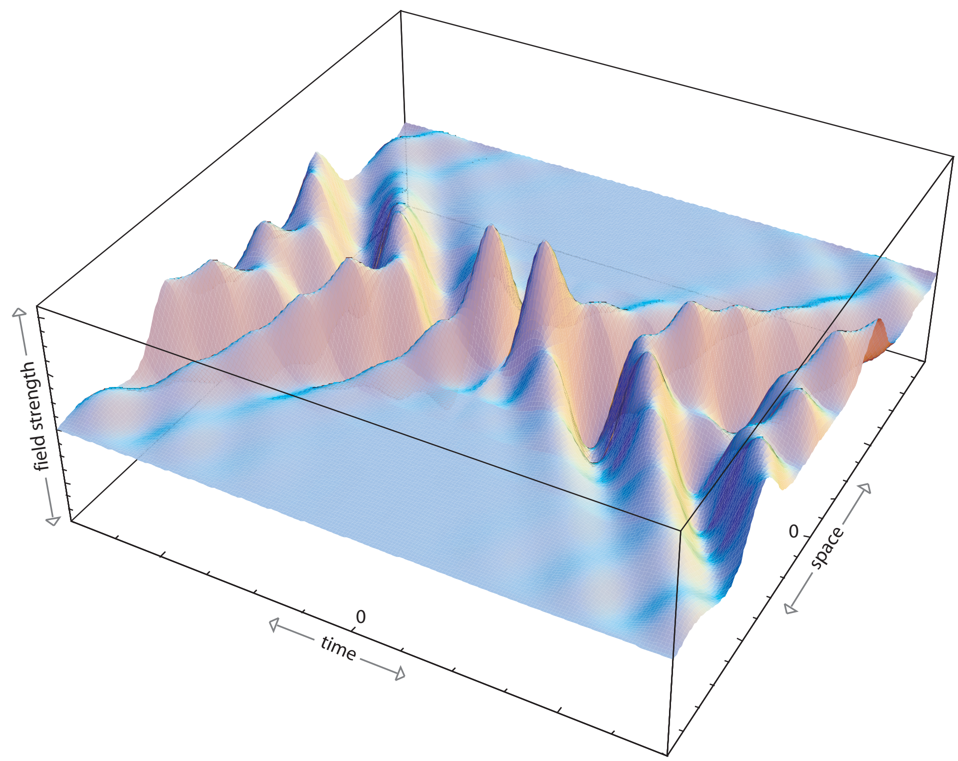
Dirk - André Deckert

Scalar Field Interaction Models



Diplomarbeit
Supervisor: Prof. Dr. Detlef Dürr
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(revised version)

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Title picture: The title picture shows a numerical simulation of how a solution of the massive Klein-Gordon field equation with the boundary conditions that the field vanishes at infinity reacts to a perturbation at $(\boldsymbol{x}, t) = 0$. The perturbation was chosen to be a scalar interaction with a source density of small but finite space- and time-like extend. Note how this perturbation creates waves that propagate with a finite velocity in the forward and backward light-cone of $(\boldsymbol{x}, t) = 0$. The simulation was written in Mathematica and its source code can be found at the author's home page *www.mathematik.uni-muenchen.de/∼deckert*.

Scalar Field Interaction Models

Dirk - André Deckert *

(revision of the December 2004 version)

October 30, 2008

Abstract

In this work we shall consider a special class of scalar field interaction models which describe the interaction of a fixed number of N particles with a scalar Klein-Gordon field. We choose two limiting cases of the relativistic dispersion relation $\sqrt{\boldsymbol{p}^2 + M^2}$ of the particles with mass M for both, the resulting classical and quantum field theory. Models of this class of field theories are known to generically produce divergent terms in the equations of motion, i.e. the Hamiltonian, as soon as one treats the particles as points. We shall analyze ways to make sense out of these ill-defined equations of motion in terms of an appropriate *renormalization theory.* One of these models, i.e. the resulting quantum field theory with the particle dispersion relation $\frac{\boldsymbol{p}^2}{2M}$, is the so-called *Nelson model.* Since for this model the abstract existence of a self-adjoint operator on a Fock-space, which can be seen as renormalized model Hamiltonian, was shown to exist we shall put our main focus on it. Unfortunately one has no explicit expression of that operator yet, nor does one know its domain or what the action of this operator on elements in its domain looks like. To these three points answers shall be given in this work.

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Revision logbook

- correction of some typos
- extension of subsection: *Fock-space*
- simplification of the integral estimates in subsection 4.5.2

Contents

1 Introduction

The aim of this work is the analysis of scalar field interaction models, which are in general the simplest possible field interaction models in classical and quantum field theories. These models describe particles, which we shall call nucleons, interacting with a scalar field. Here we will explanatory use the real Klein-Gordon field for all our considerations. Field theories of this kind are known to generically produce divergent terms in the equations of motion and therefore in the Hamiltonian as soon as we consider the nucleons as points, which we shall call the point particle limit and later on the removal of the cutoff. The emphasis lies on the analysis of these mathematical difficulties in both the classical and the quantum case and the ways how to deal with them, which are commonly known under the name renormalization theories. In the case of quantum field theories we shall always assume that the field is of a bosonic type and that the nucleons obey the Fermi statistics. Although most arguments in this work and especially Nelson's renormalization theorem are independent of the choice of the statistics of the nucleons, the choice of Fermi statistics will result in appropriate models describing e.g. nucleons interacting with a pion field or in the limit of a massless Klein-Gordon field the time-like part of the interaction of electrons with a photon field.

Section Map This work is divided into three main sections: 2, 3 and 4. Section 2 gives a definition of the Nelson model in the Fock representation, which is the most interesting model and which also brings up the central questions we address in this work. These questions are stated in its last subsection 2.8, which serves as an overview over the proceeding sections. With respect to these questions classical field theory is examined in section 3 and its quantum version in section 4. Section 5 concludes with a brief review over the most important results and a comment on today's situation in field theory.

Models When considering the interaction between nucleons and a field we will mainly look at two distinct cases. One in which a fixed number of N nucleons are nailed down at some initially chosen positions and one in which these N nucleons are able to move freely obeying an appropriate non-relativistic dispersion relation and are able to react according to the interaction. The first type we shall call the static case and the latter the dynamic one. Both are limiting cases of the relativistic nucleon dispersion relation $\sqrt{\boldsymbol{p}^2 + M^2}$. In the dynamic case we will consider the non-relativistic limit of the dispersion relation $M + \frac{\boldsymbol{p}^2}{2M}$ and in the static case only its rest mass $\sqrt{\boldsymbol{p}^2 + M^2} \approx M$. This approximation can physically be seen as the limit $\boldsymbol{p}^2 \ll M^2$ such that the interaction induces only very small momentum change compared to the rest mass of the nucleon. These models will be analyzed in the classical and the quantum regime. Although the name Nelson model is usually used only for the dynamic quantum case we like to call the resulting four models the *static classical*, the *dynamic classical*, the *static quantum* and the *dynamic quantum Nelson model*.

Notation Important formulas and results that are frequently used along the way are displayed in boxes the first time introduced. All mathematical symbols used throughout this work, which are non-standard or used in a slightly different sense than its standard versions, are explained in the appendix B.

2 Definition of the Nelson model and preliminaries

2.1 Fock-space

Let us follow the assumption that a quantum mechanical system describing particles with field-like interactions is characterized by some initial position distribution of the particles, the initial condition of the field, a ray[1] in a Hilbert-space that generates the dynamics for the particles and the field and a law that governs the time evolution of this ray in Hilbert-space. We shall call the particles nucleons. Furthermore it will be convenient to decompose the field with respect to solutions of its free field equation, the field modes, which we shall refer to as mesons. During the first definitions we speak about n-component Hilbert-spaces, where the word component stands for the *parts* of the quantum mechanical system, i.e. the nucleons or mesons respectively. A ray in a n-component Hilbert-space is able to generate the dynamics of a quantum mechanical system consisting of n of these *parts*. Since the total number of the *parts* may vary over time we need to define a bigger space, the Fock-space. In a loose speaking this Fock-space is an infinitely many-component Hilbert-space.

Definition 2.1.1 (zero-component Hilbert-space)**.** *Let V be a vector space over $\mathbb{C}$ spanned by one* ***abstract*** *vector indicated with $|0\rangle$, which we shall call the vacuum vector. The inner product we define by $(|0\rangle, |0\rangle)_V =: \langle 0|0\rangle := 1$. This abstract space is a Hilbert-space and is isomorphic to the space of complex numbers $\mathbb{C}$ with the inner product $(a, b) = a^* \cdot b$ for all $a, b \in \mathbb{C}$. We shall write $\mathcal{H}^{\otimes 0} := V$ and call it the zero-component Hilbert-space.*

Definition 2.1.2 (one-component Hilbert-space)**.** *Let $\mathcal{H}^{(1)}_{pre}$ be the set of finite $\mathbb{C}$ linear combinations of a possibly infinite but countable set of* ***abstract*** *vectors $\{|\varphi_i\rangle\}_{i\in\mathbb{N}}$. The inner product we define by $(|\varphi_i\rangle, |\varphi_j\rangle)_{\mathcal{H}^{(1)}} =: \langle\varphi_i|\varphi_j\rangle := \delta_{i,j}$ for all $i, j \in \mathbb{N}$. Let now $\mathcal{H}^{(1)}$ be the closure of $\mathcal{H}^{(1)}_{pre}$ with respect to the scalar product. In the case of an infinite basis $\mathcal{H}^{(1)}_{pre}$ this abstract space is isomorphic to $\mathcal{L}_2(\mathbb{R}^n, \mathbb{C}, d^n x)$, the space of complex valued square integrable functions over $\mathbb{R}^n$ with respect to the Lebesgue measure $d^n x$. Since a ray in that vector space shall generate dynamics that take place in the three dimensional position space the most natural choice is $n = 3$. We shall call $\mathcal{H}^{(1)}$ the one-component Hilbert-space.*

We shall write elements in $\mathcal{L}_2(\mathbb{R}^3)$ in terms of the Dirac-notation[2] $\langle \boldsymbol{x}|\varphi\rangle$ and its complex conjugate $\langle\varphi|\boldsymbol{x}\rangle$ for some $|\varphi\rangle \in \mathcal{H}^{(1)}$ and $\boldsymbol{x} \in \mathbb{R}^3$ and call them one-component wave functions. This notation stresses the fact that we mean a one-to-one correspondence between an abstract vector $|\varphi\rangle$ and a wave function $\langle\boldsymbol{x}|\varphi\rangle$, which can be defined e.g. by a one-to-one mapping of elements in the $\mathcal{H}^{(1)}$ basis $\{|\varphi\rangle_i\}_{i\in\mathbb{N}}$ onto elements on some $\mathcal{L}_2$ basis $\{\langle\boldsymbol{x}|\varphi_i\rangle\}_{i\in\mathbb{N}}$. Whenever the $\mathcal{L}_2$ basis is orthonormal the mapping is called isometric. The isomorphy, the one-to-one mapping, the formal definition of the identity $\mathbb{1}^{\mathcal{H}^{(1)}}_{id} := \int d^3x \; |\boldsymbol{x}\rangle\langle\boldsymbol{x}|$ and $\langle\boldsymbol{x}|\boldsymbol{x}'\rangle := \delta^3(\boldsymbol{x} - \boldsymbol{x}')$ yield an unique representation of the abstract space $\mathcal{H}^{(1)}$ in $\mathcal{L}_2$. For the choice of an isometric one-to-one mapping, which we will assume in the future, we find for all $|\varphi\rangle, |\phi\rangle \in \mathcal{H}^{(1)}$ that

$$(|\varphi\rangle, |\phi\rangle)_{\mathcal{H}^{(1)}} =: \langle\varphi|\phi\rangle = \int d^3x \;\langle\varphi|\boldsymbol{x}\rangle\langle\boldsymbol{x}|\phi\rangle := (\langle\boldsymbol{x}|\varphi\rangle, \langle\boldsymbol{x}|\phi\rangle)_{\mathcal{L}_2} \tag{1}$$

Definition 2.1.3 (n-component Hilbert-space)**.** *Let $\mathcal{H}^{\otimes n} := \bigotimes_{i=1}^n \mathcal{H}^{(1)}$ be the n time tensor product of one-component Hilbert-spaces. This abstract space is isomorphic to $\mathcal{L}_2(\mathbb{R}^{3n}, \mathbb{C}, \prod_{i=0}^n d^3x_i)$. Again we assume the isometric one-to-one mapping of elements $\mathcal{H}^{\otimes n}$ onto $\mathcal{L}_2(\mathbb{R}^{3n})$ elements which for some $|\varphi\rangle \in \mathcal{H}^{\otimes n}$ are called n-component wave functions $\langle\boldsymbol{x}_1, ..., \boldsymbol{x}_n|\varphi\rangle$. Hence the inner product is again given by*

$$\begin{aligned} (|\varphi\rangle, |\phi\rangle)_{\mathcal{H}^{\otimes n}} =: \langle\varphi|\phi\rangle &= \int d^3x_1 \,...\int d^3x_n \;\langle\varphi|\boldsymbol{x}_1, ..., \boldsymbol{x}_n\rangle\langle\boldsymbol{x}_1, ..., \boldsymbol{x}_n|\phi\rangle && (2)\\ &:= (\langle\boldsymbol{x}_1, ..., \boldsymbol{x}_n|\varphi\rangle, \langle\boldsymbol{x}_1, ..., \boldsymbol{x}_n|\phi\rangle)_{\mathcal{L}_2(\mathbb{R}^{3n})} && (3)\end{aligned}$$

[1] A ray in a vector space over $\mathcal{K}$ associated to a vector φ in that vector space is the set $\{z\varphi | z \in \mathcal{K}\}$.

[2] In fact we should better write $\langle\cdot|\varphi\rangle$ to indicate an element of $\mathcal{L}_2(\mathbb{R}^3)$ in order to emphasize that we mean the function and not its value at $\boldsymbol{x} \in \mathbb{R}^3$ - but in general there should not occur any confusion with either notation.

We shall call $\mathcal{H}^{\otimes n}$ the n-component Hilbert-space.

Definition 2.1.4 (Fock-space)**.** *Let $\mathcal{F}_{pre}$ be the set of finite $\mathbb{C}$ linear combinations of vectors $(|\psi_n\rangle)_{n\in\mathbb{N}}$ for any $|\psi_n\rangle \in \mathcal{H}^{\otimes n}, n \in \mathbb{N}$. We define the inner product as*

$$\langle\varphi|\phi\rangle = \sum_{n=0}^{\infty} \int d^3x_1 \ldots \int d^3x_n \ \langle\varphi|\boldsymbol{x}_1, ..., \boldsymbol{x}_n\rangle \langle\boldsymbol{x}_1, ..., \boldsymbol{x}_n|\phi\rangle \tag{4}$$

and the Fock-space as closure of $\mathcal{F}_{pre}$ with respect to this scalar product, which again is a Hilbert-space. In short we write $\mathcal{F} := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$, where $\bigoplus$ is called the direct sum and means the above.

Definition 2.1.5 (the Fock-space for the model)**.** *We indicate the Fock-space of the nucleons (and the n-nucleon Hilbert-space) by $\mathcal{F}_{nuc}$ (and $\mathcal{H}_{nuc}^{\otimes n}$) and the Fock-space of the meson field (and the n-meson Hilbert-space) by $\mathcal{F}_{mes}$ (and $\mathcal{H}_{mes}^{\otimes n}$). Hereby we forget about symmetrization of the tensor product spaces which will not play any role in this work. We define $\mathcal{F}$ to be the Fock-space of the model, i.e. $\mathcal{F} := \mathcal{F}_{nuc} \otimes \mathcal{F}_{mes}$. Since the Nelson model shall have its total number of nucleons conserved we will only need a smaller space $\mathcal{F}_N := \mathcal{H}_{nuc}^{\otimes N} \otimes \mathcal{F}_{mes}$, which we shall call the N-th nucleon sector of $\mathcal{F}$.*

Remark 2.1.1. *As usual in each Hilbert-space the norm $||\cdot||_{\mathcal{H}}$ is induced by the inner product with*

$$||\,|\psi\rangle\,||_{\mathcal{H}} := \langle\psi|\psi\rangle_{\mathcal{H}} \tag{5}$$

Wherever it is clear from which space $|\psi\rangle$ is from we will usually just use $||\cdot||$.

2.2 The Nelson model Hamiltonian H_κ in the Fock representation

The model Hamiltonian H_κ is a linear operator whose closure can be defined on whole $\mathcal{F}$ for finite κ. The field operators[3] $\psi^\dagger(\boldsymbol{x}), \psi(\boldsymbol{x})$ and $a_{\boldsymbol{k}}^\dagger, a_{\boldsymbol{k}}$ are the common creation and annihilation operators of the nucleons and the meson field respectively.

$$\begin{aligned}
H_\kappa &:= H_0 + gH_{I_\kappa} && (6)\\
H_0 &:= H_{nuc} + H_{mes} && (7)\\
H_{nuc} &:= \int d^3x \ \psi^\dagger(\boldsymbol{x}) D_{\boldsymbol{x}} \psi(\boldsymbol{x}) && (8)\\
H_{mes} &:= \int d^3k \ a_{\boldsymbol{k}}^\dagger \omega_{\boldsymbol{k}} a_{\boldsymbol{k}} && (9)\\
H_{I_\kappa} &:= \int d^3x \ \psi^\dagger(\boldsymbol{x}) \int d^3k \ \gamma_\kappa(\boldsymbol{k}) \left(e^{i\boldsymbol{k}\boldsymbol{x}} a_{\boldsymbol{k}} + e^{-i\boldsymbol{k}\boldsymbol{x}} a_{\boldsymbol{k}}^\dagger\right) \psi(\boldsymbol{x}) && (10)
\end{aligned}$$

where the integral operators are understood in the weak sense[4] and are discussed in more detail in subsection 2.3, $D_{\boldsymbol{x}}$ represents the dispersion relation of the nucleons, which for the cases examined in this work is a differential operator with respect to $\boldsymbol{x}$, $g \in \mathbb{R}_+$ is the coupling constant, which

[3] Let us use the convention that the field operators in position space are written like $\psi^\dagger(\boldsymbol{x})$ and the same operator in momentum space like $\psi_{\boldsymbol{p}}^\dagger$.

