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Master Thesis

Mean-field Dynamics of a Tracer Particle in a Fermi Sea

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Abstract

In this thesis we consider the quantum mechanical time evolution of a many particle fermionic system which interacts with a so called tracer particle. We focus on the question whether it is possible to describe the tracer particle by some free, one particle evolution equation. This effective dynamics arises from the replacement of the full Hamiltonian by the conjectured mean-field Hamiltonian. Using perturbation theory, we investigate this problem on a mathematical rigorous level. It is shown that the quantum density fluctuations are partially responsible for the deviation from free evolution. In dimension one, the fluctuations of the emergent potential are shown to be constant. This is used to prove convergence of the real time evolution to its mean-field dynamics. In contrast to prior results about mean-field dynamics, the Hamiltonian considered in the theorem does not require the scaling of the potential.
Furthermore, the density fluctuations are calculated for both the three-dimensional Dirac and the nonrelativistic Schrödinger equation. For both cases, the fluctuations are always suppressed compared to classical and bosonic systems. The question whether these three-dimensional systems can be described by some mean-field dynamics cannot be answered with the methods employed in this thesis.

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Contents

Chapter 1

Introduction

The main goal of this thesis is to investigate the following question:
Is it possible to derive an effective evolution equation for a tracer particle which interacts with a sea of electrons?

The precise meaning of this question is the following: Consider a quantum mechanical system of a finite but huge number of fermions, confined to some bounded region of space. Suppose that another particle (the *tracer particle*) interacts with this so called *sea* of electrons by means of some pair potential. In order to analyse the evolution of this tracer particle we want to derive some effective one particle evolution equation.
Of course, such an equation might only exist if it is possible to encode the interaction of all fermionic particles in some collective interaction term, which, furthermore, must not depend on the exact evolution of the system. The motivation to investigate such a system originates from at least four reasons; the one being historical, two being physical and one being mathematical.

The historic motivation bases on the insight of Dirac: Shortly after deriving his famous Dirac equation describing relativistic electrons, he was facing the problem of negative energy solutions. In principle, these states may emit an infinite amount of energy, which, as a consequence, leads to the so-called radiation catastrophe.
To solve this problem, he conjectured that the vacuum is in reality a collection of an infinite number of electrons with negative energy. This ingenious concept prevents the electron sitting "on top" of the sea to emit radiation by lowering its energy. Yet, there is no evidence of a cloud of charged particles everywhere around us, so, at first sight, this idea seems inconsistent. Dirac was of course aware of this problem and provided the following explanation:
Due to the Pauli principle, the electrons tend to arrange in such a way that the sea has an almost uniform density. This implies that the net force exerted on any particle should be zero on average. It is in analogy to e.g. gravity where the gravitational force in the center of earth vanishes.
So, in the reading of Dirac, his one particle Dirac equation emerges as an effective equation which originates from a multi particle fermionic system. By taking this model seriously, he predicted the existence of positrons, which were shortly after discovered experimentally by Anderson. Here, the positrons should be thought to be identical to the missing electrons in the sea. This approach is usually called "hole theory".

Dirac illustrated the mechanism which should prevent the sea to be detected in most experiments [1] , but he provided no further evidence to support this conjecture. So, even if we *would* know that his idea is not in agreement with physical reality, it would still be worthwhile to develop his thoughts further for the sake of a complete understanding of his theory. This knowledge would open up the possibility to compare the Dirac sea theory with some other, perhaps more advanced one.

Yet, we do not believe his ansatz to be ruled out by quantum field theory as some people claim. This takes us to the second motivation, which can be seen as a pursuit of Dirac's ideas, but this time physically motivated. Nowadays, Dirac's explanation is widely considered to be outdated and superseded by the field formalism. While physical textbooks usually "derive" QED by quantising some classical field, another option is to think of it as the generalisation of quantum theory with particle creation and annihilation. The latter option means that one starts with some quantum mechanical system (e.g. the Dirac Hamiltonian) and lifts it to Fock space, which is the infinite copy of N-particle Hilbert spaces. Afterwards, one can introduce interactions which will create or destroy particles. Ultimately, these two derivations result in the same theory, yet they are different regarding their physical motivation.

The first one takes a field to be the basic constituent of the theory. In this reading, particles should be thought as lumps or excitations of the field. The other ansatz, starting from Fock space, emphasizes the analogy to standard quantum theory. We will not try to argue here for one interpretation or the other, but want to make another observation for which the latter picture is more suited. As commonly known, field theories are plagued with divergences at every level of the theory. To better understand this peculiarity, one needs to analyse the underlying mathematical construction of the theory.

While commonly not stated, it is possible to formulate quantum field theories using the Dirac sea picture. In order to recognise this, remember the method one usually introduces field operators [2]: To render the system stable, the annihilation operator of a negative energy electron is reinterpreted as the creation operator of a positron. This implies that the two theories above are related by a simple Bogoliubov transformation. Moreover, the normal ordering scheme, which loosely speaking subtracts the "infinite constants", is also hinting at Dirac's picture. These "constants" are indeed a manifestation of the sea particles. Of course, these terms *are* problematic, and one may wonder how things could improve by simply renaming terms. We do not try to give an answer here, but refer to the mathematical analysis of external field QED. In [Deckert et al., 2010] it was proven that the time evolution in external field QED is describable by a system consisting of an infinite number of electrons. For this proof it is more convenient and natural to start from the Fock space construction. After identifying the electrons which belong to the Dirac sea, this formalism can be used to calculate e.g. cross sections or the $g-2$ correction.

Furthermore, the Dirac sea picture has gained recent interest, both in the physical and mathematical literature. In [Hainzl et al., 2007], the Dirac sea picture is used to prove a static mean-field theorem. Also [Finster, 2011] uses this formulation to describe the full time evolution of quantum fields, both perturbatively and exactly.

While we are indecisive if Dirac's idea may be ultimately correct, we do not believe that field

[1] Except, of course, in experiments where vacuum effects have to taken into account, as is the case for, e.g., pair creation or the Unruh effect. These effects may be thought of as arising from interaction of particles or fields with the sea.

[2] This discussion is for example found in [Schweber, 1961].

theories provide a more fundamental or different picture of nature, at least not at their current status. The ultimate goal should be a mathematically well defined theory with a reasonable physical interpretation. For this, it is important to understand possible solutions to cure the various problems field theories have. By investigating Dirac's proposal, it might be possible to gain further insight into these peculiarities.

So far, we considered a quite fundamental question. However, the question of this thesis is also important for many applications. Here, we will focus on the semiconducter. In order to describe the system efficiently, one considers excited electrons and holes only, neglecting the full interaction of all other electrons. This very simple picture may be used to explain the semiconductor in laymen's terms, but it is also employed in theoretical models. Considering a one particle equation, it is for example possible to model conductance properties by lattice impurities caused by the imperfect structure of atomic solids [3].
Explicitly, the emergence of a one particle equation is often presented like this:

> *If the electron-electron interaction is averaged, we can regard any deviation from this average as a small perturbation. Thus we replace the repulsion term as follows:*
>
> $$\sum_{i,j} \frac{e^2/4\pi\epsilon_0}{|\mathbf{r}_i - \mathbf{r}_j|} = H_{e0} + H_{ee}$$
>
> *where H_{e0} contributes a constant repulsive component to the electronic energy and H_{ee} is a fluctuating electron-electron interaction, which can be regarded as small. If H_{ee} is disregarded each electron reacts independently with the lattice of ions.* [Ridley, 2000, p.3-4]

One may wonder how a justification of such a tremendous simplification looks like and how one should understand the averaging of the interaction exactly. A more sophisticated analysis (performed for example in [Markowich et al., 1990]) will first perform this averaging explicitly. That is, the real interaction is replaced by its mean field, which is generated by all electrons. If this collective field turns out to be almost constant, the statement from above might be valid. As a matter of fact, we will also rely on this method for our analysis. Yet, it is important to notice in this context that it is not enough to replace the real interaction by its mean field, but one needs to justify it.

This leads us to the mathematical motivation. While physically strongly interacting fermions are usually treated using approximative methods, only few papers treat this problem on a mathematical rigorous basis. As we will argue, the *mathematical* justification of the mean-field method is important insofar that in principle an approximative treatment may break down.
Given the considerations above, we want to rephrase our question from the introduction:

Is it possible to derive a free evolution equation for a particle interacting with a sea of electrons?

In this thesis we were able to prove the following [4]

[3] This is done by some one particle random Schrödinger equation.
[4] throughout this thesis, we set $\hbar = 1$.

Theorem 1.0.1. *Let*

$$H = \sum_{k=0}^{N}(-\Delta_{x_k}) - \Delta_y + u(y) + \sum_{k=0}^{N}(v(x_k - y) + A(x_k)) \tag{1.1}$$

be the generator of the time evolution defined on the Hilbert Space $\mathcal{H} = L^2(dy;\mathbb{T}) \otimes \bigwedge_{k=0}^{N} L^2(dx^k;\mathbb{T})$ *of* $N+1$ *electrons and one tracer particle.* $\mathbb{T}$ *denotes the underlying one-dimensional space of a torus of length* L*. Let*

$$|\psi_0\rangle\rangle = |\chi_0\big) \otimes |\Lambda\rangle \tag{1.2}$$

be the inital wave function, where $|\Lambda\rangle \in \bigwedge_{k=0}^{N} L^2(dx^k;\mathbb{T})$ *denotes the fermionic ground state of the non-interacting theory i.e., the ground state of the free Schrödinger equation, without external fields.* $|\chi_0\big) \in L^2(dy;\mathbb{T})$ *is the initial wave function for the tracer particle. Let furthermore*

$$H^f = \sum_{k=0}^{N}(-\Delta_{x_k}) - \Delta_y + u(y) + constant \tag{1.3}$$

denote the effective time evolution. The constant will be determined later and is equal to the expectation value of the field strength exerted on the particles. Denote by $\lim_{TD}$ *the thermodynamic limit* $N \to \infty, L \to \infty; \rho := \frac{N+1}{L} = constant.$
Then, for smooth, compactly supported potentials, the following holds:

$$\lim_{\rho\to\infty} \lim_{TD} \left| \langle\langle U_t\psi_0 | U_t^f \psi_0\rangle\rangle \right|^2 = 1 \tag{1.4}$$

More explicit, there exists a constant B_t*, depending on* t*, such that*

$$1 - \lim_{TD} \left| \langle\langle U_t\psi_0 | U_t^f \psi_0\rangle\rangle \right|^2 \leq \frac{B_t}{\sqrt{\rho}} \tag{1.5}$$

The theorem states that, for high densities, the tracer particle moves freely, or said differently, that it behaves as if no fermionic sea was present. The free Hamiltonian H^f is the *mean-field* Hamiltonian, hence the theorem actually shows that the real interaction felt by the tracer particle is in very good approximation equal to the mean-field potential, which turns out to be constant. Note that this result, here stated for a one-dimensional, nonrelativistic system, is in accordance with the conjecture made by Dirac.

In this thesis we want to study the emergence of a mean-field equation using both mathematical rigour and physical insight. As it will turn out, the physical understanding of our system will also lead us to the proof of the theorem above.
We want to emphasize already at this stage that the analysis will rely on the rigidity of the fermionic sea. That is, the potential exerted on the tracer particle is so homogeneous that the tracer particle moves freely. For a physical understanding of the system, the reader is encouraged to skim over section 4.4 before start reading the detailed proof.
The thesis is structured as follows:

Chapter 2 In order to illustrate the mean-field idea, we will analyse the Hartree equation. We will focus especially on the heuristic justification of the mean-field treatment and identify the density fluctuations of the potential to be an important indicator of the validity of this method.

Chapter 3 This chapter will list a short summary of known results which are of interest for our thesis.

Chapter 4 This chapter contains the main part of this thesis. We will analyse the one-dimensional system mentioned in the introduction, calculate the density fluctuations and prove the theorem above. We will also consider an ultrarelativistic model and list the differences compared to the nonrelativistic case.

Chapter 5 In this chapter the density fluctuations for both the Schrödinger and the Dirac equation in three dimensions are calculated. We will emphasize the different structure of these two systems and compare our findings to a similar calculation performed in [Colin and Struyve, 2007].

Chapter 6 This chapter summarises the methods employed in the one and three-dimensional system. As it will turn out, our treatment can be generalised to arbitrary dimensions.

Chapter 7 The last chapter lists some topics which are important for further investigations.

Chapter 2

The Mean-field Idea

2.1 Mean field for bosonic systems

This thesis will focus on the mean-field idea, which was developed on physical grounds in order to deal with a huge collection of interacting particles. The main idea can be stated as follows: The interaction felt by each constituent of the system can be approximated by the expected field generated by all other particles. Since this idea is crucial, we will illustrate it using the well understood Hartree equation.
Consider the following quantum mechanical system which describes N interacting bosons:

$$\begin{cases} \mathrm{i}\frac{d}{dt}\psi(x_1,...,x_N,t) = \left(\sum_{k=1}^{N}(-\Delta_{x_k}) + \frac{1}{N}\sum_{i<j}^{N} v(x_i - x_j)\right)\psi(x_1,...,x_N,t) \\ \psi(x_1,...,x_N,0) = \prod_{k=1}^{N}\varphi(x_k) \end{cases} \tag{2.1}$$

The corresponding one particle mean-field equation, called Hartree equation, is given by:

$$\begin{cases} \mathrm{i}\frac{d}{dt}\phi(x,t) = \left(-\Delta_x + (v * |\phi(\cdot,t)|^2)(x)\right)\phi(x,t) \\ \phi(x,0) = \varphi(x) \end{cases} \tag{2.2}$$

It is known that, under some regularity assumptions on the potential $v(x)$ and on the initial state $\varphi(x)$, the one particle reduced density matrix $\gamma^{(1)}_{\psi_t} := \mathrm{tr}_{x_2,...,x_N}|\psi_t\rangle\langle\psi_t|$ converges to $|\phi_t\rangle\langle\phi_t|$, that is

$$\|\gamma^{(1)}_{\psi_t} - |\phi_t\rangle\langle\phi_t|\|_{\mathrm{tr}} \to 0 \tag{2.3}$$

as $N \to \infty$ (see e.g. [Knowles and Pickl, 2009] for a proof).
The convergence of the one particle marginal to the Hartree equation can be shown to be equivalent to the statement that almost all particles in the condensate evolve according to (2.2) [Knowles and Pickl, 2009]. Equation (2.3) demonstrates the validity of the following semiclassical model, which explains the structure of the Hartree equation physically:
In order to determine the evolution of any component $\varphi(x_i)$ of the wave function $\psi(x_1,...,x_N,0)$, we may consider φ to interact with $N-1$ classical particles which are distributed according to

$|\psi|^2$. This is in accordance with common physical intuition: Given a huge number of particles, the interaction felt by each constituent should be approximately equal to the potential which would arise if all the other particles were classical ones, distributed according to Born's law. Consequently, the potential felt by e.g. $\varphi(x_1)$ should be approximately equal to its classical counterpart, given by the expectation value

$$\mathbb{E}_{\rho(x_2,..,x_N)}\left(\frac{1}{N}\sum_{j=2}^{N} v(x_j - x_1)\right) = \frac{N-1}{N}(v * |\varphi|^2)(x_1) \approx (v * |\varphi|^2)(x_1) \tag{2.4}$$

where

$$\rho(x_2,..,x_N) = \prod_{k=2}^{N} |\varphi(x_k)|^2 = \int dx_1 |\psi(x_1,...,x_N,0)|^2 \tag{2.5}$$

denotes the joint probability measure of all remaining particles.
In order to construct an approximative time evolution, we may employ a self-consistent mean-field picture where the statistical distribution of the classical particles is given by $|\phi(x,t)|^2$. In return, this quantity is used to calculate the expected field strength on each component at any moment of time and therefore also determines the evolution of $\phi(x,t)$:

$$\underbrace{\mathrm{i}\frac{d}{dt}\phi(x,t)}_{\text{Evolution of each tensor component}} = \big(-\Delta_x + \underbrace{(v * |\phi(\cdot,t)|^2)(x)}_{\text{Expected potential at time t}} \big)\phi(x,t) \tag{2.6}$$

We know this semiclassical treatment to hold for a huge number of bosons in the sense of (2.3). This motivates the following question:

Why is it legitimate to replace the real interaction by its expected one, as it is done in (2.2) *?*

We may think of Brownian motion as an illuminating example where such a treatment fails. While the mean field, which in this case is just the average force exerted on the Brownian particle, is constant, the particle undergoes an erratic motion. The dynamical analysis, first performed by Einstein and Smoluchowski, reveals the density fluctuations to be responsible for this behaviour. So, at one moment of time, there might be more particles on the left, which cause the particle to move to the right. At some later time, it might get kicked back a bit since now there may be more particles coming form the right. This behaviour is far away from the conjectured mean-field dynamics where the net interaction is zero. This classical system demonstrates the importance to analyse the validity of a mean-field equation. Following this line of argumentation, we propose the variance of the potential to be a first indicator for the legitimacy of a mean-field equation in general. It would be in fact questionable to consider only the expected potential if the real one was fluctuating strongly. As a rule of thumb, we expect the mean-field model to hold true if fluctuations can be neglected.

In our system we can calculate these fluctuations explicitly:
Owing to the special initial condition (which can be relaxed, see [Knowles and Pickl, 2009]), the joint probability measure $\rho(x_2,..,x_N)$ is factorised, i.e. we have $N-1$ i.i.d. random

variables. We obtain[1] :

$$\mathrm{Var}_{\rho(x_2,..,x_N)}\left(\frac{1}{N}\sum_{j=2}^{N}v(x_j-x_1)\right)=\frac{1}{N^2}\mathrm{Var}_{\rho(x_2,..,x_N)}\left(\sum_{j=2}^{N}v(x_j-x_1)\right)$$
$$=\frac{1}{N^2}\sum_{j=2}^{N}\mathrm{var}_{|\varphi|^2}\left(v(x_j-x_1)\right)=\mathcal{O}\left(\frac{1}{N}\right) \tag{2.7}$$

where $\mathrm{var}_{|\varphi|^2}\left(v(x_j-x_1)\right)$ denotes the one particle variance. The mean deviation scales as $\frac{1}{\sqrt{N}}$ and is thus negligible. By the law of large numbers this result also holds for more general initial conditions. This analysis reveals two important observations:

Used scaling: In (2.1), the interaction was scaled such that it gets weaker as N increases. Our analysis indicates this to be necessary in order to obtain a meaningful mean-field equation. If one cannot neglect fluctuations, further evidence is needed why nevertheless the mean-field idea should work. As we will see, for a fermionic system the variance is always smaller than it would be for bosons or classical particles. This allows us to weaken the scaling substantially. Moreover, it is possible to resolve the fluctuations in more detail and to give an improved indicator for the validity of the mean-field picture.

Different scaling: It is commonly stated that the scaling can be determined on mathematical grounds, namely by the requirement that the potential of the full Hamiltonian has to be chosen of the same order as the kinetic energy. To this end consider the kinetic and the potential energy of each particle to be of order 1:

$$\sum_{k=1}^{N}(-\Delta_{x_k}) \approx N\mathcal{O}(1)=\mathcal{O}(N) \tag{2.8}$$

$$\frac{1}{N}\sum_{i<j=1}^{N}v(x_i-x_j) \approx \frac{1}{N}N^2\mathcal{O}(1)=\mathcal{O}(N) \tag{2.9}$$

Thus, for a bosonic system, the total energy scales as N, which means that each particle contributes equally to the energy. This is also reflected in the Hartree Hamiltonian which is of order 1. Yet, it should be apparent that the *physical* properties of the system dictate the scaling behaviour. A detailed analysis conducted in [Michelangeli, 2007] explains for example that the mean-field Gross-Pitaevskii equation, which arises from a different scaling than the one presented here, corresponds to a dilute gas of bosons. So, by carefully tuning the parameters in the experiment, one can *prepare* a physical system whose evolution is described by the corresponding scaled Hamiltonian. Yet, for our fermionic system, we seek a description which stays the same at the *microscopic* level. This is reflected in the Hamiltonian (1.1), where no scaling at all is used.

[1] See also section 4.6.

Chapter 3

A Short Summary of Known Results

As we have outlined in the introduction, the analysis of a system, governed for example by the Hamiltonian (1.1), is motivated by two physical systems, namely the Dirac equation and the semiconductor. While it is very easy to derive mean-field equations by simply replacing the real interaction by its expected one , the ultimate justification needs to be grounded on a mathematical proof. This is especially important for a system such as (1.1), where no scaling is introduced. Thus, the analysis we will perform will be mathematically rigorous.
Keeping this in mind, we will now have a short look on the existing literature and summarise the main results which are related to our discussion.

Mean-field Dynamics

The system we consider will neglect the mutual electron-electron interaction, but will only focus on the interaction of the tracer particle with the sea. A Hamiltonian similar to (2.1), but now defined to act upon antisymmetric states, would describe mutually interacting fermions and was first treated rigorously by [Bardos et al., 2002] (see also [Fröhlich and Knowles, 2011] for the case of the Coulomb-potential). Under the assumption that the potential scales as $\frac{1}{N}$, they were able to derive the so called Hartree-Fock equations, which are the analogue to the Hartree equation. In [Elgart et al., 2004] and [Benedikter et al., 2013] a different scaling was considered that is coupled to a semiclassical limit. Note that the methods employed in the proofs rely on the fact that the interaction is scaled with the number of particles. We want to emphasize once more that for our system, which only considers interactions with the tracer particle, no scaling is introduced. As a consequence, we cannot employ the methods which are presented in these works.
We further like to remark that in [Deckert et al., 2012] a similar model to ours was considered for bosons. There, a tracer particle interacts with a sea of bosons. As a matter of fact, the mass of the tracer particle needs to be scaled such that it gets heavier as N increases. This is a consequence of the discussion above, that is, for bosonic systems some physical parameter has to be scaled for the mean-field picture to hold. While the system considered there has some similarities to our model, the method of the proof is substantially different to the one presented in this work.

Fluctuations and the Dirac Sea

As we tried to motivate, one important indicator of the validity of mean-field dynamics is the statistical deviation of the expected potential from the real one. This indicator is closely related, but not equal to the density fluctuations of the mean-field constituents. Thus, in order to elaborate on Dirac's idea, one needs to calculate the density fluctuations which arise in the Dirac equation. While one often finds statements about "vacuum fluctuations", there are little concise calculations available in the literature. The most comprehensive survey was conducted by [Colin and Struyve, 2007].
In this work, Colin and Struyve calculate the density fluctuations of the fermionic vacuum using the standard model with UV-cutoff Λ. While their ultimate goal is related to ours, that is they also want to reformulate quantum field theories using the Dirac sea, their calculations were performed using the standard textbook quantum field formalism. Explicitly, they were able to estimate the density fluctuations of all fermionic constituents of the standard model to be given by $\mathrm{Var}_{\mathrm{particles}}(V) \approx \Lambda V^{1/3}$, where V is some spherical region in space. As $\Lambda \gg 1$, this quantity is much smaller than the expected variance for uncorrelated particles, which scales like $\Lambda^3 V$ [1]. Henceforth, this estimate states that fluctuations are strongly suppressed, which confirms Dirac's idea of an almost uniform charge distribution.
As we found out in our thesis, the authors use an implicit assumption in their model. By dropping this assumption (see chapter 5), the fluctuations turn out to be stronger.

Before analysing the Dirac equation, we will first investigate the one-dimensional system. At this point we want to thank Sören Petrat once more for making an unpublished paper available to us [Petrat, 2012]. In this work, while using a slightly different derivation than the one we will present later, he was able to derive the explicit formula for the fluctuations (see formula (4.30)) and to give the same asymptotic behaviour for the particle fluctuations (see equation (4.47)). Moreover, he comments on the validity of the thermodynamic limit and on the fluctuations of a bosonic system in detail. Since bosons are not our main concern here, this topic will only be sketched in section 4.6.
We like to remark in this context that the magnitude of density fluctuations are an important indicator of the validity of the mean-field idea, yet, in order to proof the theorem stated in the introduction, a more profound analysis will be required.

We have also commented on the fact that the Dirac sea picture is equivalent to normal QED. At least, in external field QED it can be proven rigorously that the S-matrix, derived from the standard formalism, is equivalent to the S-matrix one computes by means of the Dirac sea picture [Deckert et al., 2010]. It is important to note here that quantum field calculations are usually only interested in the S-matrix, so this theorem is quite strong. Since this theorem does not apply to mutually interacting fermions, it would be desirable to investigate such system in detail. Of course, this analysis is tremendously complicated for various reasons. Most importantly, one needs to consider the Dirac sea to consist of only finitely many electrons with negative energy since otherwise a meaningful analysis is probably not possible. While we employ this picture in our thesis, it is not known whether such a system is stable or how it behaves. We refer to [Deckert et al., 2010] and [Finster, 2011] for further discussion. In this context, we also want to mention [Hainzl et al., 2007], where the emergence of a self-consistent, static mean-field equation was proven, using the Dirac sea picture.

[1] This scaling will also be derived in this thesis, c.f. the discussion in section 5.3.

Semiconductor Physics

The semiconductor is such an important constituent of electronic devices, so the literature about it is tremendously big. Consequently, there are a couple of different ways how this solid state system is treated. One approach was already cited in the introduction. Without giving a detailed justification, one directly studies the one particle Schrödinger equation of a particle which only feels the potential of the nuclei (see e.g. [Ridley, 2000]). Such a treatment is ultimately justified by comparison with experiment, yet, it does not scrutinize on the physical explanation of its success. A more profound method begins with the full Hamiltonian and directly employs some mean field or related technique (see e.g. [Markowich et al., 1990]). As already mentioned, a mathematical rigorous proof usually requires the scaling of the interaction. Without this scaling, it is unclear if the mean-field description holds or if it breaks down as in the example of Brownian motion. So, starting from an unscaled system, the analysis presented may be physically correct in some cases, yet, a profound understanding is still missing.

There exists a different approach which can handle strongly correlated systems. Starting from some very specific Hamiltonian, it is sometimes possible to construct the solution explicitly. An example would be the well-known Luttinger model. We will give another simple example for such a procedure in section 4.9. A clear disadvantage of this method is that it works only with very specific Hamiltonians.

We like to remark that we found no conclusive, mathematical sound justification why it should be possible to neglect electron-electron interactions or to treat them perturbatively.

Chapter 4

The One-dimensional System

4.1 Notation

First, we will fix the notation we will use throughout this chapter. Most definitions will also be valid in three dimension upon small redefinitions which will be indicated. As mentioned in the introduction, we will consider N+1 nonrelativistic, spinless electrons interacting with a tracer particle on a one-dimensional torus $\mathbb{T} = [-L/2; L/2]$. We choose units where $\hbar = 1; m = 1/2$. The reader is encouraged to use this list as a reference, whenever the precise definition of the various norms we will use is needed.

$\mathcal{H}_{\text{sea}}^{N+1}$: The *sea Hilbert space* is defined as the space of all antisymmetric wave functions:

$$\mathcal{H}_{\text{sea}}^{N+1} := \bigwedge_{k=0}^{N} L^2(dx_k; \mathbb{T})$$

$|\Lambda\rangle$: A vector in $\mathcal{H}_{sea}^{N+1}$ will be denoted by $|\Lambda\rangle \in \mathcal{H}_{sea}^{N+1}$.
In some calculations we will write $\Lambda(x_0, ..., x_N)$ and treat $|\Lambda\rangle$ as an everywhere defined function of its $N+1$ variables.

$\|\cdot\|_\infty$: In this context, we will compute $\|\Lambda\|_\infty$, which is the infinity norm of all variables. In general, $\|\cdot\|_\infty$ will denote the supremum norm of a function.
In this thesis the L^2-norm on $\mathcal{H}_{sea}^{N+1}$ will never be used and hence we will not introduce it.

$|\varphi_m^k\rangle$: An orthonormal basis for each component $L^2(dx_k; \mathbb{T})$ is given by:

$$|\varphi_m^k\rangle = \frac{1}{\sqrt{L}} e^{\frac{2\pi \mathrm{i}}{L} m x_k} \quad m \in \mathbb{Z}\ , x_k \in \mathbb{T}$$

The superscript index indicates the position variable x_k, the lowerscript index m the quantized momentum. Note that we use the same ket notation for a vector in $L^2(dx_k; \mathbb{T})$ and in $\mathcal{H}_{\text{sea}}^{N+1}$. This should not cause any confusion since it will be obvious at any stage of the thesis to which Hilbert space the vector belongs.

$|\varphi_m\rangle$: If a distinction between different variables is not need, we will use solely $|\varphi_m\rangle$, as for example in the expression $\langle\varphi_l|v(x-y)|\varphi_m\rangle$.

$\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m)$: In lemma 4.3.1 the following operator, defined on $\mathcal{H}_{sea}^{N+1}$, will be introduced:

$$\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m) := \sum_{k=0}^{N} |\varphi_l^k\rangle\langle\varphi_m^k|$$

$\mathcal{H}_{\text{particle}}$: The *tracer particle Hilbert space* is defined as:

$$\mathcal{H}_{\text{particle}} := L^2(dy; \mathbb{T})$$

$|\chi)$ and $\|\cdot\|_y$: A vector in $\mathcal{H}_{\text{particle}}$ will be denoted by $|\chi)$; the $L^2(dy;\mathbb{T})$-norm by $\|\cdot\|_y$. In order to keep the notation handy, we may write $\|\chi\|_y$ instead of $\||\chi)\|_y$.

$\mathcal{H}$: The *full Hilbert Space* $\mathcal{H}$ is simply:

$$\mathcal{H} := \mathcal{H}_{\text{particle}} \otimes \mathcal{H}_{\text{sea}}^{N+1}$$

$|\psi\rangle\rangle$ and $\|\cdot\|$: A vector in $\mathcal{H}$ will be denoted by $|\psi\rangle\rangle = |\chi) \otimes |\Lambda\rangle \in \mathcal{H}$, its corresponding L^2-norm by $\|\cdot\|$. Again, we may write $\|\psi\|$ instead of $\||\psi\rangle\rangle\|$.

$\mathcal{F}(v)$: The *Fourier transform* of the potential $v \in C(\mathbb{T})$ is defined as:

$$\mathcal{F}(v)(k) := \int_{\mathbb{R}} dx\; v(x)e^{2\pi i kx} = \int_{-L/2}^{L/2} dx\; v(x)e^{2\pi i kx}$$

$\|\cdot\|_1$: In some bounds the L^1-norm of the potential is needed. This norm will be denoted by $\|v\|_1$.

ρ: The particle density ρ is defined as $\rho := \frac{N+1}{L}$.

$\lim_{\text{TD}}$: In order to disregard boundary effects, the *thermodynamic limit* will be performed. We will express this limit by $\lim_{\text{TD}}$. To this end, consider $f(N+1;L)$ to be a regular enough function. Then

$$\lim_{\text{TD}} f(N+1;L) := \lim_{\substack{N+1\to\infty \\ \rho=\text{const}}} f\left(N+1; \frac{N+1}{\rho}\right)$$

This will be a function of ρ, whenever this limit exists. In particular

$$\lim_{\text{TD}} \sum_{k=0}^{N} \frac{1}{L} g\left(\frac{k}{L}\right) = \int_0^\rho dk\; g(k)$$

holds for any Riemann integrable function g.

$\sim$: In our estimates we want to bound expressions by some function of ρ. Since ρ will be very big, we are only interested in bounding this expressions asymptotically in ρ, neglecting constants and lower order terms. For example we may write

$$\int_0^\rho (x + x^3)dx \sim \rho^4$$

The exact definition is the following:

$$f(\rho) \sim g(\rho) \Leftrightarrow \lim_{\rho\to\infty} \frac{g(\rho)}{f(\rho)} = \mathcal{O}(1)$$

$\gtrsim$ and $\lesssim$: In the same way, we can bound expressions asymptotically. For example, we may write $\frac{1}{\rho} \lesssim \frac{\ln(\rho)}{\rho}$. In general, the following holds:

$$f(\rho) \lesssim g(\rho) \Leftrightarrow \lim_{\rho\to\infty} \frac{f(\rho)}{g(\rho)} = \begin{cases} 0 & \text{or} \\ \mathcal{O}(1) \end{cases}$$

$$f(\rho) \gtrsim g(\rho) \Leftrightarrow \lim_{\rho\to\infty} \frac{g(\rho)}{f(\rho)} = \begin{cases} 0 & \text{or} \\ \mathcal{O}(1) \end{cases}$$

In the upper case, if $\lim\limits_{\rho\to\infty} \frac{f(\rho)}{g(\rho)} = 0$, $f(\rho)$ is strictly smaller than $g(\rho)$, as in the example above. The case $\lim\limits_{\rho\to\infty} \frac{f(\rho)}{g(\rho)} = \mathcal{O}(1)$ corresponds to $f(\rho) \sim g(\rho)$. Note that $f(\rho) \gtrsim g(\rho)$ and $f(\rho) \lesssim g(\rho)$ implies $f(\rho) \sim g(\rho)$.

C: The letter C in our estimates will denote a constant which will *always* be of order one. This constant will be unspecified and its exact value may change during the estimates.

$\Theta(t)$: The *Heavyside step function* will be denoted by $\Theta(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$.

$\delta^{\perp}_{l,m}$: We will introduce the *"anti Kronecker symbol"* $\delta^{\perp}_{l,m}$, which is defined as:

$$\delta^{\perp}_{l,m} := \begin{cases} 0 & \text{if } l = m \\ 1 & \text{if } l \neq m \end{cases}$$

$|\cdot|$: The absolute value of a function will be denoted by $|\cdot|$.

*: The complex conjugate will be denoted by *.

Further symbols will be introduced whenever they are needed during the computations.

4.2 Definition of the nonrelavistic, one-dimensional system

Now we can finally study the one-dimensional system. While for both the semiconductor and the Dirac sea the electrons are indistinguishable, we will consider a slightly modified system where the tracer particle has a distinguished role. We choose a simple Hamiltonian where the sea particles only interact with the tracer particle, while the sea particles don't interact among each other. This is a reasonable simplification, since the mutual interaction among sea particles would complicate the analysis tremendously. One should think of this system as follows: A charged particle (maybe a tau-lepton) moves in the sea of electrons which should be thought to be in some equilibrium state. While the structure of this state is not known in general, we assume that it is sufficient for our analysis to model this state using the ground state of non-interacting fermions. Whether mutual electron-electron interaction might change our result, remains of course an open and intersting question we would like to discuss later.

First, only interactions between the tracer particle and the sea will be considered. The theorem stated in the introduction, where in addition an external potential is present, then follows as a corollary. For the sake of simplicity, assume the interaction potential $v \in C^\infty(\mathbb{T}) \cap C_0^\infty(\mathbb{R})$ [1]. That is, for $v \in C^\infty(\mathbb{T})$, the *embedded* potential with support $\mathbb{T} \subset \mathbb{R}$ is in $C_0^\infty(\mathbb{R})$. The need for this additional requirement will be apparent later within in the proof. Thus, the full Hamiltonian is given by:

$$H = \sum_{k=0}^{N} (-\Delta_{x_k}) - \Delta_y + \sum_{k=0}^{N} v(x_k - y) \tag{4.1}$$

which is essentially self adjoint on $C^\infty(dy; \mathbb{T}) \otimes \bigwedge_{k=0}^{N} C^\infty(dx_k; \mathbb{T})$. First, we will study this nonrelativistic Hamiltonian in detail. Afterwards, in section 4.9 we will apply our results to an ultrarelativistic system and in chapter 5 to the Dirac equation.

The mean-field Hamiltonian is given by:

$$H^f = \sum_{k=0}^{N} (-\Delta_{x_k}) - \Delta_y + \rho \mathcal{F}(v)\,(0) \tag{4.2}$$

The derivation of the constant mean field $\rho \mathcal{F}(v)\,(0)$ will be performed in the next section. By standard Kato-Rellich, we conclude that $\mathcal{D}(H) = \mathcal{D}(H^f)$.

