Mathematical treatment of Adiabatic Pair Creation in Laser Fields

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Erklärung:
Ich versichere hiermit, dass ich die hier vorliegende Arbeit selbständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Overview:
This is a review focusing on some recent work on the phenomenon of pair creation in external fields. Both the mathematical treatment and the experimental verification of this effect are discussed. The emphasis lies on the description of planned experiments using lasers.

The review consists of three parts:

First: Introduction to pair creation
In the first part two things are done:
First, the general ideas about the phenomenon of pair creation in external fields are presented. In particular the questions *What is meant when one says "Pair Creation"?* and *What does it mean that the field is external?* are answered and simple examples of pair creation based on "early calculations" are given.
Second, the adiabatic hypothesis is stated and it is explained how one can formulate a theorem (the *adiabatic theorem*) out of it.
The purpose of the first part is to give a better understanding of what this review is about, to mention what it is not about and to clarify the most important concepts which are needed. Not all of this part is mathematically rigorous. Rather intuitive formulations which are commonly used in the physics literature are (partly) used and non rigorous calculations are reviewed.

Second: Pair creation as an adiabatic phenomenon
The second part deals with the rigorous mathematical treatment both of the adiabatic theorem and of pair creation in external electromagnetic classical backgrounds.[1] The concept of the adiabatic limit is explained and the general ideas of the proofs of the important theorems are discussed.
The main points of this second part are first the explanation why pair creation in the way it is discussed is an adiabatic phenomenon in the sense of the adiabatic theorem[2] and second the discussion of the existence of adiabatic pair creation.

Third: Adiabatic Pair Creation in laserfields
The third part is the main part of this review. Adiabatic pair creation in laserfields is discussed there. The arguments on which the predictions of this phenomenon are usually based are reviewed and it is argued that some of these are highly questionable.
Finally, two questions are discussed. First, how can the insights of the mathematical treatment be used for a better argumentation? Second, how do these points change the predictions for the planned experiments?

[1] In particular it is argued why such a mathematical treatment is necessary at all in order to get reasonable physical results.

[2] Note that there is also pair creation which is not adiabatic. However, "non-adiabatic pair creation" is not the topic of this review.

Notation:
All scalar products are denoted by $<,>$ (norms by $||.||$). Only when it is not clear from the context which scalar product (norm) is meant this is explicitly stated.

Contents

1 Introduction to pair creation

1.1 Pair creation in external fields

1.1.1 About particle creation

Particle creation is roughly speaking the effect that particles are created out of fields.[3]

In this explication a problem arises immidiatly. To understand this it is important to point out that there are differences between the concept of a field and the concept of a particle. Two obvious ones are: First, the number of degrees of freedom is different: Whereas the degrees of freedom of a physical system described with the concept of particles are always countable, the degrees of freedom of a physical system described with the concept of fields form a continuum. Second, there are concepts which can be used for describing properties of particles but which are not applicable for fields and vice versa. These differences show that the concepts of particles and fields are a priori not consubstantial[4]. Naivly one can conclude that a field (or part of a field) can never simply change to be a particle, so particle creation is impossible. If particle creation nevertheless happens the main question is: How can a particle come out of a field (although the two concepts are not consubstantial)? This is one kind of manifestation of a problem which has been called the *problem of matter*.[5]

Nowadays particle creation can be described consistently in many parts of physics.[6]

The occurance of particle creation in the physical literature can basically be separated into four groups: First, particle creation is considered in the evolution of the universe where it happens due to the time dependence of the background. Second, particle creation occurs in black hole evaporation due to the existence of the black hole horizon as well as in other astrophysical situations. Third, in

[3]Strictly speaking this is wrong. The reason (that the particles are not created out of fields but in some sense with the help of fields out of the vacuum) is explained in **1.1.2**.

[4]german: wesensgleich

[5]see e.g. [1], p. 183. Usually what e.g. Weyl calls in [1] “problem of matter“ is slightly different than here in the context of particle creation: In [1] the problem is formulated in a context where it is assumed that the only way particles can arise is out of fields because it is assumed that a particle is just the manifestation of a very specific field configuration. In this sense the concept of a field is assumed to be more fundamental than the concept of a particle. So if a theory of particles under this assumption exists, the question how the concepts of particle and field transform into each other (in other words: how the special field configurations can arise) occurs. This is then called “the problem of matter“. The way the problem of matter is formulated here in the context of particle creation does not assume that particles are just specific field configurations. This would already be part of the answer to the problem (and i.e. it turned out that it is part of one answer (**1.1.2**)). With this in mind the basic question itself “How can particles arise out of fields?“ is the same in the context of [1] and here in the context of particle creation.

[6]Models solving the problem of matter (i.e. answering the question how particles can come out of a field) are discussed in **1.1.2**.

models of heavy ions particle creation is present and fourth, particle creation is considered in strong time dependent external electromagnetic fields.
This review only deals with the fourth point. However, one should note that some of the insight which will be needed in **2.2** has been developed in the discussion of particle creation in fields of heavy ions (for a detailed review see e.g. [2]).

1.1.2 Models of pair creation

Historically consistent models of particle creation solving the problem of matter in the way it was formulated in **1.1.1** were succesfully introduced in the discussion of quantum theory.
One model allowing for particle creation is the so called Dirac sea model which was introduced in order to interpret some physical phenomena which are predicted by the Dirac equation.[7] The Dirac sea model is quite similar to the model of a decaying atom. The creation of particles out of fields is reinterpreted as kind of decay process. It is argued that (since there is just the field, i.p. nothing to decay into particles) if created particles (as decay products) exist, they must come from the vacuum. That is the essential point of the Dirac sea picture. A specific structure of the vacuum is introduced: The vacuum should be thought of as something consisting of infinitly many (bound) particles (that is like the bound state of the decaying atom), sometimes called *the Dirac sea*. Now a present field can use its energy to do work and to take one or more particles out of the vacuum. This process is called particle creation in the Dirac sea picture. The missing particle(s) in the vacuum is (are) interpreted as so called hole(s) or antiparticle(s). That is why one usually does not speak of particle creation but of *pair creation*. It is very important that in this picture particles are not created out of fields but fields are used to create pairs of particles out of the vacuum. So in fact there is no conversion of two nonconsubstantial concepts into each other.
One immidiate question is: Why do not all particles fall into the vacuum ("pair annihilation") if the energy is lower there? It turned out that this will not be possible if the particles fulfill the Pauli principle. So it is because of the Pauli principle why (fermionic) relativistic matter can be stable.
There is one further important point: Even if no pairs are created charged particles can produce electric forces which polarize the vacuum (i.e. the Dirac sea). This phenomenon is well known as *vacuum polarization*.

Now it is formulated more quantitatively how the structure of the vacuum arises:
The free Dirac equation reads[8]

$$i\frac{\partial}{\partial t}\Psi(t,x) = H_0\Psi(t,x) \tag{1}$$

[7] see e.g. [7]
[8] Units s.t. i.p. $c = 1$ and $\hbar = 1$ will be used from now on.

where $t \in \mathbb{R}$ is the time variable, $x \in \mathbb{R}^3$ the space variable and H_0 the free Dirac operator (essentially self-adjoint on $C_0^\infty(\mathbb{R}^3 \backslash \{0\})$[9])

$$H_0 = -i \sum_{i=1}^{3} \alpha_i \frac{\partial}{\partial x_i} + \beta m \tag{2}$$

with α_i ($i \in \{1, 2, 3\}$) and β the usual hermitian matrices in $\mathbb{C}^4$ (see e.g. [3]). One important point is that Ψ is here interpreted as a one particle wavefunction in the Hilbertspace $L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$ where each "component wavefunction" satisfies the (one particle) Klein Gordon equation. Thus the given free Dirac equation is also called the *free one particle Dirac equation* and the constant m in the free Dirac operator is the mass of the described particle.

The Dirac sea picture is represented in the spectrum σ of H_0 which is purely absolutely continuous[10]:

$$\sigma(H_0) = (-\infty, -m] \cup [m, \infty) \tag{3}$$

Here $(-\infty, -m]$ represents the vacuum (with negative energy)[11] which is disjoint from $[m, \infty)$, the other part of the spectrum (with positive energy). Now the creation of one pair can be formulated like this: Take a field and use it to lift one state of the negative part of the spectrum into the positive part. That this is indeed predicted to be possible by the one particle Dirac equation, i.e. that a solution of the one particle Dirac equation describing such a situation exists, will be reviewed in **2.2.3**.

But how to formulate the creation of many pairs? One possibility to implement many particle systems in quantum mechanics is the concept of second quantization, i.e. one constructs the Fock space for the many particle system. Although one may argue that also for many particles the Dirac sea picture can be a useful intuition the second quantized formalism in the way it is usually used[12] is conceptually different, i.e. the notion of the Dirac sea is replaced by another notion of *vacuum*. What this means and how this works in detail for fermions is shortly outlined:[13]

Let $\mathcal{H}$ be the Hilbertspace of the (one particle) Dirac equation. In the free case $\mathcal{H}$ can be split into two orthogonal spectral subspaces of the free Dirac operator ($\mathcal{H}_+$ and $\mathcal{H}_-$)[14]:

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \tag{4}$$

[9] On its domain $H^1(\mathbb{R}^3) \otimes \mathbb{C}^4$ H_0 is self-adjoint (see e.g. [3], Theorem 1.1).

[10] see e.g. [3], Theorem 1.1

[11] More precisely: The vacuum is the state in which all states which exist in this part of the spectrum are occupied and all other states are not occupied.
This also means that antiparticles in the Dirac sea picture have negative (kinetic) energy.

[12] in particular with normal ordered creation and annihilation operators

[13] For reference and more details see e.g. [3], Chapter 10.

[14] This can be violated for Dirac operators minimally coupled to external fields (see **1.1.3**) when the external field is time dependent. For a detailed discussion of this point see e.g. [3], Chapter 10.

Define the Hilbertspaces for the particle and the antiparticle: $\mathcal{F}_+ := \mathcal{H}_+$ and $\mathcal{F}_- := C\mathcal{H}_-$ where C is the charge conjugation operator.[15] Define the n-particle (n-antiparticle) Hilbertspace as antisymmetrized version of $\mathcal{F}_+^{(n)} := \otimes_{i=1}^n \mathcal{F}_+$ ($\mathcal{F}_-^{(n)} := \otimes_{i=1}^n \mathcal{F}_-$). Now the Fock space $\mathcal{F}$ which is used to describe an arbitrary number of particles and antiparticles is defined as

$$\mathcal{F} := \oplus_{l,m=0}^{\infty} \left(\mathcal{F}_+^{a(l)} \otimes \mathcal{F}_-^{a(m)} \right) \tag{5}$$

Here ^{a} stands for the antisymmetrized version. Elements in $\mathcal{F}$ are sequences of functions $((\Psi)_{l,m})_{l,m \in \mathbb{N}}$ where for fixed l, m $\Psi_{l,m}$ depends on $(x_1^p, ...x_l^p, x_1^{ap}, ..., x_m^{ap})$ where $x^p \in \mathbb{R}^3$ denotes a coordinate for a particle and $x^{ap} \in \mathbb{R}^3$ one for an antiparticle. In particular the vacuum (the concept which replaces the Dirac sea) here is a "physical state without anything" (i.p. without particles) and is given by the Fock space element $(e^{iW}, 0, 0, 0, ...)$ where $W \in \mathbb{R}$.[16] By defining the usual creation and annihilation operators both for particles (a and a^+) and for antiparticles (b and b^+) both satisfying anticommutation relations (because of the Pauli principle) in a normal ordered way, it is possible to use $\mathcal{F}$ to implement pair creation and annihilation processes in many particle systems s. t. both particles and antiparticles have positive (kinetic) energy.[17] By defining a field operator Ψ as a linear combination of a and b^+, Ψ exactly acts as operator which creates a pair in an analogous sense of the Dirac sea picture by mapping the subspace $\mathcal{F}_+^{a(l)} \otimes \mathcal{F}_-^{a(m)}$ of $\mathcal{F}$ into the other subspace $\mathcal{F}_+^{a(l-1)} \otimes \mathcal{F}_-^{a(m+1)}$ for some $l, m \in \mathbb{N}$. For pair creation in the sense it was explained above (i.e. for pair creation out of the vacuum) one always starts with the vacuum, so the number of particles and the number of antiparticles always coincide. Mathematically pair creation in Fock space is usually formulated as a scattering process (see **1.1.4**).

If one consults todays physics literature one sees that the Dirac sea model is not very modern and instead the described formalism of second quantization is usually used. Schwinger once said: "The picture of an infinite sea of negative energy electrons is now best regarded as a historical curiosity, and forgotten." (J. Schwinger, [4], 1973). It is considered as more natural to look at the vacuum as the "physical state without anything" (i.p. without particles). In order to take care of this one should just use the second quantized formalism mentioned above

[15] In particular: $\mathcal{F}_+$ and $\mathcal{F}_-$ are both subspaces of $L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$.

[16] Note that the definition of the vacuum as "physical state without anything" implies that the creation an annihilation operators which have to be introduced are normal ordered. This normal ordering i.p. leads to a non vanishing vacuum energy which one may put into correspondence with the energy of the Dirac sea. This shows that the notion of the vacuum still is a subtle one.

[17] This means i. p. that the eigenvalues of the Dirac operator become bounded from below. This is due to the introduction of two different sets of creation and annihilation operators in a normal ordered way which is just possible because one deals with fermions. For more details see e.g. [2], Chapter 3.

just refering to many particle systems.[18]
Nevertheless since there is a formalism which can often be used to bring results based on the "one particle" Dirac equation in second quantized form (see e.g. [3], Chapter 10) the one particle Dirac equation is still useful but may be interpreted in another way.

Nowadays phycisists usually use models of quantum field theory "reinterpreting" the one particle Dirac equation. The Dirac equation is interpreted as a classical field equation, i.p. it has nothing to do with quantum mechanics. The function $\Psi(x,t)$ occuring in the Dirac equation represents a classical field.[19] In particular nature is described without using the concept of a particle at all[20].[21] This is why the question how two nonconsubstantial concepts transform into each other does not arise. Actually the term *particle creation* (or *pair creation*) is misleading because there are no particles in this theory.
Here "pair creation" is roughly described like follows: It is assumed that to get an appropriate description of nature the classical fields (i.p. the classical Dirac field described by the Dirac equation) have to be quantized.[22] The "wavefunction" in the quantized theory naturally is a functional because it is an object depending on

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[18] This is due to the fact that in the Fock space formalism the vacuum is indeed a "physical state without anything" (see above). I.p. the negative energy states of the Dirac sea model correspond to antiparticles (here particles with opposite charge) with positive energy which contradicts the Dirac sea model. For more details see [3], Chapter 10.