[4] That means the following. Let the linear operator T on $\mathcal{F}$ be formally given by $T := \int_M d\mu A(\mu)$ and A a mapping from M to the set of linear operators on $\mathcal{F}$ then T is defined by its $\mathcal{L}_2$ representation as

$$\langle\boldsymbol{x}_1, ..., \boldsymbol{x}_n|T|\psi\rangle := \int_M d\mu \, \langle\boldsymbol{x}_1, ..., \boldsymbol{x}_n|A(\mu)|\psi\rangle \tag{11}$$

if the integral exits for all $n \in \mathbb{N}$ and some $|\psi\rangle \in \mathcal{F}$. The set of $|\psi\rangle \in \mathcal{F}$ for which this is true naturally forms the domain of the operator.

represents the strength of the coupling and $M \in \mathbb{R}_+$ is the nucleon mass. Moreover the Klein-Gordon field amplitude γ_κ and the dispersion relation $\omega_{\boldsymbol{k}}$ for a meson field with mass $\mu \in \mathbb{R}_+ := \{x \in \mathbb{R} | x > 0\}$ is given by

$$\gamma_\kappa(\boldsymbol{k}) := \frac{\hat{\rho}_\kappa(\boldsymbol{k})}{\sqrt{2\omega_{\boldsymbol{k}}}} \tag{12}$$

$$\omega_{\boldsymbol{k}} := \sqrt{k^2 + \mu^2} \tag{13}$$

where $\hat{\rho}_\kappa(\boldsymbol{k})$ is the source density of the interaction and represents for $\kappa < \infty$ the so-called ultraviolet cutoff in the model. $\hat{\rho}_\kappa(\boldsymbol{k})$ has the following properties

$$\hat{\rho}_\kappa(\boldsymbol{k}) := \int đ^3x \; \rho_\kappa(\boldsymbol{x}) e^{i\boldsymbol{kx}} \tag{14}$$

$$\lim_{\kappa \to \infty} \rho_\kappa(\boldsymbol{x}) = \delta^3(\boldsymbol{x}) \tag{15}$$

The limit $\kappa \to \infty$ is called the removal of the ultraviolet cutoff or simply the point particle limit. Soon we shall see that in this limit the Hamiltonian remains a merely formal expression. Giving a mathematical meaning to it shall be one of the main objectives in this work. We postpone this issue and continue with finite κ. As said for the special case of the dynamic quantum Nelson model we restrict ourselves to the N nucleon sector $\mathcal{F}_N$, since the total number of nucleons shall be a conserved quantity. Furthermore we take $D_{\boldsymbol{x}}$ to be the free Schrödinger operator. We can then rewrite two terms of the Hamiltonian restricted to $\mathcal{F}_N$ with $\widehat{\boldsymbol{x}}_1, ..., \widehat{\boldsymbol{x}}_N$ being the nucleon position and $\widehat{\boldsymbol{p}}_1, ..., \widehat{\boldsymbol{p}}_N$ the momentum operators fulfilling the common commutation relations

$$[\widehat{x}_i^a, \widehat{p}_j^b] = i\delta_{a,b}\delta_{i,j} \text{ for all } a, b \in \{1, 2, 3\} \text{ and } i, j \in \{1, ..., N\} \tag{16}$$

$$\text{for } \widehat{\boldsymbol{x}}_i =: \begin{pmatrix} \widehat{x}_i^1 \\ \widehat{x}_i^2 \\ \widehat{x}_i^3 \end{pmatrix} \text{ and } \widehat{\boldsymbol{p}}_j =: \begin{pmatrix} \widehat{p}_j^1 \\ \widehat{p}_j^2 \\ \widehat{p}_j^3 \end{pmatrix} \text{ for all } i, j \in \{1, ..., N\} \tag{17}$$

and $\restriction \mathcal{F}_N$ denoting the restriction to $\mathcal{F}_N$, in the following way

$$H_{nuc} \restriction \mathcal{F}_N := \sum_{n=1}^{N} \frac{\widehat{\boldsymbol{p}}_n^2}{2M} \tag{18}$$

$$H_{I_\kappa} \restriction \mathcal{F}_N := \sum_{n=1}^{N} \int d^3k \; \gamma_\kappa(\boldsymbol{k}) \left(e^{i\boldsymbol{k}\widehat{\boldsymbol{x}}_n} a_{\boldsymbol{k}} + e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}_n} a_{\boldsymbol{k}}^\dagger \right) \tag{19}$$

Hence $H_\kappa \restriction \mathcal{F}_N$ then reads

$$\boxed{H_\kappa \restriction \mathcal{F}_N := \sum_{n=1}^{N} \frac{\widehat{\boldsymbol{p}}_n^2}{2M} + \int d^3k \; a_{\boldsymbol{k}}^\dagger \omega_{\boldsymbol{k}} a_{\boldsymbol{k}} + g \sum_{n=1}^{N} \int d^3k \; \gamma_\kappa(\boldsymbol{k}) \left(e^{i\boldsymbol{k}\widehat{\boldsymbol{x}}_n} a_{\boldsymbol{k}} + e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}_n} a_{\boldsymbol{k}}^\dagger \right)} \tag{20}$$

We shall indicate vectors $|\psi\rangle \in \mathcal{F}_N$ decomposed with respect to the generalized eigenfunctions of the total number operator $\mathcal{N}_{mes}^1 := \int d^3k \; a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}$ as

$$\begin{aligned} |\psi\rangle = \sum_{n=0}^{\infty} \int d^3x_1 \, ... \int d^3x_N \; \int d^3k_1 \, ... \int d^3k_n \\ \langle \boldsymbol{x}_1, ..., \boldsymbol{x}_N; \boldsymbol{k}_1, ..., \boldsymbol{k}_n | \psi \rangle \, |\boldsymbol{x}_1, ..., \boldsymbol{x}_N; \boldsymbol{k}_1, ..., \boldsymbol{k}_n\rangle \end{aligned} \tag{21}$$

where $\boldsymbol{x}_1, ..., \boldsymbol{x}_N$ are the nucleon coordinates and behind the semi-colon follow the meson coordinates $\boldsymbol{k}_1, ..., \boldsymbol{k}_n$. In this sense we refer to $\langle \boldsymbol{x}_1, ..., \boldsymbol{x}_N; \boldsymbol{k}_1, ..., \boldsymbol{k}_n | \psi \rangle$ as n-meson wave function of the N-nucleon Fock-vector $|\psi\rangle$. Furthermore Fock-vectors with a meson vacuum in $\mathcal{F}_N$ will be written like $|...; 0\rangle$.

2.3 Operator and domain definitions

In this subsection we specify the appearing operators and their domains. Hereby we shall focus on their restrictions to $\mathcal{F}_N$. A commonly used position space representation of (16) is $\widehat{\boldsymbol{x}}_i := \boldsymbol{x}_i$ and[5] $\widehat{\boldsymbol{p}}_j := \frac{\nabla_j}{i}$, where ∇_i (∇_i^2) is the gradient (Laplacian) on $\mathcal{F}_N$ in the i-th nucleon position coordinate.

$$\nabla_i := \mathbb{1}_{id}^{\mathcal{H}_{nuc}^{\otimes 1}} \otimes ... \otimes \underbrace{\nabla}_{\text{i-th pos.}} \otimes ... \underbrace{\mathbb{1}_{id}^{\mathcal{H}_{nuc}^{\otimes 1}}}_{\text{N-th pos.}} \otimes \mathcal{F}_{mes} \tag{22}$$

$$\tag{23}$$

Now for our choice $D_{\boldsymbol{x}} = \frac{\widehat{\boldsymbol{p}}_n^2}{2M} = -\frac{\nabla_n^2}{2M}$. The domain of these operators are

$$\begin{aligned} \mathcal{D}(\widehat{\boldsymbol{p}}_j) &= \{|\psi\rangle \in \mathcal{F}_N | \text{ all } \nabla_j \langle \boldsymbol{x}_1, ..., \boldsymbol{x}_j, ..., \boldsymbol{x}_N; ...|\psi\rangle \text{ are again in } \mathcal{L}_2\} && (24) \\ \mathcal{D}(\widehat{\boldsymbol{p}}_j^2) &= \left\{|\psi\rangle \in \mathcal{F}_N | \text{ all } \nabla_j^2 \langle \boldsymbol{x}_1, ..., \boldsymbol{x}_j, ..., \boldsymbol{x}_N; ...|\psi\rangle \text{ are again in } \mathcal{L}_2\right\} && (25) \end{aligned}$$

where the word *all* relates to all n-meson wave functions of the $\mathcal{F}_N$ element.

In order that the meson field fulfills the Bose statistics we impose the bosonic commutation relations for the meson field operators

$$\begin{aligned} \left[a_{\boldsymbol{k}}, a_{\boldsymbol{l}}^\dagger\right] &= \delta^3(\boldsymbol{k} - \boldsymbol{l}) && (26) \\ \left[a_{\boldsymbol{k}}^\dagger, a_{\boldsymbol{l}}^\dagger\right] &= \left[a_{\boldsymbol{k}}, a_{\boldsymbol{l}}\right] = 0 && (27) \end{aligned}$$

which formally act like the following

$$\begin{aligned} \langle ...; \boldsymbol{k}_1 ... \boldsymbol{k}_n | a_{\boldsymbol{k}} | \psi\rangle &:= \sqrt{n+1}\, \langle ...; \boldsymbol{k}_1 ... \boldsymbol{k}_n, \boldsymbol{k} | \psi\rangle && (28) \\ \langle ...; \boldsymbol{k}_1 ... \boldsymbol{k}_n | a_{\boldsymbol{k}}^\dagger | \psi\rangle &:= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \delta^3(\boldsymbol{k}_j - \boldsymbol{k})\, \langle ...; \boldsymbol{k}_1 ..., \widehat{\boldsymbol{k}}_j, ... \boldsymbol{k}_n | \psi\rangle && (29) \end{aligned}$$

where the hat on $\widehat{\boldsymbol{k}}_j$ means the $\boldsymbol{k}_j$ variable is omitted. Please note that in our notation the n-meson wave functions are now automatically symmetric in the meson coordinates $\boldsymbol{k}_1, ..., \boldsymbol{k}_n$.

We generalize the total number of mesons operator $\mathcal{N}_{mes}^1$ by $\mathcal{N}_{mes}^f := \int d^3k\ a_{\boldsymbol{k}}^\dagger f(\boldsymbol{k}) a_{\boldsymbol{k}}$ for any measurable function f, formally acting like

$$\langle ...; \boldsymbol{k}_1 ... \boldsymbol{k}_n | \mathcal{N}_{mes}^f | \psi\rangle = \sum_{i=1}^{n} f(\boldsymbol{k}_i)\, \langle ...; \boldsymbol{k}_1 ... \boldsymbol{k}_n | \psi\rangle \tag{30}$$

for all $|\psi\rangle \in \mathcal{D}(\mathcal{N}_{mes}^f) \subset \mathcal{F}_N$ with

$$\mathcal{D}(\mathcal{N}_{mes}^f) := \left\{ |\psi\rangle \in \mathcal{F}_N | \sum_{i=1}^{\infty} \int d^3k_1 \ldots \int d^3k_n\ |f(\boldsymbol{k}_i)\, \langle ...; \boldsymbol{k}_1, ..., \boldsymbol{k}_n | \psi\rangle\,|^2 < \infty \right\} \tag{31}$$

Respecting the recent operator definitions we find

$$\boxed{\mathcal{D}(H_0 \upharpoonright \mathcal{F}_N) = \bigcap_{j=1}^{N} \mathcal{D}(\boldsymbol{p}_j^2) \bigcap \mathcal{D}(\mathcal{N}_{mes}^\omega)} \tag{32}$$

[5] Please excuse the ambiguity of the index i in $\boldsymbol{x}_i$ and the imaginary number i in $\frac{\nabla_j}{i}$.

Now we turn to the integral operators, which we wanted to look at in more detail. For any measurable function f the operators $\int d^3k\ f(k)a_{\boldsymbol{k}}$ and $\int d^3k\ f(k)a^\dagger_{\boldsymbol{k}}$ formally act like

$$\begin{aligned} \langle ...;\boldsymbol{k}_1,...,\boldsymbol{k}_n| \int d^3k\ f(\boldsymbol{k})a_{\boldsymbol{k}}|\psi\rangle &= \sqrt{n+1}\int d^3k\ f(\boldsymbol{k})\,\langle ...;\boldsymbol{k}_1,...,\boldsymbol{k}_n,\boldsymbol{k}|\psi\rangle && (33)\\ \langle ...;\boldsymbol{k}_1,...,\boldsymbol{k}_n| \int d^3k\ f(\boldsymbol{k})a^\dagger_{\boldsymbol{k}}|\psi\rangle &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n} f(\boldsymbol{k}_j)\,\langle ...;\boldsymbol{k}_1...,\widehat{\boldsymbol{k}}_j,...\boldsymbol{k}_n|\psi\rangle && (34)\end{aligned}$$

That means, for any $f\in\mathcal{L}_2(\mathbb{R}^3)$ both operators are well-defined on $\mathcal{D}(\sqrt{\mathcal{N}^1_{mes}})$ by the Schwartz inequality.

$$\begin{aligned} ||\int d^3k\ f(\boldsymbol{k})a_{\boldsymbol{k}}\,|\psi\rangle\,|| &\le ||f||_{\mathcal{L}_2}\cdot||\sqrt{\mathcal{N}^1_{mes}}\,|\psi\rangle\,|| && (35)\\ ||\int d^3k\ f(\boldsymbol{k})a^\dagger_{\boldsymbol{k}}\,|\psi\rangle\,|| &\le ||f||_{\mathcal{L}_2}\cdot||\sqrt{\mathcal{N}^1_{mes}+1}\,|\psi\rangle\,|| && (36)\end{aligned}$$

If $f\notin\mathcal{L}_2(\mathbb{R}^3)$ it is easy to see that $\int d^3k\ f(\boldsymbol{k})a^\dagger_{\boldsymbol{k}}|\psi\rangle\notin\mathcal{F}_N$ regarding the finite norm condition of the Fock-space. In contrary the annihilation operator $\int d^3k\ f(\boldsymbol{k})a_{\boldsymbol{k}}$ can be defined for all $|\psi\rangle\in\mathcal{F}_N$ such that the integral on the right hand side of (33) exists. Since $\gamma_\kappa(\boldsymbol{k})$ is only in $\mathcal{L}_2(\mathbb{R}^3)$ for finite κ it immediately follows that the operator H_{I_κ} is only well-defined for finite κ with the domain

$$\mathcal{D}(H_{I_\kappa})=\mathcal{D}(\sqrt{\mathcal{N}^1_{mes}})\ \ \forall\kappa<\infty \tag{37}$$

Thus the model Hamiltonian can be defined as an operator on

$$\boxed{\mathcal{D}(H_\kappa\restriction\mathcal{F}_N)=\mathcal{D}(H_0\restriction\mathcal{F}_N)\bigcap\mathcal{D}(\sqrt{\mathcal{N}^1_{mes}})\ \ \forall\kappa<\infty} \tag{38}$$

only for finite κ. In fact we shall see from the next subsection that $H_\kappa\restriction\mathcal{F}_N$ is self-adjoint on $\mathcal{D}(H_0\restriction\mathcal{F}_N)$.

2.4 Self-adjointness of H_κ

Theorem 2.4.1. *For $\kappa<\infty$ the operator $H_\kappa\restriction\mathcal{F}_N$ is self-adjoint on $\mathcal{D}(H_0\restriction\mathcal{F}_N)$.*

Proof. H_0 being the sum of two positive commuting self-adjoint operators H_{nuc} and H_{mes} is again self-adjoint. By Kato's theorem [13] $H_\kappa\restriction\mathcal{F}_N$ is self-adjoint on $\mathcal{D}(H_0\restriction\mathcal{F}_N)$ if there exist positive constants $a<1$ and $ab<\infty$ such that for $|\psi\rangle\in\mathcal{D}(H_0\restriction\mathcal{F}_N)$

$$||gH_{I_\kappa}\,|\psi\rangle\,||\le a||H_0\,|\psi\rangle\,||+b||\,|\psi\rangle\,|| \tag{39}$$

If so H_{I_κ} is called a small perturbation of H_0 in the sense of Kato. To show this is straight forward for $\kappa<\infty$ because by Schwartz inequality for any $|\psi\rangle\in\mathcal{D}(H_0\restriction\mathcal{F}_N)$

$$\begin{aligned} ||gH_{I_\kappa}\,|\psi\rangle\,|| &= |g|\cdot||\int d^3k\ \gamma_\kappa(k)(a_{\boldsymbol{k}}e^{i\boldsymbol{k}\widehat{\boldsymbol{x}}}+a^\dagger_{\boldsymbol{k}}e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}})\,|\psi\rangle\,|| && (40)\\ &\le |g|\cdot||\gamma_\kappa||_{\mathcal{L}_2}\left(||\sqrt{\mathcal{N}^1_{mes}}\,|\psi\rangle\,||+||\sqrt{\mathcal{N}^1_{mes}+1}\,|\psi\rangle\,||\right) && (41)\\ &\le 2|g|\cdot||\gamma_\kappa||_{\mathcal{L}_2}\cdot||\sqrt{\mathcal{N}^1_{mes}+1}\,|\psi\rangle\,|| && (42)\\ &=: C_\kappa\cdot||\sqrt{\mathcal{N}^1_{mes}+1}\,|\psi\rangle\,|| && (43)\end{aligned}$$

We now follow [15]. For all $\epsilon > 0 \; \exists \; b_\epsilon < \infty$ such that

$$
\begin{aligned}
||\sqrt{\mathcal{N}^1_{mes} + 1}\, |\psi\rangle\, || &\leq \epsilon || \mathcal{N}^1_{mes} \, |\psi\rangle \, || + b_\epsilon || \, |\psi\rangle \, || && (44)\\
&= \frac{\epsilon}{\mu} || \mathcal{N}^\mu_{mes} \, |\psi\rangle \, || + b_\epsilon || \, |\psi\rangle \, || && (45)\\
&\leq \frac{\epsilon}{\mu} || \mathcal{N}^\omega_{mes} \, |\psi\rangle \, || + b_\epsilon || \, |\psi\rangle \, || && (46)\\
&= \frac{\epsilon}{\mu} || H_{mes} \, |\psi\rangle \, || + b_\epsilon || \, |\psi\rangle \, || && (47)\\
&\leq \frac{\epsilon}{\mu} || H_0 \, |\psi\rangle \, || + b_\epsilon || \, |\psi\rangle \, || && (48)
\end{aligned}
$$

because $\mu \in \mathbb{R}_+$ and $\omega_{\boldsymbol{k}} \geq \mu \; \forall \; \boldsymbol{k} \in \mathbb{R}^3$. Choosing $\epsilon < \mu / C_\kappa$ we yield $a = C_\kappa \epsilon / \mu < 1, b = b_\epsilon < \infty$ and (39) holds. □

2.5 Translational invariance of H_κ

Theorem 2.5.1. *For $\kappa < \infty$ the operator H_κ on $\mathcal{D}(H_\kappa)$ is translation invariant, i.e. it conserves the total momentum.*

Proof. In order to show translation invariance we use the common $\mathbb{R}^3$-valued total momentum operator $\mathcal{P}$ on $\mathcal{F}$

$$\boxed{\mathcal{P} := \int d^3k \, \left(\psi^\dagger_{\boldsymbol{k}} \boldsymbol{k} \psi_{\boldsymbol{k}} + a^\dagger_{\boldsymbol{k}} \boldsymbol{k} a_{\boldsymbol{k}} \right)} \tag{49}$$

and calculate $e^{i\mathcal{P}\boldsymbol{d}} H_\kappa e^{-i\mathcal{P}\boldsymbol{d}}$ for any $\boldsymbol{d} \in \mathbb{R}^3$. First we rewrite H_κ in a more convenient way.

$$
\begin{aligned}
H_\kappa &= \int d^3k \; \psi^\dagger_{\boldsymbol{k}} \frac{\boldsymbol{k}^2}{2M} \psi_{\boldsymbol{k}} + \int d^3k \; a^\dagger_{\boldsymbol{k}} \omega_{\boldsymbol{k}} a_{\boldsymbol{k}} + && (50)\\
&+ \int d^3q \int d^3k \; \gamma_\kappa(\boldsymbol{k}) \left(\psi^\dagger_{\boldsymbol{q}+\boldsymbol{k}} a_{\boldsymbol{k}} + \psi^\dagger_{\boldsymbol{q}-\boldsymbol{k}} a^\dagger_{\boldsymbol{k}} \right) \psi_{\boldsymbol{q}} && (51)
\end{aligned}
$$