Let

$$\left|\Lambda\right\rangle := \left|\varphi_{-\frac{N}{2}} \wedge \dots \wedge \varphi_{\frac{N}{2}}\right\rangle \tag{4.3}$$

denote the fermionic ground state of the free Hamiltonian H^f.
In order to keep the notation simple, we will always assume that $N/2$ is an integer. Due to the zero mode φ_0, the total number of sea particles is $N + 1$. This explains the seemingly awkward choice we made. Within the calculations, this choice will simplify the notation.
As mean-field limits are only sensible for large systems, one should think of N and L as being

[1] Much less regular potentials would also work, yet we want to simplify the proof as much as possible.

very large and being determined by nature. Moreover, for both the semiconductor and the Dirac sea, the particle density ρ is also very large.

The initial state $|\psi_0\rangle\rangle$ is given by:

$$|\psi_0\rangle\rangle = |\chi_0\big) \otimes |\Lambda\rangle \tag{4.4}$$

where $|\chi_0\big) \in \mathcal{D}(-\Delta_y)$ is some arbitrary initial state for the tracer particle. As for the potential, we need to assume that this state is regular enough which means the following: The embedded state in $\mathbb{R}$ has finite kinetic energy. This excludes cases where the embedded function $|\chi_0\big)$ is not differentiable at the boundary [2]. While we choose a very specific initial fermionic state, the proof will actually hold for very general wave functions. We will comment on this in section 4.8.

We will need the following time evolutions:

$$|\psi_t\rangle\rangle := U_t|\psi_0\rangle\rangle := e^{-\mathrm{i}tH}|\psi_0\rangle\rangle \tag{4.5}$$
$$|\psi_t^f\rangle\rangle := U_t^f|\psi_0\rangle\rangle := e^{-\mathrm{i}tH^f}|\psi_0\rangle\rangle \tag{4.6}$$
$$|\chi_t\big) := U_t^y|\chi_0\big) := e^{-\mathrm{i}t(-\Delta_y)}|\chi_0\big) \tag{4.7}$$

4.3 Derivation of the mean-field Hamiltonian

In order to derive (4.2), we employ the method outlined in chapter 2, that is we replace the real interaction by its mean field. Starting from the static picture at $t = 0$, the joint probability density $\rho(x_0, ..., x_N, 0) = |\Lambda(x_0, ..., x_N)|^2$ determines the expected potential exerted on the tracer particle initially.
Abbreviate the real potential by

$$W(\vec{x}, y) := \sum_{k=0}^{N} v(x_k - y) \tag{4.8}$$

where $\vec{x} = (x_0, ..., x_N)$.
Using

$$\langle\varphi_l|v(x-y)|\varphi_m\rangle = \frac{1}{L}\mathcal{F}(v)\left(\frac{m-l}{L}\right) e^{\frac{2\pi \mathrm{i}}{L}(m-l)y} \tag{4.9}$$

the expected potential, denoted by $\mathbb{E}_{N+1}^v(y)$, which is exerted on the tracer particle at point y, reads:

$$\begin{aligned}\mathbb{E}_{N+1}^v(y) :=& \langle\Lambda|W(\vec{x}, y)|\Lambda\rangle = \sum_{k=0}^{N}\langle\Lambda|v(x_k - y)|\Lambda\rangle \\ =& \sum_{k=0}^{N}\langle\Lambda|\left(\sum_{l=-\infty}^{\infty}|\varphi_l^k\rangle\langle\varphi_l^k|v(x_k-y)\sum_{m=-\infty}^{\infty}|\varphi_m^k\rangle\langle\varphi_m^k|\right)|\Lambda\rangle\end{aligned}$$

[2]Since $\mathcal{D}(-\Delta_y) = H^2(\mathbb{T})$, by the Sobolev embedding theorem, the function $|\chi_0\big)$ is in $C^1(\mathbb{T})$.

$$
\begin{aligned}
&= \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{L} \mathcal{F}(v) \left(\frac{m-l}{L}\right) e^{\frac{2\pi \mathrm{i}}{L}(m-l)y} \langle \Lambda | \left(\sum_{k=0}^{N} |\varphi_l^k\rangle\langle\varphi_m^k| \right) |\Lambda\rangle \\
&=: \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{L} \mathcal{F}(v) \left(\frac{m-l}{L}\right) e^{\frac{2\pi \mathrm{i}}{L}(m-l)y} \langle \Lambda | \hat{a}^\dagger(\varphi_l) \hat{a}(\varphi_m) |\Lambda\rangle \qquad (4.10)
\end{aligned}
$$

We introduced a suggestive notation for the operator $\left(\sum_{k=0}^{N} |\varphi_l^k\rangle\langle\varphi_m^k|\right)$. As one might already conjecture, this operator replaces a particle in the state φ_m by φ_l. Note that we are *not* working on Fock-space. Thus, one should keep in mind that $\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m)$ has to be understood as *one* operator. Henceforth, at every stage of the proof, we will *always* consider the combination $\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m)$. Of course, we might have formulated everything in the second quantised formalism, but this is simply not *necessary.*
The following lemma proves the equivalence:

Lemma 4.3.1.

$$
\begin{aligned}
&\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m) : \mathcal{H}_{sea}^{N+1} \to \mathcal{H}_{sea}^{N+1} \\
&\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m) = \sum_{k=0}^{N} |\varphi_l^k\rangle\langle\varphi_m^k|
\end{aligned}
$$

is a bounded operator with operator norm 1 which replaces a particle in the state φ_m by φ_l.

Proof.
By linearity it suffices to check the proposition on basis vectors, i.e. on all Slater determinants. Moreover, for a vector which does not contain the state φ_m, the operator vanishes. We therefore need to show that the following expression vanishes:

$$
\left(\sum_{k=0}^{N} |\varphi_l^k\rangle\langle\varphi_m^k| \right) |\varphi_m \wedge \varphi_{i_1} \wedge ... \wedge \varphi_{i_N}\rangle - |\varphi_l \wedge \varphi_{i_1} \wedge ... \wedge \varphi_{i_N}\rangle \qquad (4.11)
$$

This expression vanishes iff

$$
\langle \Phi_a | \left(\left(\sum_{k=0}^{N} |\varphi_l^k\rangle\langle\varphi_m^k| \right) |\varphi_m \wedge \varphi_{i_1} \wedge ... \wedge \varphi_{i_N}\rangle - |\varphi_l \wedge \varphi_{i_1} \wedge ... \wedge \varphi_{i_N}\rangle \right) = 0
$$
$$
\forall \Phi_a,\ \{\Phi_a\} \text{ is an orthonormal basis of } \mathcal{H}_{\text{sea}}^{N+1} \qquad (4.12)
$$

Since $\sum_{k=0}^{N} |\varphi_l^k\rangle\langle\varphi_m^k|$ is symmetric under the exchange of particle labels, we get the following equality:

$$
\begin{aligned}
&\langle \Phi_a | \left(\sum_{k=0}^{N} |\varphi_l^k\rangle\langle\varphi_m^k| \right) |\varphi_m \wedge \varphi_{i_1} \wedge ... \wedge \varphi_{i_N}\rangle \\
=&\sqrt{(N+1)!} \langle \Phi_a | \left(\sum_{k=0}^{N} |\varphi_l^k\rangle\langle\varphi_m^k| \right) |\varphi_m^0\rangle \otimes |\varphi_{i_1}^1\rangle \otimes ... \otimes |\varphi_{i_N}^N\rangle \\
=&\sqrt{(N+1)!} \langle \Phi_a | \left(|\varphi_l^0\rangle \otimes |\varphi_{i_1}^1\rangle \otimes ... \otimes |\varphi_{i_N}^N\rangle \right) \\
=&\langle \Phi_a | \varphi_l \wedge \varphi_{i_1} \wedge ... \wedge \varphi_{i_N}\rangle \qquad (4.13)
\end{aligned}
$$

□

By this lemma, we conclue that

$$\langle\Lambda|\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m)|\Lambda\rangle = \delta_{l,m}\Theta(N/2-|l|) \tag{4.14}$$

holds, so the final result reads:

$$\begin{aligned}\mathbb{E}^v_{N+1}(y) &= \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{L}\mathcal{F}(v)\left(\frac{m-l}{L}\right)e^{\frac{2\pi\mathrm{i}}{L}(m-l)y}\delta_{l,m}\Theta(N/2-|l|)\\ &=\frac{N+1}{L}\mathcal{F}(v)\,(0)=\rho\mathcal{F}(v)\,(0) &(4.15)\\ &=\mathbb{E}_{N+1}(0)=:W^f &(4.16)\end{aligned}$$

Note that a factor of ρ appears in the expression of W^f. This expresses the fact that the tracer particle interacts with its ρ neighbours around it. In this context, we may think of $|\chi_0\big)$ to describe some well localised particle which interacts by means of the short range potential v. Moreover, as expected, the mean field grows with the number of particles and does not stay constant as in the example of the Hartree equation.
As the free evolution $|\psi^f_t\rangle\rangle$ does not change the fermionic state, but only introduces a global phase, $\rho(x_0,...,x_N,t)=\rho(x_0,...,x_N,0)=|\Lambda(x_0,...,x_N)|^2$ holds. The expected field exerted on the tracer particle is therefore constant for all y and all t and is exactly the constant we use in (4.2). Consequently, the mean-field dynamics of the tracer particle and the sea are decoupled.

Remember, however, that this derivation was conducted on heuristic grounds, so at this stage we do not have any insight if the conjectured Hamiltonian provides an adequate description of the interacting system. For a bosonic system, a simple calculation, which will be done later, shows the mean field to be constant, too. This result is simply a consequence of the homogeneity of the system. Yet, thinking of Brownian motion, we cannot expect the mean field to provide an adequate description there. This point will be further elaborated later.

We can choose

$$v(x-y)=\mathbb{1}_B(x-y)=\begin{cases}1 & \text{if } x-y\in B\\ 0 & \text{else}\end{cases} \tag{4.17}$$

to calculate the expected number of particles in any line segment B, whose length we also denote by B. Not very surprisingly it is given by:

$$\mathbb{E}_{\text{particle}}(B):=\frac{N+1}{L}\mathcal{F}(\mathbb{1}_B)\,(0)=\rho B \tag{4.18}$$

which reflects Dirac's idea of a uniform density distribution.

4.4 Heuristic discussion of the system

Before proving the theorem stated in the introduction, we want to discuss the method we are going to employ.
In theorem 4.6.1 it will be proven that the difference between the real and the mean-field time evolution can be estimated by the variance of the potential. Explicitly, it will be shown that

$$\|\psi_t - \psi_t^f\|^2 \leq \; t^2 \,\mathrm{Var}_{N+1}^v(0) \tag{4.19}$$

holds, where

$$\mathrm{Var}_{N+1}^v(y) := \langle \Lambda|(W(\vec{x};y) - W^f)^2|\Lambda\rangle \tag{4.20}$$

denotes the variance of the potential. As already discussed, this theorem states that density fluctuations are indeed some indicator of the validity of the mean-field description.
Furthermore, it is possible to give an explicit formula for $\mathrm{Var}_{N+1}^v(y)$. Namely, this expression is the sum of all possible transitions from any occupied state φ_m to any unoccupied state φ_l. This means:

$$\mathrm{Var}_{N+1}^v(0) = \sum_{|m|\leq N/2} \sum_{|l|>N/2} |\langle \varphi_l|v(x)|\varphi_m\rangle|^2 \tag{4.21}$$

A deviation from free motion will thus be caused by transitions of a sea particles to excited states by means of the potential.
Heuristically, this statement is obvious. If *all* sea particles remain in the ground state, then the tracer particle will move freely, too. If however one electron interacts with the tracer particles and gets excited, it will cause the tracer particle to loose energy. Hence, neither the sea, nor the tracer particle will move freely.
The variance sums up all these *possible* transitions. It will turn out that this sum stays bounded for all densities, that is

$$\sup_{\rho>0} \lim_{\mathrm{TD}} \mathrm{Var}_{N+1}^v(0) = \text{constant} \tag{4.22}$$

It should be noted in this context that both W^f and $W(\vec{x};y)$ are of the order ρ, which means:

$$\mathcal{O}(1) = \mathrm{Var}_{N+1}^v(0) \ll W^f = \mathcal{O}(\rho) \tag{4.23}$$

So, in contrast to i.i.d. random variables, in a fermionic system the variance of the potential is much smaller than its expectation. Owing to this result, the mean field is indeed a good approximation of the real potential and one might conjecture that it also provides an approximative time evolution. Note however, inequality (4.19) does not prove the validity of the mean-field picture. While the variance is strongly suppressed, the estimate given above does not yet show closeness of the two time evolutions.

For this proof, we need to employ a dynamical picture, which not only sums up all *possible* transitions $|\langle\varphi_l|v(x)|\varphi_m\rangle|^2$, but treats them also dynamically. This will be done in section 4.7.1 and 4.7.2.

Let us therefore analyse (4.21) in more detail. We may rewrite

$$\begin{aligned}\mathrm{Var}^v_{N+1}(0) &= \sum_{|m|\leq N/2}\sum_{|l|>N/2} |\langle\varphi_l|v(x)|\varphi_m\rangle|^2 \\ &= \sum_{|m|\leq N/2}\sum_{|l|>N/2} \frac{1}{L^2}\left|\mathcal{F}(v)\left(\frac{m-l}{L}\right)\right|^2 \end{aligned} \tag{4.24}$$

Since the potential v is smooth and has compact support, the Fourier transform $\mathcal{F}(v)\left(\frac{m-l}{L}\right)$ decays in $\frac{m-l}{L}$ faster than any polynomial. This means that only transitions where the momenta differ by $|m-l| \sim L$ give a significant contribution to the variance. Such transitions are only possible near the Fermi surface (which in one dimension consists of the two points $\pm N/2$), so only states where $N - L \lessapprox m \lessapprox N$ holds are of interest for our discussion.
This result is again in accordance with physical intuition. Only particles near the Fermi surface can get excited. All other particles are "stuck", since they would need to absorb a tremendous amount of energy to overcome the energy gap. This is a direct consequence of the Pauli principle.
Yet, the particles near the Fermi surface have energy $E_F \sim \frac{N^2}{L^2} = \rho^2$ and move around fast. As a consequence, any interaction with the tracer particle, which might excite a sea particle, only happens at very short time scales $t \approx \frac{1}{\rho}$. This results in the observation that any interaction which, in principle, might alter the free time evolution is so short that it can be neglected. Explicitly, we show that we can provide an improved bound using first-order perturbation theory.
Heuristically, the following expansion can be derived:

$$\begin{aligned}1 - |\langle\langle\psi_t|\psi_t^f\rangle\rangle|^2 \approx &\sum_{|m|\leq N/2}\sum_{|l|>N/2} \frac{L^2}{l^2-m^2}\frac{1}{L^2}\left|\mathcal{F}(v)\left(\frac{m-l}{L}\right)\right|^2 \\ &+\text{higher order terms}\end{aligned}$$

The quantity $\frac{L^2}{l^2-m^2}$ arises from the dynamics and is exactly the inverse of the energy difference between the unoccupied state φ_l and the occupied state φ_m. The precise estimate looks a bit more complicated, but has essentially the same structure. As

$$\frac{L^2}{l^2-m^2} = \frac{L^2}{(|l|-|m)(|l|+|m|)} \leq \frac{1}{\rho}\frac{2L}{|l|-|m|} \tag{4.25}$$

holds for $|l| > N/2$, the contribution arising in first-order perturbation theory vanishes.
Note that we have actually put the analysis upside-down: In order to show that the tracer particle moves freely, we actually prove that the sea remains unperturbed by the tracer particle. While the tracer particle has of course little influence on the whole sea, it might in principle interact with ρ sea particles around it and lift some of them, destroying the mean-field picture. We show that this behaviour is not realised.
In the proof we are able to estimate the closeness of the two time evolved vectors, one describing the real evolution, the other one arising from the mean-field dynamics, in L^2 sense. Usually, this convergence is much stronger than the convergence of reduced density matrices in trace norm, as it us usually done in the literature. Yet, for our system, these two, while mathematically not equal, should be thought to be equally valid. If, for example, we would

know that only the tracer particle moves freely, but not the sea, we would conclude that the sea is in some excited state. Explicitly, this means that some of the sea particles are excited and have more kinetic energy. Using energy conservation *and* the fact that the tracer particle moves freely, the sea particles must arrange such that they have more kinetic energy but less potential energy without interfering with the tracer particle. Such a behaviour would indeed be strange. This explanation might be useful to understand why we are able to prove a mean-field limit in norm.

What remains is to bound the terms which arise in higher order perturbation theory. As we will see, we can estimate them by considering the contribution which arises in second-order perturbation theory only. This quantity vanishes by the same line of argumentation.
Remember, in second-order perturbation theory, the process happened at first-order now happens twice.
Thus, at first order, a sea particle below the Fermi surface gets excited. Afterwards, either the already excited or another particle interacts with the tracer particle again. In analogy to (4.21), we will show that the sum of all these possible transitions stays bounded. Using the dynamical estimate for the first transition, it will be possible to get the same bound obtained in first-order perturbation theory.
We like to remark that this strategy works only in one dimension and only for a Hamiltonian with a nonrelativistic energy-momentum relation. We will comment on this further in section 4.9 and in chapter 6.

4.5 Structure of the proof

We will prove the following

Theorem 4.5.1. *For the system described above the following estimate holds:*

$$\lim_{TD}\left(1-\left|\langle\langle\psi_t^f|\psi_t\rangle\rangle\right|^2\right)\leq\frac{B_t}{\sqrt{\rho}} \tag{4.26}$$

where B_t depends only on the potential v, on $|\chi_0)$ and grows at most like t^2.

The proof will be structured as follows:

1. In the next section, we will calculate the variance and prove inequality (4.19). Moreover, this section contains some estimates on the particle fluctuations of the sea and a short discussion of the bosonic case. These last two topics are included for a comprehensive understanding of the underlying picture but will not be needed for the proof.
2. The explicit formula derived for the variance will be used to proceed in first-order perturbation theory. This will be done in section 4.7.1.
3. Finally, section 4.7.2 estimates the remaining terms coming from higher order perturbation theory.

The theorem stated above then follows from the estimates given in 4.7.1 and 4.7.2.
In section 4.8 we list some corollaries to the proof.

4.6 The variance of a fermionic system

As already mentioned, in order to analyse the dynamics of the system, we cannot employ the powerful methods usually employed when proving mean-field limits. These rely on the fact

that one needs to weaken the interaction as the number of particles increases. Consequently, we cannot circumvent the detailed analysis of the fully interacting system. As one knows, the rigorous analysis of a many body problem can be very hard. Yet, experience tells us that perturbation theory might give us an accurate picture and might hint at how one should think about the system. By using Cook's lemma, the main factors which determine the evolution of the coupled dynamical system can be spotted. As one might guess, the variance of the potential will serve as a first indication if the replacement of the potential by its mean field is justified.
For sake of completeness let us phrase

Lemma 4.6.1. *(Cook)*
Let $(H;\mathbb{D}(H))$ and $(H';\mathbb{D}(H'))$ be two self adjoint Hamiltonians with $\mathbb{D}(H) = \mathbb{D}(H')$. Let $U_t = e^{-\mathrm{i}tH}$ and $U'_t = e^{-\mathrm{i}tH'}$. Then, $\forall\psi \in \mathbb{D}(H)$ the following is true:

$$(U_t - U'_t)\psi = -\mathrm{i}\int_0^t dt' U_{t-t'}(H - H')U'_{t'}\psi \tag{4.27}$$

Proof.
Formally,

$$-\mathrm{i}\int_0^t dt' U_{t-t'}(H - H')U'_{t'} = -\int_0^t dt' \frac{d}{dt'}(U_{t-t'}U'_{t'}) = -U'_t + U_t \tag{4.28}$$

holds. Note that $U'_t\psi \in \mathbb{D}(H)$, so we can use strong differentiability of $U_{t-t'}$ and U'_t on $\mathbb{D}(H)$ to conclude (4.28). □

Remark: As already mentioned, within this thesis we shall only consider compactly supported, smooth potentials. Hence, by Kato- Rellich, $\mathbb{D}(H) = \mathbb{D}(H^f)$ holds. Of course, it might be very well possible to consider less regular and even singular potentials, but this generalisation will not be our concern here.

We will apply Cook's method to obtain the following

Theorem 4.6.1. *Let $v \in C^\infty(\mathbb{T}) \cap C_0^\infty(\mathbb{R})$. Then the following bound holds:*

$$\|\psi_t - \psi_t^f\|^2 \leq \; t^2 \; Var^v_{N+1}(0) \tag{4.29}$$

where

$$\begin{aligned} Var^v_{N+1}(y) =& Var^v_{N+1}(0) := \langle\Lambda|(W(\vec{x};0) - W^f)^2|\Lambda\rangle \\ =& \sum_{|m|\leq N/2}\sum_{|l|>N/2} |\langle\varphi_l|v(x)|\varphi_m\rangle|^2 \end{aligned} \tag{4.30}$$

does not depend on y and satisfies:

$$\lim_{\rho\to\infty}\lim_{TD} Var^v_{N+1}(y) =: \; Var^v_\infty(y) < \infty \tag{4.31}$$

So, the quantum fluctuations of the potential are finite and do not increase with higher densities.

Corollary 4.6.1. *Denote by* $v_{\rho,\delta}(x_k - y) := \frac{1}{\rho^\delta} v(x_k - y)$ *the scaled two particle interaction. Then ,* $\forall \delta > 0$*, the following equality holds:*

$$\lim_{\rho\to\infty} \lim_{TD} \|\psi_t - \psi_t^f\|^2 = 0 \ \forall t \tag{4.32}$$

Proof.
Use the theorem above with $\mathrm{Var}^v_{N+1}(y) \to \frac{1}{\rho^{2\delta}} \mathrm{Var}^v_{N+1}(y)$. □

While this result already gives a good hint about the validity of the mean-field Hamiltonian, the *physically* relevant microscopic description of the system is given for $\delta = 0$ and hence requires more effort.

Proof. of (4.6.1):
Let us contemplate a bit about this theorem and understand what it actually states:
Actually two statements are made: The first one is that the variance of the potential is indeed some indicator of the validity of the mean-field description. Secondly, these fluctuations are highly suppressed, i.e. stay asymptotically constant at high densities. That is, the Pauli exclusion principle yields to an effective repulsion of the electrons (not mediated by some potential), which arranges the particles in a homogeneous way. A small calculation we will do afterwards shows that density fluctuations behave classically when looking at bosons, i.e. the variance there is of order ρ.

1.Proof of (4.29)**:** Using Cook's lemma, the following estimate holds:

$$\begin{aligned}\|(U_t - U_t^f)\psi_0\| &= \left\| \int_0^t dt' U_{t-t'}(H - H^f) U_{t'}^f \psi_0 \right\| \\ &= \left\| \int_0^t dt' U_{-t'}(W(\vec{x}, y) - W^f) U_{t'}^f \psi_0 \right\| \\ &\le \int_0^t dt' \left\| (W(\vec{x}, y) - W^f) U_{t'}^f \psi_0 \right\| \end{aligned} \tag{4.33}$$

Note that the free evolution $U_{t'}^f$ acts as an multiplication operator on $|\Lambda\rangle$, i.e.

$$U_{t'}^f |\psi_0\rangle\rangle = \Phi_{t'} |U_{t'}^y \chi_0\big) \otimes |\Lambda\rangle \tag{4.34}$$

where

$$\Phi_{t'} := e^{-\mathrm{i}t' \left(\sum\limits_{|k| \le N/2} \left(2\pi \frac{k}{L}\right)^2 + \rho \mathcal{F}(v)(0) \right)} \tag{4.35}$$

This allows us to compute the integrand explicitly:

$$||(W(\vec{x}, y) - W^f) U_{t'}^f \psi_0||^2 = \big(\chi_{t'}| \otimes \langle\Lambda|(W(\vec{x}, y) - W^f)^2 |\chi_{t'}\big) \otimes |\Lambda\rangle \tag{4.36}$$

We will integrate over the sea components first. The result we will obtain will not depend further on y, which shows that

$$\big(\chi_{t'}| \otimes \langle\Lambda|(W(\vec{x}, y) - W^f)^2 |\chi_{t'}\big) \otimes |\Lambda\rangle = \langle\Lambda|(W(\vec{x}, y) - W^f)^2|\Lambda\rangle = \mathrm{Var}^v_{N+1}(0) \tag{4.37}$$

Let us first rewrite $W(\vec{x},y)-W^f$ using the same techniques as above:

$$\begin{aligned} W(\vec{x},y)-W^f =& \sum_{k=0}^{N}\left(\sum_{l=-\infty}^{\infty}|\varphi_l^k\rangle\langle\varphi_l^k|\left(v(x_k-y)-\frac{1}{L}\mathcal{F}(v)\,(0)\right)\sum_{m=-\infty}^{\infty}|\varphi_m^k\rangle\langle\varphi_m^k|\right)\\ =& \left(\sum_{l=-\infty}^{\infty}\sum_{\substack{m=-\infty\\ m\neq l}}^{\infty}\frac{1}{L}\mathcal{F}(v)\left(\frac{m-l}{L}\right)e^{\frac{2\pi i}{L}(m-l)y}\left(\sum_{k=0}^{N}|\varphi_l^k\rangle\langle\varphi_m^k|\right)\right)\\ =& \sum_{l=-\infty}^{\infty}\sum_{\substack{m=-\infty\\ m\neq l}}^{\infty}\frac{1}{L}\mathcal{F}(v)\left(\frac{m-l}{L}\right)e^{\frac{2\pi i}{L}(m-l)y}\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m) \end{aligned} \tag{4.38}$$

Keeping the fermionic structure in mind, $(W(\vec{x},y)-W^f)|\Lambda\rangle$ gives a vector where *any* particle from the ground state is lifted to *any* excited state.

$$(W(\vec{x},y)-W^f)|\Lambda\rangle = \sum_{|m|\leq N/2}\sum_{|l|>N/2}\frac{1}{L}\mathcal{F}(v)\left(\frac{m-l}{L}\right)e^{\frac{2\pi i}{L}(m-l)y}\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m)|\Lambda\rangle \tag{4.39}$$

Taking the squared norm of this expression, two of the appearing sums will be cancelled since

$$\langle\hat{a}^\dagger(\varphi_{l_1})\hat{a}(\varphi_{m_1})\Lambda|\hat{a}^\dagger(\varphi_{l_2})\hat{a}(\varphi_{m_2})\Lambda\rangle = \delta_{l_1,l_2}\delta_{m_1,m_2} \tag{4.40}$$

holds, whenever $|m_1|,|m_2|\leq N/2$, $|l_1|,|l_2|>N/2$.
We plug the explicit formula $(W(\vec{x},y)-W^f)|\Lambda\rangle$ in the expression for the variance and obtain:

$$\begin{aligned} \mathrm{Var}^v_{N+1}(y) =& \langle\Lambda|(W(\vec{x},y)-W^f)(W(\vec{x},y)-W^f)|\Lambda\rangle\\ =& \sum_{|m|\leq N/2}\sum_{|l|>N/2}\frac{1}{L^2}\left|\mathcal{F}(v)\left(\frac{m-l}{L}\right)\right|^2\\ =& \sum_{|m|\leq N/2}\sum_{|l|>N/2}|\langle\varphi_l|v(x)|\varphi_m\rangle|^2\\ =& \mathrm{Var}^v_{N+1}(0) \end{aligned} \tag{4.41}$$

Consequently, the variance of the potential is equal to the transition amplitude from any occupied to any unoccupied state and furthermore does not depend on y.
In total, we obtain the desired bound:

$$||(U_t-U_t^f)\psi_0|| \leq \int_0^t dt'\sqrt{\mathrm{Var}^v_{N+1}(0)} = t\sqrt{\mathrm{Var}^v_{N+1}(0)} \tag{4.42}$$

2. Proof of (4.31)**:** Next, we want to estimate (4.30) for regular enough potentials. As already mentioned, we do not seek the weakest regularity condition but choose for simplicity $v\in C^\infty(\mathbb{T})\cap C_0^\infty(\mathbb{R})$. The intersection is necessary to exclude cases where the potential is chosen to be smooth for one specific L' (just take $v=$const. in L' as one example), but fails to be even differentiable for any $L>L'$. Stated differently, the support of the smooth potential should be smoothly embedded in any L containing its support . Our estimates will rely on the Paley-Wiener theorem (see [Reed and Simon, 1975, p.16]) which states:

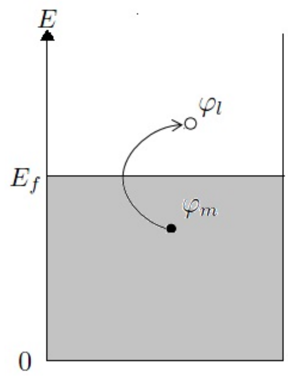

Figure 4.1: The variance sums up all transitions from φ_m to φ_l.

Theorem 4.6.2. *(Paley-Wiener)*
An entire analytic function $g(\vec{k})$ of n complex variables $\vec{k} = (k_1, ..., k_n)$ is the Fourier transform of a $C_0^\infty(\mathbb{R}^n)$ function with support in the ball $\{\vec{x} = (x_1, ..., x_n) | |\vec{x}| \le R\}$ if and only if for each $p \ge 0$ there is a D_p (depending on g), so that

$$\left|g(\vec{k})\right| \le \frac{D_p e^{R|\operatorname{Im}\vec{k}|}}{(1+|\vec{k}|)^p} \tag{4.43}$$

We will use this theorem extensively in our work and we will always assume p big enough, such that all estimates considered in this work are optimal.
Thereupon, the Fourier transform of our potential v has rapid decrease, which can be used to conclude that the transition amplitude between states which differ significantly in their energies is highly suppressed.
In order to estimate (4.30), we will first perform the thermodynamic limit. Upon taking this limit, all Riemann sums will be replaced by the corresponding integrals. We like to emphasize that we will use this limit in order to simplify our analysis. Technically, it is possible to fix N and L at every stage of our proof with the clear disadvantage that the estimates will read more complicated.
Heuristically, it should be clear that for fixed and sufficiently big values of N, L and ρ, the estimates provided here are also valid. A more profound (and probably cumbersome) analysis would be required to prove this explicitly. It was for example shown in [Petrat, 2012] that in the case of $v(x) = \mathbb{1}_B$ (we will consider this case soon), the error one makes upon switching to the thermodynamic limit is, for both N, L large, proportional to $\frac{N}{L^2}$ and is thus negligible for high densities.
So, taking this limit, we obtain:

$$\begin{aligned}\lim_{\text{TD}} \operatorname{Var}^v_{N+1}(0) =& \int_{|m|\le\rho/2} dm \int_{|l|\ge\rho/2} dl \, |\mathcal{F}(v)\,(m-l)|^2 \\ =& 2\int_0^{\rho/2} dm \int_{\rho/2}^\infty dl \, |\mathcal{F}(v)\,(m-l)|^2 + 2\int_{-\rho/2}^0 dm \int_{\rho/2}^\infty dl \, |\mathcal{F}(v)\,(m-l)|^2\end{aligned} \tag{4.44}$$

Choosing $p \geq 2$ and $C = \mathcal{O}(1)$, the first contribution can be estimated as follows:

$$\begin{aligned}\int_0^{\rho/2} dm \int_{\rho/2}^{\infty} dl \, |\mathcal{F}(v)\,(m-l)|^2 \leq & \int_0^{\rho/2} dm \int_{\rho/2}^{\infty} dl \frac{D_p^2}{(1+|m-l|)^{2p}} \\ = & \int_0^{\rho/2} dm \int_m^{\infty} dl \frac{D_p^2}{(1+l)^{2p}} \\ = & \int_0^{C} dm \int_m^{\infty} dl \frac{D_p^2}{(1+l)^{2p}} + \int_C^{\rho/2} dm \int_m^{\infty} dl \frac{D_p^2}{(1+l)^{2p}} \\ \leq & C \int_0^{\infty} dl \frac{D_p^2}{(1+l)^{2p}} + \frac{1}{2p-1} \int_C^{\infty} dm \frac{D_p^2}{m^{2p-1}} < \infty \end{aligned} \tag{4.45}$$

By the same means, the second term can be shown to be bounded. Due to the separation of m and l by at least $\rho/2$, it will eventually go to zero as $\rho \to \infty$.

$$\begin{aligned}& \int_{-\rho/2}^{0} dm \int_{\rho/2}^{\infty} dl \, |\mathcal{F}(v)\,(m-l)|^2 \\ = & \int_{-\rho/2}^{0} dm \int_{\rho/2}^{\infty} dl \, |\Theta\left(|m-l| - \rho/2\right) \mathcal{F}(v)\,(m-l)| \, |\mathcal{F}(v)\,(m-l)| \\ \leq & \frac{D_p}{(1+\rho/2)^p} \underbrace{\int_{-\rho/2}^{0} dm \int_{\rho/2}^{\infty} dl \frac{D_p}{(1+|m-l|)^p}}_{\text{finite for all } \rho} \xrightarrow{\rho\to\infty} 0 \end{aligned} \tag{4.46}$$

□

Of course, the theorem just proven does not state that our mean-field picture is correct. In order to prove this, we need also consider the dynamical evolution of the system. We can nevertheless conclude that the deviation from free evolution can only happen if some state φ_m of the sea is lifted to an excited state φ_l by interacting with the tracer particle. This transition amplitude is proportional to $\left|\mathcal{F}(v)\left(\frac{m-l}{L}\right)\right|^2$, which essentially means that only transitions near the Fermi edge need to be considered. As we have already discussed, these states are expected to interact only weakly with the tracer particle. In order to illustrate this behaviour, consider the interaction of some one-dimensional wave packet with some bounded potential, such as the square well. If the kinetic energy of this particle is very high, the particle moves approximately freely and almost all of the wave gets transmitted. We can expect the very same behaviour to be true in our system, that is, the particles near the Fermi surface do not interact with the tracer particle significantly, but pass it without mutual transfer of energy. On the other hand, the particles with slow momentum are "stuck" due to the Pauli principle, i.e. they cannot be excited. Exactly this behaviour will be apparent in first-order perturbation theory.

4.6.0.1 Density fluctuations of the sea

Before we present the proof, let us pause one moment and determine the particle fluctuations in our model and compare them to a bosonic system. The estimates given here will not be needed further. The reader who is mainly interested in the proof can therefore skip this section. As already mentioned, this calculation was essentially performed in [Petrat, 2012]. We will give a different presentation here which takes the thermodynamic limit first. By doing this, we cannot control the error we make upon switching to this limit, but, as already mentioned, this error was shown to be very small for high densities.
While we can calculate the density fluctuations using (4.44) and $v(x-y) = \mathbb{1}_B(x-y)$, this time we have to perform the integral explicitly. As it will turn out, the non-differentiability of $\mathbb{1}_B(x-y)$ changes the result slightly, as its Fourier transform does not decay rapidly enough.