[19] The symbol Ψ is now misleading because this symbol is conventionally used for wavefunctions whereas in this point of view the field Ψ is not a wavefunction.

[20] What is meant with particles like "photons" in this viewpoint is shortly explained below.

[21] In particular m in the Dirac operator is not the mass of a particle but a (meaningless) constant.

[22] Formally *quantizing a system* here basically means the following: Take a classical theory, write down its Hamiltonean form. Write down the Hamilton equations of motion. Declare the state variables to be linear operators (e.g. in a field algebra) on an appropriate seperable Hilbert space and postulate the usual commutation or anticommutation relations for the canonically conjugated operators. (By declaring the quantities to be operators the Hamilton equations of motion e.g. of classical mechanics become the Heisenberg equations of motion.)

In this formal procedure many difficulties arise. The most important one is that it is unclear (or at least highly discussed) if and how this procedure changes the "correspondence of the theory to reality". Other difficulties are of mathematical nature: How to define an appropriate Hilbert space? How to choose the domains of the operators? How to order the operators which may not commute? How to implement the classical time evolution in a unitary way? and so on. The simplest example to see that the mathematics is much more involved in a theory quantized in such a way (leading to almost all of the mathematical questions) is a classical system with one particle. Here after quantizing the system i.p. the commutation relation $[r,p]=i$ is postulated for the operators r and $p=-i\nabla_x$. Taking formally the trace on both sides of the commutation relation shows that one has to deal with infinite dimensions (where not all operators are of trace class) in the quantized form of the theory.

It is important to note two further things: First, all this does not mean that every classical theory can be quantized and Second, not every quantum theory arises from quantizing a classical theory.

the state variables representing the physical degrees of freedom of the system (to be understood as the degrees of freedom in configuration space) which in the case of fields form a continuum, so the state variables are fields. This wavefunctional cannot vanish identically without violating the (anti)commutation relations postulated in the quantization process. That is why even in the ground state the energy of the quantized field does not vanish identically. This energy can be used to create so called “virtuell pairs“. (These pairs basically correspond to the vacuum polarization of the Dirac sea picture.) Now external fields[23] can lose energy, i.e. do work, to move the virtuell particles away from each other such that they become real. (At least an amount of energy $E = 2m$ is needed.) Thus, a (real) pair is created.
For a complete understanding of this quantumfieldtheoretical viewpoint basically four questions remained:[24] First, what does one mean when one says “particle“ in this model which just uses the field concept to describe nature? Second, Why can't the wave functional and the energy of the ground state vanish identically? Third, Why does one need antiparticles and what is their meaning when there is nothing like a Dirac sea? and Fourth, What is a virtuell pair and how can it become real?
The first question is roughly answered as follows: The mode expansion of the field Ψ solving the Dirac equation reads

$$\Psi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3p \left(u(p)e^{ipx}a(p) + v(p)e^{-ipx}b^+(p) \right) \tag{6}$$

where u, v are the usual Dirac spinors. When quantizing the field a and b^+ become operators, so by formal analogy with the second quantized formalism discussed above one may interpret this framework as a framework describing a system of many particles. (In this context it is said that particles are “excitations of fields“.) That is why one sometimes speaks of particles even in this model, a model without particles. In terms of these “particles“ pair creation can be formulated as a scattering process (see **1.1.4**) which is formally equivalent to the formulation of pair creation in the Fock space of the many particle system (after second quantization) as a scattering process which was mentioned above.
The other three questions are not answered here. However, the answers can be found in the literature of quantum field theory (some good explanations can be found in [5]). Mathematical difficulties are also not mentioned here.

To summarize, there are different models and formalisms describing pair creation: The one particle Dirac equation, the second quantized formalism and models

[23]What is meant with “external fields“ is shortly explained in **1.1.3**.

[24]Note that some of these questions already arise in the discussion of the phenomenon of pair creation in a many particle framework using Fock space formalism with normal ordered creation and annihilation operators (see above).

of quantum field theory. The main difference is that the models are based on different concepts of *vacuum* and different fundamental objects (i.e. a given model is about particles, about fields or about both concepts). So the mathematical quantities are interpreted in different ways.[25]

It is important to note that all of these models solve the problem of matter in the way it was formulated in **1.1.1**. But it is not discussed in this review which model should be prefered and i.p. why quantum field theory today is seen as the “correct“ model describing pair creation by most phycisists. In particular the concept of a particle is used without further specifying what is meant.[26]

The only distinction which is made is the distinction between pair creation described by the one particle Dirac equation in the sense of the Dirac sea picture and pair creation described by a formalism in terms of second quantization. How to describe pair creation in these formalisms in detail is discussed in **1.1.4**.

1.1.3 External classical fields and the concept of minimal coupling

Pair creation is usually formulated as a phenomenon occuring in strong external fields. This basically means that the backreaction of the created pairs on the external field is neglected when formalizing the phenomenon. On top of that it is assumed that the external field can be described classically, i.e. that no quantum description is needed. That is why the external (classical) field is also referred to as *classical background*. The external field considered in this review is the classical Maxwell field described by the 4-potential A^μ. Note that one deals in this notation with equivalence classes of fields where one equivalence class corresponds to a physical field. Thus for calculations it may be useful to choose a representative (to “fix a gauge“).

The important question now is: How to incorporate the classical field in a quantum formalism? This is usually done by the prescription of minimal coupling. For Maxwell fields one uses the prescription

$$p^\mu \longrightarrow p^\mu - eA^\mu \tag{7}$$

In particular the Dirac equation minimally coupled to an external Maxwell field reads

$$i\frac{\partial}{\partial t}\Psi(t,x) = H\Psi(t,x) \tag{8}$$

[25] One main difference here is the role of the wavefunction. Whereas for example in the one particle Dirac equation the wavefunction (“describing“ particles) is the function $\Psi(x,t)$ appearing in the Dirac equation in “standard quantum field theory“ the wave function (“describing“ fields) is a functional.

[26] In this context many more questions arise (i.p. if quantum field theory really is a theory about fields or rather a theory of many particles as discussed in terms of second quantization). It is not part of this review to try to give answers to these questions.

where

$$H := \left(-i \sum_{i=1}^{3} \alpha_i \frac{\partial}{\partial x_i} + \beta m - eA^0 \mathbf{1} + e \sum_{i=1}^{3} \alpha_i A^i \right) \tag{9}$$

The use of the 4-potential A^μ (the "A^μ-field") in the prescription of minimal coupling (instead of the local E- and B-fields) has far reaching consequences. One standard example showing this is the Aharonov-Bohm effect. To what consequences the use of the A^μ-field leads here is roughly outlined: When investigating the Aharonov-Bohm effect one considers a situation with such a (very long) solonoid that a constant magnetic field is present inside the solonoid whereas outside the solonoid no electromagnetic field $F^{\mu\nu}$ is present. So by local observations (outside the solonoid) the magnetic flux present in the solonoid is not observable. However, if one uses the Schrödinger equation minimally coupled to the present electromagnetic field, i.e. if one uses the potential A^μ which is pure gauge outside of the solonoid (i.e. A^μ is such that it does not "produce" any electromagnetic field outside the solonoid) in the way it was described above for the Dirac equation the theory predicts that one can detect the flux by global observations. This is meant in the following sense. Choosing a gauge s.t. $A^\mu \propto \partial^\mu \theta$ outside the solonoid (i.e. A^μ is proportional to a total derivative leading to $F^{\mu\nu} = 0$) by Stokes theorem $\int A_\mu dx^\mu$ (the integral is taken along a path enclosing the solonoid) is a nonvanishing quantity which is independent from the chosen contour[27]. Practically this means that taking a particle around the solonoid (i.p. in such a way that it never touches the magnetic field) leads to an observable effect (a phase shift in the action) which is independent of the path the particle took. In this way the measurable effect is a global effect.[28]
A similar point will be the most important part when criticizing the "usual argumentation" which is given when considering pair creation in external laserfields in **3.2**.

Further, three important mathematical questions concerning the prescription of minimal coupling arise here:
First, Under which conditions on A is H self-adjoint on a suitable domain?[29]
Second, How does the existence of A change the spectrum of the Dirac operator? (i.e. what is the spectrum of H?) Third, Is it still possible to go to the many

[27] More precisely: It just depends on how often the chosen contour encloses the solonoid.

[28] Indeed this means that (in contrast to classical Maxwell theory) the physically redundant variable parametrizing the elements of the equivalence classes which describe a physical field cannot be eleminated without giving up the local nature of the theory. In other words, in order to be able to describe global effects one has to introduce an additional variable leading to a gauge theory.

[29] This is important when one assumes Ψ in the one particle Dirac equation to be a wave function which can be interpreted in a probabilistic way. Then due to "conservation of probability" w.r.t. time and due to Stone's theorem ([26], Theorem VIII.8) any infinitesimal generator of a time translation must be self-adjoint.

particle framework using second quantization i.p. in such a way that one has a unitary time evolution there?
A sufficient condition answering the first question for constant external fields (constant w.r.t. time) is given by the Rellich-Kato theorem ([27], Theorem X.12) and some simple Corollaries ([3], Theorem 4.2, Theorem 4.3). The second question will be shortly discussed in **2.2** and a necessary and sufficient condition answering the third question is given by the Shale-Stinespring criterion ([3], Chapter 10). However if the external field is time dependent some conceptional difficulties arise, i.p. it is well known that the notion of vacuum generically depends on time in such situations.[30]

1.1.4 How to formulate the phenomenon of pair creation mathematically

There are different approaches which can be used when formulating the process of pair creation mathematically for a given external field. Some of them are reviewed here.
One way using the intuition of the Dirac sea model is to find (or to show the existence of) solutions of the minimally coupled one particle Dirac equation in such a way that for some time t_0 one has a state with energy in the negative part of the spectrum of the free Dirac operator and for some time $t_1 > t_0$ the state is in the positive part of the spectrum (and stays there for all times $t > t_1$), i.e. for $t < t_0$:

$$< \Psi, P_{-}\Psi >= 1 \tag{10}$$

and for $t > t_1$:

$$< \Psi, P_{+}\Psi >= 1 \tag{11}$$

where P_{-} and P_{+} are the projectors corresponding to the two disconnected parts of the spectrum of the free Dirac operator.[31] For the Dirac sea picture this means that one pair is created because one particle is lifted by the external field from the Dirac sea to the positive energy spectrum (and stays there).[32] Note that i.p. in the second quantized formalism (without the Dirac sea) there is no interpretation of such a solution of the Dirac equation in terms of a particle. However, if one is able to “generalize“ the result to hold also in second quantized form[33] the solution makes sense even as a many particle process (see **1.1.2**).
Another way to formulate the process of pair creation is to directly (i.e. without considering the one particle Dirac equation) consider pair creation as a scattering

[30] Pair creation in the way it will be formulated in **1.1.4** as a scattering process can be seen as a consequence of this fact.

[31] More precisely: The described situation corresponds to “pair creation with probability one“.

[32] see **1.1.2**

[33] How such a “generalization“ can work in a mathematical rigorous way can be found in [3], Chapter 10.

process of many particles using the Fock space formalism. It is to show that the absolute value of the so called vacuum persistence amplitude

$$| < 0_{out}, 0_{in} > | \tag{12}$$

is smaller than one.[34] Here the concept of "in-" and "out-states" is used (to be understood in the Heisenberg picture).[35] This is equivalent to the vacuum matrix element of the scattering operator. It can be roughly interpreted as follows: $| < 0_{out}, 0_{in} > | < 1$ means that the vacuum in the in-region and the vacuum in the out-region are different (because some scattering processes happen during the time evolution), i.e. when the vacuum of the in-region evolves with time it evolves in such a way that when evolved to the out-region it contains particles.[36] Quantitatively this is sometimes shown as follows: By setting $| < 0_{out}, 0_{in} > | = e^{iW}$ with $W \in \mathbb{C}$ (sometimes called the *vacuum action*[37]) one gets $| < 0_{out}, 0_{in} > |^2 = e^{2Im(W)}$, so (for nonvanishing W) $Im\,(W)$ is a quantity representing the probability of pair creation. An explicit calculation to see how this can work is outlined in **1.1.6**.

A mathematical question which arises when formulating the phenomenon of pair creation as a scattering problem in Fock space is the question if the scattering operator is implementable in Fock space at all. (More details and answers to this question can for example be found in [6].) However, in the physics literature which is reviewed in **1.1** this is not discussed.

1.1.5 Some "early calculations" based on the Dirac sea picture

Historically a first formal example indicating that pair creation may really happen in an idealised situation described by the one particle Dirac equation was given by Klein in [8] and interpreted in terms of the Dirac sea picture as pair creation process for example by Hund in [17].

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[34] Note that for any reasonable physical theory $| < 0_{out}, 0_{in} > | \leq 1$ holds.

[35] For reference see e.g. [5].

For a concrete formulation in Fock space it is important that one usually uses the free Fock space to implement the scattering process.

A realistic situation is to define the in-vacuum as the groundstate at a time where no field is present, then to turn on the field adiabatically, switch it off again at a later time and define the out-vacuum as the groundstate at a time after the field has switched off.

[36] For this intuition the Schrödinger picture is useful.

In Fock space formalism this can be easily understood: Since for different times the Dirac operator (minimally coupled to an external field) may be "separated" in different spectral subspaces, for different times different Fock spaces are needed. Thus the vacua of the different Fock spaces may not coincide (i.e. what the vacuum at a given point of time is depends on the time dependent background) leading to a vacuum persistence amplitude smaller than one.

[37] It is well known that W is a lorentzinvariant.