Now we calculate the action of $\mathcal{P}$ on each operator appearing in the terms of H_κ. Since $e^{i\mathcal{P}\boldsymbol{d}}$ forms an unitary group we can expand $e^{i\mathcal{P}\boldsymbol{d}} T e^{-i\mathcal{P}\boldsymbol{d}}$ in a Taylor series around $d = 0$ for any operator T defined on an appropriate domain and yield

$$e^{i\mathcal{P}\boldsymbol{d}} T e^{-i\mathcal{P}\boldsymbol{d}} = T + \frac{i\boldsymbol{d}}{1!} [\mathcal{P}, T] + \frac{(i\boldsymbol{d})^2}{2!} [\mathcal{P}, [\mathcal{P}, T]] + ... \tag{52}$$

For $T := \psi^\dagger_{\boldsymbol{p}}$ we get

$$
\begin{aligned}
[\mathcal{P}, \psi^\dagger_{\boldsymbol{p}}] &= [\int d^3k \; \psi^\dagger_{\boldsymbol{k}} \boldsymbol{k} \psi_{\boldsymbol{k}} + \int d^3k \; a^\dagger_{\boldsymbol{k}} \boldsymbol{k} a_{\boldsymbol{k}}, \psi^\dagger_{\boldsymbol{p}}] && (53)\\
&= [\int d^3k \; \psi^\dagger_{\boldsymbol{k}} \boldsymbol{k} \psi_{\boldsymbol{k}}, \psi^\dagger_{\boldsymbol{p}}] && (54)\\
&= \int d^3k \; \psi^\dagger_{\boldsymbol{k}} \boldsymbol{k} [\psi_{\boldsymbol{k}}, \psi^\dagger_{\boldsymbol{p}}] && (55)\\
&= \int d^3k \; \psi^\dagger_{\boldsymbol{k}} \boldsymbol{k} \delta^3(\boldsymbol{k} - \boldsymbol{p}) = \boldsymbol{p} \psi^\dagger_{\boldsymbol{p}} && (56)
\end{aligned}
$$

hence

$$
\begin{aligned}
[\mathcal{P}, [\mathcal{P}, \psi^\dagger_{\boldsymbol{p}}]] &= \boldsymbol{p}^2 \psi^\dagger_{\boldsymbol{p}} && (57)\\
&\vdots && (58)\\
[\mathcal{P}, [\mathcal{P}, ... [\mathcal{P}, \psi^\dagger_{\boldsymbol{p}} \underbrace{] ...]]}_{n \text{ parentheses}} &= \boldsymbol{p}^n \psi^\dagger_{\boldsymbol{p}} && (59)
\end{aligned}
$$

and

$$e^{i\mathcal{P}\boldsymbol{d}}\psi^\dagger_{\boldsymbol{p}}e^{-i\mathcal{P}\boldsymbol{d}} = T + \frac{i\boldsymbol{d}}{1!}\big[\mathcal{P},T\big] + \frac{(i\boldsymbol{d})^2}{2!}\big[\mathcal{P},\big[\mathcal{P},T\big]\big] + ... = \psi^\dagger_{\boldsymbol{p}}e^{i\boldsymbol{p}\boldsymbol{d}} \tag{60}$$

similarly

$$e^{i\mathcal{P}\boldsymbol{d}}\psi_{\boldsymbol{p}}e^{-i\mathcal{P}\boldsymbol{d}} = \psi_{\boldsymbol{p}}e^{-i\boldsymbol{p}\boldsymbol{d}} \tag{61}$$
$$e^{i\mathcal{P}\boldsymbol{d}}a^\dagger_{\boldsymbol{k}}e^{-i\mathcal{P}\boldsymbol{d}} = a^\dagger_{\boldsymbol{k}}e^{i\boldsymbol{k}\boldsymbol{d}} \tag{62}$$
$$e^{i\mathcal{P}\boldsymbol{d}}a_{\boldsymbol{k}}e^{-i\mathcal{P}\boldsymbol{d}} = a_{\boldsymbol{k}}e^{-i\boldsymbol{k}\boldsymbol{d}} \tag{63}$$

We plug these term into $e^{i\mathcal{P}\boldsymbol{d}}H_\kappa e^{-i\mathcal{P}\boldsymbol{d}}$ and get

$$e^{i\mathcal{P}\boldsymbol{d}}H_\kappa e^{-i\mathcal{P}\boldsymbol{d}} = H_\kappa \tag{64}$$

This is the statement of the translational invariance that goes hand in hand with the conservation of total momentum since

$$e^{i\mathcal{P}\boldsymbol{d}}H_\kappa e^{-i\mathcal{P}\boldsymbol{d}} = H_\kappa + \frac{i\boldsymbol{d}}{1!}\big[\mathcal{P},H_\kappa\big] + \frac{(i\boldsymbol{d})^2}{2!}\big[\mathcal{P},\big[\mathcal{P},H_\kappa\big]\big] + ... = H_\kappa \tag{65}$$

for any $\boldsymbol{d} \in \mathbb{R}^3$ and thus $\big[\mathcal{P},H_\kappa\big] = 0$. □

2.6 Removing the ultraviolet cutoff in $\mathcal{F}_N$

Remark 2.6.1. *In the future we will restrict all our arguments to $\mathcal{F}_N$ if not noted otherwise. Hence all appearing operators are the ones restricted to $\mathcal{F}_N$ and naturally all domains are subsets of $\mathcal{F}_N$.*

We have seen that the Hamiltonian is not well-defined for $\kappa \to \infty$ since $\gamma_\infty(\boldsymbol{k}) \notin \mathcal{L}_2(\mathbb{R}^3)$. As long as $\mu > 0$ the field amplitude γ_κ is non-singular and we only need to introduce this so-called ultraviolet cutoff κ in order to obtain a well-defined model Hamiltonian H_κ. We shall refer to it as the *cutoff*. In the case of $\mu = 0$ one had also to fix the integral operators over $\gamma_\kappa(\boldsymbol{k})d^3k$ at small wave numbers $\boldsymbol{k}$ in order to have the existence of a ground state[6]. A cutoff doing that is called infrared cutoff. In position space this infrared cutoff would have the effect to cut the integrals off for big $|\boldsymbol{x}|$. One could argue that the finiteness of the universe imposes a natural infrared cutoff on our model but on the other hand for a theory on quantum scales the extend of the universe is almost infinite and on a mathematical grounding it remains an interesting question to remove the infrared cutoff as well. However in this work we shall always assume $\mu \in \mathbb{R}_+$. By a theorem of Nelson [15] we can give the Hamiltonian $H_\kappa \restriction \mathcal{F}_N$ for $\kappa \to \infty$ a mathematical meaning in the following sense.

Theorem 2.6.1 (Nelson's theorem)**.** *There exists a unique self-adjoint operator $\widehat{H}$ on $\mathcal{F}_N$ and a family of real constants R_κ*

$$\boxed{R_\kappa := -g^2\int d^3k\ \frac{\gamma_\kappa^2(\boldsymbol{k})}{\frac{\boldsymbol{k}^2}{2M}+\omega_{\boldsymbol{k}}}} \tag{66}$$

such that, for all $t \in \mathbb{R}$ and $|\psi\rangle \in \mathcal{F}_N$

$$\lim_{\kappa\to\infty} e^{-it(H_\kappa - NR_\kappa)}|\psi\rangle = e^{-it\widehat{H}}|\psi\rangle \tag{67}$$

The operator $\widehat{H}$ is bounded below.

Proof. We shall sketch the proof in [15] below. □

[6] Physically speaking the time evolution is such that its ground states carry a cloud of infinite many mesons with wave numbers close to zero without violating the energy conservation. To see this e.g. in the static Nelson Model simply compute the average of the total number of mesons for the groundstate (362). It diverges logarithmically.

A sketch of Nelson's proof In order to understand the construction of $\widehat{H}$ in the following we shall go through Nelson's proof in brief. The separation of the R_κ term is done via the Gross transformation. This is an unitary transformation implemented by $e^{\pm T_\kappa}$ whose generator is given by

$$T_\kappa := \sum_{n=1}^{N} \int d^3k\; \beta_\kappa(\boldsymbol{k}) \left(a_{\boldsymbol{k}} e^{i\boldsymbol{k}\widehat{\boldsymbol{x}}_n} - a^\dagger_{\boldsymbol{k}} e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}_n} \right) \tag{68}$$

$$\beta_\kappa(\boldsymbol{k}) := -g \frac{\gamma_\kappa(\boldsymbol{k})}{\frac{\boldsymbol{k}^2}{2M} + \omega(\boldsymbol{k})} (1 - \chi_K(\boldsymbol{k})) \tag{69}$$

The factor of $(1 - \chi_K(\boldsymbol{k}))$ in $\beta_\kappa(\boldsymbol{k})$ with $\chi_K(\boldsymbol{k})$ being the characteristic function - identity for $|\boldsymbol{k}| < K$ and zero for $|\boldsymbol{k}| > K$ - is a technical detail that allows a convenient way to show semi-boundness and closure properties of the quadratic form (91). Since $\beta_\kappa(\boldsymbol{k}) \in \mathcal{L}_2(\mathbb{R}^3)$ $\forall \kappa \leq \infty$ it is easy to show that T_κ are skew-adjoint operators on $\mathcal{D}(\sqrt{\mathcal{N}^1_{mes}})$ for all $\kappa \leq \infty$ and we define T_κ to be the closure on $\mathcal{F}_N$ of them respectively. Hence $e^{\pm T_\kappa}$ implements an unitary transformation on $\mathcal{F}_N$ for all $\kappa \leq \infty$. The action of this transformation on $\widehat{\boldsymbol{p}}_m$, $a^\dagger_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}$ is[7]

$$e^{T_\kappa} \widehat{\boldsymbol{p}}_m e^{-T_\kappa} = \widehat{\boldsymbol{p}}_m - \boldsymbol{A}_{m,\kappa} - \boldsymbol{A}^*_{m,\kappa} \tag{70}$$

$$e^{T_\kappa} a^\dagger_{\boldsymbol{k}} e^{-T_\kappa} = a^\dagger_{\boldsymbol{k}} + \sum_{m=1}^{N} \beta_\kappa(\boldsymbol{k}) e^{i\boldsymbol{k}\widehat{\boldsymbol{x}}_m} \tag{71}$$

$$e^{T_\kappa} a_{\boldsymbol{k}} e^{-T_\kappa} = a_{\boldsymbol{k}} + \sum_{m=1}^{N} \beta_\kappa(\boldsymbol{k}) e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}_m} \tag{72}$$

$$\tag{73}$$

where

$$\boldsymbol{A}_{m,\kappa} := \int d^3k\; \boldsymbol{k} \beta_\kappa(\boldsymbol{k}) a_{\boldsymbol{k}} e^{i\boldsymbol{k}\widehat{\boldsymbol{x}}_m} \tag{74}$$

$$\boldsymbol{A}^*_{m,\kappa} := \int d^3k\; \boldsymbol{k} \beta_\kappa(\boldsymbol{k}) a^\dagger_{\boldsymbol{k}} e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}_m} \tag{75}$$

and only for finite κ. That can be seen by the following computation. The operators $e^{\pm T_\kappa}$ are unitary for finite κ. Hence $e^{T_\kappa d}$ for $d \in \mathbb{R}$ forms a unitary one parameter group, which we may expand in a Taylor series around $d = 0$. So for any operator L with an appropriate domain we find

$$e^{T_\kappa} L e^{-T_\kappa} = e^{T_\kappa \boldsymbol{d}} L e^{-T_\kappa \boldsymbol{d}} \Big|_{d=1} = L + \frac{\boldsymbol{d}}{1!}[\mathcal{P}, L] + \frac{\boldsymbol{d}^2}{2!}[\mathcal{P}, [\mathcal{P}, L]] + ... \Big|_{d=1} \tag{76}$$

$$= L + \frac{1}{1!}[\mathcal{P}, L] + \frac{1}{2!}[\mathcal{P}, [\mathcal{P}, L]] + ... \tag{77}$$

[7] It seems that for some reason Nelson used $\boldsymbol{p}_m = i\nabla_m$ instead of the convention $\boldsymbol{p}_m = \frac{\nabla_m}{i}$, which we shall use throughout. That is why formulas (70) and (86) have a different sign compared to the one Nelson used in [15]. That of course is not essential to Nelson's proof.

The commutators for $L := \widehat{\boldsymbol{p}}_m$, $a^\dagger_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}$ are

$$\begin{aligned}
\left[T_\kappa, \widehat{\boldsymbol{p}}_m\right] &= -A_{m,\kappa} - A^*_{m,\kappa} && (78)\\
\left[T_\kappa, \left[T_\kappa, ...\left[T_\kappa, \widehat{\boldsymbol{p}}_m\right]...\right]\right] &= 0 && (79)\\
\left[T_\kappa, a^\dagger_{\boldsymbol{k}}\right] &= \sum_{m=1}^N \beta_\kappa(\boldsymbol{k}) e^{i\boldsymbol{k}\widehat{\boldsymbol{x}}_m} && (80)\\
\left[T_\kappa, \left[T_\kappa, ...\left[T_\kappa, a^\dagger_{\boldsymbol{k}}\right]...\right]\right] &= 0 && (81)\\
\left[T_\kappa, a_{\boldsymbol{k}}\right] &= \sum_{m=1}^N \beta_\kappa(\boldsymbol{k}) e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}_m} && (82)\\
\left[T_\kappa, \left[T_\kappa, ...\left[T_\kappa, a_{\boldsymbol{k}}\right]...\right]\right] &= 0 && (83)
\end{aligned}$$

Using this we compute the transformed Hamiltonian

$$\begin{aligned}
e^{T_\kappa} H_\kappa e^{-T_\kappa} &= H_{nuc} + H_{mes} + && (84)\\
&\quad + \frac{1}{2M}\sum_{m=1}^N \big(\boldsymbol{A}^2_{m,\kappa} + \boldsymbol{A}^{*2}_{m,\kappa} + 2\boldsymbol{A}^*_{m,\kappa}\boldsymbol{A}_{m,\kappa} + && (85)\\
&\qquad -2(\boldsymbol{p}_m \boldsymbol{A}_{m,\kappa} + \boldsymbol{A}^*_{m,\kappa}\boldsymbol{p}_m)\big) + && (86)\\
&\quad + \sum_{m\neq l}^N \int d^3k\ V_\kappa e^{-i\boldsymbol{k}(\widehat{\boldsymbol{x}}_m - \widehat{\boldsymbol{x}}_l)} + && (87)\\
&\quad + H_{IK} + N(R_\kappa - R_K) && (88)\\
=: &\quad H'_\kappa + N(R_\kappa - R_K) && (89)
\end{aligned}$$

where the potential $V_\kappa(\boldsymbol{k})$ is given by

$$V_\kappa(\boldsymbol{k}) := \quad -g\tfrac{\gamma^2_\kappa(\boldsymbol{k})}{\omega_k}\left[2\left(\tfrac{\omega_k}{D_k+\omega_k}\right) - \left(\tfrac{\omega_k}{D_k+\omega_k}\right)^2\right] \qquad (90)$$

We will discuss the nature of this potential in detail for the case of infinitely heavy nucleons in subsections 3.3 and 4.4. Nelson now showed that (89) holds as an operator equation on the set $\mathcal{D}(H_0) = \mathcal{D}(H_\kappa) = \mathcal{D}(H'_\kappa)$ but since $\boldsymbol{k}\beta_\infty(\boldsymbol{k}) \notin \mathcal{L}_2(\mathbb{R}^3)$, which appears in both integral operators $A_{m,\infty}$ and $A^*_{m,\infty}$ defined in line (74) and (75), the Hamiltonian H'_∞ has essentially the same problem like H_∞ and so also is not well-defined on $\mathcal{F}_N$. Now since we extracted the for $\kappa \to \infty$ divergent term R_κ the operator H'_κ behaves a bit better than H_κ in the following sense. For finite κ the self-adjoint operator H'_κ induces a quadratic form $q_{H'_\kappa}(|\varphi\rangle, |\phi\rangle)$ on the form domain $\mathcal{Q}(H_0) = \mathcal{D}(\sqrt{H_0})$ with the properties

1. $\langle\varphi|H'_\kappa|\phi\rangle = q_{H'_\kappa}(|\varphi\rangle, |\phi\rangle) := \langle\varphi|\overleftarrow{H_0^{1/2}}\overrightarrow{H_0^{1/2}}|\,|\phi\rangle\rangle + B_\kappa(|\varphi\rangle, |\phi\rangle)$ for all $|\varphi\rangle, |\phi\rangle \in \mathcal{D}(H_0)$ and $\kappa < \infty$

$$\begin{aligned}
B_\kappa(|\psi\rangle, |\psi\rangle) \quad := \quad & \frac{1}{M}\mathfrak{Re}\sum_{m=1}^N \Bigg(\langle\psi|\overleftarrow{(\mathcal{N}^1_{mes}+1)^{1/2}}\overrightarrow{(\mathcal{N}^1_{mes}+1)^{-1/2}\boldsymbol{A}^2_{m,\kappa}}|\psi\rangle + && (91)\\
& + \langle\psi|\overleftarrow{\boldsymbol{A}_{m,\kappa}}\overrightarrow{\boldsymbol{A}_{m,\kappa}}|\psi\rangle + \langle\psi|\overleftarrow{\boldsymbol{p}_m}\overrightarrow{\boldsymbol{A}_{m,\kappa}}|\psi\rangle \Bigg) + && (92)\\
& + \langle\psi|\sum_{m\neq l}^N \int d^3k\ V_\kappa e^{-i\boldsymbol{k}(\boldsymbol{x}_m - \boldsymbol{x}_l)}|\psi\rangle + && (93)\\
& + \langle\psi|\overrightarrow{H_{IK}}|\psi\rangle && (94)
\end{aligned}$$

2. $B_\kappa(|\psi\rangle, |\psi\rangle$ is well-defined for all $\kappa \le \infty$ and $|\psi\rangle \in \mathcal{D}(\sqrt{H_0})$.

3. There exist $\epsilon > 0$, $aK < \infty$ and $ab < \infty$ such that
$|B_\kappa(|\psi\rangle, |\psi\rangle)| \le \epsilon \langle\psi|\overleftarrow{H_0^{1/2}}\overrightarrow{H_0^{1/2}}|\psi\rangle + b\langle\psi|\psi\rangle$ for all $\kappa \le \infty$.
(this was the technical detail with K mentioned above)

4. $\lim_{\kappa\to\infty} B_\kappa(|\psi\rangle, |\psi\rangle) = B_\infty(|\psi\rangle, |\psi\rangle)$
uniformly on any set of $|\psi\rangle$ in $\mathcal{D}(\sqrt{H_0})$ for which $||H_0^{1/2}|\psi\rangle|| + |||\psi\rangle||$ is bounded.