Theorem 4.6.3. *For $v(x-y) = \mathbb{1}_B(x-y)$ the variance is asymptotically given by*

$$\lim_{TD} Var^{\mathbb{1}_B}_{N+1}(0) \sim \frac{1}{\pi^2} \ln(\pi B\rho/2) \tag{4.47}$$

that is

$$\lim_{\rho\to\infty} \frac{\pi^2 \lim_{TD} Var^{\mathbb{1}_B}_{N+1}(0)}{\ln(\pi B\rho/2)} = 1 \tag{4.48}$$

Proof.
We have already seen that for fermions fluctuations of the potential are suppressed. The theorem shows this to hold also for the particle density fluctuations. The difference between this result and the constant result in the case of smooth potentials may be explained as follows: Using the semiclassical picture once again, the variance of the potential is caused by density fluctuations of the classical particles which are distributed according to Born's law. While small fluctuations of the constituents do not change the collective potential significantly, the particles at the edge of B may in principle leave or enter the line segment under consideration causing the number of particles inside B to change. Therefore, the variance of the particle number is in general bigger than the variance of the potential.
For the proof we will only consider the first contribution in (4.44). The other one can again be neglected for high densities.

$$\begin{aligned}
\lim_{\text{TD}} \text{Var}^{\mathbb{1}_B}_{N+1}(0) \sim& 2\int_0^{\rho/2} dm \int_m^\infty dl\, |\mathcal{F}(\mathbb{1}_B)(l)|^2 \\
=& 2\int_0^{\rho/2} dm B^2 \int_m^\infty dl \frac{\sin^2(\pi Bl)}{(\pi Bl)^2} \\
=& \frac{2B}{\pi}\int_0^{\rho/2} dm \int_{m\pi B}^\infty dl \frac{\sin^2(l)}{l^2} \\
=& \frac{2B}{\pi}\int_0^{\rho/2} dm \left(\frac{d}{dm} m\right) \int_{m\pi B}^\infty dl \frac{\sin^2(l)}{l^2} \\
=& \rho\frac{B}{\pi}\int_{\rho\pi B/2}^\infty dl \frac{\sin^2(l)}{l^2} + 2B^2 \int_0^{\rho/2} dm\, m \frac{\sin^2(\pi Bm)}{(\pi Bm)^2} \\
\leq& \rho\frac{B}{\pi}\int_{\rho\pi B/2}^\infty dl \frac{1}{l^2} + \frac{2B}{\pi}\int_0^{\rho/2} dm \pi Bm \frac{\sin^2(\pi Bm)}{(\pi Bm)^2}
\end{aligned}$$

$$=2+\frac{2B}{\pi}\int_{\frac{C}{B\pi}}^{\rho/2} dm\ \pi Bm\frac{\sin^2(\pi Bm)}{(\pi Bm)^2}+\underbrace{\frac{2B}{\pi}\int_0^{\frac{C}{B\pi}} dm\ \pi Bm\frac{\sin^2(\pi Bm)}{(\pi Bm)^2}}_{\text{independent of }\rho} \tag{4.49}$$

The leading contribution is clearly the second term, which behaves roughly as $1/m$, so we expect the integral to grow logarithmically with ρ. For further convenience, the function was cut out around zero. Again, $\frac{C}{B\pi}$ should be thought to be of order one.

$$\begin{aligned}
&\frac{2B}{\pi}\int_{\frac{C}{B\pi}}^{\rho/2} dm\ \pi Bm\frac{\sin^2(\pi Bm)}{(\pi Bm)^2}\\
=&\frac{2}{\pi^2}\int_C^{\pi B\rho/2} dm\frac{\sin^2(m)}{m}\\
=&\frac{2}{\pi^2}\int_C^{\pi B\rho/2} dm\frac{\sin^2(m)-1/2}{m}+\frac{1}{\pi^2}\int_C^{\pi B\rho/2} dm\frac{1}{m}\\
=&\frac{1}{\pi^2}\ln(\pi B\rho/2)+\underbrace{\frac{2}{\pi^2}\int_C^{\pi B\rho/2} dm\frac{\sin^2(m)-1/2}{m}-\frac{\ln(C)}{\pi^2}}_{\text{bounded for all }\rho>0}
\end{aligned} \tag{4.50}$$

The variance is therefore asymptotically given by:

$$\lim_{\text{TD}} \text{Var}_{N+1}^{\mathbb{1}_B}(0) \sim \frac{1}{\pi^2}\ln(\pi B\rho/2) \tag{4.51}$$

It remains to show the boundedness of the remaining term in (4.50). The basic idea is that for high values of m the function $1/m$ only changes slightly. Hence, we expect that

$$\int_{x_1}^{x_2} dm \sin^2(m)\frac{1}{m} \approx \frac{1}{2}\int_{x_1}^{x_2} dm\frac{1}{m} \tag{4.52}$$

holds for $x_1, x_2 \gg 1$. The precise version is stated in the next

Lemma 4.6.2. *For $C>1$ the following estimate holds:*

$$\frac{2}{\pi^2}\int_C^{\pi B\rho/2} dm\frac{\sin^2(m)-1/2}{m} \leq \frac{\gamma}{\pi^2} \tag{4.53}$$

where γ is the Euler-Mascheroni constant.

Proof.
Without loss of generality choose C such that $C + s\pi = \pi B\rho/2$ for some $s \in \mathbb{N}$. Then

$$\begin{aligned}
\int_C^{\pi B\rho/2} dm \frac{\sin^2(m) - 1/2}{m} =& \sum_{k=0}^{s-1} \int_{C+k\pi}^{C+(k+1)\pi} dm \left(\frac{\sin^2(m)}{m} - \frac{1}{2m} \right) \\
\leq& \frac{1}{2} \sum_{k=0}^{s-1} \left(\frac{\pi}{C + k\pi} - \int_{C+k\pi}^{C+(k+1)\pi} dm \frac{1}{m} \right) \\
=& \frac{1}{2} \sum_{k=0}^{s-1} \int_{C+k\pi}^{C+(k+1)\pi} dm \left(\frac{1}{\lfloor m \rfloor} - \frac{1}{m} \right) \\
\leq& \frac{1}{2} \int_1^{\infty} dm \left(\frac{1}{\lfloor m \rfloor} - \frac{1}{m} \right) = \frac{1}{2}\gamma \approx 0.577/2 \qquad (4.54)
\end{aligned}$$

holds, where $\lfloor m \rfloor$ denotes the floor function. □

□

This result tells us that within a region B we are almost sure to find ρB particles. The deviation scales as $\ln(\rho)$ and is thus much smaller than its expectation. Again, we reproduce the same picture discussed above: The particle density we expect in a fermionic system is almost constant. Small density fluctuations are caused by fast particles at the Fermi surface, which in return do not interact with the tracer particle, but just fly trough.

We like to remark that one should not think of the wave function $|\Lambda\rangle$ itself, defined on configuration space [3] $\mathcal{L} \subset \mathbb{R}^{N+1}$, to be constant in some sense. Instead, it fluctuates rapidly as the following lemma illustrates.

Lemma 4.6.3. *For $|\Lambda\rangle$ the following holds:*

$$\|\Lambda\|_{L^\infty} = \left(\frac{(N+1)^{N+1}}{L^N (N+1)!} \right)^{1/2} \qquad (4.55)$$

The value is attained for $x_0 = 0, x_1 = \frac{L}{N+1}, x_2 = \frac{2L}{N+1}, ..., x_N = \frac{NL}{N+1}$ and its permutations.

Proof.
By the homogeneity of the norm and the structure of $|\Lambda\rangle$, it suffices to prove the lemma for $L = 1$. We can rewrite $|\Lambda\rangle$ as a determinant of its one particle wave functions $|\varphi_m^k\rangle$:

$$\Lambda(x_0, ..., x_N) = \frac{1}{\sqrt{(N+1)!}} \begin{vmatrix} \varphi_{-N/2}(x_0) & ... & \varphi_{N/2}(x_0) \\ ... & ... & ... \\ \varphi_{-N/2}(x_N) & ... & \varphi_{N/2}(x_N) \end{vmatrix} \qquad (4.56)$$

By Hadamard's lemma we can bound $|\Lambda\rangle$ pointwise as:

$$\begin{aligned}
|\Lambda(x_0, ..., x_N)| \leq& \frac{1}{\sqrt{(N+1)!}} \prod_{l=0}^{N} \left(\sum_{m=-N/2}^{N/2} |\varphi_m(x_l)|^2 \right)^{1/2} \\
=& \frac{1}{\sqrt{(N+1)!}} (N+1)^{\frac{N+1}{2}} \qquad (4.57)
\end{aligned}$$

[3]For the precise meaning of $\mathcal{L}$ c.f. [Dürr and Teufel, 2009]

On the other hand, the ground state can be rewritten (up to a complex phase) as (see [Sutherland, 2004]):

$$\Lambda(x_0, ..., x_N) = \frac{1}{\sqrt{(N+1)!}} \prod_{0\leq k\leq l\leq N} 2\sin\left((x_l - x_k)\pi\right) \mod U(1) \tag{4.58}$$

Choosing $x_0 = 0, x_1 = \frac{1}{N+1}, x_2 = \frac{2}{N+1}, ..., x_N = \frac{N}{N+1}$, one can confirm the following equality:

$$\begin{aligned}\Lambda\left(0, \frac{1}{N+1}, ...\right) &= \frac{1}{\sqrt{(N+1)!}} 2\sin\Big(\underbrace{\frac{1}{N+1}}_{=x_0-x_1}\pi\Big)...2\sin\Big(\underbrace{\frac{N-1}{N+1}}_{=x_0-x_{N-1}}\pi\Big)2\sin\Big(\underbrace{\frac{N}{N+1}}_{=x_0-x_N}\pi\Big)\\ &\quad\times 2\sin\Big(\underbrace{\frac{1}{N+1}}_{=x_1-x_2}\pi\Big)...2\sin\Big(\underbrace{\frac{N-1}{N+1}}_{=x_1-x_N}\pi\Big)\\ &\quad\times ...\\ &\quad\times 2\sin\Big(\underbrace{\frac{1}{N+1}}_{=x_{N-1}-x_N}\pi\Big) \mod U(1)\\ &= \frac{1}{\sqrt{(N+1)!}} \prod_{k=1}^{N} \left(2\sin\left(\frac{k}{N+1}\pi\right)\right)^{N+1-k} \mod U(1)\end{aligned} \tag{4.59}$$

Squaring this expression and using $\sin\left(\frac{N+1-k}{N+1}\pi\right) = \sin\left(\frac{k}{N+1}\pi\right)$, we can complete the triangular form in (4.59) to a square, yielding $N+1$ equal terms.

$$\begin{aligned}\left|\Lambda\left(0, \frac{1}{N+1}, ...\right)\right|^2 &= \frac{1}{(N+1)!}\Bigg(\underbrace{\prod_{k=1}^{N} 2\sin\left(\frac{k}{N+1}\pi\right)}_{=N+1}\Bigg)^{N+1}\\ &= \frac{(N+1)^{N+1}}{(N+1)!}\end{aligned} \tag{4.60}$$

Consequently, Hadamard's inequality gets sharp for equally distributed particles. □

Thus, varying for example x_1 from 0 to $\frac{L}{N+1} = \frac{1}{\rho} \ll 1$, while keeping all other variables as above, the wave functions takes values from 0 to $\|\Lambda\|_{L^\infty} \approx \frac{e^{N/2}}{L^{N/2}}$.
Yet, the particle density on *physical* space is almost constant in every region B.

4.6.0.2 The bosonic system

We want to compare our result to a system composed of bosons. For simplicity, we assume all of them to be in the same state $|\varphi\rangle$. Hence, the $N+1$ particle wave function, now defined to be symmetric, is given by:

$$|\Lambda^B\rangle := \prod_{k=0}^{N} |\varphi^k\rangle \tag{4.61}$$

The expectation value is the same as for the fermionic state:

$$\begin{aligned}\mathbb{E}_{N+1}(y) = \langle\Lambda^B|W(\vec{x},y)|\Lambda^B\rangle &= \sum_{k=0}^{N}\langle\Lambda^B|v(x_k-y)|\Lambda^B\rangle \\ &= \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{L}\mathcal{F}(v)\left(\frac{m-l}{L}\right)e^{\frac{2\pi \mathrm{i}}{L}(m-l)y}\langle\Lambda^B|\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m)|\Lambda^B\rangle\end{aligned} \tag{4.62}$$

Since we have this time $N+1$ copies of the same wavefunction (let us define $|\varphi\rangle = |\varphi_1\rangle$), we obtain:

$$\mathbb{E}_{N+1}(y) = \frac{N+1}{L}\mathcal{F}(v)\,(0) \tag{4.63}$$

Thus, if we considered bosons and used the mean-field picture without any justification, the result would be that the tracer particle moves freely, too. Yet, we know that the classical analogy of Brownian motion shows this claim to be questionable. Also here we expect the mean-field picture to break down. Explicitly, we show the density fluctuations to grow like ρ, which implies that the mean field is not a very good approximation to the real potential.
Using

$$W(\vec{x},y) - W^f = \sum_{l=-\infty}^{\infty}\sum_{m\neq l=-\infty}^{\infty}\frac{1}{L}\mathcal{F}(v)\left(\frac{m-l}{L}\right)e^{\frac{2\pi \mathrm{i}}{L}(m-l)y}\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m) \tag{4.64}$$

once again, we obtain:

$$(W(\vec{x},y) - W^f)|\Lambda^B\rangle = \sum_{|l|>1}\frac{1}{L}\mathcal{F}(v)\left(\frac{l-1}{L}\right)e^{\frac{2\pi \mathrm{i}}{L}(1-l)y}\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_1)|\Lambda^B\rangle \tag{4.65}$$

and hence

$$\langle\Lambda^B|(W(\vec{x},y) - W^f)^2|\Lambda^B\rangle = (N+1)\sum_{|l|>1}\left|\frac{1}{L}\mathcal{F}(v)\left(\frac{l-1}{L}\right)\right|^2 \tag{4.66}$$

Taking the thermodynamic limit, this expression reads

$$\lim_{\mathrm{TD}}\langle\Lambda^B|(W(\vec{x},y) - W^f)^2|\Lambda^B\rangle = \rho\int_{\mathbb{R}} dl\,|\mathcal{F}(v)\,(l)|^2 \tag{4.67}$$

Note that

$$\sum_{|l|>1}\left|\frac{1}{L}\mathcal{F}(v)\left(\frac{l-1}{L}\right)\right|^2 = \langle\varphi_1|(v(x-y)-\mathcal{F}(v)(0))^2|\varphi_1\rangle \tag{4.68}$$

is actually the variance of the one particle state $|\varphi_1\rangle$. The variance of the full state $|\Lambda\rangle$ is consequently just the sum of the corresponding one particle variances. This reflects the fact that the state is factorised and no correlations are present. Furthermore, this characteristics holds for arbitrary dimensions.
Actually, any bosonic state has fluctuations at least of order ρ. This can be seen easily by noting that it is possible for any state $|\varphi_m\rangle$ to hop to any neighbouring state.

4.7 The proof of the validity of the mean-field picture

We will employ Cook's lemma to perform a perturbation expansion of

$$\alpha_t := \langle\langle\psi_t|Q|\psi_t\rangle\rangle := \langle\langle\psi_t|\left(\mathbb{1} - |\psi_t^f\rangle\rangle\langle\langle\psi_t^f|\right)|\psi_t\rangle\rangle \tag{4.69}$$

$$= 1 - \left|\langle\langle\psi_t|\psi_t^f\rangle\rangle\right|^2 \tag{4.70}$$

α_t gives the projection onto the orthogonal complement of the free time evolution. If this quantity vanishes, the two evolutions are the same up to some global $U(1)$ phase. That is, if $\alpha_t = 0$ would be true, then $|\psi_t^f\rangle\rangle = e^{i\phi}|\psi_t\rangle\rangle$.
Since $|\psi_{t=0}^f\rangle\rangle = |\psi_{t=0}\rangle\rangle$ holds, we expect by continuity that $\phi = 0$, i.e.

$$\|\psi_t^f - \psi_t\| = 0 \Leftrightarrow \alpha_t = 0 \tag{4.71}$$

Of course, α_t will not be exactly zero, yet, according to theorem 4.5.1, which we are going to prove now,

$$\lim_{\rho\to\infty}\lim_{\mathrm{TD}} \alpha_t = 0 \tag{4.72}$$

holds. Therefore, we may conclude

$$\lim_{\rho\to\infty}\lim_{\mathrm{TD}} \|\psi_t^f - \psi_t\| = 0 \tag{4.73}$$

which shows norm convergence of the two time evolutions.
We like to emphasize that our proof only shows the smallness of α_t. While we do indeed expect norm convergence, we could not provide an explicit proof of this. However, since quantum states which only differ by some global phase cannot be distinguished, it is actually *suffcient* to show the smallness of α_t.
We can replace Q by $q^y := \mathbb{1} - |\chi_t\big)\big(\chi_t|$. Following the discussion in [Knowles and Pickl, 2009] , this quantity is a measure of the closeness of the reduced density matrix, describing the tracer particle only, to its free evolution $|\chi_t\big)$ in trace norm. This type of convergence is, as it was already mentioned, normally considered in the literature.

For the subsequent proof we need the following

Lemma 4.7.1. *Let α_t be defined as above. Then*

$$\alpha_t \leq \underbrace{\left\|\int_0^t dt' U_{-t'}^f \left(W(\vec{x};y) - W^f\right) U_{t'}^f \psi_0\right\|}_{=:E_1} \tag{4.74}$$

$$+ \underbrace{\left|\int_0^t dt' \langle\langle U_{-t} Q\psi_t|(U_{-t'} - U_{-t'}^f)\left(W(\vec{x};y) - W^f\right) U_{t'}^f|\psi_0\rangle\rangle\right|}_{=:R} \tag{4.75}$$

Proof.
Using Cook's lemma, we obtain:

$$\begin{aligned}
\langle\langle\psi_t|Q|\psi_t\rangle\rangle =& \langle\langle\psi_t|Q(U_t - U_t^f)|\psi_0\rangle\rangle \\
=& -\mathrm{i}\int_0^t dt' \langle\langle\psi_t|QU_{t-t'}\left(W(\vec{x};y) - W^f\right)U_{t'}^f|\psi_0\rangle\rangle \\
=& -\mathrm{i}\int_0^t dt' \langle\langle U_{-t}Q\psi_t|U_{-t'}\left(W(\vec{x};y) - W^f\right)U_{t'}^f|\psi_0\rangle\rangle \\
=& -\mathrm{i}\langle\langle U_{-t}Q\psi_t|\int_0^t dt'(U_{-t'} - U_{-t'}^f)\left(W(\vec{x};y) - W^f\right)U_{t'}^f|\psi_0\rangle\rangle \\
& -\mathrm{i}\langle\langle U_{-t}Q\psi_t|\int_0^t dt' U_{-t'}^f\left(W(\vec{x};y) - W^f\right)U_{t'}^f|\psi_0\rangle\rangle
\end{aligned} \tag{4.76}$$

Using Cauchy Schwarz and the triangle inequality yields the desired result. □

E_1 is the estimate we would get from first-order perturbation theory and will be treated now. In section 4.7.2, the remainder R will be estimated using second-order perturbation theory.

4.7.1 First-order perturbation theory

In this section we will estimate E_1, which is the first order of a perturbation expansion. Until now, our estimate (4.29) can be considered to be static. It bounds the deviation from the free evolution by the sum of all possible transitions. Yet, if we also consider the dynamics, these *possible* transitions may actually be suppressed. That is, during the actual time evolution, the fermionic state remains approximately in its ground state.
After we have treated E_1 successfully, we will bound the remainder R using the very same techniques we will develop now. The next theorem states that E_1 can be bounded by $\frac{1}{\sqrt{\rho}}$ and hence vanishes for high densities.

Theorem 4.7.1.

$$\lim_{TD} E_1^2 \leq \frac{I_1 + tI_2 + t^2 I_3}{\rho} \tag{4.77}$$

where I_1, I_2, I_3 are given in (4.90),(4.91) and (4.92), respectively. They depend on v, on $|\chi_0\rangle$ and stay bounded for all ρ.

Proof.
E_1^2 is given by

$$\begin{aligned} E_1^2 &= \left\| \int_0^t dt' U^f_{-t'} (W(\vec{x}, y) - W^f) U^f_{t'} \psi_0 \right\|^2 \\ &= \left\| \int_0^t dt' U^f_{-t'} \sum_{|l|>N/2} \sum_{|m|\leq N/2} \frac{1}{L} \mathcal{F}(v) \left(\frac{m-l}{L} \right) e^{\frac{2\pi \mathrm{i}}{L}(m-l)y} \hat{a}^\dagger(\varphi_l) \hat{a}(\varphi_m) U^f_{t'} \psi_0 \right\|^2 \end{aligned} \tag{4.78}$$

Note that the free time evolution with parameter $-t'$ acts on the time evolved state $e^{\frac{2\pi \mathrm{i}}{L}(m-l)y} \hat{a}^\dagger(\varphi_l) \hat{a}(\varphi_m) U^f_{t'} |\psi_0\rangle\rangle$. Since the free time evolution of the sea components can be evaluated explicitly, we conclude that:

$$\begin{aligned} & U^f_{-t'} e^{\frac{2\pi \mathrm{i}}{L}(m-l)y} \hat{a}^\dagger(\varphi_l) \hat{a}(\varphi_m) U^f_{t'} |\chi_0^y\rangle \otimes |\Lambda\rangle \\ =& e^{-\mathrm{i}t' \left(\sum_{|k|\leq N/2} \left(2\pi \frac{k}{L}\right)^2 + \rho \mathcal{F}(v)(0) \right)} U^f_{-t'} \hat{a}^\dagger(\varphi_l) \hat{a}(\varphi_m) \left(e^{\frac{2\pi \mathrm{i}}{L}(m-l)y} U^y_{t'} \right) |\psi_0\rangle\rangle \\ =& e^{\frac{4\pi^2 \mathrm{i}}{L}(l^2 - m^2)t'} \hat{a}^\dagger(\varphi_l) \hat{a}(\varphi_m) \left(U^y_{-t'} e^{\frac{2\pi \mathrm{i}}{L}(m-l)y} U^y_{t'} \right) |\psi_0\rangle\rangle \end{aligned} \tag{4.79}$$

Hence $U^f_{-t'}$ sets all the acquired phases back, expect the ones from φ_l and φ_m.
E_1^2 is therefore given by:

$$E_1^2 = \left\| \int_0^t dt' \sum_{|l|>N/2} \sum_{|m|\leq N/2} \frac{1}{L} \mathcal{F}(v) \left(\frac{m-l}{L} \right) \hat{a}^\dagger(\varphi_l) \hat{a}(\varphi_m) e^{\frac{4\pi^2 \mathrm{i}}{L^2}(l^2 - m^2)t'} U^y_{-t'} e^{\frac{2\pi \mathrm{i}}{L}(m-l)y} U^y_{t'} \psi_0 \right\|^2 \tag{4.80}$$

In order to evaluate this norm, note that we already calculated a similar expression in (4.39). We can carry the result over here, obtaining:

$$\begin{aligned}E_1^2 =& \int_0^t dt' \int_0^t dt'' \sum_{|l|>N/2} \sum_{|m|\leq N/2} \frac{1}{L^2} \left| \mathcal{F}(v) \left(\frac{m-l}{L} \right) \right|^2 \; e^{\frac{4\pi^2 \mathrm{i}}{L^2}(l^2-m^2)(t'-t'')} \\ &\times \left(U^y_{-t''} \; e^{\frac{2\pi \mathrm{i}}{L}(m-l)y} U^y_{t'} \chi_0 | U^y_{-t'} \; e^{\frac{2\pi \mathrm{i}}{L}(m-l)y} U^y_{t'} \chi_0 \right) \\ =& \sum_{|l|>N/2} \sum_{|m|\leq N/2} \frac{1}{L^2} \left| \mathcal{F}(v) \left(\frac{m-l}{L} \right) \right|^2 \left\| \int_0^t dt' e^{\frac{4\pi^2 \mathrm{i}}{L^2}(l^2-m^2)t'} U^y_{-t'} \; e^{\frac{2\pi \mathrm{i}}{L}(m-l)y} U^y_{t'} \chi_0 \right\|_y^2 \end{aligned} \tag{4.81}$$

In the thermodynamic limit, which will be easier to handle, this expression reads:

$$\lim_{\mathrm{TD}} E_1^2 = \int_{|l|>\rho/2} dl \int_{|m|<\rho/2} dm \, |\mathcal{F}(v)\,(m-l)|^2 \; \left\| \int_0^t dt' e^{4\pi^2 \mathrm{i}(l^2-m^2)t'} U^y_{-t'} \; e^{2\pi \mathrm{i}(m-l)y} U^y_{t'} \chi_0 \right\|_y^2 \tag{4.82}$$

The structure of (4.82) reflects our heuristic discussion. The validity of our mean-field description relies on the variance being small and *in addition* on the fact that transitions near the Fermi edge are also suppressed by the dynamics.
The variance expresses the static picture where every possible transition is weighted by $|\mathcal{F}(v)\,(m-l)|^2$. The fast oscillating phase $e^{4\pi^2 \mathrm{i}(l^2-m^2)t'}$, integrated over time, also prohibits transitions near the Fermi surface. These transition could not be excluded solely using the static analysis.
In order to estimate (4.82) by some stationary phase argument, we need to separate the critical point $m^2 = l^2$. To this end, let us split up the integral into two parts:

$$\begin{aligned}\lim_{\mathrm{TD}} E_1^2 =& \int_{|l|>\rho/2} dl \int_{|m|<\rho/2} dm \, |\mathcal{F}(v)\,(m-l)|^2 \; \left\| \int_0^t dt' e^{4\pi^2 \mathrm{i}(l^2-m^2)t'} U^y_{-t'} \; e^{2\pi \mathrm{i}(m-l)y} U^y_{t'} \chi_0 \right\|_y^2 \\ =& \int_{|l|>\rho/2} dl \int_{|m|<\rho/2} dm \; \Theta\left(\frac{1}{\sqrt{\rho}} - |m-l| \right) |\mathcal{F}(v)\,(m-l)|^2 \; \left\| \int_0^t dt' e^{4\pi^2 \mathrm{i}(l^2-m^2)t'} U^y_{-t'} \; e^{2\pi \mathrm{i}(m-l)y} U^y_{t'} \chi_0 \right\|_y^2 \\ &+ \int_{|l|>\rho/2} dl \int_{|m|<\rho/2} dm \; \Theta\left(|m-l| - \frac{1}{\sqrt{\rho}} \right) |\mathcal{F}(v)\,(m-l)|^2 \; \left\| \int_0^t dt' e^{4\pi^2 \mathrm{i}(l^2-m^2)t'} U^y_{-t'} \; e^{2\pi \mathrm{i}(m-l)y} U^y_{t'} \chi_0 \right\|_y^2 \end{aligned} \tag{4.83}$$

We will estimate each term separately. The first one will be small itself, while the other one can be estimated by a stationary phase argument. Using $\|\mathcal{F}(v)\|_\infty \leq \|v\|_1$, we obtain

$$\begin{aligned}&\int_{|l|>\rho/2} dl \int_{|m|<\rho/2} dm \; \Theta\left(\frac{1}{\sqrt{\rho}} - |m-l| \right) |\mathcal{F}(v)\,(m-l)|^2 \; \left\| \int_0^t dt' e^{4\pi^2 \mathrm{i}(l^2-m^2)t'} U^y_{-t'} \; e^{2\pi \mathrm{i}(m-l)y} U^y_{t'} \chi_0 \right\|_y^2 \\ \leq& \|v\|_1^2 t^2 \int_{|l|>\rho/2} dl \int_{|m|<\rho/2} dm \; \Theta\left(\frac{1}{\sqrt{\rho}} - |m-l| \right) \\ \leq& \frac{2\|v\|_1^2}{\rho} t^2 \end{aligned} \tag{4.84}$$

In order to estimate the remaining term, first note that the fast oscillating phase can be bounded from below. To this end, consider both l and m be located at the gap $\rho/2$. Since

$|l| - |m| > (\rho)^{-1/2}$ and $|l| > \rho/2$ holds, we can give the following upper bound:

$$\begin{aligned} l^2 - m^2 =& |l|^2 - |m|^2 = (|l| - |m|)(|l| + |m|) \\ \geq& \frac{1}{\sqrt{\rho}} \rho/2 \\ =& \frac{\sqrt{\rho}}{2} \end{aligned} \tag{4.85}$$

Consequently, the phase oscillates at least like $\sqrt{\rho}/2$. One should notice that this estimate is a direct consequence of the *nonrelativistic* energy-momentum relation. For a relativistic system, this estimate no longer holds.

Integrating by parts, the following bound is obtained:

$$\begin{aligned} &\left\| \int_0^t dt' e^{4\pi^2 \mathrm{i}(l^2 - m^2)t'} U^y_{-t'}\, e^{2\pi \mathrm{i}(m-l)y} U^y_{t'} \chi_0 \right\|_y \\ =& \frac{1}{4\pi^2(l^2 - m^2)} \left\| \int_0^t dt' \left(\frac{d}{dt'} e^{4\pi^2 \mathrm{i}(l^2 - m^2)t'} \right) U^y_{-t'}\, e^{2\pi \mathrm{i}(m-l)y} U^y_{t'} \chi_0 \right\|_y \\ \leq& \frac{1}{4\pi^2(l^2 - m^2)} \left\| e^{4\pi^2 \mathrm{i}(l^2 - m^2)t} U^y_{-t}\, e^{2\pi \mathrm{i}(m-l)y} U^y_t \chi_0 - e^{2\pi \mathrm{i}(m-l)y} \chi_0 \right\|_y \\ +& \frac{1}{4\pi^2(l^2 - m^2)} \left\| \int_0^t dt' e^{4\pi^2 \mathrm{i}(l^2 - m^2)t'} \frac{d}{dt'} \left(U^y_{-t'}\, e^{2\pi \mathrm{i}(m-l)y} U^y_{t'} \chi_0 \right) \right\|_y \\ \leq& \frac{1}{4\pi^2 \sqrt{\rho}} \left(2 + \int_0^t dt' \left\| \frac{d}{dt'} \left(U^y_{-t'}\, e^{2\pi \mathrm{i}(m-l)y} U^y_{t'} \chi_0 \right) \right\|_y \right) \end{aligned} \tag{4.86}$$

Using Stone's theorem, we can calculate the last term:

$$\begin{aligned} &\mathrm{i} \frac{d}{dt'} \left(U^y_{-t'}\, e^{2\pi \mathrm{i}(m-l)y} U^y_{t'} |\chi_0\rangle \right) \\ =& \Delta_y \left(U^y_{-t'}\, e^{2\pi \mathrm{i}(m-l)y} U^y_{t'} |\chi_0\rangle \right) + U^y_{-t'}\, e^{2\pi \mathrm{i}(m-l)y} (-\Delta_y) U^y_{t'} |\chi_0\rangle \\ =& 4\pi^2 (m-l)^2 U^y_{-t'}\, e^{2\pi \mathrm{i}(m-l)y} U^y_{t'} |\chi_0\rangle + 4\pi (m-l) U^y_{-t'}\, e^{2\pi \mathrm{i}(m-l)y} U^y_{t'} \hat{p}_y |\chi_0\rangle \end{aligned} \tag{4.87}$$

where $\hat{p}_y := -\mathrm{i}\partial_y$. Note in passing that the norm $\|\hat{p}_y \chi_0\|_y$ is finite. As for the potential, we excluded cases where $|\chi_0\rangle$ is not differentiable at the embedded boundaries of the torus.

Plugging everything together, we get the following estimate :

$$\begin{aligned} \lim_{TD} E_1^2 \leq& \frac{2\|v\|_1^2}{\rho} t^2 \\ &+ \frac{1}{16\pi^4 \rho} \int_{|l| > \rho/2} dl \int_{|m| < \rho/2} dm \, |\mathcal{F}(v)\,(m-l)|^2 \left(2 + 4\pi |m-l| \|\hat{p}_y \chi_0\|_y \, t + 4\pi^2 (m-l)^2 t \right)^2 \end{aligned} \tag{4.88}$$

$$=: \frac{I_1 + t I_2 + t^2 I_3}{\rho} \tag{4.89}$$

where

$$I_1 = \frac{1}{4\pi^4} \int_{|l|>\rho/2} dl \int_{|m|<\rho/2} dm \, |\mathcal{F}(v)\,(m-l)|^2 = \frac{1}{4\pi^4} \lim_{\mathrm{TD}} \mathrm{Var}^v_{N+1}(y) \tag{4.90}$$

$$I_2 = \frac{1}{\pi^4} \int_{|l|>\rho/2} dl \int_{|m|<\rho/2} dm \, |\mathcal{F}(v)\,(m-l)|^2 \left(\pi|m-l|\|\hat{p}_y\chi_0\|_y + \pi^2(m-l)^2\right) \tag{4.91}$$

$$I_3 = \frac{1}{\pi^4} \int_{|l|>\rho/2} dl \int_{|m|<\rho/2} dm \, |\mathcal{F}(v)\,(m-l)|^2 \left(\pi|m-l|\|\hat{p}_y\chi_0\|_y + \pi^2(m-l)^2\right)^2 + 2\|v\|_1^2 \tag{4.92}$$

I_1, I_2, I_3 are bounded for all ρ, since for smooth, compactly supported potentials the Fourier transform decays much faster than any polynomial.

Lemma 4.7.2. *Let $\alpha > 0, \beta > 1$. Then:*

$$\sup_{\rho>0} \left(\int_{|m|\leq\rho/2} dm \int_{|l|>\rho/2} dl |\mathcal{F}(v)\,(m-l)\,|^\beta |m-l|^\alpha \right) < \infty \tag{4.93}$$

Proof.
Upon splitting the integration into two parts, we obtain:

$$\begin{aligned}
&\int_{|m|\leq\rho/2} dm \int_{|l|>\rho/2} dl |\mathcal{F}(v)\,(m-l)\,|^\beta |m-l|^\alpha \\
=&2 \int_0^{\rho/2} dm \int_{\rho/2}^{\infty} dl |\mathcal{F}(v)\,(m-l)\,|^\beta |m-l|^\alpha \\
+&2 \int_{-\rho/2}^{0} dm \int_{\rho/2}^{\infty} dl |\mathcal{F}(v)\,(m-l)\,|^\beta |m-l|^\alpha
\end{aligned} \tag{4.94}$$

Consider the first term. It is bounded as the following estimate shows:

$$\begin{aligned}
&\int_0^{\rho/2} dm \int_{\rho/2}^{\infty} dl |\mathcal{F}(v)\,(m-l)\,|^\beta |m-l|^\alpha \\
\leq &\int_0^{\rho/2} dm \int_{\rho/2}^{\infty} dl \left(\frac{D_P}{(1+|m-l|)^p}\right)^\beta |m-l|^\alpha \\
= &\int_0^{\rho/2} dm \int_{m}^{\infty} dl \left(\frac{D_p}{(1+l)^p}\right)^\beta l^\alpha \\
= &\int_0^{C} dm \int_{m}^{\infty} dl \left(\frac{D_p}{(1+l)^p}\right)^\beta l^\alpha + \int_C^{\rho/2} dm \int_{m}^{\infty} dl \left(\frac{D_p}{(1+l)^p}\right)^\beta l^\alpha \\
\leq &C \int_0^{\infty} dl \left(\frac{D_p}{(1+l)^p}\right)^\beta l^\alpha + \int_C^{\rho} dm \int_{m}^{\infty} dl \left(\frac{D_p}{l^{p\beta-\alpha}}\right) \\
= &\underbrace{C \int_0^{\infty} dl \left(\frac{D_p}{(1+l)^p}\right)^\beta l^\alpha}_{<\infty} + \frac{1}{(1+\alpha-p\beta)(2+\alpha-p\beta)} \left(\frac{1}{C^{p\beta-\alpha-2}} - \frac{1}{\rho^{p\beta-\alpha-2}}\right)
\end{aligned} \tag{4.95}$$

Both terms are bounded choosing p big enough. As similar estimates shows that the second term vanishes as $\rho \to \infty$. □

□

4.7.2 Second-order perturbation theory

In order to estimate R, we will split up the proof into a series of small lemmas. While the computation is a bit lengthy, the underlying idea can be stated easily:
R can be controlled using second-order perturbation theory. Consequently, while before only one particle could be lifted, now we have to control the transition of two particles. We will use the results from first-order perturbation theory in order to show that the first transition is suppressed and will then estimate the subsequent amplitude to be finite again. Explicitly, we prove the following

Theorem 4.7.2.

$$\lim_{TD} R^2 \leq \frac{tI_4 + t^3 I_5 + t^4 I_6}{\rho} \tag{4.96}$$

where I_4, I_5, I_6 depend on v, on $|\chi_0\rangle$ and stay bounded for all ρ.

