[88] H. Sazdjian. Relativistic wave equations for the dynamics of two interacting particles. *Phys. Rev. D*, 33:3401–3424, 1986.

[89] H. Sazdjian. The connection of two-particle relativistic quantum mechanics with the Bethe-Salpeter equation. *J. Math. Phys.*, 28:2618, 1987.

[90] H. Sazdjian. The scalar product in two-particle relativistic quantum mechanics. *J. Math. Phys.*, 29:1620, 1988.

[91] H. Sazdjian. N-body bound state relativistic wave equations. *Ann. Phys.*, 191(1):52–77, 1989.

[92] S. Schweber. *An Introduction to Relativistic Quantum Field Theory.* Dover, 2005. Originally published in 1961.

[93] J. Schwinger. Quantum Electrodynamics. I. A Covariant Formulation. *Phys. Rev.*, 74(2162):1439–1461, 1948.

[94] E. C. Svendsen. The Effect of Submanifolds upon Essential Self-Adjointness and Deficiency Indices. *Journal of Mathematical Analysis and Applications*, 80:551–565, 1981.

[95] B. Thaller. *The Dirac Equation.* Springer, 2010.

[96] S. Tomonaga. On a Relativistically Invariant Formulation of the Quantum Theory of Wave Fields. In J. Schwinger, editor, *Selected Papers on Quantum Electrodynamics*, pages 156–168. Dover, 1958.

[97] R. Tumulka. A Relativistic Version of the Ghirardi-Rimini-Weber Model. *J. Stat. Phys.*, 125:821–840, 2006. arXiv:quant-ph/0406094v2.

[98] R. Tumulka. The 'unromantic pictures' of quantum theory. *J. Phys. A*, 40:3245–3273, 2007.

[99] L. Vaidman. Many-Worlds Interpretation of Quantum Mechanics. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Spring 2015 edition. `http://plato.stanford.edu/archives/spr2015/entries/qm-manyworlds/`.

[100] P. Van Alstine and H. W. Crater. Exact parapositroniumlike solution to two-body Dirac equations. *Phys. Rev. D*, 34:1932–1935, 1986.

[101] J. A. Wheeler and R. P. Feynman. Interaction with the Absorber as the Mechanism of Radiation. *Rev. Mod. Phys.*, 17:157–181, 1945.

[102] J. A. Wheeler and R. P. Feynman. Classical Electrodynamics in Terms of Direct Interparticle Action. *Rev. Mod. Phys.*, 21:425–433, 1949.

[103] A. N. Whitehead. *The Concept of Nature*. Dover, 2004. Originally published in 1920. Freely available at Project Gutenberg, see `http://www.gutenberg.org/ebooks/18835`.

[104] A. S. Wightman. Quantum Field Theory in Terms of Vacuum Expectation Values. *Phys. Rev.*, 101(2):860–866, 1956.