From functional analysis [15, Appendix] - or [17, VIII.6] for a more modern approach - we infer that for each quadratic form $q_{H'_\kappa}$ with the above properties (2),(3) and (4) there exists a self-adjoint operator for all $\kappa \le \infty$, which we shall call H'_κ on the domain $\mathcal{D}(H'_\kappa) \subset \mathcal{D}(\sqrt{H_0})$. For $\kappa < \infty$ we know these operators already by the construction (1.) with the associated domain $\mathcal{D}(H'_\kappa) = \mathcal{D}(H_0)$. If we perform the limit $\kappa \to \infty$ we yield a new self-adjoint operator H'_∞ with an unknown domain $\mathcal{D}(H'_\infty) \subset \mathcal{D}(\sqrt{H_0})$ whose matrix elements are given by (1.)

$$\begin{aligned}\langle\varphi|H'_\infty|\phi\rangle &= q_{H'_\infty}(|\varphi\rangle, |\psi\rangle) &(95)\\ &= \langle\varphi|\overleftarrow{H_0^{1/2}}\overrightarrow{H_0^{1/2}}|\phi\rangle + \lim_{\kappa\to\infty} B_\kappa(|\varphi\rangle, |\phi\rangle) &(96)\end{aligned}$$

for all $|\varphi\rangle, |\phi\rangle \in \mathcal{D}(H'_\infty)$. So we have a family of self-adjoint operators H'_κ for all $\kappa \le \infty$ and hence by the unitarity of the transformation

$$e^{-it(H_\kappa - NR_\kappa)} = e^{-T_\kappa} e^{-it(H'_\kappa - NR_K)} e^{T_\kappa} \tag{97}$$

can be shown to converge strongly as $\kappa \to \infty$. Like in [15] We shall indicate the closure of $e^{-T_\infty}(H'_\infty - NR_K)e^{T_\infty}$ on $\mathcal{F}_N$ with the symbol $\widehat{H}$.

2.7 Why is the existence of $\widehat{H}$ not enough?

Up to here we know that there exists a self-adjoint operator $\widehat{H}$ on $\mathcal{F}_N$ with the following properties

1. $\lim_{\kappa\to\infty} \langle\varphi|(H_\kappa - NR_\kappa)|\phi\rangle = \langle\varphi|\widehat{H}|\phi\rangle$ for all $|\phi\rangle, |\varphi\rangle \in \mathcal{Q}(\widehat{H}) = \mathcal{Q}(H_0) = \mathcal{D}(\sqrt{H_0})$

2. $\mathcal{D}(\widehat{H}) \subset \mathcal{D}(\sqrt{H_0})$

3. $s - \lim_{\kappa\to\infty} e^{-it(H_\kappa - NR_\kappa)} = e^{-it\widehat{H}}$

So to say even though we have no explicit expression for the self-adjoint operator $\widehat{H}$ we know all its matrix elements. Unfortunately that does not get us far if we want to learn about how $\widehat{H}$ acts as an operator on Fock-vectors. The reason why we can not infer information about the action of $\widehat{H}$ from its quadratic form $\langle\cdot|\widehat{H}|\cdot\rangle$ is because the quadratic form could only be defined with all creation operators acting to the left as annihilation operators. Recall that annihilation operators are slightly better behaved than creation operators, which we have discussed in subsection 2.3. Here is a simple example that shows how the operator associated to the quadratic form of the Laplacian can simply be read from the expression of the quadratic form and why this in general can not similarly be done for the case of smeared out field operators.

Example 2.7.1. *Let $\mathcal{H}$ be a Hilbert-space and $q_{-\triangle}(\cdot,\cdot)$ the quadratic form of the Laplacian $(-\triangle)$*

$$q_{-\triangle}(\varphi, \phi) := \langle\varphi|\overleftarrow{\partial}\overrightarrow{\partial}|\phi\rangle = \int d^3x\ (\partial_{\boldsymbol{x}}\langle\varphi|\boldsymbol{x}\rangle)(\partial_{\boldsymbol{x}}\langle\boldsymbol{x}|\phi\rangle) \tag{98}$$

with the form domain $\mathcal{Q}(q_{-\triangle}) = \{|\varphi\rangle \in \mathcal{H}|\ \langle\boldsymbol{x}|\varphi\rangle \in \mathcal{C}^1_c(\mathbb{R}^3, \mathbb{C})\}$, where $\mathcal{C}^1_c$ is the space of one time differentiable functions $\mathbb{R}^3 \to \mathbb{C}$ with compact support. If we would only know the quadratic *form then we could easily take a first guess about what the corresponding operator looks like by dragging one ∂ operator to the other side. Hence on the smaller operator domain $\mathcal{D}(-\triangle) = \{|\varphi\rangle \in \mathcal{H}|\, \langle \boldsymbol{x}|\varphi\rangle \in \mathcal{C}_c^2\}$*

$$q_{-\triangle}(\varphi,\phi) \quad := \quad -\int d^3x\ \langle\varphi|\boldsymbol{x}\rangle\,(\triangle_{\boldsymbol{x}}\,\langle\boldsymbol{x}|\phi\rangle) = \langle\varphi|\cdot(-\triangle\,|\phi\rangle) \tag{99}$$

$$= \quad -\int d^3x\ (\triangle_{\boldsymbol{x}}\,\langle\varphi|\boldsymbol{x}\rangle)\,\langle\boldsymbol{x}|\phi\rangle = (-\,\langle\varphi|\,\overleftarrow{\triangle})\cdot|\phi\rangle \tag{100}$$

and we can associate the operator $(-\triangle)$ on the domain $\mathcal{D}(-\triangle) \subset \mathcal{Q}(q_{-\triangle})$ to the quadratic form $q_{-\triangle}$ on $\mathcal{Q}(q_{-\triangle})$. As already discussed below line (33) the situation changes for creation and annihilation operators in quantum field theories. Let f be a measurable function not (!) in $\mathcal{L}_2(\mathbb{R}^3)$ like $\gamma_\infty(\boldsymbol{k})$ - or $\boldsymbol{k}\beta_\infty(\boldsymbol{k})$ as it is the case for $q_{H'_\infty}$ - and $\mathcal{F}$ the Fock-space spanned by $a^\dagger_{\boldsymbol{k}}, a_{\boldsymbol{k}}$ for $\boldsymbol{k} \in \mathbb{R}^3$ then the quadratic form

$$q_f(\varphi,\phi) \quad := \quad \int d^3k \int d^3l\ f(\boldsymbol{k})f(\boldsymbol{l})\,\langle\varphi|\overleftarrow{a^\dagger_{\boldsymbol{k}}}a_{\boldsymbol{l}}|\phi\rangle \tag{101}$$

$$= \quad \sum_{n=0}^{\infty}\int d^3k_1 \ldots \int d^3k_n \int d^3k \int d^3l\ \times \tag{102}$$

$$\times f(\boldsymbol{k})f(\boldsymbol{l})\,\langle\varphi|\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n,\boldsymbol{k}\rangle\,\langle\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n,\boldsymbol{l}|\phi\rangle \tag{103}$$

is by Schwartz inequality well-defined on

$$\mathcal{Q}(q_f) = \{|\varphi\rangle \in \mathcal{F} \text{ such that } \int d^3k\ f(\boldsymbol{k})\,\langle\varphi|\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n,\boldsymbol{k}\rangle \in \mathcal{L}_2(\mathbb{R}^{3n})\} \tag{104}$$

which can be shown to be dense. Unfortunately we cannot drag the $a^\dagger_{\boldsymbol{k}}$ operator to the right because for our choice of f we have $\mathcal{D}(\int d^3k\ f(\boldsymbol{k})a^\dagger_{\boldsymbol{k}}) = \{0\}$ only and so $\int d^3k\ f(\boldsymbol{k})a^\dagger_{\boldsymbol{k}}$ has no meaning as an operator on the Fock-space. On the other hand e.g. in the case of the quadratic form (91) it can be shown that despite this problem there indeed exists an operator, the one we called H'_∞, which can still be associated to its quadratic form.

Hence we need to get additional information about $\widehat{H}$ in order to learn about its action on Fock-vectors. We will argue in subsection 4.3 that the information missing is the unknown domain $\mathcal{D}(\widehat{H})$.

2.8 Fundamental questions

In the preceding subsections we have come across the typical problem of any relativistic interaction in field theory, which is that the model Hamiltonian fails to exist. At this stage naturally four questions arise

1. Why are the model Hamiltonians in contemporary field theory written down in the way they are?

2. Why do these model Hamiltonians in general fail to exist?

3. Since they are generally only well-defined with a cutoff κ, why is it necessary to consider the removal of the cutoff, i.e. $\kappa \to \infty$?

4. And if it is necessary, how shall we in general give a mathematical sense to these model Hamiltonians in the limit $\kappa \to \infty$?

To give an answer to the first three questions we will step back and reconsider the classical field theory in the proceeding section 3 before we turn to its quantum version. There we motivate the introduction of the field as a mathematical tool for describing relativistic interactions, motivate the model Hamiltonians, discuss the meaning of the cutoff, why it is desirable to remove this cutoff and which mathematical difficulties arise by doing it. With the classical field theory as a basis we proceed with the motivation of its quantum version in subsection 4.1 and 4.2. By comparison of the classical and quantum static Nelson model we shall explicitly see that beside new problems essentially the same mathematical difficulties found in the classical field theory reappear in the quantum version when the cutoff is removed. The main intention of this work however is to answer the question number four. How can we give these in the limit $\kappa \to \infty$ generally ill-defined model Hamiltonians a mathematical sense? We have already seen that this can be done in the dynamic quantum Nelson model by the renormalization concept that Nelson applied in his theorem 2.6.1. However, recalling subsections 2.6 and 2.7, we are left only with the abstract existence of a self-adjoint operator $\widehat{H}$ as expressed by (67) that is supposed to be the renormalized model Hamiltonian. In order to work with the renormalized model we obviously need to clarify what is meant by $\widehat{H}$, i.e. we need to get an explicit expression for $\widehat{H}$. Hence for the dynamic quantum Nelson model case the fourth question branch up into

1. What does $\widehat{H}$ look like?[8]

2. What does $\mathcal{D}(\widehat{H})$ look like?

3. What does the action of $\widehat{H}$ on elements in $\mathcal{D}(\widehat{H})$ look like?

These questions are investigated in the section 4. We come up with a good-natured toy model in subsection 4.3 that in some sense is a close relative to the Nelson model. To this toy model Nelson's renormalization concept can be applied and all three questions are comprehensively answered. This subsection is essential for an understanding of the computations done for the static and dynamic quantum case in the rest of section 4. Especially in the dynamic quantum Nelson model the answers can not be given as easily anymore since the computations get more complicated. Without the comparison to what we have done in the toy model steps necessary for answering the above questions in the dynamic quantum Nelson model may at a first glance be hidden from the readers' eyes.

Much work has already been done on both the massless ($\mu = 0$) and the massive ($\mu \in \mathbb{R}_+$) Nelson model. So the following references are far from being complete but give a good starting point when investigating literature about the Nelson model. In [7], among many other properties of the spectrum and other persistent models, the existence of what is commonly called a dressed state has been proven without imposing bounds on the coupling constant. In [12] the ground state properties in the massless case have been investigated via the method of functional integration. In [4] the ground state of a two nucleon massless Nelson model was proven to exist and the ground state and binding energy have been investigated. In [1] non-Fock representations of the ground state for the massless case were established. In [16] an iterative construction of the generalized eigenvectors and scattering states are presented for the dynamic quantum Nelson model with ultraviolet but without infrared cutoff.

[8] This question was already remarked by Nelson in the end of [15]: "[...] It would be interesting to have a direct description of the operator $\widehat{H}$. [...]"

3 Classical field theory

3.1 Why fields?

The question why one would want to deal with fields should be answered at first in order to give a motivation for all the mathematical work that lies in front of us. Until now this will only be partly possible since we have not even defined what we mean with the word field. We will try to give a more complete answer when introducing interactions of the field with nucleons but until then we have to be satisfied with the following explanation. One way to introduce interactions in a model about a fixed number of N nucleons is to insert a pair potential $V := V(\boldsymbol{x}, \boldsymbol{x}')$ into the Hamiltonian. This has often been successfully done but has one problem, the interaction is a priori instantaneous - or commonly called non-local. That means if one of the N particles moves the information about its position change will instantaneously be spread out through the pair potential term in the Hamiltonian and reach all other $(N-1)$ nucleons which then react accordingly. Since most yet observed interactions have a time-like retarded or advanced[9] character, commonly called local or relativistic, and an appropriate model has to take this behavior into account. As we shall see, fields obeying field equations like the Klein-Gordon one have properties that make them interesting mathematical tools for writing down such local interactions. So let us for now assume that fields are the way to go and postpone a more complete answer to 3.2.2.

3.2 The classical Klein-Gordon field

At first we shall briefly go through a classical motivation for the Klein-Gordon field, which was also one of the starting points for the early quantum field theories [21]. We shall start with a N particle system of harmonically and pairwise coupled oscillators and in the end perform a continuous limit to an infinitely many particle system, which shall be called a field. This Klein-Gordon field shall explanatorily serve as a prototype field for our considerations about field interactions. In the one dimensional or scalar case it is said to model the nucleon interaction (Yukawa theory or Pion model) and in the four dimensional case the electromagnetic interaction. Even in the classical field theory, as we are already used to from electromagnetism, we have to face generic divergences as soon as we treat the particles as points. We briefly step through these standard results and show some ways how one can make sense out of the in the point particle limit ill-defined equations of motion.

3.2.1 The free field

As said before we consider a classical N particle system of pairwise, harmonically coupled oscillators in one dimension. The harmonic coupling will be characterized by a Hook's spring constant, here called Ω^2. In addition to the pairwise coupling we introduce a second coupling of each particle to its equilibrium position, which shall also be harmonically and characterized by another Hook's spring constant Ω_0^2. We shall call the distance between two neighboring equilibrium positions the lattice constant a and the distance between the n-th particle and its equilibrium position the relative elongation q_n - see figure 1.

In order to preserve the finiteness of the system we can impose a periodic boundary condition, e.g. $q_i := q_{(i \mod n)}$ for all $i \in \mathbb{N}$, where mod i denotes the modulus function applied to the natural number i. Newton's force law for the n-th particle with a mass set to one is then given by

$$\ddot{q}_n = \dot{p}_n = \Omega^2 \left(q_{n+1} + q_{n-1} - 2q_n\right) + \Omega_0^2 q_n \tag{105}$$

[9] Usually only a retarded character but this strongly depends on the point of view because e.g. in the case of the electromagnetic field the Maxwell equations give no reason to prefer the retarded solution over the advanced one. Indeed there are theories like the Feynman-Wheeler action-at-distance theory which treat both solutions on equal footing.

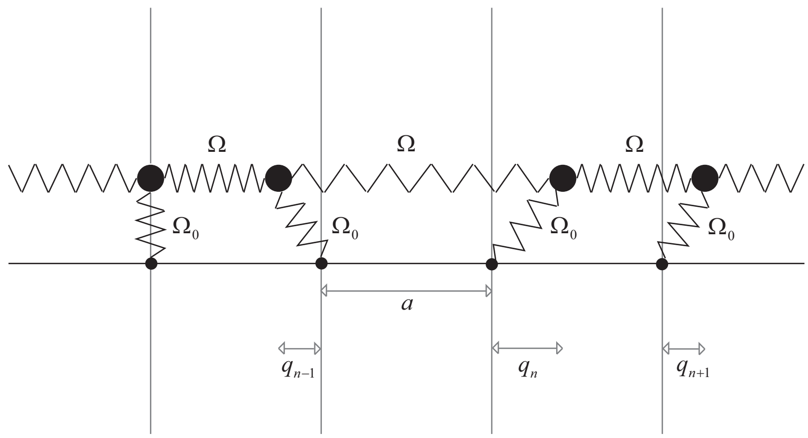

Figure 1: A string of harmonic oscillators. The springs which couple each oscillator to its equilibrium position are just symbolically drawn along a second dimension. In the case we consider all couplings lie in only one dimension.

where the dots denote partial derivatives with respect to the time t. So the whole N particle system is completely described by the first N coupled differential equations. Analogously in the Lagrange (and Hamilton) formalism the Lagrangian (and the Hamiltonian) read

$$L = \frac{1}{2}\sum_{n=1}^{N}\left[\dot{q}_n^2 - \Omega^2(q_n - q_{n+1})^2 - \Omega_0 q_n\right] \tag{106}$$

$$H = \frac{1}{2}\sum_{n=1}^{N}\left[p_n^2 + \Omega^2(q_n - q_{n+1})^2 + \Omega_0 q_n\right] \tag{107}$$

where by Legendre-transformation $p_n = \dot{q}_n$ is the canonical conjugate momentum. Although symmetry properties like the Lorentz-invariance are hidden in the Hamiltonian formulation we shall mainly use it instead of the Lagrangian formulation, which will at a later point ease the step to a quantum version. A common way to solve this system of coupled differential equations with respect to q_n and p_n is to perform a canonical transformation into Fourier space, where these equations decouple, solve them algebraically and transform the so found solution back to position space. Since we are only interested in a general understanding of the classical motivations for the Klein-Gordon field we will not go into that any deeper. As it stands the model described here is a microscopical model of a harmonic string and we are more interested to perform a change of scale to a macroscopic point of view where the string is continuous and single particles can no longer be resolved separately. This change of scale can be achieved via the following limit process

1. $N \to \infty$, take the limit to infinitely many particles

2. $a \to 0$, remove the lattice grid and turn it into a continuum

3. $aN = const =: L \in \mathbb{R}_+$, constrain the length and so the mass of the string

4. $\Omega a = const =: c \in \mathbb{R}_+$, constrain the stiffness of the string

We are now faced with infinitely many degrees of freedom and therefore it is convenient to label the elongations q_n and its derivatives with a variable $x \in \mathbb{R}$ instead of $n \in \mathbb{N}$. That x shall be the distance between one end of the string and the oscillator, i.e.

$$q_n \quad \to \quad q(x = na) \tag{108}$$
$$p_n \quad \to \quad p(x = na) \tag{109}$$

where again we may assume periodic boundary conditions $q(x + L) = q(x)$. Taking the limit of the above Newton's force law the yields

$$\ddot{q}(x) = \lim_{\substack{a \to 0 \\ N \to \infty}} \left[\Omega^2 a^2 \frac{\frac{q((n+1)a) - q(na)}{a} - \frac{q(na) - q((n-1)a)}{a}}{a} + \Omega_0^2 q(na) \right] \tag{110}$$
$$= c^2 \frac{\partial^2 q(x)}{\partial^2 x} + \Omega_0^2 q(x) \tag{111}$$

which is known as the Klein-Gordon field equation and analogously the Hamilton reads

$$H = \frac{1}{2} \int_0^L dx \, \left[p^2(x) + c^2 \left(\frac{\partial q(x)}{\partial x} \right)^2 + \Omega_0^2 q(x) \right] \tag{112}$$

Again this resulting field equation can be solved algebraically with the help of the Fourier transformation, see e.g. [11] for a comprehensive discussion. We will look at solutions of the homogeneous and inhomogeneous Klein-Gordon equation in more detail later in this section. Until then we consider some more general issues and terminology.

Since a single oscillator can not be resolved anymore we call $q : \mathbb{R} \to \mathbb{R}$ a field of oscillators or simply a field. The physical meaning of $q(x)$ in this model is clear. It is the relative elongation of the oscillator whose equilibrium position is at x. Unfortunately some fields considered to be important in quantum field theory are said not to have any underlying classical model, which could help writing down a defining field equation. In fact the motivation of field equations can only be given in terms of basic symmetry considerations, e.g. the Poincaré group, homogeneity and isotropy of the space-time continuum, simplicity arguments, etc. Finally one has to say that these arguments are just hints and motivations for one type of field equations or another rather than derivations. Mainly this is why it is so hard to assign a correspondence of the mathematical object $q(x)$, the field, in the theory to an *element of reality*[10]. Nevertheless the above classical model of an oscillating continuum, i.e. a Lorentz-invariant ether so to speak, seems to be a valuable picture for questions relating to its physical meaning.