Proof.
The structure of the proof is as follows:
As just mentioned, we will copy the proof from first-order perturbation theory. That is, we will first separate the critical point by splitting up the Fourier transform of the potential into two contributions.
For the main contribution, by means of the stationary phase lemma, a factor of $\frac{1}{\sqrt{\rho}}$ will be obtained. Since the stationary phase lemma requires integration by parts, three terms will arise. Lemma 4.7.3 will prove the corresponding bound:

$$\begin{aligned} R \leq & \frac{1}{2\pi^2\sqrt{\rho}} \int_0^t dt'' \Big\| \sum_{|l|>N/2} \sum_{|m|\leq N/2} \Big| \frac{1}{L} \mathcal{F}(\tilde{v}) \left(\frac{m-l}{L}\right); \sigma_{t;t'':\frac{l}{L};\frac{l}{L}} \Big\rangle\Big\rangle \Big\| \\ & + \frac{1}{2\pi^2\sqrt{\rho}} \int_0^t dt' \Big\| \sum_{|l|>N/2} \sum_{|m|\leq N/2} \Big| \frac{1}{L} \mathcal{F}(\tilde{v}) \left(\frac{m-l}{L}\right); \sigma_{t';t':\frac{l}{L};\frac{l}{L}} \Big\rangle\Big\rangle \Big\| \\ & + \frac{1}{2\pi^2\sqrt{\rho}} \int_0^t dt' \int_0^{t'} dt'' \Big\| \sum_{|l|>N/2} \sum_{|m|\leq N/2} \Big| \frac{1}{L} \mathcal{F}(\tilde{v}) \left(\frac{m-l}{L}\right); \tilde{\sigma}_{t';t'':\frac{l}{L};\frac{l}{L}} \Big\rangle\Big\rangle \Big\| \\ & + \int_0^t dt' \int_0^{t'} dt'' \Big\| \sum_{|l|>N/2} \sum_{|m|\leq N/2} \Big| \frac{1}{L} \mathcal{F}(w) \left(\frac{m-l}{L}\right); \sigma_{t';t'':\frac{l}{L};\frac{l}{L}} \Big\rangle\Big\rangle \Big\| \end{aligned} \tag{4.97}$$

The potential

$$\mathcal{F}(\tilde{v})(k) = \begin{cases} \mathcal{F}(v)(k) & \text{if } k \geq \rho^{-1/2} \\ 0 & \text{if } k < \rho^{-1/2} \end{cases} \tag{4.98}$$

is just the normal potential with the critical point $l^2 = m^2$ removed. In analogy

$$\mathcal{F}(w)(k) = \begin{cases} \mathcal{F}(v)(k) & \text{if } k \leq \rho^{-1/2} \\ 0 & \text{if } k > \rho^{-1/2} \end{cases} \tag{4.99}$$

describes the contribution from the gap. We already used these potentials implicitly in the estimate of E_1.

The exact definition of the vectors used above will be given in lemma 4.7.3. Essentially, they will arise in second-order perturbation theory and, in analogy to the variance, will sum up all transitions which are possible if the potential acts two times on the ground state.
Lemma 4.7.4 will show that these vectors can be estimated by a sum of functions which are relatively easy to handle. Explicitly, in lemma 4.7.4 the following bound will be given:

$$\Big\| \sum_{|l|>N/2} \sum_{|m|\leq N/2} \Big| g\left(\frac{m-l}{L}\right); \chi_{t_a;t_b;\frac{l}{L};\frac{m}{L}} \Big\rangle\Big\rangle \Big\| \leq \sqrt{2M_1 + 2M_2} + \sqrt{M_1} + \sqrt{M_3} + \sqrt{M_4} \tag{4.100}$$

The vector $|g\left(\frac{m-l}{L}\right); \chi_{t_a;t_b;\frac{l}{L};\frac{m}{L}}\rangle\rangle$ represents the three different vectors from the estimate above. The functions M_i will still depend on the density and thus need to be bounded.
This will be done in lemma 4.7.6 and 4.7.7.
For the first three vectors, which sum up all transitions except the ones which happen directly at the gap, it will be shown in lemma 4.7.6 that

$$\sup_\rho M_i < \infty \tag{4.101}$$

Finally, we will need to control the transitions arising directly at the gap. This will be done in lemma 4.7.7. Due to the smallness of the support of the potential w, it will be possible to estimate

$$\int_0^t dt' \int_0^{t'} dt'' \Big\| \sum_{|l|>N/2} \sum_{|m|\leq N/2} \Big| \frac{1}{L}\mathcal{F}(w)\left(\frac{m-l}{L}\right); \sigma_{t';t'':\frac{l}{L};\frac{l}{L}} \Big\rangle\Big\rangle \Big\| \leq t^2 \frac{C}{\sqrt{\rho}} \tag{4.102}$$

Combining the lemmas 4.7.3, 4.7.4, 4.7.6 and 4.7.7 then proves the theorem. □

Let us first separate the critical point $l^2 = m^2$. To this end we define

$$\mathcal{F}(v)(k) = \mathcal{F}(w)(k) + \mathcal{F}(\tilde{v})(k) \tag{4.103}$$

where

$$\mathcal{F}(\tilde{v})(k) = \begin{cases} \mathcal{F}(v)(k) & \text{if } k \geq \rho^{-1/2} \\ 0 & \text{if } k < \rho^{-1/2} \end{cases} \tag{4.104}$$

$$\mathcal{F}(w)(k) = \begin{cases} \mathcal{F}(v)(k) & \text{if } k \leq \rho^{-1/2} \\ 0 & \text{if } k > \rho^{-1/2} \end{cases} \tag{4.105}$$

We want to make a small remark here:
The potentials $\tilde{v}$ and w cannot have compact support in $\mathbb{R}$ since the Fourier transform of w has. However, this fact will not be of any importance. Explicitly, $\tilde{v}$ and w are given by:

$$\begin{aligned} w(x) =& \mathcal{F}^{-1}\left(\mathcal{F}(v)(\cdot)\Theta\left(\frac{1}{\sqrt{\rho}} - |\cdot|\right)\right)(x) \\ =& v * \mathcal{F}^{-1}\left(\Theta\left(\frac{1}{\sqrt{\rho}} - |\cdot|\right)\right)(x) \end{aligned} \tag{4.106}$$

$$\tilde{v}(x) = v(x) - w(x) \tag{4.107}$$

Let us now prove the following

Lemma 4.7.3. *The following inequality is true:*

$$\begin{aligned} R \leq & \frac{1}{2\pi^2\sqrt{\rho}} \int_0^t dt'' \Big\| \sum_{|l|>N/2} \sum_{|m|\leq N/2} \Big| \frac{1}{L}\mathcal{F}(\tilde{v})\left(\frac{m-l}{L}\right); \sigma_{t;t'':\frac{l}{L};\frac{l}{L}} \Big\rangle\Big\rangle \Big\| \\ & + \frac{1}{2\pi^2\sqrt{\rho}} \int_0^t dt' \Big\| \sum_{|l|>N/2} \sum_{|m|\leq N/2} \Big| \frac{1}{L}\mathcal{F}(\tilde{v})\left(\frac{m-l}{L}\right); \sigma_{t';t':\frac{l}{L};\frac{l}{L}} \Big\rangle\Big\rangle \Big\| \\ & + \frac{1}{2\pi^2\sqrt{\rho}} \int_0^t dt' \int_0^{t'} dt'' \Big\| \sum_{|l|>N/2} \sum_{|m|\leq N/2} \Big| \frac{1}{L}\mathcal{F}(\tilde{v})\left(\frac{m-l}{L}\right); \tilde{\sigma}_{t';t'':\frac{l}{L};\frac{l}{L}} \Big\rangle\Big\rangle \Big\| \\ & + \int_0^t dt' \int_0^{t'} dt'' \Big\| \sum_{|l|>N/2} \sum_{|m|\leq N/2} \Big| \frac{1}{L}\mathcal{F}(w)\left(\frac{m-l}{L}\right); \sigma_{t';t'':\frac{l}{L};\frac{l}{L}} \Big\rangle\Big\rangle \Big\| \end{aligned} \tag{4.108}$$

where:

$$\Big| g\left(\frac{m-l}{L}\right); \chi_{t_a;t_b;\frac{l}{L};\frac{m}{L}} \Big\rangle\Big\rangle := \left(W(\vec{x};y) - W^f\right) g\left(\frac{m-l}{L}\right) \hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m)|\chi_{t_a;t_b;\frac{l}{L};\frac{m}{L}}\big) \otimes |\Lambda\rangle \tag{4.109}$$

$$|\sigma_{t_a;t_b;\frac{l}{L};\frac{m}{L}}\big) := |e^{\frac{4\pi^2 \mathrm{i}}{L^2}(l^2-m^2)(t_a-t_b)} U^y_{t_b-t_a}\, e^{\frac{2\pi\mathrm{i}}{L}(m-l)y} U^y_{t_a}\chi_0\big) \tag{4.110}$$

$$|\tilde{\sigma}_{t_a;t_b;\frac{l}{L};\frac{m}{L}}\big) := |e^{\frac{4\pi^2 \mathrm{i}}{L^2}(l^2-m^2)(t_a-t_b)} U^y_{t_b} \frac{d}{dt'}\, U^y_{-t_a} e^{\frac{2\pi\mathrm{i}}{L}(m-l)y} U^y_{t_a}\chi_0\big) \tag{4.111}$$

Proof.
In this estimate, the first three terms arise from integration by parts. The last term describes the transitions at the Fermi surface.
Performing Cook's method again, we obtain:

$$
\begin{aligned}
R =& \left| \int_0^t dt' \langle\langle (U_{t'} - U^f_{t'}) U_{-t} Q \psi_t | \left(W(\vec{x};y) - W^f \right) U^f_{t'} \psi_0 \rangle\rangle \right| \\
=& \left| \int_0^t dt' \int_0^{t'} dt'' \langle\langle U^f_{t'-t''} \left(W(\vec{x};y) - W^f \right) U_{t''-t} Q \psi_t | \left(W(\vec{x};y) - W^f \right) U^f_{t'} | \psi_0 \rangle\rangle \right| \\
=& \left| \int_0^t dt' \underbrace{\langle\langle \int_0^{t'} dt'' U^f_{-t''} \left(W(\vec{x};y) - W^f \right) U_{t''-t} Q U_t \psi_0 |}_{=:\langle\langle \tilde{\psi}_{t'} |} U^f_{-t'} \left(W(\vec{x};y) - W^f \right) U^f_{t'} | \psi_0 \rangle\rangle \right| \\
=& \left| \int_0^t dt' \sum_{|l|>N/2} \sum_{m=-N/2}^{N/2} \frac{1}{L} \mathcal{F}(v) \left(\frac{m-l}{L} \right) e^{\frac{4\pi^2 \mathrm{i}}{L^2}(l^2-m^2)t'} \langle\langle \tilde{\psi}_{t'} | \hat{a}^\dagger(\varphi_l) \hat{a}(\varphi_m) U^y_{-t'} \, e^{\frac{2\pi \mathrm{i}}{L}(m-l)y} U^y_{t'} | \psi_0 \rangle\rangle \right|
\end{aligned}
\tag{4.112}
$$

Note, already at this stage, we can identity the structure of the proof. We obtain the vector $U^f_{-t'} \left(W(\vec{x};y) - W^f \right) U^f_{t'} | \psi_0 \rangle\rangle$, which was also present in first-order perturbation theory. Again, this term can be used for the stationary phase argument, which yields a factor of $\frac{1}{\sqrt{\rho}}$. This time, however, we need to estimate the norm of a vector where we act *two* times with the potential. Note that it is not possible to simply estimate the last line using Cauchy Schwarz. If we would do so, we could not apply the stationary phase method subsequently. Moreover, it would be unclear how we should estimate $|\tilde{\psi}_{t'}\rangle\rangle$, since for doing so, we would need to have some information about the *real* time evolution.
We will now integrate by parts and separate the contribution arising at the gap. Note that $|\tilde{\psi}_0\rangle\rangle = 0$. Therefore:

$$
\begin{aligned}
&R \\
=& \left| \int_0^t dt' \sum_{|l|>N/2} \sum_{|m|\le N/2} \frac{1}{L} \mathcal{F}(v) \left(\frac{m-l}{L} \right) e^{\frac{4\pi^2 \mathrm{i}}{L^2}(l^2-m^2)t'} \langle\langle \tilde{\psi}_{t'} | \hat{a}^\dagger(\varphi_l) \hat{a}(\varphi_m) U^y_{-t'} \, e^{\frac{2\pi \mathrm{i}}{L}(m-l)y} U^y_{t'} | \psi_0 \rangle\rangle \right| \\
\le& \frac{1}{4\pi^2} \left| \int_0^t dt' \sum_{|l|>N/2} \sum_{|m|\le N/2} \frac{1}{L} \mathcal{F}(\tilde{v}) \left(\frac{m-l}{L} \right) \frac{\frac{d}{dt'} e^{\frac{4\pi^2 \mathrm{i}}{L^2}(l^2-m^2)t'}}{\left(\frac{l}{L}\right)^2 - \left(\frac{m}{L}\right)^2} \langle\langle \tilde{\psi}_{t'} | \hat{a}^\dagger(\varphi_l) \hat{a}(\varphi_m) U^y_{-t'} \, e^{\frac{2\pi \mathrm{i}}{L}(m-l)y} U^y_{t'} | \psi_0 \rangle\rangle \right| \\
+& \left| \int_0^t dt' \sum_{|l|>N/2} \sum_{|m|\le N/2} \frac{1}{L} \mathcal{F}(w) \left(\frac{m-l}{L} \right) e^{\frac{4\pi^2 \mathrm{i}}{L^2}(l^2-m^2)t'} \langle\langle \tilde{\psi}_{t'} | \hat{a}^\dagger(\varphi_l) \hat{a}(\varphi_m) U^y_{-t'} \, e^{\frac{2\pi \mathrm{i}}{L}(m-l)y} U^y_{t'} | \psi_0 \rangle\rangle \right|
\end{aligned}
$$

$$\leq \underbrace{\frac{1}{4\pi^2}\left|\sum_{|l|>N/2}\sum_{|m|\leq N/2}\frac{1}{L}\mathcal{F}(\tilde{v})\left(\frac{m-l}{L}\right)\frac{e^{\frac{4\pi^2\mathrm{i}}{L^2}(l^2-m^2)t}}{\left(\frac{l}{L}\right)^2-\left(\frac{m}{L}\right)^2}\langle\langle\tilde{\psi}_t|\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m)U^y_{-t}\, e^{\frac{2\pi\mathrm{i}}{L}(m-l)y}U^y_t|\psi_0\rangle\rangle\right|}_{=:R_1}$$
$$+\underbrace{\frac{1}{4\pi^2}\left|\int_0^t dt'\sum_{|l|>N/2}\sum_{|m|\leq N/2}\frac{1}{L}\mathcal{F}(\tilde{v})\left(\frac{m-l}{L}\right)\frac{e^{\frac{4\pi^2\mathrm{i}}{L^2}(l^2-m^2)t'}}{\left(\frac{l}{L}\right)^2-\left(\frac{m}{L}\right)^2}\langle\langle\frac{d}{dt'}\tilde{\psi}_{t'}|\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m)U^y_{-t'}\, e^{\frac{2\pi\mathrm{i}}{L}(m-l)y}U^y_{t'}|\psi_0\rangle\rangle\right|}_{=:R_2}$$
$$+\underbrace{\frac{1}{4\pi^2}\left|\int_0^t dt'\sum_{|l|>N/2}\sum_{|m|\leq N/2}\frac{1}{L}\mathcal{F}(\tilde{v})\left(\frac{m-l}{L}\right)\frac{e^{\frac{4\pi^2\mathrm{i}}{L^2}(l^2-m^2)t'}}{\left(\frac{l}{L}\right)^2-\left(\frac{m}{L}\right)^2}\langle\langle\tilde{\psi}_{t'}|\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m)\frac{d}{dt'}\left(U^y_{-t'}\, e^{\frac{2\pi\mathrm{i}}{L}(m-l)y}U^y_{t'}\right)|\psi_0\rangle\rangle\right|}_{=:R_3}$$
$$+\underbrace{\left|\int_0^t dt'\sum_{|l|>N/2}\sum_{|m|\leq N/2}\frac{1}{L}\mathcal{F}(w)\left(\frac{m-l}{L}\right)e^{\frac{4\pi^2\mathrm{i}}{L^2}(l^2-m^2)t'}\langle\langle\tilde{\psi}_{t'}|\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m)U^y_{-t'}\, e^{\frac{2\pi\mathrm{i}}{L}(m-l)y}U^y_{t'}|\psi_0\rangle\rangle\right|}_{=:R_4} \tag{4.113}$$

Let us first bound R_1 using Cauchy Schwarz. By this, the perturbation expansion is effectively truncated at second order. Reinserting the definition of $|\tilde{\psi}_{t'}\rangle\rangle$, we obtain:

$$\begin{aligned}
R_1 =&\frac{1}{4\pi^2}\Bigg|\int_0^t dt''\sum_{|l|>N/2}\sum_{|m|\leq N/2}\frac{1}{L}\mathcal{F}(\tilde{v})\left(\frac{m-l}{L}\right)\frac{e^{\frac{4\pi^2\mathrm{i}}{L^2}(l^2-m^2)t}}{\left(\frac{l}{L}\right)^2-\left(\frac{m}{L}\right)^2}\\
&\times\langle\langle U_{t''-t}QU_t\psi_0|\left(W(\vec{x};y)-W^f\right)U^f_{t''}\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m)U^y_{-t}\, e^{\frac{2\pi\mathrm{i}}{L}(m-l)y}U^y_t|\psi_0\rangle\rangle\Bigg|\\
=&\frac{1}{4\pi^2}\Bigg|\int_0^t dt''\langle\langle\Phi_{-t''}U_{t''-t}QU_t\psi_0|\sum_{|l|>N/2}\sum_{|m|\leq N/2}\frac{e^{\frac{4\pi^2\mathrm{i}}{L^2}(l^2-m^2)(t-t'')}}{\left(\frac{l}{L}\right)^2-\left(\frac{m}{L}\right)^2}\\
&\times\left(W(\vec{x};y)-W^f\right)\frac{1}{L}\mathcal{F}(\tilde{v})\left(\frac{m-l}{L}\right)\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m)U^y_{t''-t}\, e^{\frac{2\pi\mathrm{i}}{L}(m-l)y}U^y_t|\psi_0\rangle\rangle\Bigg|\\
\leq&\frac{1}{2\pi^2\sqrt{\rho}}\int_0^t dt''\Bigg\|\sum_{|l|>N/2}\sum_{|m|\leq N/2}\left(W(\vec{x};y)-W^f\right)\\
&\times\frac{1}{L}\mathcal{F}(\tilde{v})\left(\frac{m-l}{L}\right)e^{\frac{2\pi^2\mathrm{i}}{L^2}(l^2-m^2)(t-t'')}\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m)U^y_{t''-t}\, e^{\frac{2\pi\mathrm{i}}{L}(m-l)y}U^y_t\psi_0\Bigg\|\\
=:&\frac{1}{2\pi^2\sqrt{\rho}}\int_0^t dt''\Bigg\|\sum_{|l|>N/2}\sum_{|m|\leq N/2}\Bigg|\frac{1}{L}\mathcal{F}(\tilde{v})\left(\frac{m-l}{L}\right);\sigma_{t;t'':\frac{l}{L};\frac{l}{L}}\Bigg\rangle\Bigg\rangle\Bigg\|
\end{aligned} \tag{4.114}$$

In the second line we evaluated the free time evolution acting on the test particle and on the perturbed sea. Explicitly, we used that

$$\begin{aligned}
&U^f_{t''}\left(|U^y_{-t'}\, e^{\frac{2\pi \mathrm{i}}{L}(m-l)y}U^y_{t'}\chi^y_0\rangle \otimes |\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m)\Lambda\rangle\right)\\
=&\exp\left(-\mathrm{i}t''\left(\sum_{\substack{|k|\leq N/2\\ k\neq m}}\left(2\pi\frac{k}{L}\right)^2+\left(2\pi\frac{l}{L}\right)^2+\rho\mathcal{F}(v)\,(0)\right)\right)|U^y_{t''-t'}\, e^{\frac{2\pi \mathrm{i}}{L}(m-l)y}U^y_{t'}\chi^y_0\rangle \otimes |\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m)\Lambda\rangle\\
=&\Phi_{t''}\, e^{-\frac{4\pi^2\mathrm{i}}{L^2}\left(l^2-m^2\right)t''}|U^y_{t''-t'}\, e^{\frac{2\pi \mathrm{i}}{L}(m-l)y}U^y_{t}\chi^y_0\rangle \otimes |\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m)\Lambda\rangle
\end{aligned}\tag{4.115}$$

holds, where

$$\Phi_{t''}=e^{-\mathrm{i}t''\left(\sum\limits_{|k|\leq N/2}\left(2\pi\frac{k}{L}\right)^2+\rho\mathcal{F}(v)(0)\right)}\tag{4.116}$$

is a global phase which was already present in first-order perturbation theory.
Note that we estimated

$$\frac{1}{\left(\frac{l}{L}\right)^2-\left(\frac{m}{L}\right)^2}\leq\frac{2}{\sqrt{\rho}}\tag{4.117}$$

using $\mathcal{F}(\tilde{v})\left(\frac{m-l}{L}\right)=0$ if $\frac{|m-l|}{L}\leq\frac{1}{\sqrt{\rho}}$.

By the very same steps the three remaining terms can be estimated identically. Explicitly, for R_2 the same bound is obtained upon replacing:

$$\begin{aligned}
t \to& t'\\
t'' \to& t'
\end{aligned}\tag{4.118}$$

The third term R_3 has two time integrals. Hence, we also integrate over t' and make the following replacements in comparison with R_1:

$$\begin{aligned}
t \to& t'\\
|e^{\frac{4\pi^2\mathrm{i}}{L^2}(l^2-m^2)(t-t'')}U^y_{t''-t}\, e^{\frac{2\pi\mathrm{i}}{L}(m-l)y}U^y_t\chi_0\rangle \to& |e^{\frac{4\pi^2\mathrm{i}}{L^2}(l^2-m^2)(t-t'')}U^y_{t''}\frac{d}{dt'}U^y_{-t'}\, e^{\frac{2\pi\mathrm{i}}{L}(m-l)y}U^y_{t'}\chi_0\rangle
\end{aligned}\tag{4.119}$$

The fourth term can be estimated the same way as the third term upon replacing:

$$\begin{aligned}
\mathcal{F}(\tilde{v})\left(\frac{m-l}{L}\right) \to& \mathcal{F}(w)\left(\frac{m-l}{L}\right)\\
|e^{\frac{4\pi^2\mathrm{i}}{L^2}(l^2-m^2)(t-t'')}U^y_{t''}\frac{d}{dt'}U^y_{-t'}\, e^{\frac{2\pi\mathrm{i}}{L}(m-l)y}U^y_{t'}\chi_0\rangle \to& |e^{\frac{4\pi^2\mathrm{i}}{L^2}(l^2-m^2)(t-t'')}U^y_{t''-t}\, e^{\frac{2\pi\mathrm{i}}{L}(m-l)y}U^y_t\chi_0\rangle
\end{aligned}\tag{4.120}$$

□

Until now, we have separated the gap, integrated by parts and estimated R in second-order perturbation theory.
The crucial step is the following

Lemma 4.7.4. *Let* $\left|g\left(\frac{m-l}{L}\right); \chi_{t_a;t_b;\frac{l}{L};\frac{m}{L}}\right\rangle\!\Big\rangle$ *be defined as above. Then*

$$\Big\| \sum_{|l|>N/2} \sum_{|m|\leq N/2} \left|g\left(\frac{m-l}{L}\right); \chi_{t_a;t_b;\frac{l}{L};\frac{m}{L}}\right\rangle\!\Big\rangle \Big\| \leq \sqrt{2M_1 + 2M_2} + \sqrt{M_1} + \sqrt{M_3} + \sqrt{M_4} \tag{4.121}$$

where $M_i = M_i(\eta; g; v; N; L)$ *and* η *are defined below.*

Proof.
This lemma bounds the norms obtained in the previous lemma by various functions which are easier to handle. This procedure can be seen in analogy to first-order perturbation theory where we estimated $\lim_{\mathrm{TD}} E_1^2$ by $\frac{I_1+tI_2+t^2I_3}{\rho}$.
Using Stone's theorem, as in equation (4.87), we can bound

$$\|\chi_{t_a;t_b;\frac{l}{L};\frac{m}{L}}\|_y \leq \eta\left(\frac{m-l}{L}\right) := \begin{cases} 1 & \text{if } |\chi_{t_a;t_b;\frac{l}{L};\frac{m}{L}}\rangle = |\sigma_{t_a;t_b;\frac{l}{L};\frac{m}{L}}\rangle \\ 4\pi\frac{|m-l|}{L}\|\hat{p}_y\chi_0\|_y + 4\pi^2\left(\frac{m-l}{L}\right)^2 & \text{if } |\chi_{t_a;t_b;\frac{l}{L};\frac{m}{L}}\rangle = |\tilde{\sigma}_{t_a;t_b;\frac{l}{L};\frac{m}{L}}\rangle \end{cases} \tag{4.122}$$

So $\|\chi_{t_a;t_b;\frac{l}{L};\frac{m}{L}}\|_y$ depends solely on the difference $\frac{m-l}{L}$, which will be used in the following.

Let us reinsert the definition of $|g\left(\frac{m-l}{L}\right);\chi_{t_a;t_b;\frac{l}{L};\frac{m}{L}}\rangle\rangle$:

$$\begin{aligned}
&\Big\| \sum_{|l|>N/2}\sum_{|m|\le N/2} \Big|g\left(\frac{m-l}{L}\right);\chi_{t_a;t_b;\frac{l}{L};\frac{m}{L}}\Big\rangle\Big\rangle \Big\| \\
=&\Big\| \sum_{|l_1|>N/2}\sum_{|m_1|\le N/2} \left(W(\vec{x};y)-W^f\right) g\left(\frac{m_1-l_1}{L}\right)\hat{a}^\dagger(\varphi_{l_1})\hat{a}(\varphi_{m_1})|\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_1}{L}}\rangle\otimes|\Lambda\rangle \Big\| \\
=&\Big\| \sum_{|l_1|>N/2}\sum_{|m_1|\le N/2}\sum_{l_2=-\infty}^{\infty}\sum_{\substack{m_2=-\infty\\ m_2\neq l_2}}^{\infty} \frac{1}{L}\mathcal{F}(v)\left(\frac{m_2-l_2}{L}\right) g\left(\frac{m_1-l_1}{L}\right) \\
&\times\hat{a}^\dagger(\varphi_{l_2})\hat{a}(\varphi_{m_2})\hat{a}^\dagger(\varphi_{l_1})\hat{a}(\varphi_{m_1})|e^{\frac{2\pi\mathrm{i}}{L}(m_2-l_2)y}|\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_1}{L}}\rangle\otimes|\Lambda\rangle \Big\| \\
=&\Big\| \sum_{|l_1|>N/2}\sum_{|m_1|\le N/2}\left(\sum_{|l_2|>N/2}+\sum_{l_2=-\infty}^{\infty}\delta_{l_2,m_1}\right)\left(\sum_{|m_2|\le N/2}\delta^\perp_{m_2,l_2}+\sum_{m_2=-\infty}^{\infty}\delta_{m_2,l_1}\delta^\perp_{m_2,l_2}\right) \\
&\times\frac{1}{L}\mathcal{F}(v)\left(\frac{m_2-l_2}{L}\right) g\left(\frac{m_1-l_1}{L}\right)\hat{a}^\dagger(\varphi_{l_2})\hat{a}(\varphi_{m_2})\hat{a}^\dagger(\varphi_{l_1})\hat{a}(\varphi_{m_1})|e^{\frac{2\pi\mathrm{i}}{L}(m_2-l_2)y}|\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_1}{L}}\rangle\otimes|\Lambda\rangle \Big\|
\end{aligned} \tag{4.123}$$

Here, $\delta^\perp_{m_2,l_2}$ is defined to be 0 if $m_2=l_2$ and 1 otherwise.
The last line was obtained by observing that, after acting with the operator $\hat{a}^\dagger(\varphi_{l_1})\hat{a}(\varphi_{m_1})$ on the ground state, we have the following options:
The annihilation operator $\hat{a}(\varphi_{m_2})$ can either annihilate a particle in the ground state not equal to φ_{m_1}, or it can annihilate the excited particle φ_{l_1}. Afterwards, by means of $\hat{a}^\dagger(\varphi_{l_2})$, we can create either a particle in some excited state or use the existing hole φ_{m_1} to fill it up again. Note, by the restriction $m_2\neq l_2$, we cannot create a particle in the state φ_{m_2}. This line of argumentation treats the operator $\hat{a}^\dagger(\varphi_{l_2})\hat{a}(\varphi_{m_2})$ as if it were two operators. We have emphasized before that this reading is technically not correct if we work on a fixed number Hilbert space. Yet, it is very helpful to actually think of this operator in this way. Since the operator acts *the very same way* on the wavefunction as the corresponding composition of the creation and annihilation operator in Fock-space, the argumentation just made yields the correct result.
Using the triangle inequality, we get four different terms, labelled T_1 till T_4:

$$\Big\| \sum_{|l|>N/2}\sum_{|m|\le N/2} \Big|g\left(\frac{m-l}{L}\right);\chi_{t_a;t_b;\frac{l}{L};\frac{m}{L}}\Big\rangle\Big\rangle \Big\| \le T_1+T_2+T_3+T_4 \tag{4.124}$$

The first term T_1 arises from the sum $\sum_{|l_2|>N/2}\sum_{|m_2|\le N/2}\delta^\perp_{m_2,l_2}$. We drop the restriction $\delta^\perp_{m_2,l_2}$, hereby obtaining the following bound:

$$\begin{aligned}
T_1 :=&\Big\| \sum_{|l_1|>N/2}\sum_{|m_1|\le N/2}\sum_{|l_2|>N/2}\sum_{|m_2|\le N/2}\frac{1}{L}\mathcal{F}(v)\left(\frac{m_2-l_2}{L}\right) g\left(\frac{m_1-l_1}{L}\right) \\
&\times\hat{a}^\dagger(\varphi_{l_2})\hat{a}(\varphi_{m_2})\hat{a}^\dagger(\varphi_{l_1})\hat{a}(\varphi_{m_1})|e^{\frac{2\pi\mathrm{i}}{L}(m_2-l_2)y}\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_1}{L}}\rangle\otimes|\Lambda\rangle \Big\|
\end{aligned} \tag{4.125}$$

The second term T_2 arises from $\sum_{|l_2|>N/2}\sum_{m_2=-\infty}^{\infty}\delta_{m_2,l_1}\delta^{\perp}_{m_2,l_2}$.

$$\begin{aligned}T_2 :=&\Big\| \sum_{|l_1|>N/2}\sum_{|m_1|\le N/2}\sum_{|l_2|>N/2}\delta^{\perp}_{l_1,l_2}\frac{1}{L}\mathcal{F}(v)\left(\frac{l_1-l_2}{L}\right)g\left(\frac{m_1-l_1}{L}\right)\\&\times\hat a^\dagger(\varphi_{l_2})\hat a(\varphi_{l_1})\hat a^\dagger(\varphi_{l_1})\hat a(\varphi_{m_1})|e^{\frac{2\pi \mathrm{i}}{L}(l_1-l_2)y}\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_1}{L}})\otimes|\Lambda\rangle\Big\|\\ \le&\Big\| \sum_{|l_1|>N/2}\sum_{|m_1|\le N/2}\sum_{|l_2|>N/2}\frac{1}{L}\mathcal{F}(v)\left(\frac{l_1-l_2}{L}\right)g\left(\frac{m_1-l_1}{L}\right)\\&\times\hat a^\dagger(\varphi_{l_2})\hat a(\varphi_{l_1})\hat a^\dagger(\varphi_{l_1})\hat a(\varphi_{m_1})|e^{\frac{2\pi \mathrm{i}}{L}(l_1-l_2)y}\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_1}{L}})\otimes|\Lambda\rangle\Big\| \end{aligned}\tag{4.126}$$

The third term T_3 arises from the sum $\sum_{l_2=-\infty}^{\infty}\delta_{l_2,m_1}\sum_{|m_2|\le N/2}\delta^{\perp}_{m_2,l_2}$. By dropping the restriction $\delta^{\perp}_{m_2,l_2}$, we obtain the bound:

$$\begin{aligned}T_3 :=&\Big\| \sum_{|l_1|>N/2}\sum_{|m_1|\le N/2}\sum_{|m_2|\le N/2}\frac{1}{L}\mathcal{F}(v)\left(\frac{m_2-m_1}{L}\right)g\left(\frac{m_1-l_1}{L}\right)\\&\times\hat a^\dagger(\varphi_{m_1})\hat a(\varphi_{m_2})\hat a^\dagger(\varphi_{l_1})\hat a(\varphi_{m_1})|e^{\frac{2\pi \mathrm{i}}{L}(m_2-m_1)y}\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_1}{L}})\otimes|\Lambda\rangle\Big\| \end{aligned}\tag{4.127}$$

The last contribution T_4 comes from the sum $\sum_{l_2=-\infty}^{\infty}\delta_{l_2,m_1}\sum_{m_2=-\infty}^{\infty}\delta_{m_2,l_1}\delta^{\perp}_{m_2,l_2}$. The constraint $\delta^{\perp}_{m_2,l_2}$ is satisfied automatically.

$$\begin{aligned}T_4 :=&\Big\| \sum_{|l_1|>N/2}\sum_{|m_1|\le N/2}\frac{1}{L}\mathcal{F}(v)\left(\frac{l_1-m_1}{L}\right)g\left(\frac{m_1-l_1}{L}\right)\\&\times\hat a^\dagger(\varphi_{m_1})\hat a(\varphi_{l_1})\hat a^\dagger(\varphi_{l_1})\hat a(\varphi_{m_1})|e^{\frac{2\pi \mathrm{i}}{L}(l_1-m_1)y}\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_1}{L}})\otimes|\Lambda\rangle\Big\| \end{aligned}\tag{4.128}$$

We can interpret the different contributions pictorially:
The first step is always the same: The particle φ_{m_1} is lifted to the excited state φ_{l_1}. Afterwards there are four different possibilities:

- The first term describes the transition of the particle φ_{m_2} to the exited state φ_{l_2}.
- In T_2, the already excited particle φ_{l_1} ends in the excited state φ_{l_2}.
- In the third term, the existing hole φ_{m_1} is occupied by the particle φ_{m_2}.
- The fourth term describes the reverse process of the first step. The excited particle φ_{l_1} goes back to where it started from.