The approach which is used here is to directly find a solution of the one particle Dirac equation. To show how this works the original discussion by Klein will be sketched. For simplicity just one space dimension is considered.[38]
The example deals with an electrostatic potential (constant w.r.t. time) which is constant in space (gauged to 0) for all $x < x_0$ for some $x_0 \in \mathbb{R}$ (w.l.o.g. $x_0 = 0$) and jumps at x_0 to another constant value $C > 0$. So the potential reads in a specific gauge:

$$A^\mu(x) = C\delta^{0\mu}\theta(x) \tag{13}$$

which leads by minimal coupling to the Dirac equation

$$i\frac{\partial}{\partial t}\Psi(x,t) = \left(-i\alpha\frac{\partial}{\partial x} + \beta m + eC\theta(x)\right)\Psi(x,t) \tag{14}$$

Klein found a solution of this equation in a similar way than it is done for the Schrödinger equation for one particle in many introductory textbooks on quantum mechanics. For this consider first the regions $x < 0$ ("region I") and $x > 0$ ("region II") seperatly and make the following ansatz for a wave coming from the left:

$$\Psi_I(x) = A_I e^{i(p_I x - Et)} + A_I' e^{i(-p_I x - Et)} \tag{15}$$

$$\Psi_{II}(x) = A_{II} e^{i(p_{II} x - Et)} \tag{16}$$

where A_I, A_I' and A_{II} have four components. A_I is given whereas A_I' and A_{II} have to be determined. By continuity of the solution at $x = 0$:

$$A_I + A_I` = A_{II} \tag{17}$$

Plugging the ansatz into the Dirac equation gives:

$$EA_I e^{i(p_I x - Et)} + EA_I` e^{i(-p_I x - Et)} = A_I\left(\alpha p_I + \beta m\right)e^{i(p_I x - Et)} + A_I`\left(-\alpha p_I + \beta\right)e^{i(-p_I x - Et)} \tag{18}$$

$$EA_{II} = A_{II}\alpha p_{II} + \beta m A_{II} + eCA_{II} \tag{19}$$

These equations couple the components of A_I, the components of $A_I`$ and the components of $A_{II}$ such that four degrees of freedom ($A_I$ is known) are left. They can be determined by solving the equations $A_I + A_I` = A_{II}$. The solutions are derived in [8].
Now the main argument to show why negative energy states occur goes like follows: Assuming that A_I, $A_I`$ and A_{II} do not vanish[39] gives (by comparing coefficients in the above equations, squaring the resulting equations using the anticommutation relations for the matrices and by absorbing e in C):

$$E^2 = p_I^2 + m^2 \tag{20}$$

[38]The whole argumentation can be easily generalized to three dimensions. The detailed analysis in three dimensions can for example be found in Klein's original work [8].

[39]that such solutions are possible is derived in [8]

$$(E-C)^2 = p_{II}^2 + m^2 \tag{21}$$

Hence, for $C = E + m$ and $C = E - m$: $p_{II} = 0$, for $C < E - m$: p_{II} is real and the kinetic energy $E - C$ is positive, for $E - m < C < E + m$: p_{II} is imaginary (so Ψ_{II} is damped for $x > 0$) and for $C > E + m$: p_{II} is real and the kinetic energy is negative. This means: For $C > E + m$ pair creation in the sense it was formulated in **1.1.4** for the Dirac sea picture occurs (a detailed analysis in [8] shows that the probability for such a transition is indeed > 0).[40]

A bit more complicated situations have been discussed by Sauter in [9] and [12]. In particular he showed that for a potential of the form

$$A^\mu(x) = \delta^{0\mu}\left(C\theta(x) + Bx\theta(-x)\theta(x - x_0)\right) \tag{22}$$

where $x_0 < 0$ and B, C are constants the transition probability is relevant only for $B\lambda_C$ of order m where λ_C is the compton wavelength. This result (earlier conjectured by Bohr) is important because it shows that an observable rate of pair creation can just occur in electrostatic fields which "jump very quickly".[41] Many similar calculations were done (i.p. for particles satisfying the Klein Gordon equation by Hund ([17])) and interpreted as pair creation. What (almost) all of these "early calculations" have in common is first, that the considered external field is constant in time but not constant in space and second, that some matter has to be there to move towards the potential.[42]

1.1.6 The Schwinger effect

In the "early days of QED" the phenomena of vacuum polarization and pair creation in external electromagnetic fields which are constant in space and time

[40] Now one can ask if this process of pair creation does also make sense as a process of many particles, i.e. if one can "generalize" the result to hold true in Fock space. This question is not simple to answer. Although in the physics literature it is usually argued that such a "generalization" is indeed possible (see e.g. [13]), mathematically some difficulties concerning the question which was asked at the end of **1.1.4** arise (see e.g. [3], notes for Chapter 10): "One would expect that the Klein paradox has a resolution in quantum electrodynamics. However, the external field theory described here is not the appropriate framework for the description of this problem because it turns out that the scattering operator for a high potential step ($> 2mc^2$) cannot be implemented in Fock space [10]. It seems as if in a complete treatment of the Klein paradox the interaction of electrons and positrons should not be neglected, but I am not aware of any mathematically rigorous solution to this problem."[3]

[41] Indeed it is sometimes argued that because such "quick jumps" never happen in reality the situation has nothing to do with pair creation in reality.

[42] Hund ([17]) also discusses pair creation without "matter coming in" and notes that here the second quantized formalism is crucial.
Another important calculation (because more realistic than the other calculations which were mentioned) is the treatment by Beck, Steinwedel and Süssmann (see [11]) where a situation with a potential V of the form $V(x) = -C$ for $|x| < A$ and $V(x) = 0$ for $|x| > A$ for some $A, C \in \mathbb{R}$ is discussed (without explicitly using the Dirac sea picture).

were in particular investigated by Dirac ([14]), Heisenberg ([15]), Heisenberg and Euler ([16]) and later by Schwinger ([18], [19]). Schwinger gave an explicit, manifestly gauge invariant and exact calculation for the probability of pair creation. That is why the phenomenon of pair creation in constant external electromagnetic fields is often called the *Schwinger effect*. The calculation (sometimes called the "proper time method") used to calculate the imaginary part of the vacuum action $Im\,(W)$ is outlined here.[43]
The starting point is to consider the vacuum persistence amplitude (see **1.1.4**). Using the S-operator this amplitude can be written as[44]

$$< 0_{out}, 0_{in} >=< 0, S0 >=< 0, \mathcal{T} e^{-ie\int d^4x j(x)A(x)} 0 > \tag{23}$$

Here $\mathcal{T}$ is the time ordering operator. So with the notation of **1.1.4** and $<>$ the vacuum expectation value:

$$2W = -e\int d^4x < j(x) > A(x) \tag{24}$$

where in QED

$$j^{\mu}(x) = -\frac{1}{2}\gamma^{\mu}\left[\Psi(x), \bar{\Psi}(x)\right] \tag{25}$$

and the quantized Dirac fields Ψ defined by the Dirac equation minimally coupled to the external potential satisfy the equal time anticommutation relations

$$\{\Psi(x,t), \Psi^{+}(y,t)\} = \delta^{(3)}(x-y) \tag{26}$$

with x, $y \in \mathbb{R}^3$ and $t \in \mathbb{R}$.
Now the original treatment of Schwinger starts with the Greens function (or propagator) formalism. As usual the Greens function G can be expressed as (with $x, y \in \mathbb{R}^4$)

$$G(x,y) = i < \mathcal{T}\Psi(x)\bar{\Psi}(y) > \tag{27}$$

Noting that

$$\frac{1}{2}\left[\Psi(x), \bar{\Psi}(x)\right] = \mathcal{T}\left(\Psi(x)\bar{\Psi}(y)\right)|_{y\to x} \tag{28}$$

where $y \to x$ is to be taken after time ordering is applied yields for the vacuum polarization desity $< j^{\mu}(x) >$

$$< j^{\mu}(x) >= ieTr\left(\gamma^{\mu}G(x,x)\right) \tag{29}$$

Here Tr refers to spin space. Schwinger uses this expression to derive a "manifestly gauge invariant calculation" of W. For this purpose he introduces a manifestly

[43] A more detailed discussion can be found both in the original work by Schwinger ([18], [19]) and in [20], 10.5.

[44] In his original work ([18]) Schwinger did neither explicitly use the vacuum persistence amplitude nor the concept of the S-matrix. The use of the S-matrix he introduced in [19]. Many of the following considerations like [36] are based on [18] and [19].

gauge invariant representation of the vacuum action, i.e. using the given expressions for the vacuum polarization density and for the Greens function one can show (see [18] or [20]) that W can be written as

$$W = \int d^4x \frac{i}{2} \int_0^\infty ds \frac{1}{s} e^{-im^2 s} Tr < x, U(s)x > \tag{30}$$

where a factor of convergence ($e^{i\epsilon}$) is implicit and U is given by

$$U(s) = exp\left(i(\gamma^\mu(p_\mu - eA_\mu))^2 s\right) =: exp(-iHs) \tag{31}$$

Thus it is enough to calculate the amplitude

$$< x, U(s)y >=:< x(s), y(0) > \tag{32}$$

This notation suggests to interpret $exp(-iHs)$ as a propagator propagating the states in time. So by analogy one can use the Heisenberg equations of motion for a particle to solve the problem (that is why the method by Schwinger is sometimes called the "proper time method"):

$$\frac{dx_\mu}{dt} = -i\left[x_\mu, H\right] = 2\left(\gamma^\mu(p_\mu - eA_\mu)\right) \tag{33}$$

$$\frac{d\left(\gamma^\mu(p_\mu - eA_\mu)\right)}{dt} = 2eF_{\mu\nu}\left(\gamma^\alpha(p_{\alpha\nu} - eA_{\alpha\nu})\right) - ieF_{\mu\nu,\nu} + \frac{1}{2}e\sigma_{\lambda\nu}F_{\lambda\nu,\mu} \tag{34}$$

where $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ and $\sigma_{\mu\nu} = \frac{i}{2}\gamma_{[\mu}\gamma_{\nu]}$.
The solution can be used to determine H and thus $< x(s), y(0) >$ from which W can be calculated.
For a constant uniform electric field Schwinger calculates this explicitly which gives for the imaginary part of the vacuum action (see [18]) with α the Sommerfeld constant:

$$Im\left(W\right) = \int d^4x \frac{\alpha^2 E^2}{2\pi^2} \sum_{n=0}^{\infty} \frac{1}{n^2} e^{-\frac{n\pi m^2}{eE}} \tag{35}$$

Using Fredholm theory this result also follows from another expression which Schwinger used in [19].[45] For $Im\left(W\right) = \int w(x) d^4x$ where $w(x)$ is the probability of pair creation per unit volume and unit time the following expression is derived there

$$w(x) = Tr < x, ln\left(1 - T\rho_{(+)}\bar{T}\rho_{(-)}\right) x > \tag{36}$$

Here for fermions

$$\rho_{(\pm)} = 2\pi(P + m)\theta\left(\pm P_0\right)\delta(P^2 - m^2) \tag{37}$$

[45] For more details consider [19] or the book written by Itzykson and Zuber ([21]). This expression is mentioned here because some later calculations like [36] are based on it (see **3.1.1**).

are the operators describing the density of states and T, $\bar{T}$ are the scattering operators satisfying

$$T = eA + eA\frac{1}{P - m + i\epsilon}T \tag{38}$$

$$\bar{T} = eA + eA\frac{1}{P - m - i\epsilon}\bar{T} \tag{39}$$

From here the same expression for $Im(W)$ as given above is calculated for a constant uniform external electric field.

The result for $Im\,(W)$ for constant external electric fields shows two important points: First, for $E \to 0$ it vanishes in all orders (in E). So pair creation in constant electric fields cannot be treated perturbatively. Second, restating c an $\hbar$ to get a significant rate of pair production the field E has to exceed the critical value

$$E_{cr} = \frac{m^2c^3\pi}{e\hbar} \tag{40}$$

otherwise the effect is exponentually supressed. Such a high field strength has never been achieved in experiments, i.e. so far there is no empirical verification of the Schwinger effect.
Further, two things are important: First, Schwinger also gave an explicit integral expression for the vacuum action for arbitrary constant external electromagnetic fields.[46] This expression i.p. shows that if the gauge invariant quantities $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}\tilde{F^{\mu\nu}}$ for a constant external field vanish, no pair creation occurs in this field. Second, it is important to point out that the Schwinger calculation is done for an idealized situation where i.p. the electric field is constant in time, i.e. it acts for an infinite time. From the calculation it is not clear whether the result also holds true for more realistic configurations where i.p. the field is (adiabatically) turned on and turned off at some time. Dealing with pair creation in more realistic ways is part of the rest of this review. For this purpose the adiabatic theorem is considered first.

1.2 Adiabatic phenomena and introduction to the adiabatic theorem

1.2.1 The adiabatic hypothesis

The first idea that adiabatic processes (i.e. processes in which the change of a system happens very slow in a sense to be specified) in quantum mechanics satisfy the so called *adiabatic hypothesis* was formulated by Ehrenfest ([22]) in the

[46] The corresponding Lagrangian is the well known *Euler-Heisenberg effective Lagrangian* (it was already calculated in [16] by Euler and Heisenberg for almost constant external electromagnetic fields).

context of “old quantum mechanics“[47]. This hypothesis claims that by adiabatically changing a system the quantum numbers of a given motion do not change - they are invariant, quantum jumps do not occur.
This idea arose due to the observation that there are adiabatic processes which can be derived classically and which precisely hold true although the laws which are used in the corresponding part of physics are of quantum form. One example is Wiens law.[48] If this holds in general for all adiabatic processes this means that adiabatic processes are not influenced by quantum effects or vice versa that adiabatic processes do not cause any “quantummechanical change“. This is precisely the adiabatic hypothesis.

1.2.2 The adiabatic theorem

The main step from the “old“ adiabatic hypothesis to the adiabatic theorem is the use of wave mechanics and the probabilistic interpretation of the wave function (see [23]). In the formalism of wavemechanics the adiabatic theorem in the formulation presented in [23] and [24] claims that the probability for a quantum jump is zero when the system is changed adiabatically by an external perturbation. A general treatment of this theorem was first given by Born and Fock in [24] for a system described by the Schrödinger equation. This treatment uses the useful concept of two different time scales: A “fast time scale“ denoted by s and a “slow time scale“ denoted by t where

$$t = \epsilon s \tag{41}$$

with ϵ small ($0 < \epsilon << 1$) the so called *adiabatic parameter*. This specifies what is meant with “slow change“ of the system: Whenever there are processes happening on two time scales t and s related by $t = \epsilon s$ one speaks of “adiabatic processes“ and one can apply the adiabatic theorem.
For proving the adiabatic theorem one then has to show that on very slow time scales the transition probability for the system (i.e. the probability to change the bound state during the adiabatic perturbation or in other words the probability for a quantum jump) is neglectable.[49] Born and Fock proved this for the Hamilton point spectrum of a system described by the Schrödinger equation.[50]

[47] I.e. in mechanics where i.p. possible motions are characterized by postulating various quantization conditions.