In the following we shall consider fields on $\mathbb{R}^4$ - with the Minkowski metric - and its time-zero fields on $\mathbb{R}^3$ - with the Euclidean metric. In order to set us free from the above classical model the field will be denoted by $\phi : \mathbb{R}^4 \to \mathbb{R}$, its time-zero field by $\phi : \mathbb{R}^3 \to \mathbb{R}$. With the appropriate boundary conditions we can extend the length L of the string to infinity and distribute the (one dimensional) oscillators along all four dimensions in $\mathbb{R}^4$ and yield the Lorentz-invariant field equation, setting $c := 1$ and $\mu := \Omega_0$

$$\boxed{(\partial_\alpha \partial^\alpha + \mu^2)\phi(x^\nu) := (\partial_\nu \partial^\nu + \mu^2)\phi(x^\nu) = 0} \tag{113}$$

where $\partial^\alpha := \frac{\partial}{\partial x_\alpha}$. This equation we shall call the homogeneous Klein-Gordon equation and take it as a starting point for all following discussions. Furthermore we will need $\pi(x^\nu) := \partial^0 \phi(x^\nu)$ as we shall see, the canonical conjugated field given by the Legendre-transformation of the appropriate Lagrangian.

[10] The reader is asked to forgive the loose kind of way the term *element of reality* is used in this work. However it should be clear what is meant by that term in our context, namely that part of what we become conscious of when watching nature and therefore that part our theory shall be about - for example a particle that we may track with our eyes or the point-like blackening on some screen.

3.2.2 Scalar interaction

We have motivated a simple field equation and now consider how to introduce interactions of the field with a N particle system. We shall refer to these particles as nucleons. Because the field is scalar the type of interaction shall be called a scalar one. Every solution to the homogeneous Klein-Gordon equation (113) in $\mathbb{R}$ can be formed by a continuous superposition of field modes $e^{\pm ik_\nu x^\nu}$

$$\phi(x^\nu) = \int d^4k \; (A(k^\nu)e^{-ik_\nu x^\nu} + A^*(k^\nu)e^{ik_\nu x^\nu}) \tag{114}$$

fulfilling the dispersion relation $k^0 = \omega_{\boldsymbol{k}} := \sqrt{\boldsymbol{k}^2 + \mu^2}$ and $(k^1, k^2, k^3) = \boldsymbol{k}$ and $A : \mathbb{R}^4 \to \mathbb{C}$, which can easily be checked. In the future we shall call ϕ the field and more loosely call special types of elongations of the oscillators in that field a wave. In that sense the whole superposition of all occurring waves is the field and components, e.g. like special regions of that field or the field modes $e^{\pm ik_\nu x^\nu}$, are waves. By the separation of the integration over positive and negative k^ν field modes with $A(k^\nu)$ and its complex conjugate $A^*(k^\nu)$ it is ensured that ϕ is a real-valued function on $\mathbb{R}^4$. Further properties of such a field, e.g. charge, can be added if we extend the solution space to complex-valued ϕ. The information about the charge may then be encoded in the complex phase of the field. Since we shall be concerned with more basic properties of fields we restrict ourselves only to real-valued ones throughout this work. Each field mode propagates freely with a constant and finite momentum k^ν through space-time according to the above dispersion relation. Therefore the homogeneous Klein-Gordon equation is also called the free Klein-Gordon equation.

Now we return to the question why the field reveals to be an interesting mathematical tool for writing down local interactions. If we for example chose the initial conditions such that we set the whole field to zero except at $x^0 = 0$ where we create a wave of a very small but finite space-like support then this wave will propagate within the forward and backward Lorentz light-cone with respect to its initial support according to the Klein-Gordon equation - recall the title picture. By doing that this wave transports all the information about its initial conditions to other nucleons within the light-cones. Exactly this property makes the field a candidate for the mediation of local interactions between nucleons. So let us consider an N nucleon baby universe with an underlying field, the Lorentz-invariant ether. We couple the N nucleons to the field such that the simple circumstance that the i-th nucleon is at some position[11] q_i^ν perturbs the field for example by creating a wave centered around q_i^ν. This can be done by adding a linear source term ρ to the Klein-Gordon equation

$$(\partial_\alpha \partial^\alpha + \mu^2)\phi(x^\nu) = \sum_{i=1}^{N} \rho(q_i^\nu - x^\nu) \tag{115}$$

where q_i^ν is the position of the i-th nucleon and $\rho : \mathbb{R}^4 \to \mathbb{R}$ is a weight function that stands for the strength of the perturbation in space-time, also called source density or interaction term, and is usually thought as sharply peaked around $q_i^\nu - x^\nu = 0$. Now for all $i = 1, ..., N$ the Klein-Gordon equation propagates the created wave within the field and with it the information about the change of the position of the i-th nucleon in finite but non-zero time to all other $(N-1)$ nucleons in the forward or backward light-cone - with respect to the support of $\rho(q_i^\nu - x^\nu)$. In that way the $(N-1)$ other nucleons are able to causally react according to that change of position of the i-th nucleon given that their equation of motion depends somehow on the field around their own position. What that dependence will look like shall be discussed soon. It is known that Lorentz-invariant source densities usually yield non-integrable equations of motion in field theory. In fact even in the case of multi-particle theories the only complete theory known using a Lorentz-invariant source density, in this case a source density being non-zero only along zero

[11] Please note that now we denote by q_i^ν the actual position of a nucleon in space-time and do not mean the elongation of a field oscillator at some space-time point as in the preceding subsection.

Minkowski distances with respect to each particle, is the Wheeler-Feynman electromagnetism - see [2]. Therefore in nowadays field theories the source densities used are usually of the form $\rho(\boldsymbol{q}_i(t)-\boldsymbol{x})$, where $\boldsymbol{q}_i(t)$ is the trajectory of the i-th nucleon in $\mathbb{R}^3$ space. Clearly the introduction of a source density with a purely space-like support breaks the idea of Lorentz-invariance because it alters non-locally within the bounds of its space-like support every time the i-th nucleon moves and q_i^ν changes. The effect seems to be controllable by the extend of the support of ρ however under large Lorentz-boosts the space-like support can blow up and even for two initially distinct source densities we can find a Lorentz-transformation such that the supports have a non-causal overlapping. The only consequence in this case is to get rid of the space-like support by the formal limit $\rho(\boldsymbol{q}_i(t)-\boldsymbol{x}) \to \delta^3(\boldsymbol{q}_i(t)-\boldsymbol{x})$, which we from now on call the point particle limit and later the removal of the cutoff. This answers the question of subsection 2.8 why it is desirable to remove the cutoff. Additionally from an information theoretic point of view the whole support of $\rho(q_i^\nu - x^\nu)$ contains redundant information except for the point $\rho(q_i^\nu)$, which carries information about the position and the strength of the field perturbation at this exact position. Unfortunately this limit is exactly what causes a lot of trouble in any field theory involving a similar interaction term and will be our steady fellow.

Until now the nucleons only cause perturbations of the field depending on their position and the strength of the interaction term ρ. When we now take into account the free propagation of the N nucleons in this kind of model we will quickly see how this interaction term already performs a back reaction on the nucleons under the principle of extremal action - the extremization of the total energy. Here we will not try to keep a Lorentz-covariant formulation any longer but consider only time-zero Hamiltonians, which can be done if the system conserves its total energy. Furthermore we choose the free propagation of the nucleons to be non-relativistic. Let t be the time coordinate, then for x^ν we write $(\boldsymbol{x}, t)$ and for $(\boldsymbol{x}, t=0)$ we write $\boldsymbol{x}$. Furthermore let $\boldsymbol{q}_i$ and $\boldsymbol{p}_i$ be the position and the canonical conjugate momentum of the here non-relativistic nucleons for all $i = 1, ..., N$. Then the Hamiltonian reads

$$\boxed{H = \sum_{i=1}^{N} \frac{\boldsymbol{p}_i^2}{2M} + \frac{1}{2}\int d^3x \ \ (\pi^2(\boldsymbol{x}) + (\nabla\phi(\boldsymbol{x}))^2 + \mu^2\phi^2(\boldsymbol{x})) + g\sum_{i=1}^{N}\int d^3x \ \rho_\kappa(\boldsymbol{q}_i - \boldsymbol{x})\phi(\boldsymbol{x})} \tag{116}$$

where the coupling constant $g \in \mathbb{R}$ represents the uniform strength and the sign of the interaction term. The source density ρ was chosen to be ρ_κ defined in the introductory subsection 2.2 in line (14) and (15) - hence the point particle limit is now mathematically $\kappa \to \infty$. Obeying the principle of extremal action we find the equations of motion by the Hamilton equations

$$(\Box_{(\boldsymbol{x},t)} + \mu^2)\phi(\boldsymbol{x}, t) \quad = \quad -g\sum_{i=1}^{N}\rho_\kappa(\boldsymbol{q}_i(t) - \boldsymbol{x}) \tag{117}$$

$$\ddot{\boldsymbol{q}}_i(t) \quad = \quad -\frac{g}{M}\nabla_{\boldsymbol{q}_i(t)}\int d^3x \ \rho_\kappa(\boldsymbol{q}_i(t) - \boldsymbol{x})\phi(\boldsymbol{x}, t) \tag{118}$$

for $i = 1, ..., N$, where $\Box_{(\boldsymbol{x},t)} = \frac{\partial^2}{\partial t^2} - \nabla^2_{\boldsymbol{x}}$. Let us call the above equations of motion of the dynamic classical Nelson model from now on. We observe that the field equation depends on the trajectories $\boldsymbol{q}_i(t)$ of the N nucleons and the trajectories of the N nucleons depend on the field. Hence the introduction of the interaction term together with the principle of extremal action already yields a back reaction of the field on the nucleons. In this picture one would say that each nucleon interacts with the field by perturbing the field at its position and in return the field guides the nucleons on trajectories such that the overall action is extremized. These $(N+1)$ differential equations are highly coupled and along the way we will recognize that it is very hard to make sense out of them in the point particle limit $\kappa \to \infty$ that we discussed above.

In the next subsection we will at first look at solutions of the inhomogeneous Klein-Gordon equation and represent them by the Klein-Gordon Green's function. The two other proceeding subsections we shall be concerned with of solutions of the static and dynamic classical Nelson model.

3.2.3 Green's function

An elegant way to specify a solution of the inhomogeneous Klein-Gordon equation is to solve it for the Green's function G, which is loosely speaking the inverse of the Klein-Gordon operator. The defining equation is

$$(\partial_\alpha \partial^\alpha + \mu^2) G(x^\nu) \quad = \quad \delta^4(x^\nu) \tag{119}$$

A solution to the inhomogeneous Klein-Gordon equation (115) say for just one source at q^ν can then be build up by

$$\phi_I(x^\nu) = \int d^4x \; \rho(q^\nu - x^\nu) G(x^\nu) \tag{120}$$

plus any solution of the homogeneous Klein-Gordon equation. In this subsection we will only work out the Green's function for the Klein-Gordon operator and in the proceeding ones look at special cases of solutions. We solve (119) by the method of Fourier transformation and use

$$G(x^\nu) \quad = \quad \int đ^4k \; \widehat{G}(k^\beta) e^{-ik_\mu x^\mu} \tag{121}$$

$$\delta^4(x^\nu) \quad = \quad \int đ^4k \; e^{-ik_\mu x^\mu} \tag{122}$$

$$\tag{123}$$

in the sense of tempered distributions. Here we shall use the notation that every dash on the differential means divided by $\sqrt{2\pi}$ - see appendix B. Inserting this in the above equation (119) yields

$$(\partial_{x\nu} \partial_x{}^\nu + \mu^2) \int đ^4k \; \widehat{G}(k^\beta) e^{-ik_\mu x^\mu} \quad = \quad \int đ^4k \; (-k_\mu k^\mu + \mu^2) \widehat{G}(k^\beta) e^{-ik_\mu x^\mu} \tag{124}$$

$$= \quad \delta^4(x^\nu) = \int đ^4k \; e^{-ik_\mu x^\mu} \tag{125}$$

$$\tag{126}$$

Hence we read off the Fourier transformed Green's function

$$\widehat{G}(k^\nu) \quad = \quad \frac{1}{\mu^2 - k_\mu k^\mu} \tag{127}$$

and

$$G(x^\nu) \quad = \quad \int đ^4k \; \frac{1}{\mu^2 - k_\mu k^\mu} e^{-ik_\mu x^\mu} \tag{128}$$

Let us for the time of computation break with the covariant formalism and separate the space and time like integration. The general result however will again be covariant as we shall see.

$$G(x^\nu) \quad = \quad \int đk_{\,0} \int đ^3k \; \frac{1}{\mu^2 - k_0^2 + \boldsymbol{k}^2} e^{-ik_0 x_0} e^{i\boldsymbol{kx}} \tag{129}$$

We at first consider the time-like integration defining $\omega_{\boldsymbol{k}} = \sqrt{\boldsymbol{k}^2 + \mu^2}$ as before

$$\int đk_0 \; \frac{e^{-ik_0 x_0}}{\omega_{\boldsymbol{k}}^2 - k_0^2} \tag{130}$$

The integrand has two analytical singularities on the real axis of the k_0 plane. Because of the exponential behavior of the integrand we can close the path of integration in the complex k_0 plane without changing the value of the integral using a semicircle around the origin and of an infinite radius in the upper or lower plane depending on the value of x_0. For $x_0 \geq 0$ ($x_0 \leq 0$) the integration along the path of a semicircle in the lower (upper) plane is zero in the limit of infinite radius because $ik_0x_0 \to \infty$. We can then use the method of residues to evaluate the integral. In this sense the above integral defines a multi-valued function with a branch line between the singularities, which is not a problem but is rather expected since an inhomogeneous differential equation has no unique solution. The type of solution is controlled by the choice of the path of integration around the singularities. At first we arbitrarily start with the path of integration shown in figure 2. This choice can be seen as preserving the causality. That is because physically the Green's function corresponds to the strength of reaction of its differential equation on a perturbation by e.g. a source density. By demanding $G(x) = 0$ for $x_0 < 0$ no effect occurs before a cause.

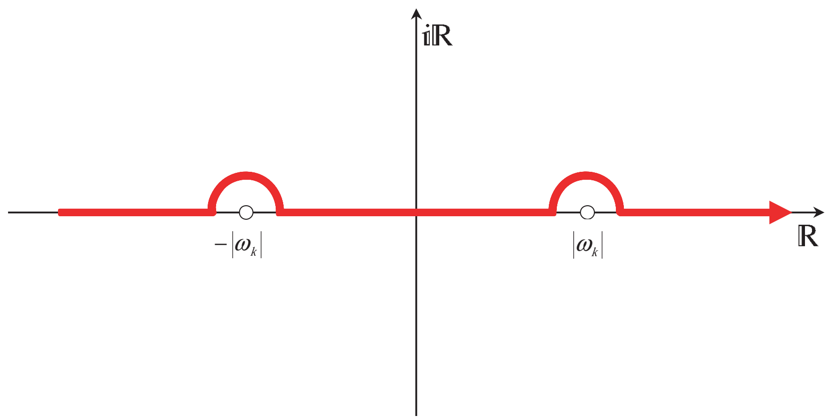

Figure 2: Path $\mathcal{C}$ around singularities in the k_0 plane

As said for $x_0 < 0$ we can close the path in the upper plane without changing the value of the integral. Now there are no singularities inside the path of integration and thus the integral is zero. For $x_0 \geq 0$ we can close the path of integration $\mathcal{C}$ in the lower plane without changing the value of the integral (this path shall be called $\mathcal{C}_l$ - please note that this path is clockwise and that is why an additional minus occurs in (132) since the closed integrals are defined anti-clockwise). That yields

$$\int \,\bar{d}k_0 \frac{e^{-ik_0x_0}}{\omega_{\boldsymbol{k}}^2 - k_0^2} = \theta(x_0) \int_{\mathcal{C}_l} \bar{d}k_0 \frac{e^{-ik_0x_0}}{\omega_{\boldsymbol{k}}^2 - k_0^2} \tag{131}$$

$$= -\theta(x_0) \oint_{\mathcal{C}_l} \bar{d}k_0 \frac{e^{-ik_0x_0}}{\omega_{\boldsymbol{k}}^2 - k_0^2} \tag{132}$$

$$= \theta(x_0) \oint_{\mathcal{C}_l} \bar{d}k_0 \frac{e^{-ik_0x_0}}{(k_0 - \omega_{\boldsymbol{k}})(k_0 + \omega_{\boldsymbol{k}})} \tag{133}$$

$$= \theta(x_0) 2\pi i \left(\frac{e^{-i\omega_{\boldsymbol{k}}x_0}}{2\omega_{\boldsymbol{k}}} + \frac{e^{i\omega_{\boldsymbol{k}}x_0}}{-2\omega_{\boldsymbol{k}}} \right) \tag{134}$$

$$= \theta(x_0) \frac{\sin(\omega_{\boldsymbol{k}}x_0)}{\omega_{\boldsymbol{k}}} \tag{135}$$

Plugging this into (129) we get

$$G_{ret}(x_0, \boldsymbol{x}) = \theta(x_0) \int \bar{d}^3k \,\frac{\sin(\omega_{\boldsymbol{k}}x_0)}{\omega_{\boldsymbol{k}}} \tag{136}$$

where 'ret' stands for retarded and denotes the special choice of the path we have chosen around the singularities. It is convenient to introduce spherical coordinates around the vector $\boldsymbol{x}$ such that[12] $\boldsymbol{kx} = kx\cos\vartheta$ and so integration over the angles gives

$$G_{ret}(x_0,\boldsymbol{x}) = \frac{\theta(x_0)}{(2\pi)^3}\int_0^{2\pi} d\phi \int_0^{\pi} d\vartheta \int_0^{\infty} dk\, \frac{k^2\sin\vartheta}{\omega_{\boldsymbol{k}}}\sin(\omega_k x_0)e^{ikx\cos\vartheta} \tag{137}$$

Substitution $u=\cos\vartheta$ and hence $du=-\sin\vartheta d\vartheta$ gives

$$G_{ret}(x_0,\boldsymbol{x}) = -\frac{\theta(x_0)}{(2\pi)^2}\int_1^{-1} du \int_0^{\infty} dk\, \frac{k^2}{\omega_{\boldsymbol{k}}}\sin(\omega_{\boldsymbol{k}} x_0)e^{ikxu} \tag{138}$$

$$= \frac{\theta(x_0)}{\pi x}\int_0^{\infty} đk\, \frac{k}{\omega_k}\sin(\omega_k x_0)\sin(kx) \tag{139}$$

For further computation it is now convenient to introduce the function $F(x)$ by

$$\frac{\partial F(x_0,\boldsymbol{x})}{\partial x} = xG_{ret}(x_0,\boldsymbol{x}) \tag{140}$$

Thus

$$F(x_0,\boldsymbol{x}) = -\frac{\theta(x_0)}{\pi}\int_0^{\infty} đk\, \frac{1}{\omega_k}\sin(\omega_k x_0)\cos(kx) \tag{141}$$

$$= -\frac{\theta(x_0)}{2\pi}\int_{-\infty}^{\infty} đk\, \frac{1}{\omega_k}\sin(\omega_k x_0)\cos(kx) \tag{142}$$

Again substituting $k=\mu\sinh\xi$ and hence $dk=\mu\cosh\xi$ gives

$$F(x_0,\boldsymbol{x}) = -\frac{\theta(x_0)}{2\pi}\int_{-\infty}^{\infty} đ\xi\, \sin(\mu\sinh\xi\, x_0)\cos(\mu\cosh\xi\, x) \tag{143}$$

$$= -\frac{\theta(x_0)}{8\pi i}\int_{-\infty}^{\infty} đ\xi\, \left(e^{i\mu(\cosh\xi\, x_0+\sinh\xi\, x)} + e^{i\mu(\cosh\xi\, x_0+\sinh\xi\, x)} + \right. \tag{144}$$

$$\left. -e^{-i\mu(\cosh\xi\, x_0+\sinh\xi\, x)} - e^{-i\mu(\cosh\xi\, x_0+\sinh\xi\, x)}\right) \tag{145}$$