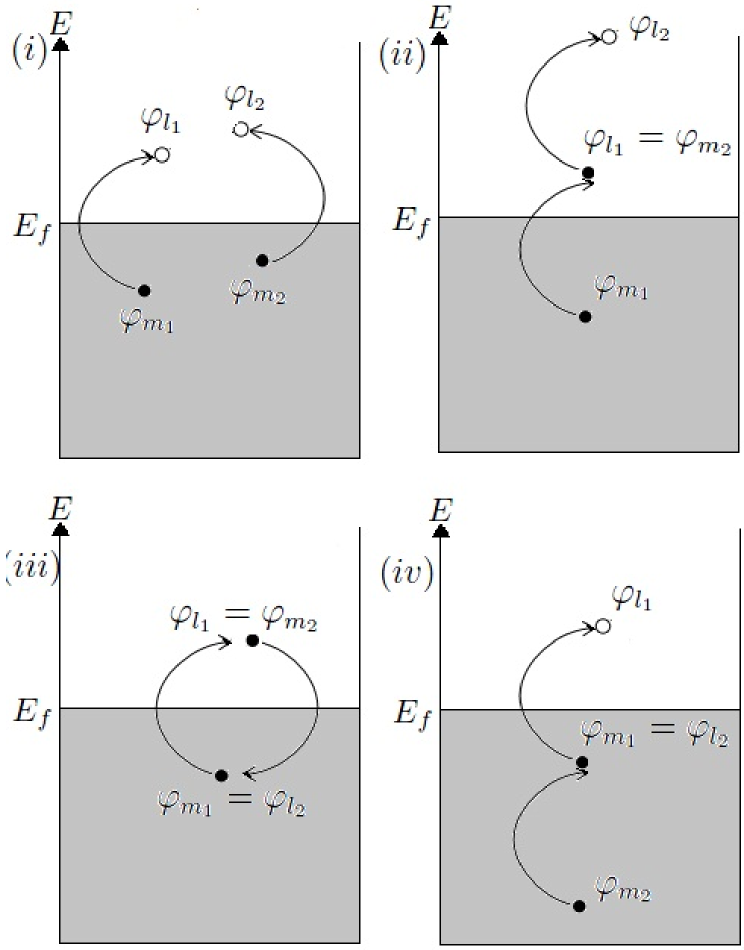

Figure 4.2: All possible transitions in second-order perturbation theory: (i) Two particles φ_{m_1} and φ_{m_2} are lifted to the excited states φ_{l_1} and φ_{l_2}. (ii) The particle φ_{m_1} is lifted twice and ends in some excited state φ_{l_2}. (iii) The particle φ_{m_1} is first lifted to the state φ_{l_1}. Afterwards, it hops back to where it started. (iv) The particle φ_{m_1} is first lifted to the state φ_{l_1}. Then, another particle φ_{l_2} jumps into the existing hole φ_{m_1}.

Each term can be estimated separately, starting with the

First term T_1: The estimate of (4.125) requires the most effort. Squaring the norm, we obtain eight sums in total, labelled by m_1, m_2, m_3, m_4 and l_1, l_2, l_3, l_4. We will first proof the following

Lemma 4.7.5. *Let* $l_i > N/2\ \forall i = 1,2,3,4$ *and let* $m_i \leq N/2\ \forall i = 1,2,3,4$. *Then*

$$\begin{aligned}
&\langle\Lambda|\hat{a}^\dagger(\varphi_{m_4})\hat{a}(\varphi_{l_4})\hat{a}^\dagger(\varphi_{m_3})\hat{a}(\varphi_{l_3})\hat{a}^\dagger(\varphi_{l_2})\hat{a}(\varphi_{m_2})\hat{a}^\dagger(\varphi_{l_1})\hat{a}(\varphi_{m_1})|\Lambda\rangle\\
=&(\delta_{m_1,m_3}\delta_{m_2,m_4}+\delta_{m_1,m_4}\delta_{m_2,m_3})(\delta_{l_1,l_3}\delta_{l_2,l_4}+\delta_{l_1,l_4}\delta_{l_2,l_3})\\
&\times\delta^\perp_{m_1,m_2}\delta^\perp_{l_1,l_2}\delta^\perp_{m_3,m_4}\delta^\perp_{l_3,l_4}\\
=&\,(\delta_{m_1,m_3}\delta_{m_2,m_4}\delta_{l_1,l_3}\delta_{l_2,l_4}+\delta_{m_1,m_4}\delta_{m_2,m_3}\delta_{l_1,l_3}\delta_{l_2,l_4})\\
&\times\delta^\perp_{m_1,m_2}\delta^\perp_{l_1,l_2}\delta^\perp_{m_3,m_4}\delta^\perp_{l_3,l_4}\\
&+(\delta_{m_1,m_4}\delta_{m_2,m_3}\delta_{l_1,l_4}\delta_{l_2,l_3}+\delta_{m_1,m_3}\delta_{m_2,m_4}\delta_{l_1,l_4}\delta_{l_2,l_3})\\
&\times\delta^\perp_{m_1,m_2}\delta^\perp_{l_1,l_2}\delta^\perp_{m_3,m_4}\delta^\perp_{l_3,l_4}
\end{aligned}\tag{4.129}$$

Proof.
The term $\delta^\perp_{m_1,m_2}\delta^\perp_{l_1,l_2}\delta^\perp_{m_3,m_4}\delta^\perp_{l_3,l_4}$ comes from the observation that it is not possible to create or remove more than one particle in the same state. For the other terms, consider the vector $\hat{a}^\dagger(\varphi_{l_2})\hat{a}(\varphi_{m_2})\hat{a}^\dagger(\varphi_{l_1})\hat{a}(\varphi_{m_1})|\Lambda\rangle$. Here, two particles *below* the Fermi edge were lifted to two *excited* states. This is due to the fact that $|l_1|, |l_2| > N/2$ and $|m_1|, m_2| \leq N/2$ holds. Consequently, $\langle\Lambda|\hat{a}^\dagger(\varphi_{m_4})\hat{a}(\varphi_{l_4})\hat{a}^\dagger(\varphi_{m_3})\hat{a}(\varphi_{l_3})$ has to be a vector where the same particles have been exchanged, otherwise the scalar product is zero. That leaves the possibilities expressed above by the Kronecker delta functions. □

Note once more that this expression is only valid since the φ_l-operators cannot create or remove a particle below the Fermi edge.
Squaring the first contribution, we obtain:

$$\begin{aligned}
T_1^2 =&\Big\| \sum_{|l_1|>N/2}\sum_{|m_1|\leq N/2}\sum_{|l_2|>N/2}\sum_{|m_2|\leq N/2}\frac{1}{L}\mathcal{F}(v)\left(\frac{m_2-l_2}{L}\right)g\left(\frac{m_1-l_1}{L}\right)\\
&\times\hat{a}^\dagger(\varphi_{l_2})\hat{a}(\varphi_{m_2})\hat{a}^\dagger(\varphi_{l_1})\hat{a}(\varphi_{m_1})|e^{\frac{2\pi\mathrm{i}}{L}(m_2-l_2)y}\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_1}{L}})\otimes|\Lambda\rangle\Big\|^2\\
=&\sum_{|l_1|>N/2}\sum_{|m_1|\leq N/2}\sum_{|l_2|>N/2}\sum_{|m_2|\leq N/2}\sum_{|l_3|>N/2}\sum_{|m_3|\leq N/2}\sum_{|l_4|>N/2}\sum_{|m_4|\leq N/2}\\
&\times g^*\left(\frac{m_4-l_4}{L}\right)\frac{1}{L}\mathcal{F}^*(v)\left(\frac{m_3-l_3}{L}\right)\frac{1}{L}\mathcal{F}(v)\left(\frac{m_2-l_2}{L}\right)g\left(\frac{m_1-l_1}{L}\right)\\
&\times\big(e^{\frac{2\pi\mathrm{i}}{L}(m_3-l_3)y}\chi_{t_a;t_b;\frac{l_4}{L};\frac{m_4}{L}}|e^{\frac{2\pi\mathrm{i}}{L}(m_2-l_2)y}\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_1}{L}}\big)\\
&\times\Big(\delta_{m_1,m_3}\delta_{m_2,m_4}\delta_{l_1,l_3}\delta_{l_2,l_4}+\delta_{m_1,m_4}\delta_{m_2,m_3}\delta_{l_1,l_3}\delta_{l_2,l_4}\\
&+\delta_{m_1,m_4}\delta_{m_2,m_3}\delta_{l_1,l_4}\delta_{l_2,l_3}+\delta_{m_1,m_3}\delta_{m_2,m_4}\delta_{l_1,l_4}\delta_{l_2,l_3}\Big)\\
&\times\delta^\perp_{m_1,m_2}\delta^\perp_{l_1,l_2}\delta^\perp_{m_3,m_4}\delta^\perp_{l_3,l_4}
\end{aligned}\tag{4.130}$$

This yields four different contributions which will again be evaluated separately.

The first term, coming from $\delta_{m_1,m_3}\delta_{m_2,m_4}\delta_{l_1,l_3}\delta_{l_2,l_4}$, reads as follows:

$$\begin{aligned}
&\sum_{|l_1|>N/2}\sum_{|m_1|\le N/2}\sum_{|l_2|>N/2,l_2\neq l_1}\sum_{|m_2|\le N/2,m_2\neq m_1} g^*\left(\frac{m_2-l_2}{L}\right)\frac{1}{L}\mathcal{F}^*(v)\left(\frac{m_1-l_1}{L}\right)\\
&\times\frac{1}{L}\mathcal{F}(v)\left(\frac{m_2-l_2}{L}\right)g\left(\frac{m_1-l_1}{L}\right)\left(e^{\frac{2\pi\mathrm{i}}{L}(m_1-l_1)y}\chi_{t_a;t_b;\frac{l_2}{L};\frac{m_2}{L}}|e^{\frac{2\pi\mathrm{i}}{L}(m_2-l_2)y}\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_1}{L}}\right)\\
&\le\left(\sum_{|l|>N/2}\sum_{|m|\le N/2}\left|\frac{1}{L}\mathcal{F}(v)\left(\frac{m-l}{L}\right)g^*\left(\frac{m-l}{L}\right)\right|\|\chi_{t_a;t_b;\frac{l}{L};\frac{m}{L}}\|_y\right)^2\\
&\le\left(\sum_{|l_1|>N/2}\sum_{|m_1|\le N/2}\left|\frac{1}{L}\mathcal{F}(v)\left(\frac{m_1-l_1}{L}\right)\right|^2\right)\left(\sum_{|l_2|>N/2}\sum_{|m_2|\le N/2}\left|g\left(\frac{m_2-l_2}{L}\right)\right|^2\|\chi_{t_a;t_b;\frac{l_2}{L};\frac{m_2}{L}}\|_y^2\right)\\
&=\left(\sum_{|l_1|>N/2}\sum_{|m_1|\le N/2}\left|\frac{1}{L}\mathcal{F}(v)\left(\frac{m_1-l_1}{L}\right)\right|^2\right)\left(\sum_{|l_2|>N/2}\sum_{|m_2|\le N/2}\left|g\left(\frac{m_2-l_2}{L}\right)\right|^2\eta\left(\frac{m_2-l_2}{L}\right)\right)\\
&=:M_1
\end{aligned}\tag{4.131}$$

The second term, coming from $\delta_{m_1,m_4}\delta_{m_2,m_3}\delta_{l_1,l_3}\delta_{l_2,l_4}$, reads as follows:

$$\begin{aligned}
&\sum_{|l_1|>N/2}\sum_{|m_1|\le N/2}\sum_{|l_2|>N/2,l_2\neq l_1}\sum_{|m_2|\le N/2,m_2\neq m_1} g^*\left(\frac{m_1-l_2}{L}\right)\frac{1}{L}\mathcal{F}^*(v)\left(\frac{m_2-l_1}{L}\right)\\
&\times\frac{1}{L}\mathcal{F}(v)\left(\frac{m_2-l_2}{L}\right)g\left(\frac{m_1-l_1}{L}\right)\left(e^{\frac{2\pi\mathrm{i}}{L}(m_2-l_1)y}\chi_{t_a;t_b;\frac{l_2}{L};\frac{m_1}{L}}|e^{\frac{2\pi\mathrm{i}}{L}(m_2-l_2)y}\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_1}{L}}\right)\\
&\le\|Lg\left(\cdot\right)\eta(\cdot)\|_\infty\|\mathcal{F}(v)\left(\cdot\right)\|_\infty\\
&\times\left(\sum_{|l_1|>N/2}\sum_{|m_1|\le N/2}\left|\frac{1}{L}g\left(\frac{m_1-l_1}{L}\right)\eta\left(\frac{m_1-l_1}{L}\right)\right|\right)\left(\sum_{|l_2|>N/2,l_2}\sum_{|m_2|\le N/2,m_2}\left|\frac{1}{L^2}\mathcal{F}(v)\left(\frac{m_2-l_2}{L}\right)\right|\right)\\
&=:M_2
\end{aligned}\tag{4.132}$$

The third term, coming from $\delta_{m_1,m_4}\delta_{m_2,m_3}\delta_{l_1,l_4}\delta_{l_2,l_3}$, reads as follows:

$$\begin{aligned}
&\sum_{|l_1|>N/2}\sum_{|m_1|\le N/2}\sum_{|l_2|>N/2,l_2\neq l_1}\sum_{|m_2|\le N/2,m_2\neq m_1}\left|\frac{1}{L}\mathcal{F}(v)\left(\frac{m_1-l_1}{L}\right)\right|^2\left|g\left(\frac{m_2-l_2}{L}\right)\right|^2\\
&\times\|e^{\frac{2\pi\mathrm{i}}{L}(m_2-l_2)y}\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_1}{L}}\|_y^2\\
&\le\left(\sum_{|l_1|>N/2}\sum_{|m_1|\le N/2}\left|\frac{1}{L}\mathcal{F}(v)\left(\frac{m_1-l_1}{L}\right)\right|^2\right)\left(\sum_{|l_2|>N/2}\sum_{|m_2|\le N/2}\left|g\left(\frac{m_2-l_2}{L}\right)\right|^2\|\chi_{t_a;t_b;\frac{l_2}{L};\frac{m_2}{L}}\|_y^2\right)\\
&=M_1
\end{aligned}\tag{4.133}$$

The fourth term, coming from $\delta_{m_1,m_3}\delta_{m_2,m_4}\delta_{l_1,l_4}\delta_{l_2,l_3}$, reads as follows:

$$\begin{aligned}
&\sum_{|l_1|>N/2}\sum_{|m_1|\leq N/2}\sum_{|l_2|>N/2,l_2\neq l_1}\sum_{|m_2|\leq N/2,m_2\neq m_1} g^*\left(\frac{m_2-l_1}{L}\right)\frac{1}{L}\mathcal{F}^*(v)\left(\frac{m_1-l_2}{L}\right)\\
&\times\frac{1}{L}\mathcal{F}(v)\left(\frac{m_2-l_2}{L}\right)g\left(\frac{m_1-l_1}{L}\right)\left(e^{\frac{2\pi\mathrm{i}}{L}(m_1-l_2)y}\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_2}{L}}|e^{\frac{2\pi\mathrm{i}}{L}(m_2-l_2)y}\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_1}{L}}\right)\\
\leq&\|Lg\left(\cdot\right)\eta(\cdot)\|_\infty\|\mathcal{F}(v)\left(\cdot\right)\|_\infty\\
&\times\left(\sum_{|l_1|>N/2}\sum_{|m_1|\leq N/2}\left|\frac{1}{L}g\left(\frac{m_1-l_1}{L}\right)\eta\left(\frac{m_1-l_1}{L}\right)\right|\right)\left(\sum_{|l_2|>N/2,l_2}\sum_{|m_2|\leq N/2,m_2}\left|\frac{1}{L^2}\mathcal{F}(v)\left(\frac{m_2-l_2}{L}\right)\right|\right)\\
=&M_2
\end{aligned}\tag{4.134}$$

So, in total, the following bound is obtained:

$$T_1\leq\sqrt{2M_1+2M_2}\tag{4.135}$$

Second term T_2: In this term, the particle φ_{m_1} gets lifted two times. Note that $\hat{a}^\dagger(\varphi_{l_2})\hat{a}(\varphi_{l_1})\hat{a}^\dagger(\varphi_{l_1})\hat{a}(\varphi_{m_1})|\Lambda\rangle=\hat{a}^\dagger(\varphi_{l_2})\hat{a}(\varphi_{m_1})|\Lambda\rangle$. This can be proven either by using the definition of the operators or by keeping in mind that the two operators $\hat{a}^\dagger(\varphi_{l_1})\hat{a}(\varphi_{m_1})$ and $\hat{a}^\dagger(\varphi_{l_2})\hat{a}(\varphi_{l_1})$ act the very same way as the corresponding operators in Fock space. This yields:

$$\begin{aligned}
T_2^2=&\Bigg\|\sum_{|m_1|\leq N/2}\sum_{|l_2|>N/2}\left(\sum_{|l_1|>N/2}\frac{1}{L}\mathcal{F}(v)\left(\frac{l_1-l_2}{L}\right)g\left(\frac{m_1-l_1}{L}\right)\right)\\
&\times\hat{a}^\dagger(\varphi_{l_2})\hat{a}(\varphi_{m_1})|e^{\frac{2\pi\mathrm{i}}{L}(l_1-l_2)y}\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_1}{L}}\rangle\otimes|\Lambda\rangle\Bigg\|^2\\
=&\sum_{|m_1|\leq N/2}\sum_{|l_2|>N/2}\left(\sum_{|l_1'|>N/2}\frac{1}{L}\mathcal{F}^*(v)\left(\frac{l_1'-l_2}{L}\right)g^*\left(\frac{m_1-l_1'}{L}\right)\right)\\
&\times\left(\sum_{|l_1|>N/2}\frac{1}{L}\mathcal{F}(v)\left(\frac{l_1-l_2}{L}\right)g\left(\frac{m_1-l_1}{L}\right)\right)\left(e^{\frac{2\pi\mathrm{i}}{L}(l_1'-l_2)y}\chi_{t_a;t_b;\frac{l_1'}{L};\frac{m_1}{L}}|e^{\frac{2\pi\mathrm{i}}{L}(l_1-l_2)y}\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_1}{L}}\right)\\
\leq&\sum_{|m_1|\leq N/2}\sum_{|l_2|>N/2}\left(\sum_{|l_1|>N/2}\left|\frac{1}{L}\mathcal{F}(v)\left(\frac{l_1-l_2}{L}\right)g\left(\frac{m_1-l_1}{L}\right)\right|\|\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_1}{L}}\|_y\right)^2\\
=&\sum_{|m_1|\leq N/2}\sum_{|l_2|>N/2}\left(\sum_{|l_1|>N/2}\left|\frac{1}{L}\mathcal{F}(v)\left(\frac{l_1-l_2}{L}\right)g\left(\frac{m_1-l_1}{L}\right)\right|\eta\left(\frac{m_1-l_1}{L}\right)\right)^2\\
=:&M_3
\end{aligned}\tag{4.136}$$

So

$$T_2\leq\sqrt{M_3}\tag{4.137}$$

Third term T_3: In this term, another particle occupies the existing hole φ_{m_1}. Note that $\hat{a}^\dagger(\varphi_{m_1})\hat{a}(\varphi_{m_2})\hat{a}^\dagger(\varphi_{l_1})\hat{a}(\varphi_{m_1})|\Lambda\rangle = -\hat{a}^\dagger(\varphi_{l_1})\hat{a}(\varphi_{m_2})|\Lambda\rangle$ holds, so we obtain:

$$\begin{aligned}
T_3^2 =& \Bigg\| \sum_{|l_1|>N/2} \sum_{|m_2|\le N/2} \left(\sum_{|m_1|\le N/2} \frac{1}{L}\mathcal{F}(v) \left(\frac{m_2-m_1}{L}\right) g\left(\frac{m_1-l_1}{L}\right)\right) \\
&\times \hat{a}^\dagger(\varphi_{l_1})\hat{a}(\varphi_{m_2}) |e^{\frac{2\pi \mathrm{i}}{L}(m_2-m_1)y}\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_1}{L}}\rangle \otimes |\Lambda\rangle \Bigg\|^2 \\
=& \sum_{|m_2|\le N/2} \sum_{|l_1|>N/2} \left(\sum_{|m_1'|\le N/2} \frac{1}{L}\mathcal{F}^*(v) \left(\frac{m_2-m_1'}{L}\right) g^*\left(\frac{m_1'-l_1}{L}\right)\right) \\
&\times \left(\sum_{|m_1|\le N/2} \frac{1}{L}\mathcal{F}(v) \left(\frac{m_2-m_1}{L}\right) g\left(\frac{m_1-l_1}{L}\right)\right) (e^{\frac{2\pi \mathrm{i}}{L}(m_2-m_1')y}\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_1'}{L}} | e^{\frac{2\pi \mathrm{i}}{L}(m_2-m_1)y}\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_1}{L}}) \\
\le& \sum_{|m_2|\le N/2} \sum_{|l_1|>N/2} \left(\sum_{|m_1|\le N/2} \left| \frac{1}{L}\mathcal{F}(v) \left(\frac{m_2-m_1}{L}\right) g\left(\frac{m_1-l_1}{L}\right)\right| \|\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_1}{L}}\|_y \right)^2 \\
=& \sum_{|m_2|\le N/2} \sum_{|l_1|>N/2} \left(\sum_{|m_1|\le N/2} \left| \frac{1}{L}\mathcal{F}(v) \left(\frac{m_2-m_1}{L}\right) g\left(\frac{m_1-l_1}{L}\right)\right| \eta\left(\frac{m_1-l_1}{L}\right) \right)^2 \\
=:& M_4
\end{aligned} \tag{4.138}$$

Therefore,

$$T_3 \le \sqrt{M_4} \tag{4.139}$$

Fourth term T_4: In this term, the particle φ_{m_1} visits the upper spectrum but then decides to come back from where it started. Note that $\hat{a}^\dagger(\varphi_{m_1})\hat{a}(\varphi_{l_1})\hat{a}^\dagger(\varphi_{l_1})\hat{a}(\varphi_{m_1})|\Lambda\rangle = |\Lambda\rangle$ holds. Hence:

$$\begin{aligned}
T_4^2 =& \Bigg\| \sum_{|l|>N/2} \sum_{|m|\le N/2} \frac{1}{L}\mathcal{F}(v) \left(\frac{l-m}{L}\right) g\left(\frac{m-l}{L}\right) \\
&\times \hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m)\hat{a}^\dagger(\varphi_l)\hat{a}(\varphi_m) |e^{\frac{2\pi \mathrm{i}}{L}(m-l)y}\chi_{t_a;t_b;\frac{l}{L};\frac{m}{L}}\rangle \otimes |\Lambda\rangle \Bigg\|^2 \\
=& \sum_{|l_1|>N/2} \sum_{|m_1|\le N/2} \frac{1}{L}\mathcal{F}^*(v) \left(\frac{l_1-m_1}{L}\right) g^*\left(\frac{m_1-l_1}{L}\right) \\
&\times \sum_{|l_2|>N/2} \sum_{|m_2|\le N/2} \frac{1}{L}\mathcal{F}(v) \left(\frac{l_2-m_2}{L}\right) g\left(\frac{m_2-l_2}{L}\right) (e^{\frac{2\pi \mathrm{i}}{L}(m_1-l_1)y}\chi_{t_a;t_b;\frac{l_1}{L};\frac{m_1}{L}}) | e^{\frac{2\pi \mathrm{i}}{L}(m_2-l_2)y}\chi_{t_a;t_b;\frac{l_2}{L};\frac{m_2}{L}}) \\
\le& \left(\sum_{|l|>N/2} \sum_{|m|\le N/2} \left| \frac{1}{L}\mathcal{F}(v) \left(\frac{l-m}{L}\right) g\left(\frac{m-l}{L}\right)\right| \|\chi_{t_a;t_b;\frac{l}{L};\frac{m}{L}}\|_y \right)^2 \\
\le& \left(\sum_{|l_1|>N/2} \sum_{|m_1|\le N/2} \left| \frac{1}{L}\mathcal{F}(v) \left(\frac{m_1-l_1}{L}\right)\right|^2 \right) \left(\sum_{|l_2|>N/2} \sum_{|m_2|\le N/2} \left| g\left(\frac{m_2-l_2}{L}\right)\right|^2 \|\chi_{t_a;t_b;\frac{l_2}{L};\frac{m_2}{L}}\|_y^2 \right) \\
=& M_1
\end{aligned} \tag{4.140}$$

This gives the bound:

$$T_4 \leq \sqrt{M_1} \tag{4.141}$$

Plugging all estimates together shows the lemma. □

This calculation was the most complicated one. Now, we will analyse the functions M_i and show them to be bounded for $\tilde{v}$ and to decay for w.

Lemma 4.7.6. *Let $M_i\left(\eta; \frac{1}{L}\mathcal{F}(\tilde{v}); v; N; L\right)$ $i = 1, 2, 3, 4$ be defined as above. Then*

$$\sup_{\rho}\left(\lim_{TD} M_i\left(\eta; \frac{1}{L}\mathcal{F}(\tilde{v}); v; N; L\right)\right) < \infty \tag{4.142}$$

Proof.
First note that we can replace $\tilde{v}$ by v, that is

$$\left|\frac{1}{L}\mathcal{F}(\tilde{v})\left(\frac{m-l}{L}\right)\right| \leq \left|\frac{1}{L}\mathcal{F}(v)\left(\frac{m-l}{L}\right)\right| \tag{4.143}$$

We will bound each function separately, using the Paley-Wiener theorem again.

Bound for M_1:

$$\begin{aligned}\lim_{\text{TD}} M_1 \leq &\underbrace{\left(\int_{|l_1|>\rho/2} dl \int_{|m_1|\leq\rho/2} dm \left|\mathcal{F}(v)\left(m_1 - l_1\right)\right|^2\right)}_{=\lim_{\text{TD}} \operatorname{Var}^v_{N+1}(0)} \\ &\times \left(\int_{|l_2|>\rho/2} dl \int_{|m_2|\leq\rho/2} dm \left|\mathcal{F}(v)\left(m_2 - l_2\right)\right|^2 \eta(m_2 - l_2)^2\right)\end{aligned} \tag{4.144}$$

This term is finite for all ρ by Lemma 4.7.2

Bound for M_2:

$$\begin{aligned}\lim_{\text{TD}} M_2 \leq &\|\mathcal{F}(v)\left(\cdot\right)\eta(\cdot)\|_\infty \|\mathcal{F}(v)\left(\cdot\right)\|_\infty \left(\int_{|l_1|>\rho/2} dl \int_{|m_1|\leq\rho/2} dm \left|\mathcal{F}(v)\left(m_1 - l_1\right)\eta\left(m_1 - l_1\right)\right|\right) \\ &\times \left(\int_{|l_2|>\rho/2} dl \int_{|m_2|\leq\rho/2} dm \left|\mathcal{F}(v)\left(m_2 - l_2\right)\right|\right)\end{aligned} \tag{4.145}$$

Using $\eta(k) \leq 4\pi\|\hat{p}_y\chi_0\|_y|k| + 4\pi^2k^2$, we can bound

$$\begin{aligned}\|\mathcal{F}(v)\left(\cdot\right)\eta(\cdot)\|_\infty \leq &\sup_{k\in\mathbb{R}} |\mathcal{F}(v)(k)(4\pi\|\hat{p}_y\chi_0\|_y|k| + 4\pi^2k^2)| \\ \leq &2\|\hat{p}_y\chi_0\|_y \int_{\mathbb{R}} dx \left|\frac{dv(x)}{dx}\right| + \int_{\mathbb{R}} dx \left|\frac{d^2v(x)}{dx^2}\right| < \infty\end{aligned} \tag{4.146}$$

By Lemma 4.7.2 the two integrals are finite for all ρ.

Bound for M_3:

$$\begin{aligned}
\lim_{\mathrm{TD}} M_3 \leq & \int_{|m|\leq\rho/2} dm \int_{|l|>\rho/2} dl \left(\int_{|l'|>\rho/2} dl' \big|\mathcal{F}(v)\left(l'-l\right)\mathcal{F}(v)\left(m-l'\right)\eta(m-l')\big| \right)^2 \\
\leq & \int_{|m|\leq\rho/2} dm \int_{|l|>\rho/2} dl \left(\int_{\mathbb{R}} dx \big|\mathcal{F}(v)\left(x-[m-l]\right)\big|\big|\mathcal{F}(v)\left(x\right)\eta(x)\big| \right)^2 \\
= & \int_{|m|\leq\rho/2} dm \int_{|l|>\rho/2} dl \left[\left(\big|\mathcal{F}(v)\big| * \big|\mathcal{F}(v)\eta\big| \right)(m-l) \right]^2 \\
=& 2\int_0^{\rho/2} dm \int_{\rho/2}^{\infty} dl \left[\left(\big|\mathcal{F}(v)\big| * \big|\mathcal{F}(v)\eta\big| \right)(m-l) \right]^2 \\
&+2\int_{-\rho/2}^{0} dm \int_{\rho/2}^{\infty} dl \left[\left(\big|\mathcal{F}(v)\big| * \big|\mathcal{F}(v)\eta\big| \right)(m-l) \right]^2
\end{aligned} \tag{4.147}$$

These two terms are bounded since the convolution has again rapid decrease. To see this, suppose first that $\eta = 1$. Then, the first term can be estimated as follows:

$$\begin{aligned}
& \int_0^{\rho/2} dm \int_{\rho/2}^{\infty} dl \left[\left(\big|\mathcal{F}(v)\big| * \big|\mathcal{F}(v)\big| \right)(m-l) \right]^2 \\
=& \int_0^{\rho/2} dm \int_{m}^{\infty} dl \left[\left(\big|\mathcal{F}(v)\big| * \big|\mathcal{F}(v)\big| \right)(l) \right]^2 \\
=& \int_0^{\rho/2} dm \left(\frac{d}{dm} m \right) \int_{m}^{\infty} dl \left[\left(\big|\mathcal{F}(v)\big| * \big|\mathcal{F}(v)\big| \right)(l) \right]^2 \\
=& \rho/2 \int_{\rho/2}^{\infty} dl \left[\left(\big|\mathcal{F}(v)\big| * \big|\mathcal{F}(v)\big| \right)(l) \right]^2 \\
&+ \int_0^{\rho/2} dm\, m \left[\left(\big|\mathcal{F}(v)\big| * \big|\mathcal{F}(v)\big| \right)(m) \right]^2
\end{aligned} \tag{4.148}$$

We treat each contribution separately. For the first term we use the estimate

$$\begin{aligned}
& \int_{\rho/2}^{\infty} dl \left[\left(\big|\mathcal{F}(v)\big| * \big|\mathcal{F}(v)\big| \right)(l) \right]^2 \\
\leq & \left(\sup_{l>\rho/2} \left(\big|\mathcal{F}(v)\big| * \big|\mathcal{F}(v)\big| \right)(l) \right) \underbrace{\int_{\mathbb{R}} dl \left(\big|\mathcal{F}(v)\big| * \big|\mathcal{F}(v)\big| \right)(l)}_{<\infty}
\end{aligned} \tag{4.149}$$

It remains to bound

$$\begin{aligned}
&\sup_{l>\rho/2}\left(\left|\mathcal{F}(v)\right| * \left|\mathcal{F}(v)\right|\right)(l)\\
&\leq \sup_{l>\rho/2}\left(\int_{\mathbb{R}} dx \left|\mathcal{F}(v)\right|(x-l/2)\left|\mathcal{F}(v)\right|(x+l/2)\right)\\
&\leq \sup_{l>\rho/2}\left(\int_{\mathbb{R}} dx \frac{D_{2p}}{(1+|x-l/2|)^{2p}}\frac{D_{2p}}{(1+|x+l/2|)^{2p}}\right)\\
&\leq \sup_{l>\rho/2}\left(\sup_{x'\in\mathbb{R}}\left[\frac{1}{(1+|x'-l/2|)^{p}}\frac{1}{(1+|x'+l/2|)^{p}}\right]\int_{\mathbb{R}} dx \frac{D_{2p}}{(1+|x-l/2|)^{p}}\frac{D_{2p}}{(1+|x+l/2|)^{p}}\right)\\
&\leq C \sup_{l>\rho/2}\sup_{x'\in\mathbb{R}}\left[\frac{1}{(1+|x'-l/2|)^{p}}\frac{1}{(1+|x'+l/2|)^{p}}\right]\\
&\sim \frac{1}{\rho^p}
\end{aligned} \tag{4.150}$$

Since this bound holds for any p, the first term vanishes as $\rho \to \infty$.
The second term in equation (4.148) is also bounded since its kernel is integrable on $\mathbb{R}$.

Now, let $\eta(k) = (4\pi\|\hat{p}_y\chi_0\|_y|k| + 4\pi^2k^2)$. Note that

$$\mathcal{F}(v)(k)\left(4\pi\|\hat{p}_y\chi_0\|_y|k| + 4\pi^2k^2\right) \tag{4.151}$$

is again decreasing rapidly in k. Hence, we can repeat the above estimate to conclude boundedness. The same line of argumentation, which we will not present here, works also for the contribution $\int_{-\rho/2}^{0} dm \int_{\rho/2}^{\infty} dl \left[\left(\left|\mathcal{F}(v)\right| * \left|\mathcal{F}(v)\eta\right|\right)(m-l)\right]^2$.

Bound for M_4: $\lim_{\mathrm{TD}} M_4$ can be bounded exactly the same way:

$$\begin{aligned}
\lim_{\mathrm{TD}} M_4 &= \int_{|m|\leq\rho/2} dm \int_{|l|>\rho/2} dl \left(\int_{|m'|<\rho/2} dm' \left|\mathcal{F}(v)(m-m')\,\mathcal{F}(v)(m'-l)\,\eta(m'-l)\right|\right)^2\\
&\leq \int_{|m|\leq\rho/2} dm \int_{|l|>\rho/2} dl \left(\int_{\mathbb{R}} dx \left|\mathcal{F}(v)(x-[m-l])\right|\left|\mathcal{F}(v)(x)\right|\eta(x)\right)^2\\
&= \int_{|m|\leq\rho/2} dm \int_{|l|>\rho/2} dl \left[\left(\left|\mathcal{F}(v)\right| * \left|\mathcal{F}(v)\eta\right|\right)(m-l)\right]^2
\end{aligned} \tag{4.152}$$

□

Finally, we treat the contribution arising from w. Since we could not apply the stationary phase method, we need to use the smallness of the support of w in order to get a factor of $\rho^{-1/2}$. This will be done in the next

Lemma 4.7.7. *Let $M_i(\eta; \frac{1}{L}\mathcal{F}(w); v; N; L)$ $i = 1, 2, 3, 4$ be defined as above. Then*

$$\lim_{TD} M_i\left(\eta; \frac{1}{L}\mathcal{F}(w); v; N; L\right) \lesssim \frac{1}{\rho} \tag{4.153}$$

holds.