[48] see e.g. [22]

[49] For doing mathematical proofs one introduces the “adiabatic limit“. This will be done in **2.1.1**. Nevertheless for understanding the basic meaning of the theorem this is not important because in realistic physical situations an “infinite slow time scale“ is not possible.

[50] The main idea of the proof is: Given the Schrödinger equation where the Hamilton operator depends on the small time scale t one can consider three systems of functions: The eigenfunctions of the Hamiltonean at one given time t_0 (w.l.o.g. $t_0 = 0$) $(\Psi_n)_{n\mathbb{N}}$, the eigenfunctions of the Hamilton operator at arbitrary time $(\Phi_n)_{n\in\mathbb{N}}$ and the functions solving the Schrödinger

2 Pair creation as an adiabatic phenomenon

2.1 Rigorous treatment of the adiabatic theorem

2.1.1 The adiabatic limit

As mentioned in **1.2.2** the adiabatic theorem deals with physical situations where two time scales t and s related by $t = \epsilon s$ can be separated in the way it was presented in **1.2.2**. The adiabatic theorem roughly claims that the smaller ϵ is the smaller the probability for a quantum jump during time evolution is (see **1.2.2**). To make a precise mathematical statement one introduces the *adiabatic limit*. That means that one claims that in the limit $\epsilon \to 0$ the probability for a quantum jump vanishes.

Conceptually this limit is very important, i.p. in the case of adiabatic pair creation (see **2.2.3**). For this it is important to note the obvious mathematical fact that in general when dealing with the concept of limit the "adiabatic case" ($lim_{\epsilon\to 0}$) and the "static case" ($\epsilon = 0$) are completely different things. Whereas "$\epsilon = 0$" represents a static situation "$lim_{\epsilon\to 0}$" is the mathematical precise formulation for an time varying situation which is "infinitesimally slow". This means i.p. that if a static physical situation differs from the situation where an arbitrary small external perturbation (w.r.t. time) acts (i.e. if there is a "jump" in the physical behaviour w.r.t. time[51]), "the case $\epsilon = 0$" models the static situation and "the case $lim_{\epsilon\to 0}$" the perturbed situation.

2.1.2 The adiabatic theorem

Here a precise mathematical formulation of a modern version of the adiabatic theorem as well as some ideas of its proof are given. The difference between "adiabatic theorems with gap" and "without gap" is explained. It is refered to [25] where i.p. more details and examples can be found.[52]

A first class of adiabatic theorems consists of some kind of generalizations of the theorem proven by Born and Fock (see **1.2.2**) in the sense that they are not restricted to the point spectrum but still deal with spectra separated by a gap

equation $(\eta_n)_{n\in\mathbb{N}}$ which coincide with $(\Psi_n)_{n\in\mathbb{N}}$ at $t = 0$. By conservation of probability and orthorgonality of the functions the three families are related to each other by unitary operators. Now the probability for a given quantum jump is given by the square of the corresponding matrix element of the unitary operator connecting the systems $(\Phi_n)_{n\in\mathbb{N}}$ and $(\eta_n)_{n\in\mathbb{N}}$. The proof shows that these vanish in adiabatic processes when connecting states with different quantum numbers.

The exact proof by Born and Fock as well as some restrictions and assumptions which have te be made can be found in [24].

[51] in other words that the behaviour is not continuous

[52] Note that in [25] it is i.p. distinguished between "time adiabatic theorems" and "space adiabatic theorems". Here just "time adiabatic theorems" are considered.

(to be specified).[53] It is because of the last point why they are sometimes called *adiabatic theorems with gap condition.* To make this precise the following definition is given:

Definition:
Let $\sigma(t)$ be the spectrum of $H(t)$ (e.g. the Dirac operator or the Hamilton operator) and let $\tilde{\sigma}(t)$ be a subset of $\sigma(t)$. $\tilde{\sigma}(t)$ is called (locally) isolated by a gap iff there exist two functions f_1 and f_2 bounded and continuous s.t. the following two conditions hold

$$\tilde{\sigma}(t) \subset [f_1(t), f_2(t)] \tag{42}$$

$$inf_{t\in\mathbb{R}} \left(dist \left([f_1(t), f_2(t)], \sigma(t) \backslash \tilde{\sigma}(t) \right) \right) > 0 \tag{43}$$

Using this definition the relevant version of the adiabatic theorem with gap can be formulated:

Theorem:
Let U^ϵ be the solution of the initial value problem[54]

$$i\epsilon \frac{d}{dt} U^\epsilon(t, t_0) = H(t) U^\epsilon(t, t_0) \tag{44}$$

$$U^\epsilon(t_0, t_0) = \mathbf{1} \tag{45}$$

and $\tilde{P}(t)$ the projector corresponding to $\tilde{\sigma}(t)$ where $\tilde{\sigma}(t)$ is assumed to be isolated by a gap. Then[55]

$$lim_{\epsilon\to 0} || \left(\mathbf{1} - \tilde{P}(t) \right) U^\epsilon(t, t_0) \tilde{P}(t_0) || = 0 \tag{46}$$

The proof of this theorem can e.g. be found in [25]. Here just the main idea is given and it is mentioned at which point the gap condition is needed in the proof[56].

[53] The notion of "no quantum jump happens during time evolution" is generalized to a so called "adiabatic decoupling of spectral subspaces" (see [25]).

For the historical development of the adiabatic theorem references can also be found in [25].

[54] Rigorous statements e.g. on the existence of the solution can be found in [27].

[55] For the proof $H \in C_b^2 \left(\mathbb{R}, \mathcal{L}_{sa}(\mathcal{H}) \right)$ is also needed.

[56] There are two points where the gap condition is needed. One point has to do with the regularity of the projector $\tilde{P}(t)$. This point is not considered here. A detailed analysis can be found in [25].

The main idea of the proof is to find a mechanism which allows you "to interchange U^ϵ and $\tilde{P}$ keeping the error of order ϵ".[57] This is done by defining another time evolution (represented by another propagator, called the "adiabatic propagator") which can be interchanged with $\tilde{P}(t)$. That time evolution has to be such that the difference between the propagator U^ϵ and the adiabatic propagator is of order ϵ. To prove that the difference between U^ϵ and the adiabatic propagator is indeed of order ϵ it can be shown that the difference can be written as an integral over a function which oscillates with a frequency proportional to ϵ^{-1}. In order to show this the gap condition is needed.

A second class of adiabatic theorems are *adiabatic theorems without gap condition*. For these theorems a spectral gap is not necessary. The corresponding proofs require a more careful analysis than presented in the idea above (see [25]).

For pair creation considered as adiabatic phenomenon (see **2.2.2**) both classes of adiabatic theorems are needed. In this discussion in **2.2.3** i.p. a sketch of a proof of an "adiabatic lemma without gap" is given when sketching the proof of the first part of the main theorem there.

2.2 Rigorous treatment of pair creation

The calculations reviewed so far (**1.1.5**, **1.1.6**) have been given using physical formalisms without refering to mathematical rigour. However, mainly in the discussion of pair creation in constant external fields produced by heavy ions it has been realized that mathematical insight is essential for understanding the phenomenon of pair creation in realistic situations. Why this is the case can be understood easily. As already mentioned in **1.1.4** one way to show that pair creation occurs with probability one is (by using the one particle Dirac equation) to show that $<\Psi, P_-\Psi>=1$ for some time $t<t_0$ and $<\Psi, P_+\Psi>=1$ for some time $t>t_1$ where $t_1>t_0$ (and then to generalize the result in such a way that it holds true in the second quantized formalism). If one wants to apply this procedure to situations with realistic external potentials (i. p. with more realistic potentials than the ones considered in **1.1.5**) knowledge about the spectrum of the Dirac operator minimally coupled to that potential is needed.[58] I.p. knowledge about the behaviour of the bound states possibly occuring in the spectrum and depending on the external potential will turn out to be essential. (That bound states may indeed occur is a fact which is commonly used in the physics of heavy ions (see e.g. [2]).) This will be argued in more details in **2.2.2** and **2.2.3**. Before, in **2.2.1**, a mathematical proof is given which shows (using second quan-

[57] A simple argument showing that $\tilde{P}$ and U^ϵ cannot simply "be interchanged" can be found in [25].

[58] I.e. one has to answer the second question which was aked at the end of **1.1.3**.

tization) that pair creation in constant external fields A^μ does not occur (under specific technical assumptions). This can be taken as a further motivation to study more realistic situations. Note that this result should not be confused with the Schwinger result (**1.1.6**). In **1.1.6** the E-field was assumed to be constant whereas here the A^μ-field is constant.
In general it is important to note that in contrast to the work considered so far the rigorous considerations in **2.2** deal with the A^μ-field. No arguments using local E- and B-fields are used.

2.2.1 Nonexistence of pair creation in static fields

In the literature on mathematical physics one can find rigourous arguments showing that pair creation (considered as a scattering process in the second quantized formalism) in external time independent fields $A^\mu(x)$ does not occur (see e.g. [28] and references therein). A theorem claiming this is formulated in [28][59]:

Theorem:
Let P_+, P_- the spectral projectors of the free Dirac operator. Let V be time independent and bounded with relative bound less than one.[60] Assuming that a unitary single particle scattering operator $S(V) : L^2(\mathbb{R}^3) \otimes \mathbb{C}^4 \longrightarrow L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$ exists the following holds:
(i) S(V) commutes with the free time evolution, i.p. $P_+S(V)P_- = 0$ and $P_-S(V)P_+ = 0$.
(ii) The unitary operator $\mathcal{S}(V) : \mathcal{F} \longrightarrow \mathcal{F}$ describing the scattering theory in the free Fock space $\mathcal{F}$[61] is uniquely determined by $S(V)$ up to a phase which can be chosen in such a way that

$$< 0, \mathcal{S}(V)0 >= 0 \tag{47}$$

if AA^* has 1 as one of its eigenvalues and otherwise

$$< 0, \mathcal{S}(V)0 >= det(\mathbf{1} + BAA^*B^*)^{-\frac{1}{2}} \tag{48}$$

where $A := P_+S(V)P_-$, $B := (P_+S(V)P_+)^{-1}$.

That the vacuum persistence amplitude for static external fields is indeed equal to one is a simple consequence of this theorem: By (i) $A = 0$ and thus by (ii) $< 0, \mathcal{S}0 >= (det(\mathbf{1}))^{-\frac{1}{2}} = 1$.

[59] For the proof it is refered to some earlier work (see [28]).
[60] In some sense this assumption means that V is small w.r.t. the free Dirac operator (see [27], X.2). It is needed in order to make shure that $H_0 + V$ is selfadjoint on the domain of H_0 by the Rellich-Kato theorem. For the purpose of pair creation the relevant potential V is given by $V(x) = -e\phi(x) + e\alpha A(x)$.
[61] How to come from S to $\mathcal{S}$ (i.e. how to implement the scattering operator in Fock space) is e.g. discussed in [3], Chapter 10 and in [6]. For the existence one needs that $P_+S(V)P_-$ and $P_-S(V)P_+$ are Hilbert Schmidt. (This answers the question raised at the end of **1.1.4**.)

Does this mean that (under the conditions of the theorem) pair creation in reality is not possible in such "static situations"? It turns out that this is not quite correct. It just means that one has to formulate the problem more carefully. This is meant in the following sense: The result is not stable, i.e. for a slowly changing strong field "pair creation happens (even) in the adiabatic limit". This will be made precise in **2.2.2** and **2.2.3**. Thus the theorem which is valid for an idealized situation (for static external fields) is not useful for making predictions about experiments because there perfect static A^μ-fields (i.p. fields which have the same value for an infinite time) are not realizable.
However, there is still something one can learn from the rigorous treatment, namely that in contrast to the Schwinger result (**1.1.6**) there is nothing like a critical field strength which is needed in order to create an observable amount of pairs. Indeed the theorem holds for arbitrary strong fields, thus the claim of the theorem holds true independently of the strength of the external field. This may be taken as a hint that the actual physical effect of pair creation (also) in more realistic situations does not (only) depend on the strength of the external field if the external field is more complicated than the one considered by Schwinger.[62]

2.2.2 Pair creation as an adiabatic phenomenon and the adiabatic switching formalism

As mentioned in **2.2.1**, since pair creation does not occur in static external A^μ-fields it is a natural question to ask whether this result is stable, i.e. if pair creation occurs in external fields which are almost constant w.r.t. time. The main intuition about how pair creation could happen in such external fields which are almost constant w.r.t. time, i.e. in external fields which change adiabatically, can be formulated in terms of the Dirac sea picture (**1.1.2**) and the adiabatic hypothesis (**1.2.1**): If one takes into account that by an external field the spectrum of the corresponding Dirac operator (in particular) consists of the disjoint absolutely continuous parts of the spectrum of the free Dirac operator $(-\infty, -m] \cup [m, \infty)$ and possibly of bound states lying between these disjoint parts one could imagine the following situation. By varying the external field adiabatically in time it could be possible that one bound state energy level moves from the lower continuum of the spectrum to the upper continuum. By the adiabatic hypothesis an electron from the Dirac sea will follow this energy level moving to a state of the upper continuum. (Thus a pair will be created.) A (time dependent) bound state of the Dirac operator which can realize such a situation is called a *gap bridge*. Here

[62] It is part of the rest of this review to point out that the different treatments (using either the A^μ-field in the consideration as it is done here in **2.2** or the local external electromagnetic fields as e.g. done by Schwinger) have to be separated. Indeed, the main point in this review will be the argumentation that for more realistic situations than the one considered by Schwinger it is crucial that one sudies the phenomenon of pair creation using the external A^μ-field instead of arguing with local E- and B-fields (see **3.2**).

it is essential that the external field is not constant w.r.t. time. However if the effect is true also in the adiabatic limit (**2.1.1**)[63], then for all practical purpose (i.e. for situations which are realizable in real experiments) one can speak of pair creation occuring in constant external A^{μ}-fields. The main new insight would (just) be that modelling the problem (from the very beginning) as a problem in constant external fields does not work (because "the description is not continuous"). Instead one has to consider the problem as a problem in an adiabatic formalism.