Let $\lambda := x_0^2 - x^2$. Since $x_0 \geq 0$ we now distinguish between two cases. The first is $x_0 < x$ for which we use the substitution $x_0 = \sqrt{-\lambda}\sinh\xi_0$ and $x_0=\sqrt{-\lambda}\cosh\xi_0$ and get

$$F(x_0,\boldsymbol{x})\Big|_{x_0<x} = -\frac{\theta(x_0)}{8\pi i}\int_{-\infty}^{\infty} đ\xi\, \left(e^{i\mu\sqrt{-\lambda}\sinh(\xi+\xi_0)} + e^{i\mu\sqrt{-\lambda}\sinh(\xi-\xi_0)} + \right. \tag{146}$$

$$\left. -e^{-i\mu\sqrt{-\lambda}\sinh(\xi+\xi_0)} - e^{-i\mu\sqrt{-\lambda}\sinh(\xi-\xi_0)}\right) \tag{147}$$

$$= -\frac{\theta(x_0)}{8\pi i}\int_{-\infty}^{\infty} đ\xi'\, \left(e^{i\mu\sqrt{-\lambda}\sinh\xi'} + e^{i\mu\sqrt{-\lambda}\sinh\xi'} + \right. \tag{148}$$

$$\left. -e^{-i\mu\sqrt{-\lambda}\sinh\xi'} - e^{-i\mu\sqrt{-\lambda}\sinh\xi'}\right) \tag{149}$$

$$= 0 \tag{150}$$

The second case is $x_0 > x$ for which we use the substitutions $x_0 = \sqrt{\lambda}\cosh\xi_0$ and $x_0 = \sqrt{\lambda}\sinh\xi_0$ and get

$$F(x_0,\boldsymbol{x})\Big|_{x_0\geq x} = -\frac{\theta(x_0)}{8\pi i}\int_{-\infty}^{\infty} đ\xi\, \left(e^{i\mu\sqrt{\lambda}\cosh(\xi+\xi_0)} + e^{i\mu\sqrt{\lambda}\cosh(\xi-\xi_0)} + \right. \tag{151}$$

$$\left. -e^{-i\mu\sqrt{\lambda}\cosh(\xi+\xi_0)} - e^{-i\mu\sqrt{\lambda}\cosh(\xi-\xi_0)}\right) \tag{152}$$

$$= -\frac{\theta(x_0)}{4\pi i}\int_{-\infty}^{\infty} đ\xi'\, \left(e^{i\mu\sqrt{\lambda}\cosh\xi'} - e^{-i\mu\sqrt{\lambda}\cosh\xi'}\right) \tag{153}$$

[12] Please excuse the ambiguity between $x \in \mathbb{R}^4$ e.g. in the argument of the Green's function and $x = |\boldsymbol{x}|$, the length for $\boldsymbol{x} \in \mathbb{R}^3$.

which is the integral representation of the Bessel function of zero-th order [8, 6.431]

$$F(x_0,\boldsymbol{x})\Big|_{x_0\geq x} = -\frac{\theta(x_0)}{4\pi}\left(\frac{1}{2}H_0^{(1)}(\mu\sqrt{\lambda})+\frac{1}{2}H_0^{(2)}(\mu\sqrt{\lambda})\right) \tag{154}$$

$$= -\frac{\theta(x_0)}{4\pi}J_0(\mu\sqrt{\lambda}) \tag{155}$$

Therefore

$$F(x_0,\boldsymbol{x}) = -\frac{\theta(x_0)\theta(x_0-x)}{4\pi}J_0(\mu\sqrt{\lambda}) \tag{156}$$

which gives

$$G_{ret}(x_0,\boldsymbol{x}) = -\frac{\theta(x_0)}{4\pi x}\frac{\partial}{\partial x}\left(\theta(x_0-x)J_0(\mu\sqrt{\lambda})\right) \tag{157}$$

$$= \frac{\theta(x_0)\delta(x_0-x)}{4\pi x}J_0(\mu\sqrt{\lambda}) - \frac{\mu\theta(x_0)\theta(x_0-x)}{4\pi\sqrt{\lambda}}J_1(\mu\sqrt{\lambda}) \tag{158}$$

or in other form with

$$\delta(x_\mu x^\mu) = \frac{1}{2|\boldsymbol{x}|}\left(\delta(x_0-|\boldsymbol{x}|)+\delta(x_0+|\boldsymbol{x}|)\right) \tag{159}$$

and noting that x_0 must be positive because of the $\theta(x_0)$ term we finally arrive at

$$G_{ret}(x^\nu) = \theta(x_0)\left(\frac{\delta(x_\mu x^\mu)}{2\pi} - \frac{\mu\theta(x_\mu x^\mu)}{4\pi\sqrt{x_\mu x^\mu}}J_1(\mu\sqrt{x_\mu x^\mu})\right) \tag{160}$$

However this is still not covariant because of $\theta(x_0)$. For the other choice of integration path we get the advanced Green's function being proportional to $\theta(-x_0)$. That path is already anticlockwise and the minus in (132) would disappear. Thus we find the relation $G_{ret}(x_0,\boldsymbol{x}) = G_{adv}(-x_0,\boldsymbol{x})$ and both Green's functions add up to one again covariant expression being defined for all $x\in\mathbb{R}^4$, because it only depends on the invariant scalar $x_\mu x^\mu$.

3.3 The static classical Nelson model

In this section we turn to the static classical Nelson model. We nail the nucleons down at positions $\boldsymbol{q}_i$ for all $i=1,...,N$ by giving them an extremely heavy rest mass so that the momentum change due to the field reacting on the nucleons can be neglected. This limit simply turns off the free propagation term of the nucleons in the Hamiltonian and we are left with

$$H = \frac{1}{2}\int d^3x\ \left(\pi^2(\boldsymbol{x})+(\nabla\phi(\boldsymbol{x}))^2+\mu^2\phi^2(\boldsymbol{x})\right) + g\sum_{i=1}^N\int d^3x\ \rho_\kappa(\boldsymbol{q}_i-\boldsymbol{x})\phi(\boldsymbol{x}) \tag{161}$$

and the Hamilton equations yield

$$(\Box_{(\boldsymbol{x},t)}+\mu^2)\phi(\boldsymbol{x},t) = -g\sum_{i=1}^N\rho_\kappa(\boldsymbol{q}_i-\boldsymbol{x}) \tag{162}$$

Since the solution is not unique we choose a special type of solution, which is a sum of a solution to the homogeneous Klein-Gordon equation, ϕ_0, and a solution to the inhomogeneous one, ϕ_I. For ϕ_0 we choose the representation given in (114). Furthermore we demand that the field vanishes at infinity as a boundary condition. Based on the last subsection 3.2.3 we can compute the inhomogeneous solution with the help of the Green's function like

$$\phi_I(\boldsymbol{x}) = -g\sum_{i=1}^N\int d^4x'\ \rho_\kappa(\boldsymbol{q}_i-\boldsymbol{x}')G(x'^\nu-x^\nu) \tag{163}$$

$$= -g\sum_{i=1}^N\int d^3x'\ \rho_\kappa(\boldsymbol{q}_i-\boldsymbol{x}')\int dt\ G(\boldsymbol{x}'-\boldsymbol{x},t'-t) \tag{164}$$

we infer from line (129) that

$$\phi_I(\boldsymbol{x}) \quad = \quad -g\sum_{i=1}^{N}\int d^3x'\ \rho_\kappa(\boldsymbol{q}_i-\boldsymbol{x}')\int dt\ \int đk_0\ \int đ^3k\ \frac{e^{-i\boldsymbol{k}(\boldsymbol{x}'-\boldsymbol{x})}e^{ik_0(t'-t)}}{\omega_{\boldsymbol{k}}^2-k_0^2} \tag{165}$$

Because of the meaning of the static source (see e.g. [11]) we can perform the dt integration before the d^3k one and do not have to worry about the branch cuts in the complex plane. The dt integration yields a $\delta(k_0)$ on which we can then easily perform the dk_0 integration and get

$$\phi_I(\boldsymbol{x}) \quad = \quad -g\sum_{i=1}^{N}\int d^3x'\ \rho_\kappa(\boldsymbol{q}_i-\boldsymbol{x}')\int đ^3k\ \frac{e^{-i\boldsymbol{k}(\boldsymbol{x}'-\boldsymbol{x})}}{\omega_{\boldsymbol{k}}^2} \tag{166}$$

$$= \quad -g\sum_{i=1}^{N}\int d^3x'\ \rho_\kappa(\boldsymbol{q}_i-\boldsymbol{x}')\frac{e^{-\mu||\boldsymbol{x}'-\boldsymbol{x})||}}{4\pi||\boldsymbol{x}'-\boldsymbol{x})||} \tag{167}$$

The Fourier integral was computed in the appendix A.1. Before we come to the point particle limit we have a look at the complete solution $\phi_0+\phi_I$ evaluated at time zero

$$\phi(\boldsymbol{x}) := \phi_0(\boldsymbol{x})+\phi_I(\boldsymbol{x}) \quad = \quad \int d^3k\ \Bigg(\Bigg(A(\boldsymbol{k})-g\sum_{i=1}^{N}\frac{\widehat{\rho}_\kappa^*(\boldsymbol{k})}{2(2\pi)^{3/2}\omega_{\boldsymbol{k}}^2}e^{-i\boldsymbol{k}\boldsymbol{q}_i}\Bigg)e^{i\boldsymbol{k}\boldsymbol{x}}+ \tag{168}$$

$$+\Bigg(A^*(\boldsymbol{k})-g\sum_{i=1}^{N}\frac{\widehat{\rho}_\kappa(\boldsymbol{k})}{2(2\pi)^{3/2}\omega_{\boldsymbol{k}}^2}e^{i\boldsymbol{k}\boldsymbol{q}_i}\Bigg)e^{-i\boldsymbol{k}\boldsymbol{x}}\Bigg) \tag{169}$$

where we simply inserted the Fourier transform of the source density in order to drag everything under only one integral. Hence it looks like the only effect the sources have on this special solution is to cause a shift in the oscillator amplitudes[13] $A(\boldsymbol{k})$ and $A^*(\boldsymbol{k})$. Taking the limit $\kappa\to\infty$ will cause the $\phi_I(\boldsymbol{q}_i)$ to be ill-defined for all $i=1,...,N$ because of the $||\boldsymbol{q}_i-\boldsymbol{x}||$ in the denominator. Since the nucleons in the model considered here do not move this divergence is ugly but not too severe because we do not necessarily need the ϕ_I to be evaluated at some $\boldsymbol{q}_i$ since they can not move anyway. This situation changes dramatically in the dynamic case where we would have to solve the equation of motion for the i-th nucleon that depends on gradient of the field ϕ_i at $\boldsymbol{q}_i$. We will find in the next subsection that the nucleon equations of motion are, because of this reason, generically ill-defined in the point particle limit. So let us now analyze what effects these divergences have. Therefore we calculate the energy of our solution $\phi=\phi_0+\phi_I$ and compare it to the energy of the homogeneous solution ϕ_0 like in [11] for the N particle case.

$$H(\phi,\pi) := \frac{1}{2}\int d^3x\ \ (\pi^2(\boldsymbol{x},t)+(\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x},t))^2+\mu^2\phi^2(\boldsymbol{x},t))+g\sum_{i=1}^{N}\int d^3x\ \rho_\kappa(\boldsymbol{q}_i-\boldsymbol{x})\phi(\boldsymbol{x},t) \tag{170}$$

We now insert the in the above computed special solution $\phi=\phi_0+\phi_I$ and yield

$$H(\phi_0+\phi_I,\pi_0+\pi_I=\pi_0) \quad = \quad H(\phi_0,\pi_0)+ \tag{171}$$

$$+\frac{1}{2}\int d^3x\ \ \Big(2\nabla_{\boldsymbol{x}}\phi_0(\boldsymbol{x},t)\nabla_{\boldsymbol{x}}\phi_I(\boldsymbol{x})+2\mu^2\phi_0(\boldsymbol{x},t)\phi_I(\boldsymbol{x}) \tag{172}$$

$$+\nabla_{\boldsymbol{x}}\phi_I(\boldsymbol{x})\nabla_{\boldsymbol{x}}\phi_I(\boldsymbol{x})+\mu^2\phi_I(\boldsymbol{x})\phi_I(\boldsymbol{x})\Big) \tag{173}$$

$$+g\sum_{i=1}^{N}\int d^3x\ \rho_\kappa(\boldsymbol{q}_i-\boldsymbol{x})\,(\phi_0(\boldsymbol{x},t)+\phi_I(\boldsymbol{x})) \tag{174}$$

[13] We will come back to this property when considering the quantum version of it in subsection 4.2. There we will be able to control this oscillator amplitude shift by the choice of representation of the commutator algebra or in a few times even by an unitary transformation. We will find that by choosing an appropriate representation the static interaction effects can then not be seen anymore.

Note that $\pi_I = \dot{\phi}_I = 0$. Having in mind the boundary conditions we can continue by partial integration and find

$$\begin{aligned} H(\phi_0+\phi_I,\pi_0+\pi_I=\pi_0) &= H(\phi_0,\pi_0)+ && (175)\\ &-\frac{1}{2}\int d^3x\ \Big(2\phi_0(\boldsymbol{x},t)(-\nabla^2_{\boldsymbol{x}}+\mu^2)\phi_I(\boldsymbol{x})+ && (176)\\ &\qquad +\phi_I(\boldsymbol{x},t)(-\nabla^2_{\boldsymbol{x}}+\mu^2)\phi_I(\boldsymbol{x})\Big) && (177)\\ &+g\sum_{i=1}^N\int d^3x\ \rho_\kappa(\boldsymbol{q}_i-\boldsymbol{x})\,(\phi_0(\boldsymbol{x},t)+\phi_I(\boldsymbol{x})) && (178)\end{aligned}$$

and since ϕ_I is a solution of the inhomogeneous Klein-Gordon equation

$$(\Box_{(\boldsymbol{x},t)}+\mu^2)\phi_I(\boldsymbol{x}) = (-\nabla^2_{\boldsymbol{x}}+\mu^2)\phi_I(\boldsymbol{x}) = \sum_{i=1}^N\rho_\kappa(\boldsymbol{q}_i-\boldsymbol{x}) \tag{179}$$

hence

$$\begin{aligned} H(\phi_0+\phi_I,\pi_0+\pi_I=\pi_0) &= H(\phi_0,\pi_0)+\frac{g}{2}\sum_{i=1}^N\int d^3x\ \rho_\kappa(\boldsymbol{q}_i-\boldsymbol{x})\phi_I(\boldsymbol{x}) && (180)\\ &= H(\phi_0,\pi_0)+ && (181)\\ &-\frac{g^2}{2}\sum_{i,j=1}^N\int d^3x\ \rho_\kappa(\boldsymbol{q}_i-\boldsymbol{x})\times && (182)\\ &\qquad\times\int d^3x'\ \rho_\kappa(\boldsymbol{q}_j-\boldsymbol{x}')\int đ^3k\ \frac{e^{-i\boldsymbol{k}(\boldsymbol{x}'-\boldsymbol{x})}}{\omega^2_{\boldsymbol{k}}} && (183)\end{aligned}$$

Let us look at the energy difference $\Delta E := H(\phi_0+\phi_I,\pi_0+\pi_I)-H(\phi_0,\pi_0)$ that is

$$\begin{aligned} \Delta E &= -\frac{g^2}{2}\sum_{i\neq j}^N\int d^3x\ \rho_\kappa(\boldsymbol{q}_i-\boldsymbol{x})\int d^3x'\ \rho_\kappa(\boldsymbol{q}_j-\boldsymbol{x}')\int đ^3k\ \frac{e^{-i\boldsymbol{k}(\boldsymbol{x}'-\boldsymbol{x})}}{\omega^2_{\boldsymbol{k}}}+ && (184)\\ &\quad -\frac{g^2}{2}\sum_{i=1}^N\int d^3x\ \rho_\kappa(\boldsymbol{q}_i-\boldsymbol{x})\int d^3x'\ \rho_\kappa(\boldsymbol{q}_i-\boldsymbol{x}')\int đ^3k\ \frac{e^{-i\boldsymbol{k}(\boldsymbol{x}'-\boldsymbol{x})}}{\omega^2_{\boldsymbol{k}}} && (185)\end{aligned}$$

In the point particle limit where $\kappa\to\infty$ the first term turns out to be the Yukawa pair potential

$$V := -\frac{g^2}{2}\sum_{i\neq j}^N\int đ^3k\ \frac{e^{-i\boldsymbol{k}(\boldsymbol{q}_i-\boldsymbol{q}_j)}}{\omega^2_{\boldsymbol{k}}} = -\frac{g^2}{2}\sum_{i\neq j}^N\frac{e^{-\mu||\boldsymbol{q}_i-\boldsymbol{q}_j||}}{4\pi||\boldsymbol{q}_i-\boldsymbol{q}_j||} \tag{186}$$

which is kind expected from the experiences in electromagnetism. The second term however diverges in the limit $\kappa\to\infty$ because the integral

$$NV_{SE} := -\frac{g^2}{2}\sum_{i=1}^N\int đ^3k\ \underbrace{\frac{1}{\omega^2_{\boldsymbol{k}}}}_{\sim|k|^{-2}} \tag{187}$$

does not exist. The symbol $\sim$ states the asymptotic behavior and is explained in the appendix B. This term is commonly called the self-energy of the field. Since energy, like any other quantity, can only be measured relatively one could now, without any loss of information, define a renormalized Hamiltonian $\widehat{H}$ by formally subtracting this divergent self-energy

$$\widehat{H} := H - NV_{SE} \tag{188}$$

and decide that $\widehat{H}$ is the Hamiltonian to work with. So to say the substraction of NV_{SE} from the Hamiltonian can be seen as an energy renormalization. It has to be remarked that exactly this term will again show up in the static quantum Nelson model in subsection 3.3.