Proof.
We will bound every contribution using the smallness of the support of $\frac{1}{L}\mathcal{F}(w)\left(\frac{m-l}{L}\right)$. Note, for the estimates considered here, we need only consider the case

$$\eta(k) = 1 \tag{4.154}$$

Bound for M_1: In analogy to the estimate provided in equation (4.84), it is possible to bound $\lim_{\mathrm{TD}} M_1$ as follows:

$$\begin{aligned}\lim_{\mathrm{TD}} M_1 \leq& \lim_{\mathrm{TD}} \mathrm{Var}^v_{N+1}(0) \left(\int_{|l|>\rho/2} dl \int_{|m|\leq\rho/2} dm \left|\mathcal{F}(w)\,(m-l)\right|^2 \eta(m-l)^2 \right) \\ \leq& \lim_{\mathrm{TD}} \mathrm{Var}^v_{N+1}(0) \frac{2\|v\|_1^2}{\rho}\end{aligned} \tag{4.155}$$

Bound for M_2: Using once more $\|\mathcal{F}(w)\|_\infty \|\mathcal{F}(v)\|_\infty \leq \|v\|_1^2$, the following estimate is true:

$$\begin{aligned}\lim_{\mathrm{TD}} M_2 \leq& \|\mathcal{F}(w)\,(\cdot)\,\|_\infty \|\mathcal{F}(v)\,(\cdot)\,\|_\infty \left(\int_{|l_1|>\rho/2} dl \int_{|m_1|\leq\rho/2} dm \left|\mathcal{F}(w)\,(m_1-l_1)\right| \right) \\ &\times \left(\int_{|l_2|>\rho/2} dl \int_{|m_2|\leq\rho/2} dm \left|\mathcal{F}(v)\,(m_2-l_2)\right| \right) \\ \leq& \underbrace{\left(\int_{|l_2|>\rho/2} dl \int_{|m_2|\leq\rho/2} dm \left|\mathcal{F}(v)\,(m_2-l_2)\right| \right)}_{<\infty} \frac{2\|v\|_1^3}{\rho}\end{aligned} \tag{4.156}$$

Bound for M_3: To this end we must estimate

$$\lim_{\mathrm{TD}} M_3 = \int_{|m|\leq\rho/2} dm \int_{|l|>\rho/2} dl \left(\int_{|l'|>\rho/2} dl' \left|\mathcal{F}(v)\,(l'-l)\,\right|\left|\mathcal{F}(w)\,(m-l')\,\right| \right)^2 \tag{4.157}$$

Using

$$\mathrm{supp}(\mathcal{F}(w)) \in \left[-\frac{1}{\sqrt{\rho}}; \frac{1}{\sqrt{\rho}}\right] \tag{4.158}$$

we conclude:

$$|m| \in \left[\rho/2 - \frac{1}{\sqrt{\rho}}, \rho/2\right] \tag{4.159}$$

Since also $|l'| > \rho$ holds, we can furthermore restrict

$$|l'| \in \left[\rho/2, \rho/2 + \frac{1}{\sqrt{\rho}}\right] \tag{4.160}$$

Hence:

$$\begin{aligned}
&\int_{|m|\le\rho/2} dm \int_{|l|>\rho/2} dl \left(\int_{|l'|>\rho/2} dl' \big|\mathcal{F}(v)\left(l'-l\right)\big|\big|\mathcal{F}(w)\left(m-l'\right)\big| \right)^2 \\
\le& 2\frac{\|v\|_1^2}{\rho} \int_{|l|>\rho/2} dl \left(\sup_{|l'|\in[\rho/2,\rho/2+\frac{1}{\sqrt{\rho}}]} \big|\mathcal{F}(v)\left(l'-l\right)\big| \right)^2 \\
\le& 4\frac{\|v\|_1^2}{\rho} \underbrace{\int_{\rho/2}^{\infty} dl \sup_{l'\in[\rho/2,\rho/2+\frac{1}{\sqrt{\rho}}]} \left(\frac{D_p}{(1+|l'-l|)^p} \right)^2}_{<\infty\ \forall\rho}
\end{aligned} \tag{4.161}$$

Bound for M_4: This term can be bounded in analogy to M_3.

$$\lim_{\text{TD}} M_4 = \int_{|m|\le\rho/2} dm \int_{|l|>\rho/2} dl \left(\int_{|m'|<\rho/2} dm' \big|\mathcal{F}(v)\left(m-m'\right)\mathcal{F}(w)\left(m'-l\right)\big| \right)^2 \tag{4.162}$$

Using

$$|m'| \in \left[\rho/2 - \frac{1}{\sqrt{\rho}}, \rho/2\right] \ \wedge\ |l| \in \left[\rho/2, \rho/2 + \frac{1}{\sqrt{\rho}}\right] \tag{4.163}$$

we obtain:

$$\begin{aligned}
&\int_{|m|\le\rho/2} dm \int_{|l|>\rho/2} dl \left(\int_{|m'|<\rho/2} dm' \big|\mathcal{F}(v)\left(m-m'\right)\mathcal{F}(w)\left(m'-l\right)\big| \right)^2 \\
\le& 2\frac{\|v\|_1^2}{\rho} \int_{|m|<\rho/2} dm \left(\sup_{|m'|\in[\rho/2-\frac{1}{\sqrt{\rho}},\rho/2]} \big|\mathcal{F}(v)\left(m-m'\right)\big| \right)^2 \\
\le& 4\frac{\|v\|_1^2}{\rho} \underbrace{\int_0^{\rho/2} dm \sup_{m'\in[\rho/2-\frac{1}{\sqrt{\rho}},\rho/2]} \left(\frac{D_p}{(1+|m-m'|)^p} \right)^2}_{<\infty\ \forall\rho}
\end{aligned} \tag{4.164}$$

□

4.8 Corollaries to the proof

Until now, we have chosen a very simple Hamiltonian in order to focus on the underlying structure of the proof. One may ask if it is possible to generalise this procedure to more complicated systems. There are two possible generalisations:
We can add some external potential which acts on the tracer particle and/or on the sea, or we may consider the mutual interaction between the sea particles. While the first option is relatively simple to include, the latter requires much more effort. As stated before, mutual interactions between fermions were for example treated in [Benedikter et al., 2013, Bardos et al., 2002, Elgart et al., 2004] and required the scaling of the interaction.
Therefore, we want to focus only on the first option, that is, we want to include some external potential. Physically, this system may describe the composed system coupled to some electric field and naturally arises in the study of semiconductor physics, where one measures the conductivity of the carrier electrons. This potential may also arise from the nuclei[4] and consequently would yield to the band structure of solids.
We first include an external potential which only acts upon the tracer particle.

Lemma 4.8.1. *Let*

$$H = \sum_{k=0}^{N}(-\Delta_{x_k}) - \Delta_y + \sum_{k=0}^{N} v(x_k - y) + u(y) \tag{4.165}$$

where $u \in C^\infty(\mathbb{T}) \cap C_0^\infty(\mathbb{R})$ *and let*

$$H^f = \sum_{k=0}^{N}(-\Delta_{x_k}) - \Delta_y + u(y) + \rho\mathcal{F}(v)\,(0) \tag{4.166}$$

Then

$$\lim_{TD}\left(1 - \langle\langle \psi_t | \psi_t^f \rangle\rangle\right) \le \frac{B_t}{\sqrt{\rho}} \tag{4.167}$$

holds, where B_t *only depends on the potential* v*, on* $|\chi_0)$ *and grows at most like* t^2*.*

Proof.
We can repeat the proof from above with minor modifications. Explicitly, using Stone's theorem, equation (4.87) now reads:

$$\begin{aligned}
&\mathrm{i}\frac{d}{dt'}\left(U^y_{-t'}\, e^{2\pi \mathrm{i}(m-l)y} U^y_{t'}|\chi_0)\right) \\
=&\,(\Delta_y - u(y))\left(U^y_{-t'}\, e^{2\pi \mathrm{i}(m-l)y} U^y_{t'}|\chi_0)\right) + U^y_{-t'}\, e^{2\pi \mathrm{i}(m-l)y}\left(-\Delta_y + u(y)\right) U^y_{t'}|\chi_0) \\
=&\,4\pi^2(m-l)^2 U^y_{-t'}\, e^{2\pi \mathrm{i}(m-l)y} U^y_{t'}|\chi_0) + 4\pi(m-l) U^y_{-t'}\, e^{2\pi \mathrm{i}(m-l)y} U^y_{t'}(-\mathrm{i}\partial_y)|\chi_0) \\
&-u(y) U^y_{-t'}\, e^{2\pi \mathrm{i}(m-l)y} U^y_{t'}|\chi_0) + U^y_{-t'}\, e^{2\pi \mathrm{i}(m-l)y} u(y) U^y_{t'}|\chi_0)
\end{aligned} \tag{4.168}$$

which can be bounded in norm by:

$$\begin{aligned}
&\left\| \mathrm{i}\frac{d}{dt'}\left(U^y_{-t'}\, e^{2\pi \mathrm{i}(m-l)y} U^y_{t'}\chi_0\right)\right\|_y \\
\le&\, 4\pi|m-l|\|p_y\chi_0\|_y + 4\pi^2(m-l)^2 + 2\|u\|_{\mathrm{op}}
\end{aligned} \tag{4.169}$$

Using this new estimate, the proof from above can be applied. □

[4] Note however that we only consider non-singular, smooth potentials with compact support.

We can also add some external potential which acts upon the sea.

Lemma 4.8.2. *Let*

$$H = \sum_{k=0}^{N}(-\Delta_{x_k}) - \Delta_y + \sum_{k=0}^{N}(v(x_k - y) + A(x_k)) + u(y) \tag{4.170}$$

where $A \in C^\infty(\mathbb{T}) \cap C_0^\infty(\mathbb{R})$ *and let*

$$H^f = \sum_{k=0}^{N}(-\Delta_{x_k}) - \Delta_y + u(y) + \rho\left(\mathcal{F}(v)(0) + \mathcal{F}(A)(0)\right) \tag{4.171}$$

Then

$$\lim_{TD}\left(1 - \langle\langle\psi_t|\psi_t^f\rangle\rangle\right) \leq \frac{B_t}{\sqrt{\rho}} \tag{4.172}$$

where B_t *only depends on the potential* v*, on* $|\chi_0\big)$ *and grows at most like* t^2*.*

Proof.
Essentially, this lemma can be proven using the techniques of our main proof. Using Cook's lemma, this time the perturbation expansion reads:

$$\begin{aligned}
\langle\langle\psi_t|Q|\psi_t\rangle\rangle =& \langle\langle\psi_t|Q(U_t - U_t^f)|\psi_0\rangle\rangle \\
=& - \mathrm{i}\int_0^t dt'\langle\langle\psi_t|QU_{t-t'}\left(W(\vec{x};y) - W^f\right)U_{t'}^f|\psi_0\rangle\rangle \\
=& - \mathrm{i}\int_0^t dt'\langle\langle U_{-t}Q\psi_t|U_{-t'}\left(W(\vec{x};y) - W^f\right)U_{t'}^f|\psi_0\rangle\rangle \\
=& - \mathrm{i}\langle\langle U_{-t}Q\psi_t|\int_0^t dt'(U_{-t'} - U_{-t'}^f)\left(W(\vec{x};y) - W^f\right)U_{t'}^f|\psi_0\rangle\rangle \\
& - \mathrm{i}\langle\langle U_{-t}Q\psi_t|\int_0^t dt' U_{-t'}^f\left(W(\vec{x};y) - W^f\right)U_{t'}^f|\psi_0\rangle\rangle
\end{aligned} \tag{4.173}$$

where now

$$\begin{aligned}
W(\vec{x};y) =& \sum_{k=0}^{N}(v(x_k - y) + A(x_k)) && (4.174)\\
W^f =& \rho\left(\mathcal{F}(v)(0) + \mathcal{F}(A)(0)\right) && (4.175)
\end{aligned}$$

We can split this sum up and estimate each contribution separately. While this time we will get more terms, the idea of the proof remains unchanged.
Explicitly, the first contribution reads:

$$\begin{aligned}
&\left|\langle\langle U_{-t}Q\psi_t|\int_0^t dt' U_{-t'}^f\left(W(\vec{x};y) - W^f\right)U_{t'}^f|\psi_0\rangle\rangle\right| \\
\leq& \left|\langle\langle U_{-t}Q\psi_t|\int_0^t dt' U_{-t'}^f\left(\sum_{k=0}^{N}A(x_k) - \mathcal{F}(A)(0)\right)U_{t'}^f|\psi_0\rangle\rangle\right| \\
+& \left|\langle\langle U_{-t}Q\psi_t|\int_0^t dt' U_{-t'}^f\left(\sum_{k=0}^{N}v(x_k - y) - \mathcal{F}(v)(0)\right)U_{t'}^f|\psi_0\rangle\rangle\right|
\end{aligned} \tag{4.176}$$

We can repeat the estimate given in section 4.7.1 for each term separately.
The remainder can be split up as follows:

$$\begin{aligned}&\left|\langle\langle U_{-t}Q\psi_t|\int_0^t dt'(U_{-t'}-U^f_{-t'})\left(W(\vec{x};y)-W^f\right)U^f_{t'}|\psi_0\rangle\rangle\right|\\ \le&\left|\langle\langle U_{-t}Q\psi_t|\int_0^t dt'(U_{-t'}-U^f_{-t'})\left(\sum_{k=0}^N v(x_k-y)-\mathcal{F}(v)(0)\right)U^f_{t'}|\psi_0\rangle\rangle\right|\\ +&\left|\langle\langle U_{-t}Q\psi_t|\int_0^t dt'(U_{-t'}-U^f_{-t'})\left(\sum_{k=0}^N A(x_k)-\mathcal{F}(A)(0)\right)U^f_{t'}|\psi_0\rangle\rangle\right| \qquad (4.177)\end{aligned}$$

Using Cook's lemma, we can rewrite $U_{-t'}-U^f_{-t'}$, yielding, after performing the triangle inequality, four terms. Each one can be treated the same way as in section 4.7.2. We will not perform this analysis in detail, but invite the reader to convince himself of the validity of the lemma. □

Note, in this lemma, for potentials with support much smaller than L, the external fields only acts upon a very small part of the sea. Me may think of this local interaction to be small in the sense that it does not perturb the sea much. This lemma does not hold for trapping potentials, since then the support of the potential would be of same order as the size of the fermionic system, which is excluded by taking the thermodynamic limit.
We can also improve the bound given by restricting the class of admissible potentials.

Lemma 4.8.3 (Potential with gap)**.** *Let $v\in C^\infty(\mathbb{T})\cap C_0^\infty(\mathbb{R})$ with $\mathcal{F}(v)(k)=0$ for $|k|\le C=\mathcal{O}(1)$. Then, for a system without an external potential* [5]*, the following holds:*

$$\lim_{TD}\left(1-\langle\langle\psi_t|\psi_t^f\rangle\rangle\right)\le\frac{B_t}{\rho} \qquad (4.178)$$

Proof.
We refer to this potential as a potential with spectral gap since it mimics a spectral gap around the Fermi edge. If our system is initially in the ground state, then the sea particles sitting at the Fermi surface can only jump to excited states with momenta $|l|\approx\rho/2+C$ [6] . While there does not exist a real gap in the spectrum of the operator, it nevertheless mimics the band structure of a semiconductor. The lemma states that most of the deviation from free evolution comes from the fluctuations directly around the gap where the particles can change their momentum infinitesimally. When we forbid these transitions, the particles sitting at the Fermi edge need to overcome the gap of order C.
The proof of this lemma is straightforward. After the thermodynamic limit, we can replace the bound given in (4.85) by:

$$\begin{aligned}l^2-m^2=&|l|^2-|m|^2\\ =&(|l|-|m|)(|l|+|m|)\\ \ge&C\rho/2 \qquad (4.179)\end{aligned}$$

Note furthermore that the potential w we used in the estimate of the remainder is equal to zero. □

[5]This will be assumed for simplicity. Of course, if also an external potential with gap is present, then the estimate will be also true.
[6]Note that $|l|\approx\rho/2+C$ holds *after* taking the thermodynamic limit.

We would like to remark that it is possible to consider much more general initial states for the sea. As long as the number of excited states is of order $o(L)$, the estimates given above can be applied. Moreover, the idea one might come up with that Dirac's idea works because of an uniform *momentum* distribution is not correct. That is, the claim that for every particle with momentum p there exists another particle with momentum $-p$ and therefore the net momentum transfer between the tracer particle and the sea cancels out, is wrong. To construct a counterexample, simply take the (unphysical) initial state where 3/4 of all particles move to the left and 1/4 of the particles move to the right [7]. Still, for such a system, our proof applies.

[7] Of course, using the plane waves, such that the mean field is constant again.

4.9 The ultra-relativistic, one-dimensional system

In this section we want to investigate the following one-dimensional system whose time evolution is generated by the Hamiltonian

$$H^{\mathrm{rel}} = \sum_{k=0}^{N}(-\mathrm{i}\partial_{x_k}) - \mathrm{i}\partial_y + \sum_{k=0}^{N} v(x_k - y) \tag{4.180}$$

The question which arises naturally is if the above discussion is also true for this model. H^{rel} may be thought to describe ultra relativistic or massless electrons. We choose this Hamiltonian because it is the simplest one using the linear energy-momentum relation of relativistic electrons [8], given by:

$$E(m) = \frac{2\pi}{L} m \, , \, m \in \mathbb{Z} \tag{4.181}$$

As we have already mentioned, the bound (4.85) uses the nonlinear energy-momentum relation. It stems from the fact that

$$\frac{1}{E(\rho/2 + C) - E(\rho/2)} \sim \frac{1}{\rho} \tag{4.182}$$

holds for the nonrelatvistic energy-momentum relation $E(m) \sim m^2$. Since

$$\frac{1}{E(\rho/2 + C) - E(\rho/2)} = \mathcal{O}(1) \tag{4.183}$$

is true for $E(m) \sim m$, we cannot apply the methods from before.
We will investigate this problem in first-order perturbation theory. If E_1 does not vanish, the possibility to prove the mean-field picture by means of a perturbation expansion fails.
Explicitly, (4.82) now reads:

$$\lim_{\mathrm{TD}} E_1^2 = \int_{m<\rho/2} dm \int_{|l|>\rho/2} dl \, |\mathcal{F}(v)\,(m-l)|^2 \, \left\| \int_0^t dt' e^{2\pi\mathrm{i}(l-m)t'} U^y_{-t'} \, e^{2\pi\mathrm{i}(m-l)y} U^y_{t'} \chi_0 \right\|_y^2 \tag{4.184}$$

where

$$U^y_{t'} = e^{-\mathrm{i}t'(-\mathrm{i}\partial_y)} = e^{-t'\partial_y} \tag{4.185}$$

is simply the shift operator. We replaced the nonrelativistic energy-momentum relation $4\pi^2 l^2$ (which holds after taking the thermodynamic limit) by its relativistic counterpart $2\pi l$.
The norm can be calculated explicitly, using the following identity:

$$e^{2\pi\mathrm{i}(m-l)y} U^y_{t'} \chi_0(y) = e^{2\pi\mathrm{i}(m-l)y} \chi_0(y - t') \tag{4.186}$$

By inserting $\mathbb{1} = \mathcal{F}^{-1}\mathcal{F}$, we obtain:

$$\left(U^y_{-t'} \, e^{2\pi\mathrm{i}(m-l)y} U^y_{t'} \chi_0\right)(y) = \mathcal{F}^{-1}\left(\mathcal{F} e^{t'\partial_y} \mathcal{F}^{-1}\right)\mathcal{F}\left(e^{2\pi\mathrm{i}(m-l)\cdot} \chi_0(\cdot - t')\right)(y) \tag{4.187}$$

[8] The one-dimensional Dirac Hamiltonian $H = -\mathrm{i}\sigma_1\partial_x + \sigma_3 m + v(x)\mathbb{1}$ would also take the spin into account.

Using

$$\left(\mathcal{F}e^{t'\partial_y}\mathcal{F}^{-1}\right) = e^{-2\pi \mathrm{i}kt'} \tag{4.188}$$

and

$$\mathcal{F}\left(e^{2\pi\mathrm{i}(m-l)\cdot}\chi_0(\cdot - t')\right)(k) = e^{2\pi\mathrm{i}kt'}\; e^{2\pi\mathrm{i}(m-l)t'}\mathcal{F}\left(\chi_0\right)(k+[m-l]) \tag{4.189}$$

yields to:

$$\begin{aligned}\left(U^y_{-t'}\; e^{2\pi\mathrm{i}(m-l)y}U^y_{t'}\chi_0\right)(y) =& e^{2\pi\mathrm{i}(m-l)t'}\int_{\mathbb{R}} dk\; e^{-2\pi\mathrm{i}ky}\mathcal{F}\left(\chi_0\right)(k+[m-l])\\ =& e^{2\pi\mathrm{i}(m-l)y}e^{2\pi\mathrm{i}(m-l)t'}\chi_0(y)\end{aligned} \tag{4.190}$$

This implies that, up to a phase, the wavefunction remains the same.
The norm is therefore given by:

$$\begin{aligned}&\left\|\int_0^t dt' e^{2\pi\mathrm{i}(l-m)t'}U^y_{-t'}\; e^{2\pi\mathrm{i}(m-l)y}U^y_{t'}\chi_0\right\|^2_y\\ =&\left\|\int_0^t dt' e^{2\pi\mathrm{i}(l-m)t'}e^{2\pi\mathrm{i}(m-l)t'}e^{2\pi\mathrm{i}(m-l)\cdot}\chi_0(\cdot)\right\|^2_y\\ =&t^2\end{aligned} \tag{4.191}$$

By this equality we obtain the following, somewhat disappointing result:

$$\begin{aligned}\lim_{\mathrm{TD}} E_1^2 =& \int_{m<\rho/2} dm \int_{|l|>\rho/2} dl\, |\mathcal{F}(v)\,(m-l)|^2 \left\|\int_0^t dt' e^{2\pi\mathrm{i}(m-l)t'}U^y_{-t'}\; e^{2\pi\mathrm{i}(m-l)y}U^y_{t'}\chi_0\right\|^2_y\\ =&t^2 \lim_{\mathrm{TD}} \mathrm{Var}^v_{N+1}(0)\end{aligned} \tag{4.192}$$

This surprising result means that perturbation theory works for the nonrelativistic system, while for the Hamiltonian discussed here the method fails. This was even true if we would only consider potentials fulfilling the gap condition. We want to make one side remark at this point. The slightly modified Hamiltonian

$$H^{\mathrm{rel}'} = \sum_{k=0}^{N}(-\mathrm{i}\partial_{x_k}) + \mathrm{i}\partial_y + \sum_{k=0}^{N} v(x_k - y) \tag{4.193}$$

can be solved exactly. As discussed by [Mattis and Lieb, 1965], the wavefunction

$$e^{\frac{\mathrm{i}}{2}\sum_{k=0}^N \int_{-L/2}^{x_k} v(x-y)dx}\psi(x_0, x_1, ..., x_N; y; t) \tag{4.194}$$

solves the corresponding Schrödinger equation for $H^{\mathrm{rel}'}$ if $\psi(x_0, x_1, ..., x_N; y; t)$ satisfies

$$\mathrm{i}\partial_t\psi(x_0, x_1, ..., x_N; y; t) = \left(\sum_{k=0}^{N}(-\mathrm{i}\partial_{x_k}) + \mathrm{i}\partial_y\right)\psi(x_0, x_1, ..., x_N; y; t) \tag{4.195}$$

which is just the free Schrödinger equation. We chose this example since it illustrates that a somewhat different procedure beyond perturbation theory can be a valuable tool. As outlined in the article mentioned above, this model can be seen as a simple example of an exactly solvable many-fermion-system. The approach is called "bosonisation-method". While this method is very powerful, it works only for very specific Hamiltonians and cannot be generalised to arbitrary systems.

Chapter 5

The Three-dimensional System

5.1 Notation

First, we would like to list the notational changes in comparison to the one-dimensional system. For this chapter the number of particles will be different. Remember that we worked with $N+1$ particles in the one-dimensional setting in order to keep the notation simple. In three dimensions, we will have either N particles (without spin) or $2N$ particles (with spin).

p: Vectors in three dimensions will always be denoted by a boldface letter.

$\mathbb{T}^3$: We will work from now on on the three-dimensional torus $\mathbb{T}^3 = [-L/2; L/2]^3$. The volume of the torus will be denoted by V.

$|\varphi_m\rangle$: The one particle wave functions are now given by:

$$\begin{aligned} |\varphi_{\mathbf{p}}\rangle =& \frac{1}{\sqrt{V}} e^{\frac{2\pi \mathrm{i}}{L}\mathbf{p}\cdot\mathbf{x}} \\ \mathbf{p} =& (p_1, p_2, p_3) \in \mathbb{Z}^3 \end{aligned}$$

For the Dirac equation, these states will be different. We will list them explicitly in the corresponding section.

ρ: The particle density ρ is defined as

$$\rho := \begin{cases} N/V & \text{for spinless electrons} \\ 2N/V & \text{for electrons with spin} \end{cases}$$

$\mathbf{p}_m$: As in one dimension, considering the system on a torus, the possible momenta are quantised. In order to keep the notation handy, we will employ the following definition: We will label all possible momenta using the countability of $\mathbb{Z}^3$. Starting from $m = 1$, the momenta will be labelled in increasing order. This means the following:

$$m_1 < m_2 \Rightarrow |\mathbf{p}_{m_1}| \leq |\mathbf{p}_{m_2}|$$

Thus, we first label the momenta closest to the origin an then go outwards. We will furthermore comment on this definition below.

$\lim_{\mathrm{TD}}$: Again we want to simplify our calculations by taking the thermodynamic limit. In order to coincide with the definitions in the literature, the spacing of an elementary cube will be given by $\frac{(2\pi)^3}{V}$. For example, we may write

$$\lim_{\mathrm{TD}} \sum_{m=1}^{N} \frac{1}{V} e^{\frac{2\pi \mathrm{i}}{L} \mathbf{p}_m \cdot (\mathbf{x}-\mathbf{y})} = \int_{|\mathbf{p}|<k_F} d^3p \frac{1}{(2\pi)^3} e^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}$$

if the $\mathbf{p}_m$ lie inside the Fermi sphere. k_F will be determined in the corresponding section and will be a function of ρ.

$\mathcal{F}(v)$: The *Fourier transform* of the potential $v \in C(\mathbb{T}^3)$ is now given by:

$$\mathcal{F}(v)\,(\mathbf{p}) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} d^3x \; v(\mathbf{x}) e^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}$$

This definition is slightly different from the one we used before.

5.2 The nonrelativistic system

First, we will consider the nonrelativistic system. In contrast to the one-dimensional system, there are two slight differences: All integrals are now defined on three-dimensional space and, in addition, the ground state is obtained filling up all energies and momenta uniformly. This procedure is well known and yields to the Fermi sphere (see for example [Maruhn and Suraud, 2010] for a detailed discussion). Explicitly, we will label the momenta of the occupied states from $\mathbf{p}_1$ till $\mathbf{p}_N$, such that in the thermodynamic limit all occupied states have momenta $|\mathbf{p}| \leq k_F$, where k_F denotes the Fermi momentum. In order to keep the notation simple, we will also label all unoccupied momenta, starting from $\mathbf{p}_{N+1}$. In the thermodynamic limit these states lie outside the Fermi sphere.
In this section, we will only consider spinless fermions. Considering spin, merely some numerical factors would change. In the next section, we will finally consider the Dirac equation. There, the scalar product of the spinors will yield additional terms, of which we need to keep track of. This chapter will focus on the fluctuations of the potential. While this quantity alone is not enough to conclude the mean-field picture, it was of central importance in the proof before. In the next chapter we will comment on the possibility to repeat the methods used for the one-dimensional system.
First of all, we will determine the Fermi momentum of our sphere. As already mentioned, our initial state describes the ground state of N non-interacting electrons [1].
The corresponding N particle state is given by:

$$|\Lambda\rangle = |\varphi_{\mathbf{p}_1} \wedge ... \wedge \varphi_{\mathbf{p}_N}\rangle \tag{5.1}$$

The $\mathbf{p}_i$ lie approximately in a sphere with radius k_F which is related to ρ as follows:
The one particle reduced density matrix

$$\gamma^{(1)} := \sum_{m \leq N} |\varphi_{\mathbf{p}_m}\rangle\langle\varphi_{\mathbf{p}_m}| \tag{5.2}$$

of our system, given in position representation

$$\gamma^{(1)}(\mathbf{x},\mathbf{y}) = \sum_{m=1}^{N} \frac{1}{V} e^{\frac{2\pi i}{L}\mathbf{p}_m\cdot(\mathbf{x}-\mathbf{y})} \tag{5.3}$$

reads in the thermodynamic limit:

$$\lim_{\mathrm{TD}} \gamma^{(1)}(\mathbf{x},\mathbf{y}) = \int_{|\mathbf{p}|<k_F} d^3p \frac{1}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \tag{5.4}$$

The Fermi momentum k_F can be determined as follows: Note, by (5.3), $\gamma^{(1)}(\mathbf{x},\mathbf{x}) = \frac{N}{L} = \rho$ holds. Thus,

$$\begin{aligned}\rho =& \lim_{\mathrm{TD}} \gamma^{(1)}(\mathbf{x},\mathbf{x}) = \int_{|\mathbf{p}|<k_F} d^3p \frac{1}{(2\pi)^3} \\ =& k_F^3 \frac{1}{6\pi^2} \\ \Rightarrow k_f =& \left(6\pi^2\rho\right)^{1/3}\end{aligned} \tag{5.5}$$

[1]Due to the fact that more than one state has the same absolute value of $\mathbf{p}$, there may exist more than one ground state. However, any of these will yield the same result.

Note in passing that, repeating this derivation for d dimensional systems, the Fermi momentum is always proportional to $\rho^{\frac{1}{d}}$.
Using formula (4.30) and our definition of how to label the momenta, the variance reads:

$$\mathrm{Var}_N^v(\mathbf{0}) = \sum_{m=1}^{N} \sum_{l=N+1}^{\infty} \left|\langle \varphi_{\mathbf{p}_l}|v(\mathbf{x})|\varphi_{\mathbf{p}_m}\rangle\right|^2 \tag{5.6}$$

We will first consider smooth potentials for which a simple estimate, using the Paley-Wiener theorem, yields the asymptotic behaviour in ρ. Afterwards, we will consider the particle fluctuations which correspond to the case $v(\mathbf{x}) = \mathbb{1}_B(\mathbf{x})$. As in one dimension, the Fourier transform of the indicator function does not decrease fast enough to yield the same result as for smooth potentials. Our result will be corrected by the factor $\ln(\rho)$, which, however, does not change the qualitative behaviour:
Also in three dimensions, density fluctuations are suppressed.
To this end remember that for bosonic systems the variance always scales like $\rho \sim k_F^3$.
Let us first take the thermodynamic limit:

$$\lim_{\mathrm{TD}} \mathrm{Var}_N^v(\mathbf{0}) = \int_{|\mathbf{p}_1|\leq k_F} d^3p_1 \int_{|\mathbf{p}_2|>k_F} d^3p_2 \left|\mathcal{F}(v)\left(\mathbf{p}_2 - \mathbf{p}_1\right)\right|^2 \tag{5.7}$$

A heuristic argument gives the correct estimate of (5.7). To this end consider first the points $\mathbf{p}_1,\ \mathbf{p}_2$ where $|\mathbf{p}_1 - \mathbf{p}_2| = \mathcal{O}(1)$ holds. For these points, the Fourier transform $\mathcal{F}(v)\left(\mathbf{p}_2 - \mathbf{p}_1\right)$ is of order one. Hence, near the Fermi surface, the function

$$g(\mathbf{p}_1) := \int_{|\mathbf{p}_2|>k_F} d^3p_2 \left|\mathcal{F}(v)\left(\mathbf{p}_2 - \mathbf{p}_1\right)\right|^2 \tag{5.8}$$

is of order one for all $|\mathbf{p}_1| \approx k_F$. Integrating over a small sphere with radius k_F and width $\approx \mathcal{O}(1)$ yields:

$$\lim_{\mathrm{TD}} \mathrm{Var}_N^v(\mathbf{0}) \gtrsim (k_F)^2 \sim \rho^{2/3} \tag{5.9}$$

Therefore, the variance grows at least as $\rho^{2/3}$. On the other hand,
$\left|\mathcal{F}(v)\left(\mathbf{p}_2 - \mathbf{p}_1\right)\right|^2 = \mathcal{O}\left(|\mathbf{p}_2 - \mathbf{p}_1|^{-\infty}\right)$ holds, so points far apart will not contribute to the asymptotics. This can be shown rigorously by the following

Theorem 5.2.1. *Let* $v \in C^\infty(\mathbb{T}^3) \cap C_0^\infty(\mathbb{R}^3)$. *Then*

$$\lim_{TD} Var_N^v(\boldsymbol{0}) \sim \rho^{2/3} \tag{5.10}$$

Proof.
We will only estimate the variance from above. As we have already argued, the variance grows at least as $\rho^{2/3}$ since all points $\mathbf{p}_1$ near the Fermi surface, whose area scales like $\rho^{2/3}$, contribute to the result.

We will use the Paley-Wiener theorem to estimate $g(\mathbf{p}_1)$:

$$\begin{aligned}\lim_{\mathrm{TD}} \mathrm{Var}_N^v(\mathbf{0}) &\leq \int_{|\mathbf{p}_1|\leq k_F} d^3p_1 \int_{|\mathbf{p}_2|>k_F} d^3p_2 \left|\frac{D_p}{(1+|\mathbf{p}_2-\mathbf{p}_1|)^p}\right|^2 \\ &\leq \int_{|\mathbf{p}_1|\leq k_F} d^3p_1 \int_{|\mathbf{p}_2|>k_F} d^3p_2 \left|\frac{D_p}{(1+||\mathbf{p}_2|-|\mathbf{p}_1||)^p}\right|^2 \\ &= \int_{|\mathbf{p}_1|\leq k_F} d^3p_1 \int_{|\mathbf{p}_2|>k_F-|\mathbf{p}_1|} d^3p_2 \left|\frac{D_p}{(1+|\mathbf{p}_2|)^p}\right|^2 \\ &\leq \int_{|\mathbf{p}_1|\leq k_F} d^3p_1 \int_{|\mathbf{p}_2|>k_F-|\mathbf{p}_1|} d^3p_2 \frac{D_p^2}{(1+|\mathbf{p}_2|^2)^p} \end{aligned} \tag{5.11}$$

Using spherical coordinates for $\mathbf{p}_2$, we can write:

$$\lim_{\mathrm{TD}} \mathrm{Var}_N^v(\mathbf{0}) \leq \int_{|\mathbf{p}_1|\leq k_F} d^3p_1 \int_{k_F-|\mathbf{p}_1|}^{\infty} dr\, r^2 \frac{4\pi D_p^2}{(1+r^2)^p} \tag{5.12}$$

In principle, this integral could be evaluated explicitly for any given p, at the price that the antiderivative reads lengthy. We will proceed differently by splitting up the domain as follows:

$$\begin{aligned}\lim_{\mathrm{TD}} \mathrm{Var}_N^v(\mathbf{0}) \leq &\underbrace{\int_{k_F-C\leq|\mathbf{p}_1|\leq k_F} d^3p_1 \int_{k_F-|\mathbf{p}_1|}^{\infty} dr\, r^2 \frac{4\pi D_p^2}{(1+r^2)^p}}_{=:\mathrm{Var}_{\mathrm{I}}} \\ &+ \underbrace{\int_{|\mathbf{p}_1|<k_F R-C} d^3p_1 \int_{k_F-|\mathbf{p}_1|}^{\infty} dr\, r^2 \frac{4\pi D_p^2}{(1+r^2)^p}}_{=:\mathrm{Var}_{\mathrm{II}}} \end{aligned} \tag{5.13}$$

First, consider $\mathrm{Var}_{\mathrm{I}}$:

$$\begin{aligned}\mathrm{Var}_{\mathrm{I}} &= \int_{k_F-C\leq|\mathbf{p}_1|\leq k_F} d^3p_1 \int_{k_F-|\mathbf{p}_1|}^{\infty} dr\, r^2 \frac{4\pi D_p^2}{(1+r^2)^p} \\ &\leq \int_{k_F-C\leq|\mathbf{p}_1|\leq k_F} d^3p_1 \int_0^{\infty} dr\, r^2 \frac{4\pi D_p^2}{(1+r^2)^p} \\ &\sim \rho^{2/3} \end{aligned} \tag{5.14}$$

In the last step we used that, for $p \geq 2$,

$$\int_0^{\infty} dr\, r^2 \frac{D_p^2}{(1+r^2)^p} < \infty \tag{5.15}$$

holds. Using $\mathbf{p}_1 \sim k_F$, we further remark that $\int_{k_F-|\mathbf{p}_1|}^{\infty} dr\, r^2 \frac{D_p^2}{(1+r^2)^p} \approx \int_0^{\infty} dr\, r^2 \frac{D_p^2}{(1+r^2)^p}$ holds. That is, the *actual* value of the integral (before the estimate) is of order one. Henceforth, the bound just given describes the asymptotic behaviour correctly. This is also a consequence of our discussion made above; that is, fluctuations around the Fermi surface are always proportional to $\rho^{2/3}$.