Note that a quantitative mathematical formulation of this intuition of pair creation might be difficult i.p. because the behaviour of the spectrum of the Dirac operator minimally coupled to a given (time dependent) external electromagnetic field w.r.t. time is not easy to handle. However such a treatment has been given recently ([29], [30]). Before reviewing this, the formalism which is used is described more carefully.[64]

For modelling this adiabatic phenomenon the first step is to introduce two time scales into the problem (see **1.1.2**). So let s be the fast and t be the slow time scale with $t = \epsilon s$. The minimal coupled one particle Dirac equation then reads:

$$i\epsilon \frac{\partial \Psi^{\epsilon}}{\partial t} = H\Psi^{\epsilon} \tag{49}$$

where Ψ^{ϵ} means that Ψ is considered on the slow time scale. Under suitable conditions[65] for H there exists a unitary operator $U^{\epsilon}(t, t_0)$ describing the time evolution of the system and satisfying

$$i\epsilon\partial_t U^{\epsilon}(t, t_0) = HU^{\epsilon}(t, t_0) \tag{50}$$

$$U^{\epsilon}(t_0, t_0) = \mathbf{1} \tag{51}$$

Now it is useful to give a few commonly used definitions (see e.g. [28], [30]) to introduce the *adiabatic switching formalism.*

Definition:

(i) A function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$, $\phi \in C^1$ is called a switching factor if ϕ' is bounded, $\phi(0) = 1$, $\phi'(0) > 0$ and $\phi(t) = 0$ for $t < t_i$, $t > t_f$ for some t_i, $t_f \in \mathbb{R}$ with $t_i < 0$ and $t_f > 0$.[66]

[63] That means (when taking into account **2.2.1**) that the behaviour for $\epsilon \longrightarrow 0$ (adiabatic limit) and the behaviour for $\epsilon = 0$ (static case) are different. I.e. the adiabatic and the static result do not coincide because the behaviour is not continuous. This shows the importance of the concept of the adiabatic limit (**2.1.1**).

[64] This formalism has been introduced by G. Nenciu in [28].

[65] see e.g. [3]

[66] The special point in time $t = 0$ is chosen because below it is assumed that the external field becomes overcritical (a notion to be explained) at $t = 0$.

(ii) A time independent external potential A is called undercritical if $\phi(t)A$[67] is such that all bound states of the spectrum of the corresponding Dirac operator stay below the upper continuum part of the spectrum for all $t \in \mathbb{R}$ (to be understood in the adiabatic limit).
(iii) A time independet external potential A is called overcritical if $\phi(t)A$ is such that for at least one bound state the eigenenergycurve (as a curve w.r.t time) crosses the upper continuum (to be understood in the adiabatic limit).

With the intuition given at the beginning of this section and the adiabatic theorem in mind the condition for the existence of pair creation is now easy to state: Pair creation exists in the adiabatic limit in a given external A^μ-field iff for some time s the external potential becomes overcritical.[68] A proof showing that this intuition is indeed correct is reviewed in **2.2.3**.
Note again that in this treatment one i.p. neglects effects of vacuum polarization on the external fields, i.e. one uses external field approximation (see **1.1.3**).

2.2.3 Existence of adiabatic pair creation in the Dirac sea picture

Now (in **2.2.3**) the main steps showing that the proof of adiabatic pair creation in the sense it was formulated in **2.2.2** really exists are reviewed. The proof which is due to [30] and [29] uses the adiabatic switching formalism. The main part is the proof of the following theorem where it is assumed that the external time dependent A^μ-field can be written as $\phi(t)A$ where A is time independent and ϕ is a switching factor, i.p. $\phi(0) = 1$ (see **2.2.2**).[69]

Theorem:
Let $t = 0$ be the time when the external A^μ-field becomes overcritical. Let $\Phi_{\phi(t_0)}$ be a bound state of H for some $t_0 < 0$. Let $\Psi^\epsilon(t)$ be the solution of the Dirac equation with $\Psi^\epsilon(t_0) = \Phi_{\phi(t_0)}$. Then for all $\eta \in L^2$ and for all $t > 0$

$$lim_{\epsilon \to 0} < \Psi^\epsilon(t), \eta >= 0 \tag{52}$$

In [30] the proof of this theorem is done in two parts: First, it is shown that a bound state with energy in the intervall $[-m, m]$ moves to the upper spectral continuum for some time (when the field becomes overcritical) in an adiabatic

[67] For an actual external time dependent field A it is assumed that it can be factorized like this, i.e. that $A(t) = \phi(t)\tilde{A}$ with $\tilde{A}$ time independent and ϕ a switching factor.
[68] Becoming overcritical at $t = 0$ means that $t = 0$ is the time where one bound state energy of an overcritical external field crosses the upper continuum of the spectrum.
[69] Some technical assumptions (see [30], Condition 2.2) have to be used.

process.[70] Second, it is shown that the state stays in the upper continuum when the field becomes undercritical again. From these two parts the claim of the theorem follows. The main steps of the proof are sketched:

First part of the proof

Denote the eigenspace of the eigenenergy E_ϕ of the Dirac operator minimally coupled to the external field $\phi(t)A$ (where $E_\phi \in [-m,m]$) by $\mathcal{N}_\phi$ and the corresponding projector by $P_{\mathcal{N}_\phi}$. Denote the Dirac operator corresponding to the external field $\phi(t)A$ by H_ϕ.

As a first step it is proven that the eigenspaces $\mathcal{N}_\phi$ converge to $\mathcal{N}_1$ when the field becomes overcritical, i.e. Lemma 6.1 of [30]:

Lemma:

(i) For any sequence $(\Phi_\phi)_\phi$, $\Phi_\phi \in \mathcal{N}_\phi$, $||\Phi_\phi|| = 1$

$$lim_{\phi\to 1}||P_{\mathcal{N}_1}\Phi_\phi|| = 1 \tag{53}$$

(ii) $dim\mathcal{N}_1 = dim\mathcal{N}_\phi$ for all ϕ close enough to one, i.e. there exists a ϕ_B s.t. $dim\mathcal{N}_1 = dim\mathcal{N}_\phi$ for all $\phi \in [\phi_B, 1)$.

For the proof of this lemma an operator

$$R_\phi : (\mathcal{N}_1\backslash\Phi_1)^\perp \longrightarrow \mathcal{N}_1^\perp \tag{54}$$

where Φ_1 is a normalized state in $\mathcal{N}_1$ s.t. for some Φ_ϕ $\Phi_\phi \in (\mathcal{N}\backslash\Phi_1)^\perp$ holds[71] is defined as

$$R_\phi\eta = \frac{1}{H_1 - E_\phi}P_{\mathcal{N}^\perp}(A\eta) \tag{55}$$

Using simple arguments it is shown in [30] that

$$< \Phi_\phi, \Phi_1 > \Phi_1 = (1-(1-\phi)R_\phi)\,\Phi_\phi \tag{56}$$

It is proven that for some $C < \infty$

$$||R_\phi||_2^{op} < C(1-\phi)^{-\frac{13}{16}} \tag{57}$$

Using this bound it is shown that $(1-(1-\phi)R_\phi)$ is invertible with inverse

$$(1-(1-\phi)R_\phi)^{-1} = \sum_{j=0}^{\infty}(1-\phi)^j R_\phi^j \tag{58}$$

[70] The essence of the proof of this first part is a proof of an adiabatic lemma without gap (as mentioned in **2.1.2**). For this purpose a preliminary lemma for sequences which are needed to prove the adiabatic lemma is proven first (using an operator R_ϕ), see below

[71] The existence of such a Φ_1 is shown in [30]. Note that Φ_1 depends on Φ_ϕ.

In particular it follows from $\Phi_\phi \in (\mathcal{N}\backslash\Phi_1)^\perp$ that E_ϕ is in the resolvent set of H_1 and that therefore the following expression is well defined.

From this together with $< \Phi_\phi, \Phi_1 > \Phi_1 = (1-(1-\phi)R_\phi)\,\Phi_\phi$ it is easy to see that with $\zeta_\phi := \frac{\Phi_\phi}{<\Phi_\phi,\Phi>}$

$$lim_{\phi\to 1}||P_{\mathcal{N}^\perp}\zeta_\phi|| = lim_{\phi\to 1}||\sum_{j=0}^{\infty}(1-\phi)^j\, R_\phi^j\Phi_1|| = 0 \tag{59}$$

Using Schwarz inequality and the fact that Φ_ϕ and Φ_1 are normalized gives

$$lim_{\phi\to 1}||P_{\mathcal{N}^\perp}\Phi_\phi|| = 0 \tag{60}$$

which proves (i) of the Lemma. The proof of (ii) can be done by contradiction (see [30]).

In a second step an adiabatic lemma without gap (lemma 7.1. in [30]) is proven. For this purpose it is first shown that for every bound state $\Phi_1 \in \mathcal{N}_1$ there exists a sequence $(\Phi_\phi)_\phi \in \mathcal{N}_\phi$ (which i.p. has the property of the lemma above) which can be used to prove the adiabatic lemma. This basically works like follows.
The sequence $\Phi_\phi := \frac{\zeta_\phi}{||\zeta_\phi||}$ has the following two properties (see lemma 6.4 of [30]):

$$< \zeta_\phi, \Phi_1 >= 1 \tag{61}$$

$$\zeta_\phi \in (\mathcal{N}_1\backslash\Phi_1)^\perp \tag{62}$$

The existence of such a sequence is proven in [30]. It is shown (by using R_ϕ) that for some $C < \infty$

$$||\partial_\phi\Phi_\phi|| \leq C(1-\phi)^{-\frac{13}{16}} \tag{63}$$

for all ϕ close enough to 1.
Using these sequences the adiabatic lemma without gap stated in the following can be proven.

Lemma (Adiabatic lemma without gap)
Let $t < 0$ be such that a bound state Φ_ϕ of $H_0 + \phi A$ with energy $E_\phi > -m$ exists. Then:

$$lim_{\epsilon\to 0}||P_{\mathcal{N}_1}U^\epsilon(0,t)\Phi_{\phi(t)}|| = 1 \tag{64}$$

The proof of this lemma uses the adiabatic theorem, i.e. that for $t_0 < 1$

$$lim_{\epsilon\to 0}||P_{\mathcal{N}_{\phi(t_0)}}U^\epsilon(t_0,t)\Phi_{\phi(t)}|| = 1 \tag{65}$$

The constructed sequences are needed to form suitable orthonormal basises for $(\mathcal{N}_\phi)_\phi$:

If $\{\Phi^i\}_{i\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{N}_1$, the constructed sequences $\{\Phi^i_\phi\}$ corresponding to Φ^i are orthonormal basises of $\mathcal{N}_\phi$ for ϕ close enough to one (but unequal one).
The proven bound for $||\partial_\phi \Phi_\phi||$ is needed to estimate the error between an approximate time evolution defined by

$$\Phi^\epsilon_{\phi(t),t_0} = e^{-\frac{i}{\epsilon}\int_{t_0}^t E_{\phi(s)}ds} \sum_{l=1}^n \alpha^\epsilon_{t_0,l} \Phi^l_{\phi(t)} \tag{66}$$

where the sum stands for the expansion in the basis $\{\Phi^l_{\phi(t)}\}$ and the true time evolution $U^\epsilon(0,t_0)\Phi_{\phi(t_0)}$. It turns out that for some $C < \infty$

$$||\Phi^\epsilon_{\phi(0),t_0} - U^\epsilon(0,t_0)\Phi_{\phi(t_0)}|| \leq \frac{13}{16} C t_0^{\frac{3}{16}} \tag{67}$$

Note that the right hand side is independent of ϵ.
With this result the adiabatic lemma can be proven easily (see [30], proof of lemma 7.1). The lemma completes the first part of the proof.

Second part of the proof
Mathematically "the state stays in the upper continuum when the A^μ-field becomes undercritical again" means that the bound states during overcriticality must leave the support of the (time dependent) external potential fast enough in order not to be taken out of the upper continuum again when the potential becomes undercritical.
The proof of this fact in [30] uses generalized eigenfunctions. The proof is partly very technical. It consists of three basic steps:
First, the time evolution of the state is controlled in the case of potentials constant in time. Several estimates are given in this first step. The one which is most intuitive is that (see Corollaries 5.2 and 7.2 in [30]) for all $\eta \in L^2$

$$||\mathbf{1}_S V^\epsilon_\phi(s,0)\eta|| \leq C_\zeta \left(||\eta|| + ||H_0 + \phi A||\right) |\Phi - 1|^{-\frac{1}{2}} t^{-\frac{3}{2}(1-\zeta)} \tag{68}$$

where $S \subset \mathbb{R}^3$ compact, V_ϕ is the unitary time evolution generated by the Dirac operator minimally coupled to a potential which is constant in time, $0 < \zeta < 1$. The proof of this first step heavily relies on an estimate for generalized eigenfunctions for Dirac operators (see [30], part 4).
Second, the time evolution of the state is controlled in the case of time dependent potentials for small times (of order 1).
Third, this result is extended to all times.
This separation of the proof between the second step and the third step is done because it turned out that in the small time considered in step 2 the state already leaves the range of the potential.

The most important parts of the procedure of controlling the time evolution of the state for time dependent external potentials (i.e. the most important steps of the second and third steps of part two of the proof) are reviewed (see 7.2. and 7.3 in [30]). For this purpose σ denotes a time of order 1. Now in step 2 estimates for times $T \in (0, \sigma]$ are proven and in step 3 estimates for times $T \in [\sigma, \infty)$. The lemma to prove where (i) corresponds to step 2 and (ii) corresponds to step 3 reads like follows (see lemma 7.3 and lemma 7.5 in [30]).