3.4 The dynamic classical Nelson model

Recalling the equations of motion for the classical Klein-Gordon field (117p) and the equation of motion for the nucleons

$$(\Box_{(\boldsymbol{x},t)} + \mu^2)\phi(\boldsymbol{x},t) = -g\sum_{i=1}^{N}\rho_\kappa(\boldsymbol{q}_i(t)-\boldsymbol{x}) \tag{189}$$

$$\ddot{\boldsymbol{q}}_i(t) = -\frac{g}{M}\nabla_{\boldsymbol{q}_i}\int d^3x\; \rho_\kappa(\boldsymbol{q}_i(t)-\boldsymbol{x})\phi(\boldsymbol{x},t) \tag{190}$$

we have already realized that these differential equations are highly coupled, not only in the nucleon positions and the field at the same time but also at different times. This four dimensional coupling is naturally caused by the desired locality of the interaction. Now that we know the Green's function from line (160) we can easily generate solutions to the inhomogeneous Klein-Gordon equation assuming given trajectories $\boldsymbol{q}_i(t)$ of the N nucleons. Such a solution could be $\phi_0 + \phi_I$, where ϕ_0 is a solution of the homogeneous Klein-Gordon solution like in line (114) and

$$\phi_I(\boldsymbol{x},t) = -g\sum_{i=1}^{N}\int dt'\int d^3x'\; \rho_\kappa(\boldsymbol{q}_i(t')-\boldsymbol{x}')G_{ret}(\boldsymbol{x}-\boldsymbol{x}',t-t') \tag{191}$$

In the point particle limit for $\kappa\to\infty$ we get

$$\lim_{\kappa\to\infty}\phi_I(\boldsymbol{x},t) = -g\sum_{i=1}^{N}\int dt'\; G_{ret}(\boldsymbol{x}-\boldsymbol{q}_i(t'),t-t') \tag{192}$$

which is by the singular behavior the Green's function ill-defined everywhere on the trajectory of each $q_i(t)$. Since for the equation of motion of the i-th nucleon we obtain

$$\lim_{\kappa\to\infty} M\ddot{\boldsymbol{q}}_i(t) = -g\sum_{i=1}^{N}\nabla_{\boldsymbol{q}_i}\left(\phi_0(\boldsymbol{q}_i(t),t) + \phi_I(\boldsymbol{q}_i(t),t)\right) \tag{193}$$

and find that $\nabla_{\boldsymbol{q}_i}\phi_I$ is exactly there ill-defined where we would need it, namely along the trajectory. There are several attempts in order to still make sense out of the equations of motion. The one suggested by Dirac for the case of $\mu = 0$ [5, 20] represents ϕ_I as

$$\phi_I(\boldsymbol{x},t) = -g\sum_{i=1}^{N}\int dt'\int d^3x'\; \rho_\kappa(\boldsymbol{q}_i(t')-\boldsymbol{x}')\times \tag{194}$$

$$\times\left[\frac{1}{2}\left(G_{ret}(\boldsymbol{x}-\boldsymbol{x}',t-t') - G_{adv}(\boldsymbol{x}-\boldsymbol{x}',t-t')\right)+\right. \tag{195}$$

$$\left.+\frac{1}{2}\left(G_{ret}(\boldsymbol{x}-\boldsymbol{x}',t-t') + G_{adv}(\boldsymbol{x}-\boldsymbol{x}',t-t')\right)\right] \tag{196}$$

Then one finds that even in the point particle limit the first term under the integral stays finite while second term is of the form $\delta M\ddot{\boldsymbol{q}}_i$ but unfortunately δM diverges. However this term can then be dragged to the left-hand side of (193). If we now choose M such that $0 < (M+\delta M) < \infty$ we get a well-defined equations of motion. Unfortunately many of its solutions have undesirable properties like pre-acceleration and run away character. These can be additionally eliminated by appropriate asymptotic boundary conditions - see [20]. We have to conclude that the equations of motions in line (117) as they stand are a merely formal expression and far away from being a complete mechanical theory. In order to get a feeling about how the total energy diverges it would be interesting to do a similar calculation for the energy difference $H(\phi_0+\phi_I,\psi_0+\pi_I) - H(\phi_0,\pi_0)$ like the one we did in the last subsection for the static model. The difficulty here is that generally $\pi_I \neq 0$, which means it will not be that easy to extract the divergent terms in the Hamiltonian as it was the case for the static model. However one could expand the Green's function in orders of $|t - t'|$ and perform the dt' integration in the resulting power series. It would be interesting to know if this energy divergence is of the same kind as the one we will find in the dynamic quantum Nelson model in subsection 4.5.1 and 4.5.3. Another interesting point is the following. When we formally write down a general solution for the field $\phi_0 + \phi_I$ we will again find an oscillator amplitude shift like we did in the static case. The only difference will be that now this shift is time dependent because it will depend on the trajectories of the nucleons. The question is, will it be possible in an appropriate Taylor expansion of the shift terms to separate a time-independent term, which carries the overall divergent behavior? This would mean we could then shift the oscillator amplitudes such that this term disappears and then observe the purely dynamical effects, e.g. radiation damping, of the interaction without being hidden by the divergent static ones. In the quantum version of it we could again perform this shift easily by an appropriate representation of the commutator algebra as mentioned in a footnote to the preceding subsection. This point will be subject of further investigation.

4 Quantum field theory

4.1 Motivation for a quantum version of the theory of fields

One starting point for quantum field theories was the observation [21] that multi-particle versions of the quantum mechanical wave equations, like the Schrödinger or the Dirac equation, involving classical field interactions give rise to contradictions, e.g. under certain circumstance the violation of Heisenberg's uncertainty principle for the position and momentum operator. Furthermore it was observed that relativistic multi-particle wave equations generically yield negative energy solutions and like for the case of the Klein-Gordon equation it is hard to find appropriate probability measures. Open problems like these led the physical community to consider the so-called *quantization* of fields. To the author's best knowledge there is no such thing like an algorithm that transforms any meaningful classical equation of motion into a quantum analog, not in the case of a multi-particle theory nor in the case of a field theory, and hence the *quantization* is to be understood as a motivation for the quantum equation of motions grounded on their classical analogs. That is not too bad because there exists no derivation of any classical force law or any classical field equation either. We are usually only left with symmetry and simplicity arguments in order to motivate a specific equation of motion or another. The only big difference in quantum field theory is that in most other theories we know which mathematical object correspond to which *element of reality*. For example in classical mechanics the position coordinate of two particles corresponds to their distance, which is considered to be *real* or at least measurable. The principle of stationarity uniquely defines the position probability measure and we are ready to do statistics on the positions of particles. For example for the Schrödinger or Dirac equation the position probability measure is given by the principle of equivariance [6], which again enables us to make statistical predictions about positions of particles. In contrary to that in most quantum field theories it is completely unclear how to define its model Hamiltonian as an operator on some space, what the equation of motion is and what the position probability measure shall be. Here is a small extract of [18, X.7] in which Simon and Reed describe the situation of interacting quantum field theories:

> *[...] Hamiltonians and fields are written down, but no Hilbert space is given on which they are well-defined operators. The matrix elements of the S-operator are then calculated by formal manipulation. These matrix elements are represented as power series whose coefficients depend on the vacuum expectation values of a related free field theory. Typically, each coefficient of this power series is given by a divergent integral. One formally cancels these infinities by making various input parameters in the theory infinite and then follows a set of prescriptions for extracting the "principle parts" of the resulting difference of divergent integrals; this is called "renormalization". In quantum electrodynamics, these procedures have produced predictions very close to experimental values. [...]*

The axiomatic quantum field theory is a promising approach to make things mathematically rigorous although as it stands the φ^4 theory is the only one which is shown to be compatible with the Wightman axioms in four dimensions and it still does not answer the question about what the theory is. Is it a probability theory about elementary particles? If not about what else than particles? It does not matter what kind of mathematical objects, e.g. particles, fields, operators, etc., the theory utilizes for computation as long as the correspondence between some of these objects with the *elements of reality* we wish to model is clear. If the theory is about elementary particles it seems natural to ask questions about their positions and how they evolve in time. Following this assumption we would need to find a position probability measure. In the Nelson model this point is also unclear but the mathematical situation is not as bad as described in the above quote. In this model the existence of a well-defined Hamiltonian and hence a well-defined time evolution was proven. In the following we briefly go through the historical field *quantization* recipe. It has to be emphasized that the purpose here is only to motivate the quantum field Hamiltonians which we are going to deal with in the proceeding subsections and to understand why they were written down in they way they were. So in the next subsection we will briefly go through the *quantization* of the free and interacting scalar Klein-Gordon field. It consists of three subsections. The first treats the free field case in a non-rigorous way - see [18] for a detailed and mathematically rigorous treatment of free fields. The second subsection is about introducing an interaction term in the found Hamiltonian, where the Nelson Hamiltonian will show up, and is again written in a non-rigorous fashion. This third subsection ends this topic with a comment on the *quantization* of interacting fields.

4.2 The quantum Klein-Gordon field

4.2.1 The free field

In the classical field section we have seen that in the continuous limit of the string of oscillators q_i and p_i become fields $q(\boldsymbol{x})$ and $p(\boldsymbol{x})$. If we had to come up with a quantum theoretical description of the system of N oscillators one could e.g. use the correspondence principle and impose the following commutator relations

$$[q_i, p_j] = i\delta_{i,j} \text{ and } [q_i, g_j] = [p_i, p_j] = 0 \text{ for all } i, j = 1, ..., N \tag{197}$$

So for the continuous system it seems natural to use the continuous analog

$$[q(\boldsymbol{x}, t), p(\boldsymbol{x}', t)] = i\delta(\boldsymbol{x} - \boldsymbol{x}') \text{ and} \tag{198}$$

$$[q(\boldsymbol{x}, t), g(\boldsymbol{x}', t)] = [p(\boldsymbol{x}, t), p(\boldsymbol{x}', t)] = 0 \text{ for all } \boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R}^3 \text{ and } t \in \mathbb{R} \tag{199}$$

except for the fact that we now call the field and its conjugate ϕ and π instead of q and p. Since the intended commutator relations, which one now likes to use, manifestly break the Lorentz-invariance while separating the time and space coordinate we will not even try to take the trouble continuing with a covariant formalism. So let us take the classical field theory outlined in last section as a starting point. There we have found the free Klein-Gordon field solutions, line (114), for $x^0 = 0$, $k^0 = \omega_{\boldsymbol{k}} := \sqrt{\boldsymbol{k}^2 + \mu^2}$ and $\boldsymbol{k} := (k^1, k^2, k^3)$ we can write the time-zero solutions like

$$\phi(\boldsymbol{x}) \;=\; = \int d^3k \; (A(\boldsymbol{k})e^{i\boldsymbol{k}\boldsymbol{x}} + A^*(k^\nu)e^{-i\boldsymbol{k}\boldsymbol{x}}) \tag{200}$$

$$\pi(\boldsymbol{x}) \;=\; \int d^3k \; (-i\omega_{\boldsymbol{k}})(A(\boldsymbol{k})e^{i\boldsymbol{k}\boldsymbol{x}} - A^*(\boldsymbol{k})e^{-i\boldsymbol{k}\boldsymbol{x}}) \tag{201}$$

We speak in plural because we still have the freedom to choose any $A : \mathbb{R}^3 \to \mathbb{C}$. The time-zero Hamiltonian was given by

$$H \;=\; \frac{1}{2}\int d^3x \; \left(\pi^2(\boldsymbol{x}) + (\nabla\phi(\boldsymbol{x}))^2 + \mu^2\phi^2(\boldsymbol{x})\right) \tag{202}$$

We plug the time-zero field ϕ and its canonical conjugate in the commutation relations (198) and (199) and yield

$$[A(\boldsymbol{k}), A^*(\boldsymbol{k}')] = \frac{1}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{k}}}\delta(\boldsymbol{k} - \boldsymbol{k}') \text{ and} \tag{203}$$

$$[A(\boldsymbol{k}), A(\boldsymbol{k}')] = [A^*(\boldsymbol{k}), A^*(\boldsymbol{k}')] = 0 \text{ for all } \boldsymbol{k}, \boldsymbol{k}' \in \mathbb{R}^3 \tag{204}$$

This relations clearly cannot be fulfilled by functions $A : \mathbb{R}^3 \to \mathbb{C}$. Instead it reminds us of the algebra of the boson creation and annihilation operators $a^\dagger_{\boldsymbol{k}}, a_{\boldsymbol{k}}$. This turns our field ϕ and its canonical conjugate into functionals of $a^\dagger_{\boldsymbol{k}}, a_{\boldsymbol{k}}$. One could say the step to *quantization* of the fields is now to find a meaningful representation of the above commutator algebra. One family of representations is

$$A(\boldsymbol{k}) \;=\; \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}}a_{\boldsymbol{k}} + f(\boldsymbol{k}) \tag{205}$$

$$A^*(\boldsymbol{k}) \;=\; \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}}a^\dagger_{\boldsymbol{k}} + f^*(\boldsymbol{k}) \tag{206}$$

for any $f : \mathbb{R}^3 \to \mathbb{C}$. Setting $f = 0$ shall be called the Fock representation. It is easily checked that for models whose Hamiltonians do not depend on $\widehat{\boldsymbol{p}}$ or for which $[\widehat{\boldsymbol{p}}, f(\boldsymbol{k})] = 0$ all representations with $f \in \mathcal{L}_2(\mathbb{R}^3)$ are unitary equivalent to $f = 0$. This means there exists a unitary transformation which maps one into the other. Even if we take these cases out there are only for this family uncountably many representations left. In fact it is this ambiguity in the choice of the representation of the commutator algebra that brings in great difficulties when we ask questions about the position probability measure. We continue and define the Hamiltonian of the free fields by plugging the chosen Fock representation for the field into the Hamiltonian. The field in the Fock representation $f = 0$ and its conical conjugate is thus given by

$$\phi(\boldsymbol{x}) \quad = \quad = \int d\!\!{}^-{}^3k \ \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}}(a_{\boldsymbol{k}}e^{i\boldsymbol{k}\boldsymbol{x}} + a^\dagger_{\boldsymbol{k}}e^{-i\boldsymbol{k}\boldsymbol{x}}) \tag{207}$$

$$\pi(\boldsymbol{x}) \quad = \quad -i\int d\!\!{}^-{}^3k \ \sqrt{\frac{\omega_{\boldsymbol{k}}}{2}}(a_{\boldsymbol{k}}e^{i\boldsymbol{k}\boldsymbol{x}} - a^\dagger_{\boldsymbol{k}}e^{-i\boldsymbol{k}\boldsymbol{x}}) \tag{208}$$

plugging this in the time-zero Hamiltonian yields

$$H = \int d^3k \ \frac{\omega_{\boldsymbol{k}}}{2}\left(a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}} + a_{\boldsymbol{k}}a^\dagger_{\boldsymbol{k}}\right) \text{“} = \text{”} \int d^3k \ \omega_{\boldsymbol{k}}\left(a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}} + \frac{\delta^3(0)}{2}\right) \tag{209}$$

Please note that the $\boldsymbol{x}$ dependence cancels out. The second term on the right-hand side is due to the sum over the zero point energies of the uncountable many, continuously distributed harmonic oscillators of the field, which obviously diverges. Physicist have therefore introduced a normal ordering symbol $: \cdot :$, which is defined such that all occurring products of operators in $\cdot$ are reordered in such that all creational operators are moved to the left and all annihilation operators to the right. Since energy can only be relatively measured one defines

$$H := \int d^3k \ \frac{\omega_{\boldsymbol{k}}}{2} \ : a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}} + a_{\boldsymbol{k}}a^\dagger_{\boldsymbol{k}} := \int d^3k \ \omega_{\boldsymbol{k}}a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}} \tag{210}$$

to be the Hamiltonian to work with. In contrary to the functionals ϕ and π this Hamiltonian can be understood as an operator on the Fock-space spanned by the representation $a^\dagger_{\boldsymbol{k}}, a_{\boldsymbol{k}}$ of the commutator algebra. Since this operator is obviously symmetric and its domain and the one of its adjoint are equal, H is even self-adjoint. Now it is the first time that we can make a picture about how our theory will look like, i.e. depending on the chosen representation we have an underlying Hilbert-space, in this case the Fock-space, on which we have a self-adjoint Hamiltonian. Unfortunately even now we can not say much more about the position operator except that a natural choice in the case of the free fields could be the representation $f = 0$ and $|| \langle x|\Psi\rangle ||^2 dx$ for the position probability measure. Let us now turn to the introduction of interactions.

4.2.2 Scalar interaction

Again we take the classical theory as a starting point and motivate a quantum version of the dynamic classical Nelson Hamiltonian (116)

$$H = \sum_{i=1}^N \frac{\boldsymbol{p}_i^2}{2M} + \frac{1}{2}\int d^3x \ \left(\pi^2(\boldsymbol{x}) + (\nabla\phi(\boldsymbol{x}))^2 + \mu^2\phi^2(\boldsymbol{x})\right) + g\sum_{i=1}^N \int d^3x \ \rho_\kappa(\boldsymbol{q}_i - \boldsymbol{x})\phi(\boldsymbol{x}) \tag{211}$$

via a representation of the commutator algebra. Beside the field ϕ and its canonical conjugate π additional variables $\boldsymbol{q}_i, \boldsymbol{p}_i$, the position and momentum of the i-th nucleon, for $i = 1, ..., N$ appear. Again using the correspondence principle we replace them by their quantum analogs, the position operator $\widehat{\boldsymbol{x}}_i$ and momentum operator $\widehat{\boldsymbol{p}}_i$ which obey the common commutation relations (16). In order to arrive at the Hamiltonian (20) we could then just plug in the quantum version of the field

ϕ and π from line (207) and (208) and hence yield

$$H = \sum_{i=1}^{N} \frac{\boldsymbol{p}_i^2}{2M} + \int d^3k\ a_{\boldsymbol{k}}^\dagger \omega_{\boldsymbol{k}} a_{\boldsymbol{k}} + g \sum_{i=1}^{N} \int d^3x\ \rho_\kappa(\widehat{\boldsymbol{x}}_i - \boldsymbol{x}) \times \tag{212}$$

$$\times \int d\!\!\!{}^-{}^3k\ \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}} (a_{\boldsymbol{k}} e^{i\boldsymbol{kx}} + a_{\boldsymbol{k}}^\dagger e^{-i\boldsymbol{kx}}) \tag{213}$$

$$= \sum_{i=1}^{N} \frac{\boldsymbol{p}_i^2}{2M} + \int d^3k\ a_{\boldsymbol{k}}^\dagger \omega_{\boldsymbol{k}} a_{\boldsymbol{k}} + g \sum_{i=1}^{N} \int d^3k\ \frac{\widehat{\rho}_\kappa(\boldsymbol{k})}{\sqrt{2\omega_{\boldsymbol{k}}}} (a_{\boldsymbol{k}} e^{i\boldsymbol{k}\widehat{\boldsymbol{x}}_i} + a_{\boldsymbol{k}}^\dagger e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}_i}) \tag{214}$$

where $\widehat{\rho}_\kappa$ is the Fourier transformed ρ_κ - see line (14). This is the dynamic quantum Nelson Model Hamiltonian, which we have already introduced in the introductory subsection 2.2 - having in mind that we defined $\gamma_\kappa(\boldsymbol{k}) := \frac{\widehat{\rho}_\kappa(\boldsymbol{k})}{\sqrt{2\omega_{\boldsymbol{k}}}}$ in line (12). For finite κ this Hamiltonian is self-adjoint on for example the same Fock-space like the free Hamiltonian was defined on - see theorem 2.4.1. For the point particle limit $\kappa \to \infty$ however H is as it stands a merely formal object. For the limiting case $\boldsymbol{p}^2 \ll M^2$ we also arrive at the static quantum Nelson model

$$H = \int d^3k\ a_{\boldsymbol{k}}^\dagger \omega_{\boldsymbol{k}} a_{\boldsymbol{k}} + g \sum_{i=1}^{N} \int d^3k\ \frac{\widehat{\rho}_\kappa(\boldsymbol{k})}{\sqrt{2\omega_{\boldsymbol{k}}}} (a_{\boldsymbol{k}} e^{i\boldsymbol{k}\widehat{\boldsymbol{x}}_i} + a_{\boldsymbol{k}}^\dagger e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}_i}) \tag{215}$$