For $\mathrm{Var_{II}}$, we can estimate the contribution as follows:

$$\begin{aligned}
\mathrm{Var_{II}} =& \int_{|\mathbf{p}_1|\leq k_F - C} d^3p_1 \int_{k_F-|\mathbf{p}_1|}^{\infty} dr\, r^2 \frac{4\pi D_p^2}{(1+r^2)^p} \\
\leq& \int_{|\mathbf{p}_1|\leq k_F - C} d^3p_1 \int_{k_F-|\mathbf{p}_1|}^{\infty} dr\, r^2 \frac{4\pi D_p^2}{r^{2p}} \\
=& \frac{16\pi^2 D_p^2}{(2p-3)} \int_0^{k_F-C} dx\, x^2 \frac{1}{(k_F-x)^{2p-3}} \\
=& \frac{16\pi^2 D_p^2}{(2p-3)} \int_C^{k_F} dx\, (x-k_F)^2 \frac{1}{x^{2p-3}} \\
=& \frac{16\pi^2 D_p^2}{(2p-3)} \int_C^{k_F} dx\, (x^{5-2p} - 2k_F x^{4-2p} + k_F^2 x^{3-2p}) \\
\sim& (k_F)^2 \sim \rho^{2/3}
\end{aligned} \tag{5.16}$$

which holds for all $p > 2$. The case $p = 2$ will be of importance for the indicator function and is not covered by this estimate. In total, we obtain:

$$\lim_{\mathrm{TD}} \mathrm{Var}_N^v(\mathbf{0}) \lesssim \rho^{2/3} \tag{5.17}$$

□

Next, we want to consider the case

$$v(\mathbf{x}) = \mathbb{1}_B(\mathbf{x}) \tag{5.18}$$

where B denotes a ball with radius $R = \mathcal{O}(1)$ centred around the origin. Now, the variance reads:

$$\begin{aligned}
\lim_{\mathrm{TD}} \mathrm{Var}_N^{\mathbb{1}_B}(\mathbf{0}) =& \int_{|\mathbf{p}_1|<k_F} d^3p_1 \int_{|\mathbf{p}_2|>k_F} d^3p_2 \left| \int_{|\mathbf{x}|<R} d^3x \frac{1}{(2\pi)^3} e^{\mathrm{i}\mathbf{x}\cdot(\mathbf{p}_2-\mathbf{p}_2)} \right|^2 \\
=& \frac{1}{4\pi^4} \int_{|\mathbf{p}_1|\leq k_F} d^3p_1 \int_{|\mathbf{p}_2|>k_F} d^3p_2 \left(\frac{\sin(R|\mathbf{p}_1-\mathbf{p}_2|) - R|\mathbf{p}_1-\mathbf{p}_2|\cos(R|\mathbf{p}_1-\mathbf{p}_2|)}{|\mathbf{p}_1-\mathbf{p}_2|^3} \right)^2
\end{aligned} \tag{5.19}$$

The question which arises naturally is whether the asymptotics derived above still holds. The answer is no, but the correction is only logarithmically and thus the heuristics is not affected by this.

Lemma 5.2.1. *Let $v = \mathbb{1}_B(\boldsymbol{x})$. Then*

$$\lim_{TD} Var_N^{\mathbb{1}_B}(\boldsymbol{0}) \sim \rho^{2/3} \ln(\rho) \tag{5.20}$$

Proof.
Again, we want to split up the domain of integration for $\mathbf{p}_1$ into two terms.
Using

$$g(\mathbf{p}_1) = \frac{1}{4\pi^4} \int_{|\mathbf{p}_2|>k_F} d^3p_2 \left(\frac{\sin(R|\mathbf{p}_1-\mathbf{p}_2|) - R|\mathbf{p}_1-\mathbf{p}_2|\cos(R|\mathbf{p}_1-\mathbf{p}_2|)}{|\mathbf{p}_1-\mathbf{p}_2|^3} \right)^2 \tag{5.21}$$

the variance reads:

$$\lim_{\mathrm{TD}} \mathrm{Var}_N^{\mathbb{1}_B}(\mathbf{0}) = \int_{|\mathbf{p}_1|<k_F} d^3p_1\, g(\mathbf{p}_1) = \underbrace{\int_{k_F-C\leq|\mathbf{p}_1|\leq k_F} d^3p_1\, g(\mathbf{p}_1)}_{=:\mathrm{Var}_{\mathrm{I}}} + \underbrace{\int_{|\mathbf{p}_1|<k_F-C} d^3p_1\, g(\mathbf{p}_1)}_{=:\mathrm{Var}_{\mathrm{II}}} \tag{5.22}$$

Estimate of $\mathrm{Var}_{\mathrm{I}}$: As above, $\mathrm{Var}_{\mathrm{I}}$ basically measures the surface of the Fermi sphere. This can be seen explicitly using

$$\int_{\mathbb{R}^3} d^3p\, |\mathcal{F}(\mathbb{1}_B)(\mathbf{p})|^2 = \frac{B^3}{6\pi^2} \tag{5.23}$$

Hence:

$$\begin{aligned}\mathrm{Var}_{\mathrm{I}} &= \int_{k_F-C\leq|\mathbf{p}_1|\leq k_F} d^3p_1\, g(\mathbf{p}_1)\\ &\leq \int_{k_F-C\leq|\mathbf{p}_1|\leq k_F} d^3p_1 \int_{\mathbb{R}^3} d^3p_2\, |\mathcal{F}(\mathbb{1}_B)(\mathbf{p}_2-\mathbf{p}_1)|^2\\ &\sim \rho^{2/3}\end{aligned} \tag{5.24}$$

Note that $g(\mathbf{p}_1) = \mathcal{O}(1)$ holds for $|\mathbf{p}_1| \approx k_F$. So, the asymptotic behaviour of $\mathrm{Var}_{\mathrm{I}}$ is indeed given by $\rho^{2/3}$.

Estimate of $\mathrm{Var}_{\mathrm{II}}$: We try to give a sharp bound for the remaining integral

$$\mathrm{Var}_{\mathrm{II}} = \int_{|\mathbf{p}_1|\leq k_F-C} d^3p_1 \int_{|\mathbf{p}_2|>k_F} d^3p_2\, |\mathcal{F}(\mathbb{1}_B)(\mathbf{p}_2-\mathbf{p}_1)|^2 \tag{5.25}$$

For the asymptotic behaviour, only the part which has the slowest decay is important. Expanding

$$\begin{aligned}|\mathcal{F}(\mathbb{1}_B)(\mathbf{p}_2-\mathbf{p}_1)|^2 =& \frac{1}{4\pi^4}\left(\frac{\sin^2(R|\mathbf{p}_2-\mathbf{p}_1|)}{|\mathbf{p}_2-\mathbf{p}_1|^6} - \frac{2R\sin(R|\mathbf{p}_2-\mathbf{p}_1|)\cos(R|\mathbf{p}_2-\mathbf{p}_1|)}{|\mathbf{p}_2-\mathbf{p}_1|^5}\right)\\ &+ \frac{1}{4\pi^4}\left(\frac{R^2\cos^2(R|\mathbf{p}_2-\mathbf{p}_1|)}{|\mathbf{p}_2-\mathbf{p}_1|^4}\right)\end{aligned} \tag{5.26}$$

we keep only the last term. The other two correspond to the estimate (5.16) above with $p = 3$ (the first term) and $p = 2.5$ (the second one) and thus grow like $\rho^{2/3}$. We will show that

$$\begin{aligned}&\int_{|\mathbf{p}_1|\leq k_F R-C} d^3p_1 \int_{|\mathbf{p}_2|>k_F R} d^3p_2 \frac{\cos^2(|\mathbf{p}_1-\mathbf{p}_2|)}{|\mathbf{p}_1-\mathbf{p}_2|^4}\\ \sim& \int_{|\mathbf{p}_1|\leq k_F R-C} d^3p_1 \int_{|\mathbf{p}_2|>k_F R} d^3p_2 \frac{1}{|\mathbf{p}_1-\mathbf{p}_2|^4} \sim \rho^{2/3}\ln(\rho)\end{aligned} \tag{5.27}$$

is true by bounding this term in both directions. We will abbreviate

$$\mathrm{Var}_2 := \frac{R^2}{4\pi^4}\int_{|\mathbf{p}_1|\leq k_F-C} d^3p_1 \int_{|\mathbf{p}_2|>k_F} d^3p_2 \frac{\cos^2(R|\mathbf{p}_1-\mathbf{p}_2|)}{|\mathbf{p}_1-\mathbf{p}_2|^4} \tag{5.28}$$

First, we will bound Var_2 from above using

$$\mathrm{Var}_2 \leq \frac{R^2}{4\pi^4} \int_{|\mathbf{p}_1| \leq k_F - C} d^3p_1 \int_{|\mathbf{p}_2| > k_F} d^3p_2 \frac{1}{|\mathbf{p}_1 - \mathbf{p}_2|^4} \tag{5.29}$$

This term can be estimated the same way as in equation (5.16) using $p = 2$.

$$\begin{aligned} \mathrm{Var}_2 \lesssim & \int_C^{k_F} dx \frac{(k_F - x)^2}{x^{3-2\cdot 2}} \\ \sim & (k_F)^2 \ln(k_F) \\ \sim & \rho^{2/3} \ln(\rho) \end{aligned} \tag{5.30}$$

Next, we bound Var_2 from below. This will be done using the positivity of the integrand. Fixing $\mathbf{p}_1$, we will calculate the part of the $\mathbf{p}_2$ integral which yields the biggest contribution. Since $\frac{\cos^2(R|\mathbf{p}_1 - \mathbf{p}_2|)}{|\mathbf{p}_1 - \mathbf{p}_2|^4}$ is essentially decaying in $\mathbf{p}_1 - \mathbf{p}_2$, we consider the following set which contains most of the points $\mathbf{p}_2$ closest to $\mathbf{p}_1$:

$$M^\alpha_{\mathbf{p}_1} := \{\mathbf{p}_2 \in \mathbb{R}^3 | |\mathbf{p}_2| > k_F; |\mathbf{p}_2 - \mathbf{p}_1| \geq \frac{k_F - |\mathbf{p}_1|}{\cos(\alpha)};\ \mathbf{p}_1 \cdot (\mathbf{p}_2 - \mathbf{p}_1) \geq |\mathbf{p}_1||\mathbf{p}_2 - \mathbf{p}_1| \cos(\alpha); \alpha < \pi/4\} \tag{5.31}$$

The following figure shows the set which was chosen such that it is possible to introduce spherical coordinates for $\mathbf{p}_2 - \mathbf{p}_1$:

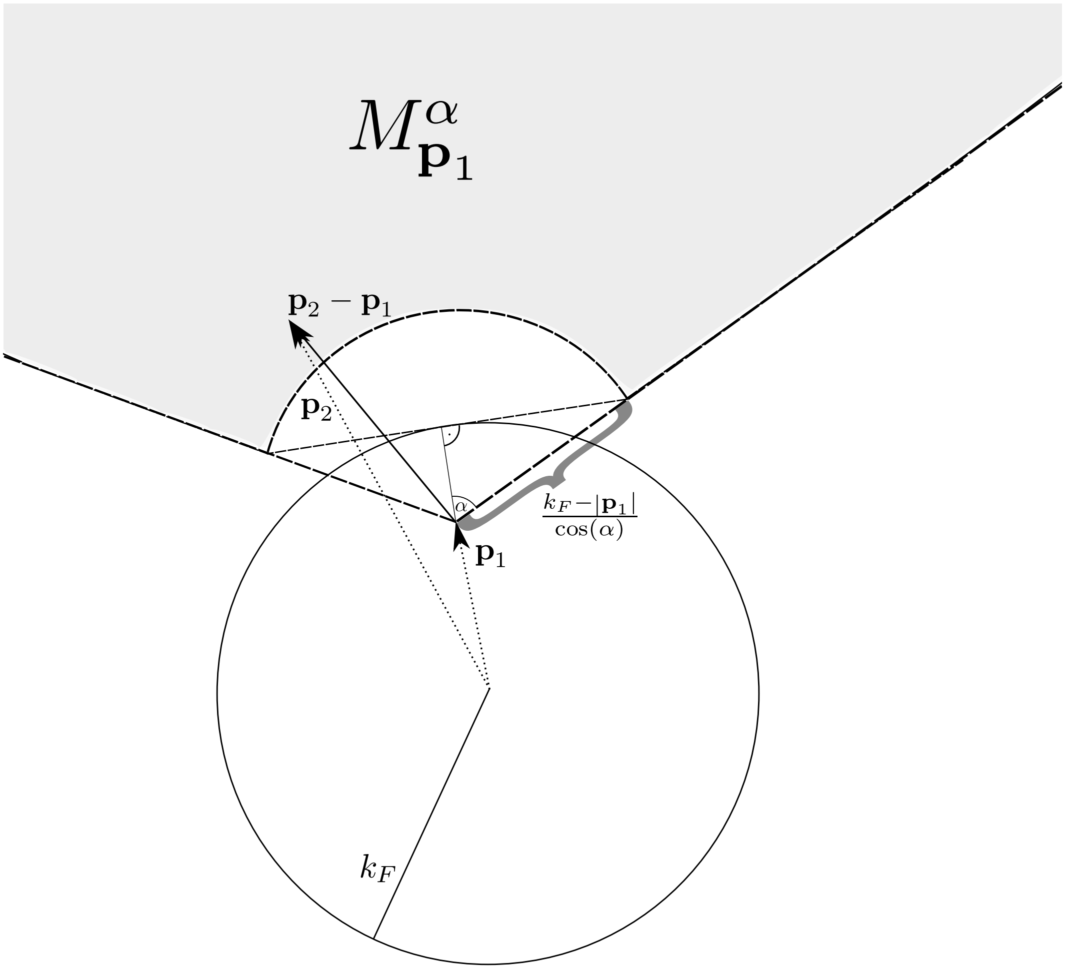

Figure 5.1: The set $M^{\alpha}_{\mathbf{p}_1}$. The vector $\mathbf{p}_2 - \mathbf{p}_1$ lies inside the shaded area. For this vector it is possible to introduce spherical coordinates.

Hence, using the positivity of the integrand, we can write:

$$\mathrm{Var}_2 \geq \frac{R^2}{4\pi^4} \int_{|\mathbf{p}_1| \leq k_F - C} d^3p_1 \int_{|\mathbf{p}_2| \in M^{\alpha}_{\mathbf{p}_1}} d^3p_2 \frac{\cos^2(R|\mathbf{p}_1 - \mathbf{p}_2|)}{|\mathbf{p}_1 - \mathbf{p}_2|^4} \tag{5.32}$$

First, we calculate:

$$\int_{|\mathbf{p}_2| \in M^{\alpha}_{\mathbf{p}_1}} d^3p_2 \frac{\cos^2(R|\mathbf{p}_1 - \mathbf{p}_2|)}{|\mathbf{p}_1 - \mathbf{p}_2|^4} \tag{5.33}$$

Using spherical coordinates $(r; \Theta; \varphi)$ (c.f. the figure above)

$$r \geq \frac{k_F - |\mathbf{p}_1|}{\cos(\alpha)} \qquad 0 \leq \Theta \leq \alpha \qquad 0 \leq \varphi \leq 2\pi \tag{5.34}$$

we obtain:

$$\begin{aligned}
&\int_{|\mathbf{p}_2|\in M^{\alpha}_{\mathbf{p}_1}} d^3p_2 \frac{\cos^2(R|\mathbf{p}_1-\mathbf{p}_2|)}{|\mathbf{p}_1-\mathbf{p}_2|^4} = \int_0^{\alpha} d\Theta \int_0^{2\pi} d\varphi \int_{\frac{k_F-|\mathbf{p}_1|}{\cos(\alpha)}}^{\infty} dr \sin(\Theta)\frac{\cos^2(Rr)}{r^2} \\
=&2\pi(1-\arccos(\alpha))\int_{\frac{k_F-|\mathbf{p}_1|}{\cos(\alpha)}}^{\infty} dr \left[\left(\frac{\cos^2(Rr)}{r^2}-\frac{1}{2r^2}\right)+\frac{1}{2r^2}\right] \\
=&\pi(1-\arccos(\alpha))\frac{\cos(\alpha)}{k_F-|\mathbf{p}_1|} \\
&+2\pi(1-\arccos(\alpha))\int_{\frac{k_F-|\mathbf{p}_1|}{\cos(\alpha)}}^{\infty} dr\left(\frac{\cos^2(Rr)}{r^2}-\frac{1}{2r^2}\right)
\end{aligned} \tag{5.35}$$

Next, we want to control the difference $\frac{\cos^2(Rr)}{r^2}-\frac{1}{2r^2}$. We expect this contribution to be of subleading order by the same line of argumentation we employed in equation (4.53). First, let us rewrite:

$$\int_z^{\infty} dr \frac{\cos^2(Rr)}{r^2} = \sum_{k=0}^{\infty}\int_{z+k\pi/R}^{z+(k+1)\pi/R} dr\frac{\cos^2(Rr)}{r^2} \tag{5.36}$$

Using

$$\int_{z+k\pi/R}^{z+(k+1)\pi/R} dr\cos^2(Rr) = \frac{\pi}{2R} \tag{5.37}$$

we can give the following bound:

$$\sum_{k=0}^{\infty}\left(\frac{R}{z+(k+1)\pi}\right)^2\frac{\pi}{2R} \leq \sum_{k=0}^{\infty}\int_{z+k\pi/R}^{z+(k+1)\pi/R} dr\frac{\cos^2(Rr)}{r^2} \leq \sum_{k=0}^{\infty}\left(\frac{R}{z+k\pi}\right)^2\frac{\pi}{2R} \tag{5.38}$$

The same estimate holds for $\int_z^{\infty} da\ \frac{1}{2a^2}$:

$$\sum_{k=0}^{\infty}\left(\frac{R}{z+(k+1)\pi}\right)^2\frac{\pi}{2R} \leq \sum_{k=0}^{\infty}\int_{z+k\pi/R}^{z+(k+1)\pi/R} dr\frac{1}{2r^2} \leq \sum_{k=0}^{\infty}\left(\frac{R}{z+k\pi}\right)^2\frac{\pi}{2R} \tag{5.39}$$

Subtracting these bounds, we conclude [2] :

$$\Rightarrow -\frac{1}{z^2}\frac{R\pi}{2} \leq \int_z^{\infty} da\left(\frac{\cos^2(Rr)}{r^2}-\frac{1}{2r^2}\right) \leq \frac{1}{z^2}\frac{R\pi}{2} \tag{5.40}$$

Hence, this term is of order $\frac{1}{(k_F-|\mathbf{p}_1|)^2}$ an thus of subleading order.
In total, it was shown that:

$$\begin{aligned}
\mathrm{Var}_2 \gtrsim &\int_{|\mathbf{p}_1|\leq k_F R-r} d^3p_1\ \pi(1-\arccos(\alpha))\frac{\cos(\alpha)}{k_f R-|\mathbf{p}_1|} \\
\sim&\rho^{2/3}\ln(\rho)
\end{aligned} \tag{5.41}$$

□

[2] As expected, a faster oscillating cosine which corresponds to a small parameter R improves this result.

This calculation confirm the result we found in one dimension. For fermionic systems, the Pauli exclusion principle yields to an effective repulsion of the electrons (not mediated by some potential) which arranges the particles in a homogeneous way. Yet, in contrast to $d = 1$, the fluctuations grow with the density. Henceforth, it could be that the perturbation expansion we used before might not work any more. We would like to comment on this in the next chapter after having calculated the fluctuations arising in the Dirac equation. This will be done now.

5.3 The Dirac equation

We will now consider relativistic electrons described by the Dirac equation. We will analyse the idea of Dirac, that is, calculate the density fluctuations of a finite Dirac sea composed of particles with negative energy. This analysis was already performed in [Colin and Struyve, 2007] using the second quantized field formalism. As it is common when using this formalism, a cut-off Λ has to be introduced. The authors themselves note that a rigorous analysis must furthermore confine the system into a box. While this remark is technically correct, this observation is nevertheless meant to be a side remark. It is assumed that a more careful analysis will not change the final result. However, we get a different result which we will explain now.
Remember: For a fermionic system, the density fluctuations are given by:

$$\lim_{\text{TD}} \text{Var}^{v}_{2N}(\mathbf{0}) = \sum_{l \in O^c} \sum_{m \in O} |\langle \varphi_m | v(\mathbf{x}) | \varphi_l \rangle|^2 \tag{5.42}$$

where O is the set of all occupied states and O^c the set of all unoccupied ones (we choose $2N$ because of the spin). Thus, considering the Dirac equation on bounded space, the sea will be a $2N$-particle wedge product of one particle states with negative energy. For the estimate of the variance two different kind of transitions have to be taken into account: A negative energy particle may fall down the sea to another state with even less energy. Furthermore, transitions into the positive spectrum are possible, which correspond to the physical effect of pair creation.
As our analysis will reveal, the cut-off Λ has two effects. First, it will define all occupied states with negative energy. Hence, Λ is equal to the Fermi momentum k_F [3]. Yet, using the cut-off, another, probably unwanted effect arises: Due the structure of the field operators, the set O^c used in equation (5.42) reads differently. In the field formalism with cutoff Λ, all states with momenta greater than Λ are not present. Hence, the variance actually calculated in [Colin and Struyve, 2007] originates from a system where one would start from bounded space and from a restricted, finite dimensional Hilbert space which contains only states with momenta smaller than Λ. As a consequence, there is no possibility for the particles to fall down the sea, or, stated differently, there are no transitions into states with even more negative energy. Moreover, only transitions to positive energy states with momenta smaller than Λ are allowed. While the transition to positive energy states with momenta greater than Λ are highly suppressed and thus are unimportant, the transitions to states with even lower energy are of the same order as in the nonrelativistic case. Consequently, the calculation performed in Coin and Struyve implicitly assumes a different system we use. We will comment on this issue in more detail later.

[3] Therefore, the fluctuations of a bosonic system, using the field formalism, would scale like Λ^3.

We will now list the spinors and definitions we use for the Dirac equation. These definitions are taken form [Bjorken and Drell, 1998] and are in accordance with the formulas used in [Colin and Struyve, 2007]. Of course, different conventions will yield the same final result. For the reader's convenience, we will list the spinors explicitly[4]:

$$
\begin{aligned}
u(\mathbf{p},+) =& \sqrt{\frac{E(\mathbf{p})+m}{2m}} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E(\mathbf{p})+m} \\ \frac{p_+}{E(\mathbf{p})+m} \end{pmatrix} \qquad & u(\mathbf{p},-) = \sqrt{\frac{E(\mathbf{p})+m}{2m}} \begin{pmatrix} 0 \\ 1 \\ \frac{p_-}{E(\mathbf{p})+m} \\ \frac{-p_z}{E(\mathbf{p})+m} \end{pmatrix} \\
v(\mathbf{p},+) =& \sqrt{\frac{E(\mathbf{p})+m}{2m}} \begin{pmatrix} \frac{p_-}{E(\mathbf{p})+m} \\ \frac{-p_z}{E(\mathbf{p})+m} \\ 0 \\ 1 \end{pmatrix} \qquad & v(\mathbf{p},-) = \sqrt{\frac{E(\mathbf{p})+m}{2m}} \begin{pmatrix} \frac{p_z}{E(\mathbf{p})+m} \\ \frac{p_+}{E(\mathbf{p})+m} \\ 1 \\ 0 \end{pmatrix}
\end{aligned} \tag{5.43}
$$

where

$$
p_\pm = p_x \pm \mathrm{i}p_y \qquad E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2} \tag{5.44}
$$

The orthogonality relations are given by:

$$
\begin{aligned}
v^\dagger(\mathbf{p},s)v(\mathbf{p},s') =& \frac{E(\mathbf{p})}{m}\delta_{s,s'} \qquad u^\dagger(\mathbf{p},s)u(\mathbf{p},s') = \frac{E(\mathbf{p})}{m}\delta_{s,s'} \\
v^\dagger(-\mathbf{p},s)u(\mathbf{p},s') =& u^\dagger(\mathbf{p},s)v(-\mathbf{p},s') = 0
\end{aligned} \tag{5.45}
$$

where $s, s' = \pm$. A solution to the free Dirac equation can be written as:

$$
\psi(\mathbf{x};t) = \int_{\mathbb{R}^3} d^3p \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{m}{E(\mathbf{p})}} \sum_{\pm s} \left[b(\mathbf{p},s)u(\mathbf{p},s)e^{-\mathrm{i}p^\mu x_\mu} + d^*(\mathbf{p},s)v(\mathbf{p},s)e^{\mathrm{i}p^\mu x_\mu} \right] \tag{5.46}
$$

Here, $b(\mathbf{p},s)$ and $d^*(\mathbf{p},s)$ are the Fourier coefficients which satisfy

$$
\int_{\mathbb{R}^3} d^3p \sum_{\pm s} \left(|b(\mathbf{p},s)|^2 + |d(\mathbf{p},s)|^2 \right) = 1 \tag{5.47}
$$

if $\psi(\mathbf{x};t)$ is normalised. Confining the particles on a three-dimensional box with periodic boundaries, we obtain again the quantised momenta $\frac{2\pi}{L}\mathbf{p}_k$, $\mathbf{p}_k \in \mathbb{Z}^3$. As before, the letter $k \in \mathbb{N}$ labels all possible momenta in increasing order. By means of the normalisation relations, the basis vectors are given by:

$$
|\varphi^{\mathrm{pos}}_{k,s}\rangle = \sqrt{\frac{1}{V}} \sqrt{\frac{m}{E(\mathbf{p}_k)}} u(\mathbf{p}_k,s) e^{\frac{2\pi\mathrm{i}}{L}\mathbf{p}_k\cdot\mathbf{x}} \tag{5.48}
$$

$$
|\varphi^{\mathrm{neg}}_{k,s}\rangle = \sqrt{\frac{1}{V}} \sqrt{\frac{m}{E(\mathbf{p}_k)}} v(\mathbf{p}_k,s) e^{-\frac{2\pi\mathrm{i}}{L}\mathbf{p}_k\cdot\mathbf{x}} \tag{5.49}
$$

The set of these vectors forms an orthonormal system for the Dirac equation on a three-dimensional torus. Note that confining the system on a torus breaks Lorentz invariance. However, also the treatment of Colin and Struyve is not Lorentz invariant since they introduce

[4]We set c=1.

a cut-off Λ. While it would be desirable to formulate everything in a covariant manner, this subtlety will not be of our concern here.
We may decompose a solution to the free Dirac equation as follows:

$$|\psi(t)\rangle = \sum_{s=\pm}\sum_{k=1}^{\infty} b(\mathbf{p}_k, s)e^{-\mathrm{i}E(\mathbf{p}_k)t}|\varphi^{\mathrm{pos}}_{k,s}\rangle + \sum_{s=\pm}\sum_{k=1}^{\infty} d^*(\mathbf{p}_k, s)e^{\mathrm{i}E(\mathbf{p}_k)t}|\varphi^{\mathrm{neg}}_{k,s}\rangle \tag{5.50}$$

Our initial Dirac sea is modelled by $2N$ particles filled up until the Fermi edge.

$$|\Lambda\rangle = |\varphi^{\mathrm{neg}}_{1,+} \wedge \varphi^{\mathrm{neg}}_{1,-} \wedge \ldots \wedge \varphi^{\mathrm{neg}}_{N,+} \wedge \varphi^{\mathrm{neg}}_{N,-}\rangle \tag{5.51}$$

As before, all occupied momenta lie inside the Fermi ball with radius proportional to $\rho^{1/3}$. Explicitly, we can repeat the calculation from above: The one particle reduced density matrix is given by:

$$\gamma^{(1)} := \sum_{s=\pm}\sum_{k\leq N} |\varphi^{\mathrm{neg}}_{k,s}\rangle\langle\varphi^{\mathrm{neg}}_{k,s}| \tag{5.52}$$

In position representation we obtain:

$$\gamma^{(1)}(\mathbf{x},\mathbf{y}) = \sum_{s=\pm}\sum_{k\leq N} |\varphi^{\mathrm{neg}}_{k,s}\rangle\langle\varphi^{\mathrm{neg}}_{k,s}| = \sum_{s=\pm}\sum_{k=1}^{N} \frac{1}{V}\frac{m}{E(\mathbf{p}_k)} v(\mathbf{p}_k, s)v^\dagger(\mathbf{p}_k, s)e^{\frac{2\pi\mathrm{i}}{L}\mathbf{p}_k\cdot(\mathbf{x}-\mathbf{y})} \tag{5.53}$$

which reads in the thermodynamic limit as follows:

$$\lim_{\mathrm{TD}} \gamma^{(1)}(\mathbf{x},\mathbf{y}) = \sum_{s=\pm}\int_{|\mathbf{p}|<k_F} d^3p \frac{1}{(2\pi)^3}\frac{m}{E(\mathbf{p})} v(\mathbf{p}, s)v^\dagger(\mathbf{p}, s)e^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \tag{5.54}$$

The Fermi momentum k_F can be determined as follows: Note that

$$\mathrm{tr}_{\mathbb{C}^4}\Big(v(\mathbf{p}, s)v^\dagger(\mathbf{p}, s)\Big) = v^\dagger(\mathbf{p}, s)v(\mathbf{p}, s) = \frac{E(\mathbf{p})}{m} \tag{5.55}$$

Thus,

$$\mathrm{tr}_{\mathbb{C}^4}\gamma^{(1)}(\mathbf{x},\mathbf{x}) = \frac{2N}{V} = \rho \tag{5.56}$$

holds and, as a consequence,

$$\begin{aligned} \lim_{\mathrm{TD}} \mathrm{tr}_{\mathbb{C}^4}\gamma^{(1)}(\mathbf{x},\mathbf{x}) &= \int_{|\mathbf{p}|<k_F} d^3p \frac{2}{(2\pi)^3} = \frac{k_F^3}{3\pi^2} \\ \Rightarrow k_F &= \left(3\pi^2\rho\right)^{1/3} \end{aligned} \tag{5.57}$$

This definition of the Fermi momentum is standard in the literature[5]. Note that if we would haven taken the thermodynamic limit differently (without the factors of 2π), k_F would read differently. However all integrals would also be defined differently (by some factors of 2π) and, of course, the final result would be the same.
As already explained, the variance is the sum of two terms; the one coming from all transitions into the negative spectrum, while the other is the sum of all transitions into the positive spectrum. We will treat each contribution separately.

[5] See also [Maruhn and Suraud, 2010].

Theorem 5.3.1. *Let $v \in C^\infty(\mathbb{T}^3) \cap C_0^\infty(\mathbb{R}^3)$. Then*

$$\lim_{TD} Var_{2N}^{v}(\boldsymbol{0}) \sim \rho^{2/3} \tag{5.58}$$

holds. Moreover, the contributions which arise only from the transitions into the positive spectrum scale as $\rho^{1/3}$.

Proof.
The variance is the sum of the transitions into the negative energy spectrum with energy smaller than the Fermi energy plus the sum of all transition amplitudes into the positive spectrum. Explicitly, we write:

$$\lim_{\text{TD}} \text{Var}_{2N}^{v}(\mathbf{y}) = \lim_{\text{TD}} \text{Var}_{2N}^{v}(\mathbf{0}) = \text{Var}_{2N}^{\text{pos},v}(\mathbf{0}) + \text{Var}_{2N}^{\text{neg},v}(\mathbf{0}) \tag{5.59}$$

where

$$\begin{aligned}
\text{Var}_{2N}^{\text{neg},v}(\mathbf{0}) &:= \sum_{s,s'=\pm} \sum_{k=1}^{N} \sum_{l=N+1}^{\infty} \left| \langle \varphi_{l,s'}^{\text{neg}} | v(\mathbf{x}) | \varphi_{k,s}^{\text{neg}} \rangle \right|^2 \\
&= \sum_{s,s'=\pm} \sum_{k=1}^{N} \sum_{l=N+1}^{\infty} \left| \frac{m}{\sqrt{E(\mathbf{p}_l)E(\mathbf{p}_k)}} \frac{1}{V} v^\dagger(\mathbf{p}_l, s') v(\mathbf{p}_k, s) \int_{\mathbb{R}^3} d^3x \; v(\mathbf{x}) e^{\frac{2\pi \mathrm{i}}{L} \mathbf{x} \cdot (\mathbf{p}_l - \mathbf{p}_k)} \right|^2 \\
&= \sum_{s,s'=\pm} \sum_{k=1}^{N} \sum_{l=N+1}^{\infty} (2\pi)^6 \left| \frac{m}{\sqrt{E(\mathbf{p}_l)E(\mathbf{p}_k)}} \frac{1}{V} v^\dagger(\mathbf{p}_l, s') v(\mathbf{p}_k, s) \mathcal{F}(v) \left(\frac{2\pi}{L} (\mathbf{p}_l - \mathbf{p}_k) \right) \right|^2
\end{aligned} \tag{5.60}$$

Taking the thermodynamic limit, we obtain[6]:

$$\begin{aligned}
\lim_{\text{TD}} \text{Var}_{2N}^{\text{neg},v}(\mathbf{0}) &= \sum_{s,s'=\pm} \int_{|\mathbf{p}_1| < k_F} d^3p_1 \int_{|\mathbf{p}_2| > k_F} d^3p_2 \left| \frac{m}{\sqrt{E(\mathbf{p}_2)E(\mathbf{p}_1)}} v^\dagger(\mathbf{p}_2, s') v(\mathbf{p}_1, s) \mathcal{F}(v) (\mathbf{p}_2 - \mathbf{p}_1) \right|^2 \\
&= \sum_{s,s'=\pm} \int_{|\mathbf{p}_1| < k_F} d^3p_1 \int_{|\mathbf{p}_2| > k_F} d^3p_2 \frac{m^2}{E(\mathbf{p}_1)E(\mathbf{p}_2)} \\
&\quad \times v^\dagger(\mathbf{p}_1, s) v(\mathbf{p}_2, s') v^\dagger(\mathbf{p}_2, s') v(\mathbf{p}_1, s) \left| \mathcal{F}(v) (\mathbf{p}_2 - \mathbf{p}_1) \right|^2
\end{aligned} \tag{5.61}$$

In analogy, $\text{Var}_{2N}^{\text{pos},v}(\mathbf{0})$ is given by:

$$\begin{aligned}
\text{Var}_{2N}^{\text{pos},v}(\mathbf{0}) &= \sum_{s,s'=\pm} \sum_{k=1}^{N} \sum_{l=1}^{\infty} \left| \langle \varphi_{l,s'}^{\text{pos}} | v(\mathbf{x}) | \varphi_{k,s}^{\text{neg}} \rangle \right|^2 \\
&= \sum_{s,s'=\pm} \sum_{k=1}^{N} \sum_{l=1}^{\infty} \left| \frac{m}{\sqrt{E(\mathbf{p}_l)E(\mathbf{p}_k)}} \frac{1}{V} u^\dagger(\mathbf{p}_l, s') v(\mathbf{p}_k, s) \int_{\mathbb{R}^3} d^3x \; v(\mathbf{x}) e^{-\frac{2\pi \mathrm{i}}{L} \mathbf{x} \cdot (\mathbf{p}_l + \mathbf{p}_k)} \right|^2 \\
&= \sum_{s,s'=\pm} \sum_{k=1}^{N} \sum_{l=N+1}^{\infty} (2\pi)^6 \left| \frac{m}{\sqrt{E(\mathbf{p}_l)E(\mathbf{p}_k)}} \frac{1}{V} u^\dagger(\mathbf{p}_l, s') v(\mathbf{p}_k, s) \mathcal{F}(v) \left(\frac{-2\pi}{L} (\mathbf{p}_l + \mathbf{p}_k) \right) \right|^2
\end{aligned} \tag{5.62}$$

[6]Again, we have to take $\frac{2\pi}{L}$ as the axes division to be in agreement with the derived expression of k_F.