Lemma
Let $U^\epsilon(t,T)$ be the solution of $i\epsilon\partial_t U^\epsilon(t,T) = HU^\epsilon(t,T)$ (see **2.2.2**) and $\eta \in L^2$ s.t. $||\eta|| = 1$, $||H_0\eta|| < \infty$ and $supp(\eta) \subset S$. Assume that $\tilde{C} \geq \partial_t\phi(t) \geq C > 0$ for $t \in (0,T]$ and $C, \tilde{C} \in \mathbb{R}$[72]. Let $0 < \zeta < \frac{1}{3}$ arbitrary. Then
(i) there exist constants $C_\zeta \in \mathbb{R}$ s.t. for all $t \in (0,T]$

$$||\mathbf{1}_S U^\epsilon(t,0)\eta|| \leq C_\zeta \left(\epsilon^{\frac{1}{2}-\frac{3}{2}\zeta} t^{-\frac{3}{2}}\right) \tag{69}$$

(ii) there exists some constant $c \in \mathbb{R}$ s.t. for all $t \in [T, \infty)$

$$||\mathbf{1}_S U^\epsilon(t,0)\eta|| \leq c\epsilon^{\frac{1}{12}-\frac{3}{4}\zeta} \tag{70}$$

Idea of the proof of the lemma
(i)
A key point to control the error is to use $V^\epsilon_{\phi(l)}$ (see above) with $l = t$. Thus

$$\begin{aligned}
\epsilon\left(U^\epsilon(t,0) - V^\epsilon_{\phi(t)}\right) &= \epsilon\int_0^t \partial_a\left(V^\epsilon_{\phi(t)}(t,a)U^\epsilon(a,0)\right)da \\
&= -i\int_0^t V^\epsilon_{\phi(t)}(t,a)\left(H_{\phi(t)} - H_{\phi(a)}\right)U^\epsilon(a,0)da \\
&= -i\int_0^t V^\epsilon_{\phi(t)}(t,a)\left(\phi(t) - \phi(a)\right)AU^\epsilon(a,0)da
\end{aligned} \tag{71}$$

Thus

$$U^\epsilon(t,0)\eta = V^\epsilon_{\phi(t)}(t,0)\eta + \frac{i}{\epsilon}\int_0^t \left(\phi(a) - \phi(t)\right)V^\epsilon_{\phi(0)}(t,a)AU^\epsilon(a,0)\eta da \tag{72}$$

Given this expression one can introduce an appropriate cutoff $\tilde{\sigma} \in (0,t)$, split the integral and estimate both resulting integrals. The necessarity of the cutoff is due to an estimate concerning generalized eigenfunctions which is needed to

[72] This assumption can always be fulfilled because of the definition of ϕ (see **2.2.2**, Definition (i)).

prove step 1 of the second part of the proof which led to a Corollary (Corollary 7.2 of [30]) used to perform the estimate for $U^\epsilon(t,0)\eta$. This corollary is essential for the proof to "go through" but it is not reviewed here because a rather long discussion on generalized eigenfunctions would be needed. The lemma follows when using this corollary carefully (see [30], proof of lemma 7.3).

(ii)
For the proof of the estimate an auxiliary time evolution $\tilde{U}^\epsilon(t)$ is introduced by defining $\tilde{U}^\epsilon(t,0) = U^\epsilon(t,0)$ for $t \le \sigma$ and $\tilde{U}^\epsilon(t,0) = V^\epsilon_{\phi(\sigma)}(t,\sigma)$ for $t > \sigma$. now i.p. with corollary 7.2 again it is shown in [30] that there exists some $\tilde{\Psi}^\epsilon_t$ s.t.

$$||\tilde{U}^\epsilon(t,0)\eta - \tilde{\Psi}^\epsilon_t|| \le C\epsilon^{\frac{1}{12}-\frac{3}{4}\zeta} \tag{73}$$

and that there exist C_ζ s.t.

$$||\mathbf{1}_S\tilde{\Psi}^\epsilon_t|| \le C_\zeta\epsilon^{\frac{2}{3}-1}t^{-2} \tag{74}$$

This is lemma 7.6 of [30].
Using this the claim of lemma (ii) follows easily: As in the proof of (i) for $t > \sigma$:

$$\epsilon\left(U^\epsilon(t,\sigma) - \tilde{U}^\epsilon(t,\sigma)\right) = -i\int_\sigma^t U^\epsilon(t,a)\left(\phi(\sigma) - \phi(a)\right)A\tilde{U}^\epsilon(a,\sigma)da \tag{75}$$

Using this and the last estimate of lemma 7.6 it is easy to show that for some $C \in \mathbb{R}$

$$||\mathbf{1}_S U^\epsilon(t,\sigma)\tilde{\Psi}^\epsilon_\sigma|| \le Ct^{-2}\epsilon \tag{76}$$

and together with the first estimate of lemma 7.6 the claim follows.

The results of the first and the second part of the proof can be combined to prove the theorem (see [30], 7.4). From the theorem and the adiabatic theorem pair creation in the formalism reviewed in **1.1.4** follows, i.e.
(i) for $t < t_i$:

$$lim_{\epsilon\to 0} < \Psi^\epsilon, P_-\Psi^\epsilon >= 1 \tag{77}$$

(ii) for $t > t_f$:

$$lim_{\epsilon\to 0} < \Psi^\epsilon, P_+\Psi^\epsilon >= 1 \tag{78}$$

Note that the theorem mentioned here in **2.2.3** is just needed to prove (ii), (i) is a direct consequence of the initial condition and the adiabatic theorem.
For the discussion of adiabatic pair creation in external laserfields (**3.2**) the most important insight from this proof will be that in order to prove the existence of adiabatic pair creation the bound states and i.p. the whole space dependence played a crucial role.

3 Adiabatic Pair Creation in laserfields

3.1 Review of the "standard arguments"

In the recent physics literature it has been discussed if pair creation in laserfields can happen (e.g. in [32], [33], [34], [35]). In this section the arguments given there are carefully reviewed.
It is argued that lasers of a new generation (X-ray lasers) can be used (when focussing them to a spot) to produce very large electric fields which can be used to create an observable amount of pairs. For example in his analysis in [32] Ringwald i.p. refers to the Schwinger calculation (**1.1.6**) and its generalization by Brezin and Itzykson ([36]) as well as to calculations by Popov and Marinov ([38], [39], [40] and references therein). So these calculations are reviewed first.
They mainly treat the phenomenon of pair creation in external fields periodic w.r.t. time (to be specified in **3.1.1** and **3.1.2**) and use quasiclassical approximations which are valid under the two conditions[73]

$$\omega_0 << m \tag{79}$$

$$eE << m^2 \tag{80}$$

where ω_0 is the period of the external field, E is the strength of the electric field and m the mass of the particle to create. These conditions seem to be plausible for quasiclassical considerations because in a quasiclassical treatment $Im(S) >> \hbar$ should hold which leads to the given conditions.

3.1.1 The treatment by Brezin and Itzykson ([36]) and an early criticism by Troup and Perlman ([37])

The paper by Brezin and Itzykson ([36]) asks the question if fields created by optical lasers are able to cause pair creation which can be observed. The discussion is based on the Schwinger calculation ([19]) which is generalized for time dependent periodic fields constant in space.[74] The generalization is meant in a way to be an "order-of-magnitude estimate"[36] which is not applicable for all situations.[75]
(In the notation used in the following capital letters (like P_0, P_1, X_0, ...) represent operators whereas p^0, p^1, ... $\in \mathbb{R}$. Bold face letters like $\mathbf{p}$, $\mathbf{x}$ represent the spatial part of some 4-vector.)
The discussion in [36] starts with a potential A^μ which is given by

$$A^\mu(t,x) = A(t)\delta^{\mu 4} \tag{81}$$

[73] This is not true for some of the work of Marinov and Popov (like [39]) where exact methods are used (see **3.1.2**).

[74] The reason why such types of fiels are considered is given below.

[75] A more detailed explanation how this is meant is given at the end of **3.1.1** when the criticism by Troup and Perlman ([37]) is discussed.

where A is periodic with a period $\omega_0 << m$, $x \in \mathbb{R}^3$ are the space variables and $t \in \mathbb{R}$ is the time variable. In a specific gauge (here temporal gauge) the given A^μ describes an electric field which is periodic w.r.t. time and constant w.r.t space.[76] It is crucial to point out that the given A^μ-field wich is used describes such an electric field just in the chosen gauge and that Brezin and Itzykson use this fixed gauge when giving physical arguments. This is different to the Schwinger calculation (reviewed in **1.1.6**) which is (manifestly) gauge invariant.
Now the detailed treatment of Brezin and Itzykson proceeds like follows: The given expression for A^μ is plugged in the general expression given by Schwinger for the probability per unit volume and unit time for pair creation $w(x)$ where (see **1.1.6**)

$$w(x) = Tr < x, ln\left(1 - T\rho_+ \bar{T}\rho_-\right) x > \tag{82}$$

The calculation starting from this expression is done for bosons where the expressions for T, $\bar{T}$ and $\rho_\pm$ are a bit different as the ones given in **1.1.6** (see [36]). This is done because it is assumed that the difference between fermions and bosons is just a counting factor (this is showed for the static case for $eE << m^2$). Brezin and Itzykson obtain the following result using i.p. $eE << m^2$:

$$w(x) \approx \frac{1}{2\pi}\frac{d}{dt}\left(\int \frac{d^3p}{(2p^0)^2} | < -p^0, T_p p^0 > |^2\right) \tag{83}$$

where $p^0 = \sqrt{\mathbf{p}^2 + m^2}$ (i.e. p^0 lies on-shell) and T_p satisfies the one dimensional Lippmann-Schwinger equation

$$T_p = V_p + V_p \frac{1}{P_0^2 - \left(p^0\right)^2 + i\epsilon} T_p \tag{84}$$

Here $V_p = -2ep_3 A(X_0) + e^2 A^2(X_0)$.
As a consequence the matrix element $< -p^0, T_p p^0 >$ describes the scattering amplitude for a process backwards in time which is identified with antiparticle scattering. The corresponding differential equation for the particle is given by

$$\left(\frac{d^2}{dt^2} + \left(p^0\right)^2 + V_p(t)\right)\Psi(t) = 0 \tag{85}$$

Now the assumption $eE << m^2$ is used (again) to justify the use of a classical approximation by a generalized WKB-method. That is because $eE << m^2$ implies a slow variation of $\left(p^0\right)^2 + V_p(t) =: w(t)$ (see [36]). By the generalized WKB-method one makes the ansatz

$$\Psi(t) = A(t)e^{-i\int_0^t ds w(s)} + B(t)e^{i\int_0^t ds w(s)} \tag{86}$$

[76] The restriction to a constant electric field is justified by Brezin and Itzykson through the following statement. "Furthermore we limit ourselves to an oscillating field constant throughout space. We expect that the space variation of the field might produce similar effects. They are neglected for the sake of simplicity in this order-of-magnitude calculation."[36]

with boundary conditions given by classical physics: $lim_{t\to\infty}A(t) = 1$, $lim_{t\to\infty}B(t) = 0$. So one has to find appropriate functions A and B. In [36] this is done perturbatively in the following sense. In a first step differential equations for A, B are found:

$$\frac{d}{dt}A = -\frac{\frac{d}{dt}w(t)}{2w(t)}\left(A - Be^{2i\int_0^t dsw(s)}\right) \tag{87}$$

$$\frac{d}{dt}B = -\frac{\frac{d}{dt}w(t)}{2w(t)}\left(B - Ae^{-2i\int_0^t dsw(s)}\right) \tag{88}$$

As a first approximation the oscillations ($e^{\pm i\int w(s)ds}$) are neglected, i.e. approximated by a constant. This is justified because by $eE << m^2$ the oscillations are very fast compared to the time variation of A and B (see [36]). So when coarse graining the situation to large times the approximation is a good one. An appropriate solution is then given by $A^{(0)}(t) = \sqrt{\frac{p^0}{w(t)}}$, $B^{(0)}(t) = 0$. The next order ($A^1(t)$, $B^1(t)$) is used to get an approximation for the pair creation per unit volume (see [36])

$$\int wdt = \int d^3p\frac{1}{(2\pi)^3}|\int_{\mathbb{R}} dt\frac{\frac{d}{dt}w(t)}{2w(t)}e^{-2i\int_0^t E(s)ds}| \tag{89}$$

Next from this expression an explicit solution for w is derived in [36]:

$$w \approx w_0\left(\frac{eE}{2w_0}\right)^2\int\frac{d^3p}{(2\pi)^2}|c|^2 \tag{90}$$

where

$$c = \int_{-\pi}^{\pi}\frac{dx}{2\pi}\frac{sin(x)\left(p_3 - \frac{eE}{w_0}cos(x)\right)}{m^2 + p_1^2 + p_2^2 + \left(p_3 - \frac{eE}{w_0}cos(x)\right)^2}e^{\frac{2i}{w_0}\int_0^x dy\left(m^2+p_1^2+p_2^2+\left(p_3-\frac{eE}{w_0}cos(y)\right)^2\right)^{\frac{1}{2}}} \tag{91}$$

This is further simplified using the method of steepest decend (see [36]) to get

$$w = \frac{1}{9}w_0\int\frac{d^3p}{(2\pi)^2}e^{-2\alpha}cos^2(\beta) \tag{92}$$

where α and β are defined by

$$-\alpha + i\beta = \frac{2i}{w_0}\int_0^{x_0} dy\left(m^2 + p_1^2 + p_2^2 + \left(p_3 - \frac{eE}{w_0}cos(y)\right)^2\right)^{\frac{1}{2}} \tag{93}$$

Here x_0 is the zero of $\left(m^2 + p_1^2 + p_2^2 + (p_3 - \frac{eE}{w_0}cos(x))^{\frac{1}{2}}\right)$ for which both the real and imaginary parts are positive.

To get a final expression estimates are made in [36]: $cos^2(\beta) = \frac{1}{2}$ (the average of cos^2) and $p_3 = 0$. Thus

$$w = \frac{\alpha E^2}{2\pi} \frac{e^{-\frac{\pi m^2}{eE} g(\frac{m w_0}{eE})}}{g(\frac{m w_0}{eE}) + \frac{1}{2}\frac{m w_0}{eE} g'\left(\frac{m w_0}{eE}\right)} \tag{94}$$

where α is the Sommerfeld constant and

$$g(x) = \frac{4}{\pi} \int_0^1 dy \left(\frac{1-y^2}{1+x^2y^2}\right)^{\frac{1}{2}} \tag{95}$$

For $\gamma := \frac{m w_0}{eE}$[77] small (i.e. $\gamma << 1$) which is i.p. the case for (optical) lasers one gets (see [36])

$$w \approx \frac{\alpha E^2}{2\pi} e^{-\frac{\pi m^2}{eE}} \tag{96}$$

From this result the following consequence which is the essence of the discussion is taken:
“The condition for observing pairs in vacuum can thus be summarized as

$$eE \geq \pi m^2 g\left(\gamma\right) \tag{97}$$

“[36]. Otherwise the effect is expected to be exponentially supressed.
Since for optical lasers such a high field strength is not achievable the result of Brezin and Itzykson is that optical lasers cannot be used to produce an observable amount of pairs.