Or, taking the thermodynamic limit:

$$\lim_{\text{TD}} \operatorname{Var}^{\text{pos},v}_{2N}(\mathbf{0}) = \sum_{s,s'=\pm} \left| \frac{m}{\sqrt{E(\mathbf{p}_2)E(\mathbf{p}_1)}} u^\dagger(\mathbf{p}_2, s')v(\mathbf{p}_1, s)\mathcal{F}(v)\left(-(\mathbf{p}_2+\mathbf{p}_1)\right)\right|^2$$
$$= \sum_{s,s'=\pm} \int_{|\mathbf{p}_1|<k_F} d^3p_1 \int_{\mathbb{R}^3} d^3p_2 \frac{m^2}{E(\mathbf{p}_1)E(\mathbf{p}_2)}$$
$$\times v^\dagger(\mathbf{p}_1, s)u(\mathbf{p}_2, s')u^\dagger(\mathbf{p}_2, s')v(\mathbf{p}_1, s)\left|\mathcal{F}(v)\left(\mathbf{p}_2+\mathbf{p}_1\right)\right|^2 \tag{5.63}$$

In order to simplify the expression derived above, we use the following identities (c.f. [Colin and Struyve, 2007], equation (B.12)):

$$\sum_{s=\pm} u_a(\mathbf{p}, s)u^\dagger_{a'}(\mathbf{p}, s) = \left(\frac{E(\mathbf{p}) + \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m}{2m}\right)_{aa'}$$
$$\sum_{s=\pm} v_a(\mathbf{p}, s)v^\dagger_{a'}(\mathbf{p}, s) = \left(\frac{E(\mathbf{p}) + \boldsymbol{\alpha}\cdot\mathbf{p} - \beta m}{2m}\right)_{aa'} \tag{5.64}$$

where $a, a' = 1, 2, 3, 4$ denote the corresponding entries of the vector or the matrix, respectively. $\boldsymbol{\alpha}$ and β are the Dirac matrices.
A small calculation yields:

$$v^\dagger(\mathbf{p}_1, s)u(\mathbf{p}_2, s')u^\dagger(\mathbf{p}_2, s')v(\mathbf{p}_1, s) = \frac{E(\mathbf{p}_1)E(\mathbf{p}_2) + \mathbf{p}_1\cdot\mathbf{p}_2 - m^2}{m^2} \tag{5.65}$$
$$v^\dagger(\mathbf{p}_1, s)v(\mathbf{p}_2, s')v^\dagger(\mathbf{p}_2, s')v(\mathbf{p}_1, s) = \frac{E(\mathbf{p}_1)E(\mathbf{p}_2) + \mathbf{p}_1\cdot\mathbf{p}_2 + m^2}{m^2} \tag{5.66}$$

So we obtain:

$$\lim_{\text{TD}} \operatorname{Var}^{\text{neg},v}_{2N}(\mathbf{0}) = \int_{|\mathbf{p}_1|<k_F} d^3p_1 \int_{|\mathbf{p}_2|>k_F} d^3p_2\; h^{\text{neg}}(\mathbf{p}_1, \mathbf{p}_2)\left|\mathcal{F}(v)\left(\mathbf{p}_2-\mathbf{p}_1\right)\right|^2 \tag{5.67}$$

where

$$h^{\text{neg}}(\mathbf{p}_1, \mathbf{p}_2) := \frac{E(\mathbf{p}_1)E(\mathbf{p}_2) + \mathbf{p}_1\cdot\mathbf{p}_2 + m^2}{E(\mathbf{p}_1)E(\mathbf{p}_2)} \tag{5.68}$$

Similar, the positive energy transitions read:

$$\lim_{\text{TD}} \operatorname{Var}^{\text{pos},v}_{2N}(\mathbf{0}) = \int_{|\mathbf{p}_1|<k_F} d^3p_1 \int_{\mathbb{R}^3} d^3p_2\; h^{\text{pos}}(\mathbf{p}_1, \mathbf{p}_2)\left|\mathcal{F}(v)\left(\mathbf{p}_2+\mathbf{p}_1\right)\right|^2 \tag{5.69}$$

where

$$h^{\text{pos}}(\mathbf{p}_1, \mathbf{p}_2) := \frac{E(\mathbf{p}_1)E(\mathbf{p}_2) + \mathbf{p}_1\cdot\mathbf{p}_2 - m^2}{E(\mathbf{p}_1)E(\mathbf{p}_2)} \tag{5.70}$$

We may also write up the variance of the density fluctuations $v(x) = \mathbb{1}_B(\mathbf{x})$ explicitly:

$$\begin{aligned}
\lim_{\text{TD}} \text{Var}_{2N}^{\text{neg},\mathbb{1}_B}(\mathbf{0}) =& \sum_{s,s'=\pm} \int_{|\mathbf{p}_1|<k_F} d^3p_1 \int_{|\mathbf{p}_2|>k_F} d^3p_2 \frac{m^2}{E(\mathbf{p}_1)E(\mathbf{p}_2)} \\
&\times \left| \int_{|\mathbf{x}|<R} d^3x \; v^\dagger(\mathbf{p}_2, s')v(\mathbf{p}_1, s) \frac{1}{(2\pi)^3} e^{\mathrm{i}\mathbf{x}\cdot(\mathbf{p}_2-\mathbf{p}_1)} \right|^2 \\
=& \sum_{s,s'=\pm} \int_{|\mathbf{p}_1|<k_F} d^3p_1 \int_{|\mathbf{p}_2|>k_F} d^3p_2 \int_{|\mathbf{x}|<R} d^3x_1 \int_{|\mathbf{x}|<R} d^3x_2 \frac{m^2}{E(\mathbf{p}_1)E(\mathbf{p}_2)} \\
&\times v^\dagger(\mathbf{p}_1, s)v(\mathbf{p}_2, s')v^\dagger(\mathbf{p}_2, s')v(\mathbf{p}_1, s) \frac{1}{(2\pi)^6} e^{\mathrm{i}(\mathbf{x}_1-\mathbf{x}_2)\cdot(\mathbf{p}_2-\mathbf{p}_1)} \\
=& \frac{1}{4\pi^4} \int_{|\mathbf{p}_1|<k_F} d^3p_1 \int_{|\mathbf{p}_2|>k_F} d^3p_2 \frac{E(\mathbf{p}_1)E(\mathbf{p}_2) + \mathbf{p}_1 \cdot \mathbf{p}_2 + m^2}{E(\mathbf{p}_1)E(\mathbf{p}_2)} \\
&\times \left(\frac{\sin(R|\mathbf{p}_1 - \mathbf{p}_2|) - R|\mathbf{p}_1 - \mathbf{p}_2| \cos(R|\mathbf{p}_1 - \mathbf{p}_2|)}{|\mathbf{p}_1 - \mathbf{p}_2|^3} \right)^2
\end{aligned} \tag{5.71}$$

and

$$\begin{aligned}
\lim_{\text{TD}} \text{Var}_{2N}^{\text{pos},\mathbb{1}_B}(\mathbf{0}) =& \sum_{s,s'=\pm} \int_{|\mathbf{p}_1|<k_F} d^3p_1 \int_{\mathbb{R}^3} d^3p_2 \frac{m^2}{E(\mathbf{p}_1)E(\mathbf{p}_2)} \\
&\times \left| \int_{|\mathbf{x}|<R} d^3x \; u^\dagger(\mathbf{p}_2, s')v(\mathbf{p}_1, s) \frac{1}{(2\pi)^3} e^{\mathrm{i}\mathbf{x}\cdot(\mathbf{p}_2+\mathbf{p}_1)} \right|^2 \\
=& \sum_{s,s'=\pm} \int_{|\mathbf{p}_1|<k_F} d^3p_1 \int_{\mathbb{R}^3} d^3p_2 \int_{|\mathbf{x}|<R} d^3x_1 \int_{|\mathbf{x}|<R} d^3x_2 \frac{m^2}{E(\mathbf{p}_1)E(\mathbf{p}_2)} \\
&\times v^\dagger(\mathbf{p}_1, s)u(\mathbf{p}_2, s')u^\dagger(\mathbf{p}_2, s')v(\mathbf{p}_1, s) \frac{1}{(2\pi)^6} e^{\mathrm{i}(\mathbf{x}_1-\mathbf{x}_2)\cdot(\mathbf{p}_2+\mathbf{p}_1)} \qquad (5.72) \\
=& \frac{1}{4\pi^4} \int_{|\mathbf{p}_1|\leq k_F} d^3p_1 \int_{\mathbb{R}} d^3p_2 \frac{E(\mathbf{p}_1)E(\mathbf{p}_2) + \mathbf{p}_1 \cdot \mathbf{p}_2 - m^2}{E(\mathbf{p}_1)E(\mathbf{p}_2)} \\
&\times \left(\frac{\sin(R|\mathbf{p}_1 + \mathbf{p}_2|) - R|\mathbf{p}_1 + \mathbf{p}_2| \cos(R|\mathbf{p}_1 + \mathbf{p}_2|)}{|\mathbf{p}_1 + \mathbf{p}_2|^3} \right)^2
\end{aligned} \tag{5.73}$$

Colin and Struyve derived expression (5.72), where, as already mentioned, in their paper the cut-off Λ is equal to our Fermi momentum k_F. There is one difference to our expression: The transitions into the positive spectrum are restricted by $|\mathbf{p}_2| \leq \Lambda$. For further estimates, this restriction is however removed, beginning with their equation (B.14), they get the same result we obtain. Thus, our analysis reveals the main difference in comparison with their result. We would get the same result if we would *initially* restrict the possible transitions only to positive states with momenta smaller than k_F, or, after dropping this restriction, only to positive momenta. We think that our result is more honest about the underlying picture. We have a finite dimensional sea in mind and want to show that, for high densities, it cannot be detected. In their model the cut-off is introduced in order to get finite expressions, however, they neglect $\lim_{\text{TD}} \text{Var}_{2N}^{\text{neg},v}(\mathbf{0})$. We want to remark that they also commit a small error. They claim that

$$\left(\frac{\sin(R|\mathbf{p}_1 + \mathbf{p}_2|) - R|\mathbf{p}_1 + \mathbf{p}_2| \cos(R|\mathbf{p}_1 + \mathbf{p}_2|)}{|\mathbf{p}_1 + \mathbf{p}_2|^3} \right)^2 \lesssim e^{-R^2|\mathbf{p}_1+\mathbf{p}_2|^2} \tag{5.74}$$

holds (cf. their equations (B.13) and (B.14)). This wrong estimate is performed without calculating the explicit Fourier transform of the indicator function, hence they are not aware of the real behaviour of this function. While their estimate is technically not true, their final estimate for $\lim_{\mathrm{TD}} \mathrm{Var}_{2N}^{\mathrm{pos},\mathbb{1}_B}(\mathbf{0})$ is the same. This equivalence is because of the following: Whether we estimate the Fourier transform of the potential using the Paley-Wiener theorem, or, as it is done in [Colin and Struyve, 2007], this function is considered to decay exponentially, in both cases, the same asymptotic behaviour is obtained.

Next, we want to estimate the two types of transitions separately.

Lemma 5.3.1. *For the transitions which arise at the negative part of the spectrum, the following asymptotic behaviour is true:*

$$\lim_{TD} Var_{2N}^{neg,v}(\boldsymbol{0}) \sim \begin{cases} \rho^{2/3}\ln(\rho) & \text{for } v(\boldsymbol{x}) = \mathbb{1}_B(\boldsymbol{x}) \\ \rho^{2/3} & \text{for } v \in C^\infty(\mathbb{T}^3) \cap C_0^\infty(\mathbb{R}^3) \end{cases} \tag{5.75}$$

Proof.
Up to the function $h^{\mathrm{neg}}(\mathbf{p}_1,\mathbf{p}_2)$, $\lim_{\mathrm{TD}} \mathrm{Var}_{2N}^{\mathrm{neg},v}(\mathbf{0})$ is exactly expression (5.19) we obtained in the spinless case. Let us therefore deliberate whether this additional function changes our result from above. For this, we use the following estimates:

$$0 < h^{\mathrm{neg}}(\mathbf{p}_1,\mathbf{p}_2) \leq 2 \tag{5.76}$$

$$\text{For } \mathbf{p}_1 \cdot \mathbf{p}_2 > 0 \Rightarrow 1 \leq h^{\mathrm{neg}}(\mathbf{p}_1,\mathbf{p}_2) \leq 2 \tag{5.77}$$

Choosing $\mathbf{p}_2 \in M^\alpha_{\mathbf{p}_1}$, we know that in this set the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ are aligned. Hence, we can use the estimate $1 \leq h^{\mathrm{neg}}(\mathbf{p}_1,\mathbf{p}_2) \leq 2$ to conclude that

$$\lim_{\mathrm{TD}} \mathrm{Var}_{2N}^{\mathrm{neg},v}(\mathbf{0}) \sim \rho^{2/3} \tag{5.78}$$

holds for smooth potentials. For the indicator function, this result is again corrected by $\ln(\rho)$. □

The interesting case are of course the transitions to the positive spectrum. Remember: The perturbation expansion for the one-dimensional system relied on the fact that transitions near the Fermi edge are suppressed due to the high energies involved. This argumentation is no longer valid here. In principle, a negative energy particle might get excited to end up in *any* positive state. Hence, the fluctuations must be suppressed even more for the mean-field picture to work. Using that

$$0 \leq h^{\mathrm{pos}}(\mathbf{p}_1,\mathbf{p}_2) \leq 2 \tag{5.79}$$

holds for all momenta $\mathbf{p}_1, \mathbf{p}_2$, the integrand is positive for all $\mathbf{p}_1$ and $\mathbf{p}_2$. Colin and Struyve claim that the integral behaves approximately as $k_F \sim \rho^{1/3}$. For this to be true, (5.79) must be very small for the values where the Fourier transform is of order one. Namely, if we would replace $h^{\mathrm{pos}}(\mathbf{p}_1,\mathbf{p}_2)$ by 1, the variance would scale like ρ.
Note that $|\mathcal{F}(v)\,(\mathbf{p}_2+\mathbf{p}_1)|^2 = \mathcal{O}(1)$ only holds if $\mathbf{p}_2 \approx -\mathbf{p}_1$. So, only transitions to positive energy states with opposite momenta are possible. This is a consequence of the fact that negative energy solutions posses their momentum opposed to their velocity, i.e. for a right moving wave the momentum points to the left. However, the spin factor $h^{\mathrm{pos}}(\mathbf{p}_1,\mathbf{p}_2) = 0$ for

$\mathbf{p}_1 = -\mathbf{p}_2$, so transitions with exactly opposite momentum are not allowed. Consequently, these two contributions have the opposite behaviour, i.e. the one gets small whenever the other gets big. Integrating over $\mathbf{p}_1$ and $\mathbf{p}_2$, the question remains how the two functions behave for all values we are integrating over. As it will turn out, the two factors yield the overall behaviour

$$\lim_{\text{TD}} \text{Var}_{2N}^{\text{pos},v}(\mathbf{0}) \sim k_F \sim \rho^{1/3} \tag{5.80}$$

which was also derived by Colin and Struyve. Actually, we will give an upper bound for the case $m = 0$ and argue afterwards why the this bound should also be satisfied for $m \neq 0$.

Lemma 5.3.2. *For $m = 0$ and $v \in C^\infty(\mathbb{T}^3) \cap C_0^\infty(\mathbb{R}^3)$, the following bound holds:*

$$\lim_{TD} Var_{2N}^{pos,v}(\boldsymbol{0}) \lesssim \rho^{1/3} \tag{5.81}$$

Proof.
For $m = 0$, $h^{\text{pos}}(\mathbf{p}_1, \mathbf{p}_2)$ simplifies to:

$$h^{\text{pos}}(\mathbf{p}_1, \mathbf{p}_2) = \frac{|\mathbf{p}_1||\mathbf{p}_2| + \mathbf{p}_1 \cdot \mathbf{p}_2}{|\mathbf{p}_1||\mathbf{p}_2|} \tag{5.82}$$

which yields to the following equation:

$$\lim_{\text{TD}} \text{Var}_{2N}^{\text{pos},v}(\mathbf{0}) = \int_{|\mathbf{p}_1| \le k_F} d^3p_1 \int_{\mathbb{R}} d^3p_2 (1 + \cos(\Theta)) \left|\mathcal{F}(v)\left(\mathbf{p}_2 + \mathbf{p}_1\right)\right|^2 \tag{5.83}$$

Here, Θ denotes the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$. Again, one should note that $(1 + \cos(\Theta)) \approx 0$ holds for $\mathbf{p}_1 \approx -\mathbf{p}_2$. We will choose spherical coordinates (r, ϕ, Θ) for $\mathbf{p}_2$ and will again estimate the Fourier transform by Paley-Wiener.

$$\begin{aligned}\lim_{\text{TD}} \text{Var}_{2N}^{\text{pos},v}(\mathbf{0}) \le & \int_{|\mathbf{p}_1| \le k_F} d^3p_1 \int_{\mathbb{R}} d^3p_2 (1 + \cos(\Theta)) \left| \frac{D_p}{(1 + |\mathbf{p}_2 + \mathbf{p}_1|)^p} \right|^2 \\ =& 2\pi \int_{|\mathbf{p}_1| \le k_F} d^3p_1 \int_0^\infty dr \int_{-1}^1 d\cos(\Theta)\; r^2 (1 + \cos(\Theta)) \\ & \times \frac{D_p^2}{\left[1 + \sqrt{|\mathbf{p}_1|^2 + r^2 + 2|\mathbf{p}_1| r \cos(\Theta)}\right]^{2p}}\end{aligned} \tag{5.84}$$

Using the estimate

$$\frac{1}{\left[1 + \sqrt{|\mathbf{p}_1|^2 + r^2 + 2|\mathbf{p}_1| r \cos(\Theta)}\right]^{2p}} \le \frac{1}{[1 + |\mathbf{p}_1|^2 + r^2 + 2|\mathbf{p}_1| r \cos(\Theta)]^p} \tag{5.85}$$

we may write:

$$\begin{aligned}\lim_{\text{TD}} \text{Var}_{2N}^{\text{pos},v}(\mathbf{0}) \lesssim & \int_{|\mathbf{p}_1| \le k_F} d^3p_1 \int_0^\infty dr \int_{-1}^1 d\cos(\theta)\; r^2 (1 + \cos(\Theta)) \\ & \times \frac{1}{[1 + |\mathbf{p}_1|^2 + r^2 + 2|\mathbf{p}_1| r \cos(\Theta)]^p}\end{aligned} \tag{5.86}$$

Using that

$$\int dx \frac{1}{(A+Bx)^p} = -\frac{1}{p-1}\frac{1}{B}(A+Bx)^{-p+1} \tag{5.87}$$

and

$$\int dx \frac{x}{(A+Bx)^p} = -\frac{1}{p-1}\frac{1}{p-2}\frac{1}{B^2}(A+Bx)^{-p+2} - \frac{1}{p-1}\frac{1}{B}x(A+Bx)^{-p+1} \tag{5.88}$$

holds for $p > 2$, we can perform the integration over Θ with

$$\begin{aligned} A =& 1 + |\mathbf{p}_1|^2 + r^2 \\ B =& 2|\mathbf{p}_1| r \end{aligned}$$

to obtain:

$$\begin{aligned} \lim_{\text{TD}} \text{Var}_{2N}^{\text{pos},v}(\mathbf{0}) \lesssim & \int_{|\mathbf{p}_1|\le k_F} d^3p_1 \int_0^\infty dr\ r^2 \Bigg(\frac{-1}{p-1}\frac{1}{r|\mathbf{p}_1|}\left[1+(r+|\mathbf{p}_1|)^2\right]^{1-p} \\ & - \frac{1}{p-1}\frac{1}{p-2}\frac{1}{(2r|\mathbf{p}_1|)^2}\left(\left[1+(r+|\mathbf{p}_1|)^2\right]^{2-p} - \left[1+(r-|\mathbf{p}_1|)^2\right]^{2-p}\right)\Bigg) \\ =& \frac{-4\pi}{p-1}\int_0^{k_F} dx \int_0^\infty dr\ rx\left[1+(r+x)^2\right]^{1-p} \\ & -\pi\frac{1}{p-1}\frac{1}{p-2}\int_0^{k_F} dx \int_0^\infty dr \left(\left[1+(r+x)^2\right]^{2-p} - \left[1+(r-x)^2\right]^{2-p}\right) \end{aligned} \tag{5.89}$$

Only the last term is of importance for our estimate. The other two are asymptotically constant:

$$\begin{aligned} & \lim_{\rho\to\infty}\int_0^{k_F} dx \int_0^\infty dr\ rx\left[1+(r+x)^2\right]^{1-p} \\ & = \int_0^\infty dx \int_0^\infty dr\ rx\left[1+(r+x)^2\right]^{1-p} < \infty \end{aligned} \tag{5.90}$$

$$\begin{aligned} & \lim_{\rho\to\infty}\int_0^{k_F} dx \int_0^\infty dr \left[1+(r+x)^2\right]^{2-p} \\ & = \int_0^\infty dx \int_0^\infty dr \left[1+(r+x)^2\right]^{2-p} < \infty \end{aligned} \tag{5.91}$$

which holds for p big enough. The important contribution reads:

$$\begin{aligned} \lim_{\text{TD}} \text{Var}_{2N}^{\text{pos},v}(\mathbf{0}) \lesssim & \int_0^{k_F} dx \int_0^\infty dr \left[1+(r-x)^2\right]^{2-p} \\ =& \int_0^{k_F} dx \int_{-x}^\infty dr \left[1+r^2\right]^{2-p} \\ \le& \int_0^{k_F} dx \int_{\mathbb{R}} dr \left[1+r^2\right]^{2-p} \sim \rho^{1/3} \end{aligned} \tag{5.92}$$

Note furthermore that $\int_{-x}^\infty dr\left[1+r^2\right]^{2-p} = \mathcal{O}(1)$ for all $x > 0$, which yields the conclusion that the asymptotics is indeed given by $\rho^{1/3}$. □

While we have estimated the variance only from above, we expect this bound to hold in general. The asymptotic behaviour is determined by high momenta $|\mathbf{p}_1| \approx k_F$. As we have argued, we have to choose $\mathbf{p}_2$ of the same order but in the opposite direction. Otherwise, the Fourier transform of the potential is very close to zero. In the regime $|\mathbf{p}_1|, |\mathbf{p}_2| \gg m$

$$h^{\text{pos}}(\mathbf{p}_1, \mathbf{p}_2) \approx \frac{|\mathbf{p}_1||\mathbf{p}_2| + \mathbf{p}_1 \cdot \mathbf{p}_2}{|\mathbf{p}_1||\mathbf{p}_2|} \tag{5.93}$$

holds to a very high accuracy. Henceforth the estimate above is indeed expected to be true and measures the growth of the fluctuations coming from the transitions of negative to positive energy states. □

It is quite remarkable that the spin structure, originally derived from relativistic invariance, helps tremendously to suppress the fluctuations and henceforth supports the mean-field idea, which describes the behaviour of very many particles. Note that this surprising fact does not origin from the Pauli principle.
Colin and Struyve obtain this result for $m \neq 0$ by performing a Taylor expansion of the spinor factor $h^{\text{pos}}(\mathbf{p}_1, \mathbf{p}_2)$ and keeping only the leading terms. Our result shows that their derivation, using the second quantised formalism, yields to the same result as the corresponding calculation in first quantisation. This behaviour is quite general in QED. There is of course no mystery involved whatsoever since the Fock space used is equivalent to an infinite copy of N-particle Hilbert spaces. So, as long as we investigate systems with a fixed particle number, the results have to coincidence. Yet, our example might weaken the questionable claim that QED describes the dynamics of quantised fields and it is not possible to give any other interpretation of it. Of course, for further analysis of the mean-field picture, we need to treat the system dynamically. This procedure will be sketched now for systems in arbitrary dimensions.

Chapter 6

The Analysis of General Systems

In this section we want to discuss the possibility to repeat the perturbation expansion for higher dimensional systems. In contrast to the one-dimensional case, it may not be possible to apply the method we used before. The main difference we have to deal with is the growth of the fluctuations. Indeed, for smooth potentials, the variance is always proportional to the surface of the Fermi sphere, that is

$$\lim_{\text{TD}} \text{Var}_N^v(\mathbf{0}) \sim k_F^{d-1} \sim \rho^{\frac{d-1}{d}} \tag{6.1}$$

In addition, for the Dirac equation another contribution arises, namely the transitions to the positive spectrum. Owing these facts, we cannot repeat the estimate given for the one-dimensional system. There, for a potential with gap (c.f. lemma 4.8.3), the estimate reads very roughly as follows:

$$\lim_{\text{TD}} \alpha_t \leq \underbrace{\frac{\sqrt{\lim_{\text{TD}} \text{Var}_{N+1}^v(0)}}{E(\rho/2+C)-E(\rho/2)}}_{\text{first order}} + \underbrace{\frac{\sqrt{\lim_{\text{TD}} \text{Var}_{N+1}^v(0) \lim_{\text{TD}} \text{Var}_{N+1}^v(0)}}{E(\rho/2+C)-E(\rho/2)}}_{\text{second order}} \tag{6.2}$$

Of course, the second order term reads differently, but due to the structure of the functions M_i used in the estimate, the asymptotics is the same. Since

$$\frac{1}{E(\rho/2+C)-E(\rho/2)} \approx \frac{1}{\rho} \tag{6.3}$$

holds for nonrelativistic Hamiltonians, we obtain an overall decay of ρ^{-1}. We already saw that this estimate was no longer true for the ultra-relativistic system.
By estimating the first-order term E_1 it is possible to infer if the perturbation expansion will still work. Explicitly, E_1 reads now in d dimensions as follows:

$$\lim_{\text{TD}} E_1^2 = \int_{|\mathbf{m}|<k_F} d^d m \int_{|\mathbf{l}|>k_F} d^d l \, |\mathcal{F}(v)(\mathbf{m}-\mathbf{l})|^2 \left\| \int_0^t dt' e^{4\pi^2 \mathrm{i}(\mathbf{l}^2-\mathbf{m}^2)t'} U_{-t'}^y \, e^{2\pi \mathrm{i}(\mathbf{m}-\mathbf{l})\cdot \mathbf{y}} U_{t'}^y \chi_0 \right\|_y^2 \tag{6.4}$$

As discussed, the most important contribution arises directly from the gap. Thus, we only consider the points $\mathbf{m}, \mathbf{l}$ where $|\mathbf{m}-\mathbf{l}| = \mathcal{O}(1)$ holds. For these points, a decay in ρ is needed for the mean-field picture to hold. On the other hand, if E_1 is small for these points, then we

can apply the Paley-Wiener theorem to conclude that the remainder also decays.
Let us first consider potentials with a gap. By integration by parts, the stationary phase argument yields a factor of:

$$\frac{1}{\mathbf{l}^2-\mathbf{m}^2}=\frac{1}{|\mathbf{l}|-|\mathbf{m}|}\frac{1}{|\mathbf{l}|+|\mathbf{m}|}\sim\frac{1}{C}\frac{1}{k_F}\sim\frac{1}{\rho^{1/d}} \tag{6.5}$$

Note that this term gets squared in the estimate of $\lim_{\text{TD}} E_1^2$. The variance, as already explained, yields a contribution $\sim\rho^{\frac{d-1}{d}}$. Therefore, the following estimate can be given:

$$\begin{aligned}\lim_{\text{TD}} E_1^2 \sim&\rho^{\frac{d-1}{d}}\frac{1}{\rho^{2/d}}\\ =&\rho^{\frac{d-3}{d}}\end{aligned} \tag{6.6}$$

Explicitly, we obtain:

$$d=1\to\lim_{\text{TD}} E_1^2\sim\rho^{-2} \tag{6.7}$$

$$d=2\to\lim_{\text{TD}} E_1^2\sim\rho^{-1/2} \tag{6.8}$$

$$d=3\to\lim_{\text{TD}} E_1^2\sim\mathcal{O}(1) \tag{6.9}$$

We conclude the following:
For a potential with a gap, the perturbation expansion might work in $d=2$ and will not work in $d=3$. For $d=2$, the proof will certainly be more complicated than the one presented for $d=1$. This is due to the fact that the fluctuations grow like $\rho^{1/2}$, so we cannot stop our perturbation expansion at second order, but must consider also higher order terms. Physically, we might think of graphene as an important example of a two dimensional system.
In $d=3$, we might expect that the mean-field model holds if we scale down the potential by some arbitrary small factor (like $\rho^{-\epsilon}$). At least, the term from first-order perturbation theory vanishes then. This result could not be obtained by solely looking at the fluctuations. Whether this scaling is physical, is of course an interesting question on its own. We will come back to this in the conclusion.

Next, consider general potentials which do not contain a gap. In order to apply the stationary phase lemma, we need to separate the critical point. As in $d=1$, we need to perform the splitting such that both terms yield the same asymptotics. To this end write

$$\begin{aligned}\lim_{\text{TD}} E_1^2=&\int_{|\mathbf{m}|<k_F} d^d m\int_{|\mathbf{l}|>k_F} d^d l\left(\Theta\left(|\mathbf{m}-\mathbf{l}|-\rho^{-\alpha}\right)+\Theta\left(\rho^{-\alpha}-|\mathbf{m}-\mathbf{l}|\right)\right)\\ &\times|\mathcal{F}(v)\,(\mathbf{m}-\mathbf{l})|^2\left\|\int_0^t dt' e^{4\pi^2\mathrm{i}(\mathbf{l}^2-\mathbf{m}^2)t'}U^y_{-t'}\,e^{2\pi\mathrm{i}(\mathbf{m}-\mathbf{l})\cdot\mathbf{y}}U^y_{t'}\chi_0\right\|_y^2\end{aligned} \tag{6.10}$$

where α will be determined now:
The last term entails the contributions arising from the gap. To estimate this term, consider first the integration over $\mathbf{l}$. We can essentially restrict the integration over the points where $k_F\le|\mathbf{l}|\le k_F+\rho^{-\alpha}$ and $|\mathbf{l}-\mathbf{m}|\approx\rho^{-\alpha}$ holds. In this region, the integrand is of order one, so we must only determine the volume over which is integrated. Consider the integration

pictorially: Fixing $\mathbf{m}$ very close to the Fermi sphere, the $\mathbf{l}$ integration takes place in this vicinity with distance to $\mathbf{m}$ of order $\rho^{-\alpha}$. This integral is of the same order as the volume of a ball with radius given by $\rho^{-\alpha}$. Next, the $\mathbf{m}$ integration takes place over a spherical area with radius k_F and width $\rho^{-\alpha}$. That is

$$\int_{|\mathbf{m}|<k_F} d^d m \int_{|\mathbf{l}|>k_F} d^d l\, \Theta\left(\rho^{-\alpha} - |\mathbf{m}-\mathbf{l}|\right) |\mathcal{F}(v)\,(\mathbf{m}-\mathbf{l})|^2 \left\| \int_0^t dt' e^{4\pi^2 \mathrm{i}(\mathbf{l}^2-\mathbf{m}^2)t'} U^y_{-t'}\; e^{2\pi \mathrm{i}(\mathbf{m}-\mathbf{l})\cdot \mathbf{y}} U^y_{t'} \chi_0 \right\|^2_y$$
$$\sim k_F^{d-1} \rho^{-\alpha} \rho^{-\alpha d} \sim \rho^{\frac{d-1}{d} - \alpha(d+1)} \tag{6.11}$$

The other contribution is proportional to the variance times the estimate of the phase:

$$\int_{|\mathbf{m}|<k_F} d^d m \int_{|\mathbf{l}|>k_F} d^d l\, \Theta\left(|\mathbf{m}-\mathbf{l}| - \rho^{-\alpha}\right) |\mathcal{F}(v)\,(\mathbf{m}-\mathbf{l})|^2 \left\| \int_0^t dt' e^{4\pi^2 \mathrm{i}(\mathbf{l}^2-\mathbf{m}^2)t'} U^y_{-t'}\; e^{2\pi \mathrm{i}(\mathbf{m}-\mathbf{l})\cdot \mathbf{y}} U^y_{t'} \chi_0 \right\|^2_y$$
$$\sim k_F^{d-1} \left(\frac{1}{|\mathbf{l}| - |\mathbf{m}|} \frac{1}{|\mathbf{l}| + |\mathbf{m}|} \right)^2 \sim \rho^{\frac{d-1}{d}} \left(\rho^{\alpha} k_F^{-1}\right)^2 \sim \rho^{\frac{d-3}{d} + 2\alpha} \tag{6.12}$$

Thus

$$\lim_{\mathrm{TD}} E_1^2 \sim \rho^{\frac{d-1}{d} - \alpha(d+1)} + \rho^{\frac{d-3}{d} + 2\alpha} \tag{6.13}$$

Using the optimal value

$$\alpha = \frac{2}{3d + d^2} \tag{6.14}$$

we obtain:

$$\lim_{\mathrm{TD}} E_1^2 \sim \rho^{\frac{d^2-5}{3d+d^2}} \tag{6.15}$$

which means:

$$d = 1 \to \lim_{\mathrm{TD}} E_1^2 \sim \rho^{-1} \tag{6.16}$$
$$d = 2 \to \lim_{\mathrm{TD}} E_1^2 \sim \rho^{-1/10} \tag{6.17}$$
$$d = 3 \to \lim_{\mathrm{TD}} E_1^2 \sim \rho^{2/9} \tag{6.18}$$

Thus, for two dimension, we still expect the mean-field picture to hold. For $d = 3$, the estimate of the first-order term does even grow with ρ, so we expect the mean-field picture to be wrong in this case.

Let us now consider the relativistic case: For fluctuations arising directly at the gap, we can no longer use the stationary phase method. That is, the perturbation expansion does not work in this case. For the transitions arising in the Dirac equation to the positive spectrum, it might be possible to pursue in perturbation theory, although this needs to be checked explicitly.
The method presented here thus works for nonrelativistic, low dimensional systems, but does not work for relativistic or high-dimensional ones. It is important to notice that this does *not* imply that the mean-field picture is wrong in the latter cases, but hints at the fact that a more profound analysis is required to decide this.

Chapter 7

Conclusion and Outlook

In this work we have shown that Dirac's idea of a uniform fermionic sea, which net effect is zero, can be justified on a rigorous basis, at least for some physical systems. Surprisingly, while people tried to argue why Dirac's idea might not be *necessary* for a second quantised *field* theory, the question of physical plausibility was never investigated on a dynamical basis. Our discussion was based on perturbation theory and a very simple Hamiltonian which only considers the interaction with one specific particle. The insight of Dirac helped us to actually prove the emergence of free motion for a nonrelativistic, one-dimensional system. Yet, we think that a more realistic model is necessary in order to decide for which fermionic systems it is possible to extract an effective evolution equation for a single particle. More explicitly, one needs to investigate the following assumptions we made:

Is it possible to model the sea as some non-interacting system?
This assumption is indeed the most severe one we made. For this statement to be true, the equilibrium or ground state, obtained from some interacting theory, needs to be similar to the ground state of the non-interacting theory for all relevant time scales. If this is not the case, then a quite different analysis would be needed to conclude the validity of the mean-field picture. Of course, an interacting system of mutually interacting fermions with repulsive potential might even help to use the picture we employed. For such a system, we might expect that the constituents are even more uniformly distributed. Consequently, this might give a partial insight to the following question:

Is it possible to introduce some scaling by some physical argument?
As we have seen, for a three-dimensional, nonrelativistic system, such a scaling might enable us to prove the validity of the mean-field picture. Yet, it is not clear why, in principle, such a scaling should be possible. If it cannot be justified, then it might nevertheless be possible to extract some mean-field dynamics out of the fully interacting system. This might be done using reduced density matrices, where a different treatment beyond perturbation theory might be needed.
Of course, the most interesting question which could not be answered in this thesis is the conjecture made by Dirac:

Is it possible to base QED on the Dirac sea picture?

While we certainly made some progress here, the questions remains highly open. The most important step, using the methods presented here, would be to consider a model similar to the one employed by [Colin and Struyve, 2007] and to estimate the first-order perturbation there. Maybe it is also possible to perform this perturbative analysis for some fully interacting system. This might yield some affirmative insight, but, as expected, for a decisive answer there remains much to be done.

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Declaration of Authorship

I hereby confirm that I prepared this master thesis independently and on my own, by exclusive reliance on the tools and literature indicated therein.

Munich, 28.10.2013 *Maximilian Jeblick*