The conclusion of Brezin and Itzykson was criticized very early in a short paper by Troup and Perlman ([37]). The two points which are criticized read:

“(1) A focused laser field will be very close to a wrenchless ($< E, B >= 0$), null ($E = B$) field (cgs units are used). Such a field is lightlike in relativistic terminology, and remains so in all Lorentz frames. The authors derive their result using a timelike, alternating electric field.
(2) The authors consider only the electric field, stating that pair creation does not occur in a pure magnetic field. In the static case, Toll (J. Toll, thesis, Princton University, 1952, unpublished) states that pair production is “completely inhibited“ if the field is wrenchless and if $E \leq B$. Thus a static null electromagnetic field will not create pairs by the “leakage through the barrier“ mechanism discussed by the authors.“[37]

The essence of this criticism is not the claim that external wrenchless null electromagnetic fields do not produce pairs. This is a fact which has already been

[77] This quantity is not a priori small because it is the ratio of two quantities which are both assumed to be small in the calculation.

known since the gauge invariant treatment by Schwinger (see [18]) and which is well known by Brezin and Itzykson[78]. (I.e. Brezin and Itzykson say that their result is (just) an "order-of-magnitude estimate"[36] and that (at least) special cases are not covered by their calculation[79].)
The essence of the criticism by Troup and Perlman is i.p. that fields created by optical lasers are indeed (approximatly) null and wrenchles (a detailed argument supporting this point can be found in [37]): "It seems to us therefore that the failure to take account of the closely null nature of the alternating magnetic field associated with the electric field, could invalidate the application of the authors' result to a focused laser field."[37] Thus, Troup and Perlman claim that the "order-of-magnitude calculation" of Brezin and Itzykson cannot be applicable for alternating fields created by optical lasers (because it is already known that for such fields no pair creation occurs).[80]
The consequence which is taken from this criticism is that in the discussion of pair creation in laserfields it is nowadays assumed that at least two (coherent) laser beams have to be superposed in order to cause pair creation: "Therefore to produce pairs it is necessary to focus at least two coherent laser beams and form a standing wave."[33] Why (If resp.) the calculation of Brezin and Itzykson is assumed to be applicable for such laser fields will be discussed in **3.1.3**.

3.1.2 The treatment by Popov and Marinov

Popov and Marinov who - at least when writing [40] - knew about the criticism of Troup an Perlman ([37])[81] basically used the "imaginary time method" for their calculation of the probability of pair production in specific external time varying electric fields. Roughly the essence of the imagniary time method is that formally one can treat the motion of a particle through the spectral gap as classical motion with imaginary values of "time" $t \in \mathbb{C}$, i.p. one can find the motion of the particle by extremizing $Im(S)$. (For a deduction of this method see [40], appendix B.) In [38] this is done for special time dependent external electric fields which are assumed to be uniform in space (the same assumption as in the Brezin-Itzykson treatment, see **3.1.1**) and pointing in one space direction.

[78] "It is, of course, well known that specific anomalies can occur; for instance, there is no pair creation in a plane wave field."[36]

[79] "Therefore, if the rate predicted by the theory were more favorable, one should pay more attention to the particular geometrical characterization. At the present stage, however, we present rather an order-of-magnitude estimate."[36]

[80] This is one fact which is also known by the authors of the resent literature discussing pair creation in laserfields (at least by some of them). Refering to the work of Troup and Perlman ([37]), Ringwald for example writes "It has been argued that fields produced in (optical) focusing of laser beams are very close to such a light-like electromagnetic field, leading to an essential suppression of pair creation."[32]

[81] see footnote on page 385 in [40]

Starting with the action (for $t \in \mathbb{C}$):

$$S(t) = \int^t L(s)ds \tag{98}$$

where $L(s) = -m\sqrt{1-\dot{x}^2} + eE(s)x$ is the lagrangefunction of the problem it is shown with the imaginary time method that

$$2Im(S) = \frac{m^2}{eE} g(\frac{mw_0}{eE}) \tag{99}$$

where $g(\frac{mw_0}{eE}) = \int_{-1}^{1} \sqrt{1-u^2} \Psi\left(\frac{mw_0}{eE}u\right) du$ with $\Psi(x) = \frac{d\tau}{dx}$. Here τ is determined by $x = \int_0^\tau f(s)ds$ where f is the time dependent factor of the external field E, i.e. $E(t) = \tilde{E}f(t)$ with $\tilde{E}$ the amplitude chosen s.t. $|f| \leq 1$. So for a special form of a time dependence of an external field the main step is to calculate Ψ. This is in [38] i.p. done for a field of the form $f(t) = cos(w_0 t)$ which is claimed to correspond to the field of a laser. The result is equivalent to the result of Brezin and Itzykson (see **3.1.1**) and leads to the same conclusion. It is assumed that intense lasers could be used to observe the phenomenon:

"The situation will change if in the future we suceed in constructing x-ray or γ-ray lasers. Thus, for $\omega = 50keV$ and $E \propto (10^{13} - 10^{14})\frac{V}{cm}$, we have $\gamma \propto (10-100)$, $N = 10$ and it is possible to attain $wVT \propto 1$ (i.e., a pair is produced during a time $T \propto 1sec$ in a volume $V \propto 1cm^3$)"[38]

"Apply the result of section V to the sinusoidal field $E(t) = Ecos(\omega t)$. Such a field may be creatd in an antinode of the standing light wave produced by a superposition of two coherent laser beams."[40]

It is important to note two points: First, the imaginary time method can also be applied for some magnetic fields, i.e. this part of the electromagnetic field does not have to be neglected. More precisely: Such a calculation is possible if the electromagnetic field is such that the classical motion of a particle in such a field can be determined. (Nevertheless such a possibility is mentioned but a calculation is not given by Marinov and Popov.) Second, Marinov and Popov do not use "gauge invariant arguments", as Brezin and Itzykson when arguing in terms of the A^μ-field their argumentation is done in one specific gauge (see **3.1.1**).

In another article ([39]) an exact method (without quasiclassical approximations like the imaginary time method) is developed. However, this method is not applicable for electromagnetic fields with nonvanishing magnetic parts.

Kind of a summary of the work of Popov and Marinov is [40].

3.1.3 Recent publications on pair creation in laserfields

Recently, a big amount of articles concerning pair creation in external fields produced by x-ray lasers has been published (e.g. [32], [33], [34], [35]). Their results are based on the calculations sketched in **3.1.1** and **3.1.2**. The claim is that x-ray lasers (which are currently under construction) will be able to cause an observable amount of pairs. The main arguments are reviewed here.
Ringwalds analysis in [32] is based on the assumption that a field of the type $E(t) = E_0 cos(\omega t)\mathbf{e}_z$, $B(t) = 0$ can be produced in "an antinode of the standing wave produced by a superposition of two coherent laser beams with wavelength $\lambda = \frac{2\pi}{\omega}$"[32]. That is why he argues that the calculations reviewed in **3.1.1** and **3.1.2** can be applied in this case exactly. Since the strength of the electric field will be large enough in x-ray lasers to be constructed he concludes that pair creation can be observed. However, it is also noted that the assumption that the laserfield is of the type given above is an approximation. Nevertheless it is assumed that this approximation yields an order-of-magnitude estimate.[82]
In [33] and [34] it is stated that a field of the type $E(t) = E_0 cos(\omega t)\mathbf{e}_z$, $B(t) = 0$ cannot be produced in realistic situations using coherent laser beams to form a standing wave. It is concluded that the treatment using such a field yields an upper bound for the production rate of pairs.[83]

3.1.4 The role of the magnetic field

Since both in the calculations by Popov and Marinov (**3.1.2**) and in the calculations by Brezin and Itzykson (**3.1.1**) concerning pair production rates in laser fields purely electric fields are considered and since it is a basic assumption made in the recent publications (**3.1.3**) that such a treatment leads to an upper bound for the pair production rate (i.e. that the magnetic part of the field is neglectable), it is useful to consider the role the magnetic field can play in pair creation processes. For this purpose the commonly used assumptions are reviewed.
Schwinger shows in [18] using the derived integral expression for the vacuum action for constant electromagnetic fields (see **1.1.6**) that constant magnetic fields

[82] "We elaborate on a model which retains the main features of the general case but nevertheless allows to obtain final expressions for the pair production rate in closed form. This should be sufficient for an order-of-magnitude estimate of the critical parameters."[32]

[83] "Subsequently we assume an "ideal experiment": Owing to the diffraction limit the spot radius of the crossing beams cannot be smaller than the wavelength, so we choose $r_\sigma \approx \lambda$; and we assume a space volume in which the electric field is nonzero but the magnetic field vanishes. Even for carefully chosen x-ray optical elements, such a situation is impossible to achieve in practice, which means that the field strength actually availabe to produce particles is weaker than the peak field value and hence our estimate of the production rate will be an upper bound."[33]

cannot produce pairs.[84] This is because for such fields the vacuum action is purely real.[85]
Brezin and Itzykson claim in [36] that magnetic fields cannot produce pairs at all. However the intuitive argument which is given in [36] explicitly refers to constant fields.[86]
In [21] a perturbative argument is given claiming that pair creation is a "purely electric effect", more precisely that if the field is weak (i.e. if the perturbative argument is claimed to work) pair creation does not occur in magnetic fields. Here it is not just refered to a constant field. The argument is based on a perturbative treatment given by Schwinger in [18]. The most important points of the result are reviewed shortly:
It is assumed that for weak fields an appropriate vacuum action can be calculated doing an expansion in powers of the Sommerfeld constant α. One then gets for the imaginary part of the vacuum action in first order (see [18] or [21])

$$Im(W) = \frac{\alpha}{12} \int_{-k^2>4m^2} d^4k \left(|E(k)|^2 - |B(k)|^2\right) \left(1 - \frac{4m^2}{-k^2}\right)^{\frac{1}{2}} \left(2 + \frac{4m^2}{-k^2}\right) \quad (100)$$

where $E(k)$, $B(k)$ are the Fouriertransforms of the electric and magnetic fields. Since in this expression the integration domain is restricted to values $-k^2 > 4m^2$ and since for such k there exists a reference frame in which $B(k) = 0$[87] one concludes that pair creation is an "electric effect".
In the recent literature (**3.1.3**) it is also assumed that pair creation is an "electric effect". Thus the conclusion that the situation of a pure periodic electric field which underlies the discussion gives an upper bound for the real situation in which a magnetic field is present seems to be justified.

3.2 Criticism of some standard arguments

3.2.1 Modelling laserfields

In the reviewed treatments in **3.1** it is (by using one specific gauge) always argued in terms of the (local) electric and magnetic fields. However one must emphasize that it is the 4-potential A^μ which is used in the formulation of the problem and which can in principle lead to global effects like the Aharonov-Bohm effect. Such effects are not taken into account if one just uses the local electric and magnetic fields in the discussion (see **1.1.3**).

[84] More precisely he shows that for any constant electromagnetic field for which a reference frame exists in which the given field is purely magnetic pair creation does not occur.

[85] The argument can also be found in [20], p. 288.

[86] "Pair creation does not occur in a pure magnetic field. This is clear since a constant field cannot transfer energy to a charged particle. It is due to the electric field that enables particles to leak through the 2m potential barrier. Hence the physically relevant case can be taken to be the one of a pure electric field."[36]

[87] Indeed the reference frame where k has just a temporal component.

Thus when modelling a laserfield to be used in the treatment of adiabatic pair creation one should use the 4-potential A^μ and when giving arguments on the physical situation one must be (at least) very careful when one uses the local electric and magnetic fields. Brezin and Itzykson mention this briefly in [36]: "Therefore, if the rate predicted by the theory were more favorable, one should pay more attention to the particular geometrical characterization."[36]
However, in the recent articles on the subject (see **3.1.3**) it is not argued for such global effects (explicitly) and it is just stated that the discussed treatment is an order-of-magnitude calculation which gives an upper bound for the actual pair production rate. The question to be discussed here is if this is justified. Here the formulation of the problem in the adiabatic switching formalism (see **2.2.2**) gives a clear answer (see [31]): A pair is created (in the adiabatic limit) iff the external potential A^μ is overcritical - a notion which depends on the behaviour of the bound states. However the question if a suitable bound state exists can just be answered if the full space dependence of the potential is known. This directly follows from the definition of a bound state.
Thus, for pair creation to exist the whole geometry of the problem is essential (when using the adiabatic switching formalism in the argumentation). Pair creation is a global effect and the local arguments given in the recent literature cannot be justified when the adiabatic switching formalism is taken into account.
To summarize: For dealing properly with the phenomenon of adiabatic pair creation taking the adiabatic switching formalism into account a global description of the laserfield is needed in terms of the 4-potential.
One further well known fact to point out (which will be essential in **3.2.2**) is the fact that laserfields do not have a Coulomb electric part, i.e. that laser fields are properly modelled by purely propagating fields which are not attached to any source. This fact in particular holds for any superposition of laser fields.[88]

3.2.2 Gap bridge for laserfields?

In the calculations considered so far (**1.1.6**, **3.1.1**, **3.1.2**) the conclusion is always that pair creation happens when the external electric field takes a critical value, i.e. the existence of pair creation depends (just) on the strength of the external electric field.
Here the treatment of pair creation as adiabatic phenomenon (**2.2.2**) gives a crucial new insight.
In the adiabatic switching formalism it is manifest that pairs are created (in the adiabatic limit) iff the external field becomes overcritical, i.e. if a gap bridge exists. A priori this has nothing to do with the field strength of the external field and is due to the fact that pair creation is a global effect (see **3.2.1**).

[88]For the case of pair creation it is most relevant that the fact also holds for a superposition of two coherent laser beams. Such a situation is usually assumed when discussing pair creation (see end of **3.1.1**).

Thus, the question whether pair creation in laserfields exists reduces to the question if a given laserfield is overcritical or not. In [31] an argument answering this question is given which crucially depends on the magnetic part of the electromagnetic field of the laser. This shows that this part is not neglectable as claimed by the standard arguments (**3.1**). This argument of [31] is reviewed here. Its essence is that a gap bridge does not exist when there is no "Coulomb electric potential part" which is the case for laserfields (see **3.2.1**).

First fix Coulomb gauge (i.e. $div\mathbf{A} = 0$). Note that in this gauge "no Coulomb electric potential part" means $A^0 = 0$. Under this assumption it can be shown that there is no eigenvalue curve crossing the value zero (i.p. no gap bridge) using the eigenvalue equation for the Dirac operator H minimally coupled to the external field, i.e. (since $A^0 = 0$)

$$\left(H_0 + e \sum_{i=1}^{3} \alpha_i A^i \right) \Phi = E\Phi \tag{101}$$

Here E is the eigenenergy corresponding to the bound state Φ. Using $\beta^2 = \mathbf{1}$, the usual anticommutation relations for the matrizes α_i ($i \in \{1, 2, 3\}$) and β (see e.g. [3]) and $\mathbf{B} = curl\mathbf{A}$ for the magnetic field $\mathbf{B}$ the following holds

$$\left(H_0 + e \sum_{i=1}^{3} \alpha_i A^i \right)^2 \Phi = \left(H_0 + e \sum_{i=1}^{3} \alpha_i A^i \right) E\Phi$$

$$[\left(-i \sum_i \alpha_i \frac{\partial}{\partial x^i} + \beta m \right)^2 + \left(e \sum_i \alpha_i A^i \right)^2 + e \left(-i \sum_i \alpha_i \frac{\partial}{\partial x^i} + \beta m \right) \left(\sum_i \alpha_i A^i \right) +$$

$$+ e \left(\sum_i \alpha_i A^i \right) \left(- \sum_i \alpha_i \frac{\partial}{\partial x^i} + \beta m \right)]\Phi = E^2 \Phi$$

$$\left(-\mathbf{1}_4 \sum_i \frac{\partial^2}{\partial {x^i}^2} + m^2 \mathbf{1}_4 + \sum_i {A^i}^2 + \left(i \sum_i \alpha_i B^i(\mathbf{x}) \right) \alpha_1 \alpha_2 \alpha_3 \right) \Phi = E^2 \Phi \tag{102}$$

For a magnetic field with constant direction[89] (i.e. $\sum_i \alpha_i B^i(\mathbf{x}) = b(\mathbf{x}) \sum_i \alpha_i b^i$ where b^i are constants and b is the field strength) this gives with the Schrödinger operator H_S:

$$\left(\mathbf{1}_4 H_S + m^2 \mathbf{1}_4 + b(\mathbf{x}) \sum_i \alpha_i b^i \alpha_1 \alpha_2 \alpha_3 \right) \Phi = E^2 \Phi \tag{103}$$

and by diagonalizing $\sum_i \alpha_i b^i \alpha_1 \alpha_2 \alpha_3$

$$\left(H_S \mathbf{1}_2 + m^2 \mathbf{1}_2 + \mu_i b(\mathbf{x}) \mathbf{1}_2 \right) \phi_i = E^2 \phi_i \tag{104}$$

[89] Note that this is a standard assumption, see e.g. **3.1.2**.

where $\mu_i = \pm 1$ are the eigenvalues of $\sum_i \alpha_i b^i \alpha_1\alpha_2\alpha_3$ and ϕ_i are the two component vectors one gets by projecting Φ on its spin parts. Note that $A^0 = 0$ was essential in this argument.

This means that the eigenenergy E of a bound state of the Dirac equation minimally coupled to an external laser field has the very special property that it enters quadratically in the stated Schrödinger equation. Because such an energycurve (crossing zero) is atypical, pair creation is not assumed to occur.
On top of that, one can use a similar argument to show even more, i.e. one can show that $||H\Phi||^2 \geq m^2$ for laserfields. Since $H\Phi = E\Phi$ would hold for a bound state Φ this means that $-m \leq E \leq m$. Thus, by minimally coupling of an external laserfield no bound states in the intervall $[-m, m]$ appear at all, i.p. no gap bridge can exist. The calculation showing this is given using the results from the previous calculation (it is again assumed that the magnetic field has constant direction):

$$\begin{aligned} ||H\Phi||^2 &= <\Phi|H^+H|\Phi> \\ &= <\Phi|\left(-\mathbf{1}_4\sum_j \frac{\partial^2}{\partial {x^i}^2} + m^2\mathbf{1}_4 + \mathbf{1}_4\sum_i (A^i)^3 + i(\sum_i \alpha_i B^i)\alpha_1\alpha_2\alpha_3\right)|\Phi> \\ &= ||\nabla\Phi||^2 + m^2 + \sum_i (A^i)^2 + <\Phi|ib(\mathbf{x})\left(\sum_i \alpha_i b^i\right)\alpha_1\alpha_2\alpha_3|\Phi> \end{aligned} \tag{105}$$

Since the last term vanishes (one sees this by diagonalizing $\sum_i \alpha_i b^i \alpha_1\alpha_2\alpha_3$ and noting that the eigenvaules are $+1$ and -1 which leads to two terms canceling each other) one has

$$||H\Phi||^2 = ||\nabla\Phi||^2 + \sum_i (A^i)^2 + m^2 \geq m^2 \tag{106}$$

3.3 Restrictions

There are a few relevant things which have been left open or which have not been considered in this review. These are briefly mentioned here.

3.3.1 External field approximation

The whole treatment was based on the assumption that the electromagnetic field can assumed to be external (see **1.1.3**). It was not discussed why this is justified. Also mathematical treatments of external field approximation in Q.E.D. have not been discussed.
However the neglection of this point does not change the criticism of the standard arguments (**3.2**) since external field approximation is always used.

3.3.2 The kinetic approach

In some of the recent work on pair creation in laserfields (e.g. in [34] and [35]) a quantum kinetic approach showing a non-Markovian character of the phenomenon of pair creation is used. This approach was not considered in this review. However, it is stated in the recent works that the approach is only valid for simple fields of the form $\mathbf{E}(t) = (0, 0, E(t))$: “This procedure is exact but it is valid only for the simplest field configurations; e.g. for a spatially-uniform time-dependent electric field with fixed direction $\mathbf{E}(t) = (0, 0, E(t))$.“[35] Thus the criticism given in **3.2** does not have to be changed because of the kinetic approach.

3.3.3 Existence of pair creation in second quantized theory

In **2.2** the proof of the existence of adiabatic pair creation in external overcritical fields was just discussed for the one particle Dirac equation. In particular it was not argued how to “generalize“ the result to hold true in the second quantized formalism. However a procedure how such a generalization can work for adiabatic pair creation is given in [29].

3.3.4 Applicability of the adiabatic theorem for laserfields

In **3.2** it was implicitly assumed that the adiabatic theorem holds for laserfields. (Otherwise it is not justified why particles from the Dirac sea cannot jump from some energy level below the upper continuum to another energy level in such a way that they never reach the upper continuum.) However, first this does not change the criticism of the standard arguments and second the adiabatic theorem may also hold true for laserfields. One has to show for this point that the adiabatic theorem holds for fields changing in space and time in the form $A(x)B(t)$ where $xt = const.$ (this is the case for simple laserfields since there $\lambda f = c = const.$).

3.4 Conclusion

As a summary from this review basically the following conclusion can be taken. For dealing with the problem of pair creation in slowly changing laserfields in a realistic way it is natural to use results from adiabatic theory.[90] It is useful to start the treatment using the one particle Dirac equation, the Dirac sea picture and the adiabatic switching formalism and afterwards to generalize the result obtained there to a second quantized formalism. When using the adiabatic switching formalism to describe pair creation one has to take into account that pair creation is a global effect and for its existence i.p. the magnetic part of the external laserfield

[90] Note that here the restriction **3.3.4** has to be taken into account.

is essential. Pair creation occurs if the external A^μ-field is overcritical. For pure laserfields this is not the case (not even bound states of the potential are present in the spectral gap). Thus recently given “order-of-magnitude calculations“ do not apply for laserfields and pair creation is not expected to occur there - at least if the treatment in external field approximation is justified. The reason that the “usual arguments“ are not applicable is that from the “order-of-magnitude calculations“ consequences are taken by using local arguments involving the local electric and magnetic fields instead of global arguments using the A^μ-field.
Because one essential part in the argument given in **3.2.3** was that the Coulomb part of the external potential has to vanish it could be possible to observe pair creation in laserfields if one additionally turns on a Coulomb external potential. Such an experiment using heavy ions to produce strong Coulomb fields is proposed in [31].

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References

[1] Raum. Zeit. Materie: Vorlesungen über allgemeine Relativitätstheorie, H. Weyl, Springer, 1921

[2] Fermions and bosons interacting with arbitrarily strong external fields, J. Rafelski, L. P. Fulcher, A. Klein, Phys. Rep. 38, 227, 1978

[3] The Dirac Equation, B. Thaller, Springer, 1992

[4] A Report on Quantum Electrodynamics, in: The Physicist's Conception of Nature, J. Schwinger, Reichel, Dordrecht, 1973

[5] The Quantum Theory of Fields Volume 1 Foundations, S. Weinberg, Cambridge University Press, 1995

[6] Particles with spin $s \leq 1$ in an external field, R. Seiler, in: lecture notes in physics Vol. 73, Springer, 1938

[7] Relativistic Quantum Mechanics, J. D. Bjorken and S. Drell, McGraw-Hill, 1964

[8] Die Reflexion von Elektronen an einem Potentialsprung nach der relativistischen Dynamik von Dirac, O. Klein, Z. Phys. 53, 157, 1929

[9] Über das Verhalten eines Elektrons im homogenen elektrischen Feld nach der relativistischen Theorie Diracs, F. Sauter, Z. Phys. 69, 742, 1931

[10] The Klein paradox as a many particle problem, P. J. M. Bongaarts and S. N. M. Rujisenaarts, Ann. Phys. 101, 289, 1976

[11] Bemerkungen zum Kleinschen Paradox, F. Beck, H. Steinwedel, G. Süssmann, Z. Phys. 171, 189, 1963

[12] Zum „Kleinschen Paradoxon", F. Sauter, Z. Phys. 73, 547, 1932

[13] Kleins Paradox, B. R. Holstein, Am. J. Phys. 66, 1998

[14] Theorie du positron, P. Dirac, Rapport du 7^e Conseil Solvay de Physique, Structure et Properietes des Noyaux Atomiques, p. 203, 1934, in: Selected Papers on Quantum Electrodynamics, Dover (ed. J. Schwinger)

[15] Bemerkungen zur Diracschen Theorie des Positrons, W. Heisenberg, Z. Phys. 90, 209, 1934

[16] Folgerungen aus der Diracschen Theorie des Positrons, W. Heisenberg, H. Euler, Z. Phys. 98, 714, 1936

[17] Materieerzeugung im anschaulichen und im gequantelten Wellenbild der Materie, F. Hund, Z. Phys. 117, 1, 1941

[18] On Gauge Invariance and Vacuum Polarization, J. Schwinger, Phys. Rev. 82, 664, 1951

[19] The Theory of Quantized Fields. V, J. Schwinger, Phys. Rev. 93, 615, 1954

[20] Quantum electrodynamics of strong fields, W. Greiner, B. Müller, J. Rafelski, Springer, 1985

[21] Quantum Field Theory, C. Itzykson and J.-B. Zuber, McGraw-Hill, 1978

[22] Adiabatische Invarianten und Quantentheorie, P. Ehrenfest, Ann. Physik 51, 327, 1916

[23] Das Adiabatenprinzip in der Quantenmechanik, M. Born, Z. Phys. 40, 167, 1926

[24] Beweis des Adiabatensatzes, M. Born and V. Fock, Z. Phys. 51, 165, 1928

[25] Adiabatic perturbation theory in quantum dynamics, S. Teufel, Springer, 2000

[26] Methods of Modern Mathematical Physics, Functional Analysis, Revised and Enlarged Edition, M. Reed and B. Simon, Elsevier

[27] Methods of Modern Mathematical Physics, Fourier Analysis and Self-Adjointness, M. Reed and B. Simon, Elsevier

[28] On the Adiabatic Limit for Dirac Particles in External Fields, G. Nenciu, Comm. Math. Phys. 76, 117, 1980

[29] Existence of Spontaneous Pair Creation, P. Pickl, Dissertation an der Fakultät für Mathematik, Informatik und Statistik der LMU München, 2005

[30] On Adiabatic Pair Creation, P. Pickl and D. Dürr, Commun. Math. Phys. 282, 161, 2008

[31] Adiabatic pair creation in heavy-ion and laser fields, P. Pickl and D. Dürr, EPL 81, 40001, 2008

[32] Pair production from vacuum at the focus of an X-ray free electron laser, A. Ringwald, Phys. Lett. B 510, 107, 2001

[33] Pair Creation and an X-Ray Free Electron Laser, R. Alkhofer, M. B. Hecht, C. D. Roberts, S. M. Schmidt and D. V. Vinnik, Phys. Rev. Lett. 87, 193902-1, 2001

[34] Pair Production in Optical Lasers, D. B. Blaschke, A. V. Prozorkevich, C. D. Roberts, S. M. Schmidt and S. A. Smolyansky, Phys. Rev. Lett. 96, 140402, 2006

[35] Dynamical Schwinger effect and high-intensity lasers, D. B. Blaschke, A. V. Prozorkevich, G. Röpke, C. D. Roberts, S. M. Schmidt, D. S. Shkirmanov, S. A. Smolyanski, plasm-ph 0811.357v1, 2008

[36] Pair Production in Vacuum by an Alternating Field, E. Brezin and C. Itzykson, Phys. Rev. D 3, 1191, 1970

[37] Pair Production in a Vacuum by an Alternating Field, G. J. Troup and H. S. Perlman, Phys. Rev. D 6, 2299, 1971

[38] Pair production in a variable external field (quasiclassical approximation), V. S. Popov, Sov. Phys. JETP 34, 709, 1972

[39] e+e- pair production in an alternating electric field, V. S. Popov and M. S. Marinov, Sov. J. Nucl. Phys. 16, 449, 1973

[40] Electron-Positron pair creation from vacuum induced by variable electric field, M. S. Marinov and V. S. Popov, Fortsch. Phys. 25, 373, 1977