4.2.3 Comment on the choice of representation of the commutator algebra

Beside the fact that we arrived at the dynamic quantum Nelson Hamiltonian in this way one could now ask why we would want to do so and plug in the field, which we have found for the free case, into a Hamiltonian which shall describe the interaction case. Recall how we arrived at the free Hamiltonian in the last subsection. We computed a general solution of the homogeneous field equation, tried to find a quantum analog by choosing a representation of the commutator algebra and plugged that into the Hamiltonian. If we take this motivation in defining the free Hamiltonian any serious then why would we not want to do the same in the case of the interaction? Let us investigate this issue in the case of the static quantum Nelson model. In subsection 3.3 the classical Hamiltonian was given by

$$H = \frac{1}{2} \int d^3x\ \ (\pi^2(\boldsymbol{x}) + (\nabla\phi(\boldsymbol{x}))^2 + \mu^2\phi^2(\boldsymbol{x})) + g \sum_{i=1}^{N} \int d^3x\ \rho_\kappa(\boldsymbol{q}_i - \boldsymbol{x})\phi(\boldsymbol{x}) \tag{216}$$

Again we take the solution $\phi_0 + \phi_I$ to the inhomogeneous field equation we computed subsection 3.3 as a starting point. Please note that because of the non-uniqueness of the solution of an inhomogeneous differential equation we get an additional ambiguity beside the representation of the commutator algebra in our theory. From line (168) we infer that

$$\phi(\boldsymbol{x}) = \phi_0(\boldsymbol{x}) + \phi_I(\boldsymbol{x}) \tag{217}$$

$$= \int d^3k \left(\left(A(\boldsymbol{k}) - g \sum_{i=1}^{N} \frac{\widehat{\rho}_\kappa^*(\boldsymbol{k})}{2(2\pi)^{3/2}\omega_{\boldsymbol{k}}^2} e^{-i\boldsymbol{k}\boldsymbol{q}_i} \right) e^{i\boldsymbol{kx}} + \right. \tag{218}$$

$$\left. + \left(A^*(\boldsymbol{k}) - g \sum_{i=1}^{N} \frac{\widehat{\rho}_\kappa^*(\boldsymbol{k})}{2(2\pi)^{3/2}\omega_{\boldsymbol{k}}^2} e^{i\boldsymbol{k}\boldsymbol{q}_i} \right) e^{-i\boldsymbol{kx}} \right) \tag{219}$$

$$\pi(\boldsymbol{x}) = \int d^3k\ (-i\omega_{\boldsymbol{k}}) \left(\left(A(\boldsymbol{k}) - g \sum_{i=1}^{N} \frac{\widehat{\rho}_\kappa^*(\boldsymbol{k})}{2(2\pi)^{3/2}\omega_{\boldsymbol{k}}^2} e^{-i\boldsymbol{k}\boldsymbol{q}_i} \right) e^{i\boldsymbol{kx}} + \right. \tag{220}$$

$$\left. + \left(A^*(\boldsymbol{k}) - g \sum_{i=1}^{N} \frac{\widehat{\rho}_\kappa^*(\boldsymbol{k})}{2(2\pi)^{3/2}\omega_{\boldsymbol{k}}^2} e^{i\boldsymbol{k}\boldsymbol{q}_i} \right) e^{-i\boldsymbol{kx}} \right) \tag{221}$$

Computing the commutator relations gives the same results as in lines (203) and (204). That is because the only degree of freedom in this type of solution lies in the choice of A. Following the recipe we make a choice for a representation of the commutator algebra, say $f = 0$ like the one we also used in the free case, replace $\boldsymbol{q}_i$ with $\widehat{\boldsymbol{x}}_i$ and $\boldsymbol{p}_i$ with $\widehat{\boldsymbol{p}}_i$, plug the field into the Hamiltonian, apply the ordering symbol $:\cdot:$ and obtain

$$H = \int d^3k\ a^\dagger_{\boldsymbol{k}}\omega_{\boldsymbol{k}}a_{\boldsymbol{k}} - g^2\sum_{i,j=1}^{N}\int d^3k\ \frac{|\widehat{\rho}_\kappa(\boldsymbol{k})|^2}{2\omega^2_\kappa(\boldsymbol{k})}e^{ik(\widehat{\boldsymbol{x}}_i-\widehat{\boldsymbol{x}}_j)} \tag{222}$$

$$= \int d^3k\ a^\dagger_{\boldsymbol{k}}\omega_{\boldsymbol{k}}a_{\boldsymbol{k}} \underbrace{-g^2\sum_{i\neq j}^{N}\int d^3k\ \frac{|\widehat{\rho}_\kappa(\boldsymbol{k})|^2}{2\omega^2_\kappa(\boldsymbol{k})}e^{ik(\widehat{\boldsymbol{x}}_i-\widehat{\boldsymbol{x}}_j)}}_{=:V} \underbrace{-N\cdot g^2\int d^3k\ \frac{|\widehat{\rho}_\kappa(\boldsymbol{k})|^2}{2\omega^2_\kappa(\boldsymbol{k})}}_{=:NV_{SE}} \tag{223}$$

$$\tag{224}$$

In the point particle limit $\kappa \to \infty$ we get

$$V = -\frac{g^2}{2}\sum_{i\neq j}^{N}\frac{e^{-\mu||\widehat{\boldsymbol{x}}_i-\widehat{\boldsymbol{x}}_j||}}{4\pi||\widehat{\boldsymbol{x}}_i-\widehat{\boldsymbol{x}}_j||} \tag{225}$$

$$\tag{226}$$

the pairwise Yukawa potential and a diverging self-energy V_{SE}, so formally

$$NV_{SE} = -N\frac{g^2}{2}\int d\!\!\!^{-3}k\ \underbrace{\frac{1}{\omega^2_\kappa(\boldsymbol{k})}}_{\sim|\boldsymbol{k}|^{-2}} \tag{227}$$

We note that both potentials are the equal to their classical analogs in lines (186) and (187) - with hats taken away of course.

So following the *quantization* recipe yields beside the free field term $\int d^3k\ a^\dagger_{\boldsymbol{k}}\omega_{\boldsymbol{k}}a_{\boldsymbol{k}}$ a N particle wave equation with a Yukawa pair potential and seems to separate the divergent terms without any additional effort - at least here in the static source case. With appropriate boundary conditions the Hamiltonian $H - NV_{SE}$ is a self-adjoint operator on for example the same Fock-space like the free Hamiltonian was defined on. Let us next consider another representation of the commutator algebra than $f = 0$. The choice

$$f(\boldsymbol{k}) := g\sum_{i=1}^{N}\frac{\widehat{\rho}_\kappa(\boldsymbol{k})}{2(2\pi)^{3/2}\omega^2_{\boldsymbol{k}}}e^{i\boldsymbol{k}\widehat{\boldsymbol{x}}_i} \tag{228}$$

will result in static quantum Nelson Hamiltonian (215). So both Hamiltonians (215) and (222) can result from the same *quantization* recipe by choosing different representations of the commutator algebra. We have argued that the Hamiltonian resulting from the representation $f = 0$ minus the divergent NV_{SE} term can be given a meaning as self-adjoint operator on the Fock-space on which the free Hamiltonian was defined on. This raises the questions if we could do the same for the Hamiltonian resulting from the latter representation (228) and if that can be done how these two representations are related to each other. The only thing that strikes at first glance is that in our case $f \notin \mathcal{L}_2(\mathbb{R}^3)$ for $\kappa \to \infty$ and hence both representations are not unitary equivalent in the point particle limit. Which one of both representations is physically relevant is a question that can not be answered until we know what the theory is about, i.e. for the case of particles what the appropriate position probability measure is. It is also conceivable that both representations with their own position probability measures respectively may yield the same predictions. In the end all this plugging and playing and putting hats on variables is an explanation why things are written down in the way they are but does not help to arrive at a complete theory nor does it answer the question about what the theory is. So the physical content of this work almost ends here because unfortunately we shall have a lot of trouble with giving all these ill-defined interaction Hamiltonians a mathematical meaning as self-adjoint operators on a Fock-space before we are able to reconsider the physics they are supposed to model.

4.3 Toy model

In the previous introduction to this section the static and dynamic quantum Nelson model Hamiltonians have been motivated and we are ready to discuss on how to apply Nelson's renormalization concept from section 2.6 on them in order to obtain self-adjoint operators on a Fock-space. Unfortunately the static quantum Nelson model can not be renormalized in this way as we shall see in subsection 4.4 and the renormalization of the dynamic quantum Nelson model involves awkward computations hiding the, in principle simple, ideas that shall answer the remaining questions about the renormalized Hamiltonian $\widehat{H}$ we have posed in subsection 2.8:

1. What does $\widehat{H}$ look like?

2. What does $\mathcal{D}(\widehat{H})$ look like?

3. What does the action of $\widehat{H}$ on elements in $\mathcal{D}(\widehat{H})$ look like?

Therefore we come up with a simple toy model for which the ideas can easily be grasped and all three question can be answered comprehensively. The toy model Hamiltonian as it stands is at first ill-defined for the same reason like the static and dynamic Nelson model Hamiltonians are. However computations and especially giving answers to the above questions will be a lot simpler than in the dynamic Nelson model[14]. Note that in the case of the here discussed toy model we shall label the Hamiltonian by T and not by H.

4.3.1 Definition of the model

The toy model Hamiltonian shall be given as the formal expression

$$\boxed{T_\kappa := \int d^3k \; a^\dagger_{\boldsymbol{k}} D(\boldsymbol{k}) a_{\boldsymbol{k}} + g \int d^3k \; I_\kappa(\boldsymbol{k}) \left(a^\dagger_{\boldsymbol{k}} + a_{\boldsymbol{k}} \right)} \tag{229}$$

with

$$D : \mathbb{R}^3 \to \mathbb{R}, \tag{230}$$
$$D(\boldsymbol{k}) \neq 0 \text{ for all } \boldsymbol{k} \in \mathbb{R}^3, \tag{231}$$
$$I_\kappa \in C_c^\infty(\mathbb{R}^3) \text{ for all } \kappa < \infty, \tag{232}$$
$$I_\infty : \mathbb{R}^3 \to \mathbb{R}, I_\infty \notin \mathcal{L}_2(\mathbb{R}^3) \text{ but} \tag{233}$$
$$I_\kappa(\boldsymbol{k})/D(\boldsymbol{k}) \in \mathcal{L}_2(\mathbb{R}^3) \text{ for all } \kappa \leq \infty \text{ (!)} \tag{234}$$

As it stands this operator is not well-defined because the creation operator is smeared out with a function that is not in $\mathcal{L}_2(\mathbb{R}^3)$ and hence is not a well-defined operator on $\mathcal{F}_{mes}$. Note that the situation here is exactly similar to the one we have examined in the dynamic quantum Nelson model - recall subsection 2.3. For the existence of an unitary time evolution we need the self-adjointness of T_κ. This is with what we start.

[14] That is due to the artificial construction of the toy model. We have chosen to combine the property of having a simple structured ground state from the static Nelson model and the property of being renormalizable adopted from the dynamic quantum Nelson model. Nevertheless this toy model can also be given a physical meaning, which shall reveal an interesting fact about renormalization concepts of the kind and is discussed in 4.3.8.

4.3.2 Self-adjointness of T_κ

The self-adjointness of T_κ for finite κ can be shown with the help of Kato's theorem [13] like we already did in the proof to theorem 2.4.1 for the dynamic quantum Nelson model. We define

$$T_0 \quad := \quad \int d^3k\; a^\dagger_{\boldsymbol{k}} D(\boldsymbol{k}) a_{\boldsymbol{k}} \tag{235}$$

$$T_{I_\kappa} \quad := \quad \int d^3k\; I_\kappa(\boldsymbol{k}) \left(a^\dagger_{\boldsymbol{k}} + a_{\boldsymbol{k}}\right) \tag{236}$$

and hence $T_\kappa = T_0 + gT_{I_\kappa}$. Since T_0 is obviously self-adjoint then by Kato's theorem T_κ is self-adjoint on $\mathcal{D}(T_0)$ if T_{I_κ} is only a small perturbation to T_0 in the sense of Kato. That means if there exist positive constants $a < 1$ and $ab < \infty$ such that for all $|\psi\rangle \in \mathcal{D}(T_0)$

$$||gT_{I_\kappa} |\psi\rangle || \leq a||T_0 |\psi\rangle || + b|| |\psi\rangle || \tag{237}$$

This computation is absolutely analogous[15] to what we did in the proof of theorem 2.4.1 and therefore we will not repeat it. By doing this computation one finds that (237) holds and T_κ is self-adjoint on $\mathcal{D}(T_0)$ for finite κ.

4.3.3 Perturbation theory

In order to apply regular perturbation theory of the ground state of an operator $T_\kappa = T_0 + gT_{I_\kappa}$ with respect to the coupling constant g we need to prove analyticity of this operator, i.e. it has to be possible to expand the perturbed ground state and its energy eigenvalue in a Taylor series in the coupling constant g with some reasonable radius of convergence. We are interested in the ground state of T_κ so following [19, XII] we need a property that is called analyticity of type (A) near $g = 0$ in the perturbation theory of linear operators. This means the following. Let $E_0 \in \mathbb{R}$ be the ground state eigenvalue of T_0 then analyticity of type (A) ensures that if E_0 is an isolated and non-degenerate point in the spectrum $\sigma(T_0)$ there exists a unique point $E_\kappa(g) \in \sigma(T_\kappa)$, the eigenvalue of the perturbed ground state, which is again isolated and non-degenerate and moreover then $E_\kappa(g)$ and the ground state itself are analytic near $g = 0$. By [19, XII.9] T_κ is an analytic family of type (A) in $g \in \mathbb{R}$ near $g = 0$ if

1. $\mathcal{D}(T_{I_\kappa}) \supset \mathcal{D}(T_0)$
2. for some $a, b \in \mathbb{R}$ and $\forall\, |\psi\rangle \in \mathcal{D}(T_0)$, $||gT_{I_\kappa}|| \leq a||T_0 |\psi\rangle || + b|| |\psi\rangle ||$

(1.) is given by $\mathcal{D}(T_{I_\kappa}) = \mathcal{D}(\sqrt{\mathcal{N}^1_{mes}}) \supset \mathcal{D}(\mathcal{N}^D_{mes}) = \mathcal{D}(T_0)$ and (2.) by line (237) for finite κ. It is left to show that the infimum of the spectrum of T_0 is an isolated and non-degenerate point in the spectrum. All generalized eigenvectors of $T_0^{\boldsymbol{p}}$ are given by $|\boldsymbol{k}_1, ..., \boldsymbol{k}_n\rangle$ for all $n \geq 0$ because of the completeness $\mathcal{F}_{mes} = \text{span}\{|\boldsymbol{k}_1, ..., \boldsymbol{k}_n\rangle\}_{n\in\mathbb{N}_0}$. By computation we find that

$$T_0 |\boldsymbol{k}_1, ..., \boldsymbol{k}_n\rangle = \sum_{i=0}^{n} D(\boldsymbol{k}_i) |\boldsymbol{k}_1, ..., \boldsymbol{k}_n\rangle \tag{238}$$

and since $D(\boldsymbol{k}) \neq 0$ for all $\boldsymbol{k} \in \mathbb{R}^3$ by (230p) $\inf \sigma(T_0) = 0$ with the corresponding generalized eigenvector $|0\rangle$. The eigenvalue $E_0 = 0$ is obviously non-degenerate and a lower bound on the gap ΔE between E_0 and the next neighboring eigenvalue in the spectrum $D(\boldsymbol{k})$ is $0 < \Delta E < \inf_{\boldsymbol{k}\in\mathbb{R}^3}\{D(\boldsymbol{k})\}$. The ground state and its eigenvalue $E_\kappa(g)$ are from regular perturbation theory given by

$$|\Psi_\kappa(g)\rangle \quad = \quad \frac{1}{2\pi i} \oint_{\mathcal{C}(E_0)} d\zeta \frac{1}{T_0 + gT_{I_\kappa} - \zeta} |0\rangle \tag{239}$$

$$E_\kappa(g) \quad = \quad \frac{\langle 0|T_\kappa|\Psi_\kappa(g)\rangle}{\langle 0|\Psi_\kappa(g)\rangle} = E_0 + g\frac{\langle 0|T_{I_\kappa}|\Psi_\kappa(g)\rangle}{\langle 0|\Psi_\kappa(g)\rangle} \tag{240}$$

[15] Since I_κ and γ_κ are both elements of $\mathcal{L}_2(\mathbb{R}^3)$ for finite κ.

where $\mathcal{C}(E_0)$ is a circle around E_0 in the complex plane with a radius smaller than ΔE. The operator

$$P_\kappa^{E_n}(g) := \frac{1}{2\pi i}\oint_{\mathcal{C}(E_n)} d\zeta \frac{1}{T_0 + gT_{I_\kappa} - \zeta} \tag{241}$$

is usually called the projector to the perturbed eigenvectors. Let $P_\kappa := P_\kappa^{E_0}(g)$ for the ease of writing. Every $P_\kappa^{E_n}(g)$ projects a generalized eigenvector $|\boldsymbol{k}_1, ..., \boldsymbol{k}_n\rangle$ of T_0 with the isolated and non-degenerate eigenvalue $E_n := \sum_{i=0}^n D(\boldsymbol{k}_i)$ to its corresponding perturbed eigenvector of T_κ. We will explanatorily compute the ground state in the following. With these formulas we can write the eigenvalue equations as

$$T_\kappa |0\rangle \;=\; E_0 |0\rangle \tag{242}$$

$$T_\kappa P_\kappa(g)|0\rangle = T_\kappa |\Psi_\kappa(g)\rangle \;=\; E_\kappa(g)|\Psi_\kappa(g)\rangle = E_\kappa(g)P_\kappa(g)|0\rangle \tag{243}$$

Since we have shown that the ground state is analytic around $g = 0$ we may expand it in a Taylor series around $g = 0$. This yields

$$|\Psi_\kappa(g)\rangle \;=\; P_\kappa(g)|0\rangle = \frac{1}{2\pi i}\oint_{\mathcal{C}(E_0)} d\zeta \sum_{n=0}^\infty \frac{(-g)^n}{T_0 - \zeta}\left(T_{I_\kappa}\frac{1}{T_0-\zeta}\right)^n |0\rangle \tag{244}$$

With the method of residues the complex integral can be directly computed and we obtain

$$|\Psi_\kappa(g)\rangle \;=\; \sum_{n=0}^\infty (-g)^n \int d^3k_1 \,...\int d^3k_n \;\times \tag{245}$$

$$\times \frac{1}{\sum_{j=1}^n D(\boldsymbol{k}_j)}\cdot\frac{1}{\sum_{j=1}^{n-1} D(\boldsymbol{k}_j)}\cdot ...\cdot\frac{1}{D(\boldsymbol{k}_1)}\prod_{i=1}^n I_\kappa(\boldsymbol{k}_i)a^\dagger_{\boldsymbol{k}_i}|0\rangle \tag{246}$$

$$\tag{247}$$

now using the symmetry in the coordinates $\boldsymbol{k}_1, ..., \boldsymbol{k}_n$ we can write the above as

$$|\Psi_\kappa(g)\rangle \;=\; \sum_{n=0}^\infty (-g)^n \int d^3k_1 \,...\int d^3k_n \;\times \tag{248}$$

$$\times\frac{1}{n!}\sum_{\mathbb{SP}^l_n\{\boldsymbol{k}_1,...,\boldsymbol{k}_n\}} \frac{1}{\sum_{j=1}^n D(\boldsymbol{k}_{l_j})}\cdot\frac{1}{\sum_{j=1}^{n-1} D(\boldsymbol{k}_{l_j})}\cdot ...\cdot\frac{1}{D(\boldsymbol{k}_{l_1})}\times \tag{249}$$

$$\times\prod_{i=1}^n I_\kappa(\boldsymbol{k}_i)a^\dagger_{\boldsymbol{k}_i}|0\rangle \tag{250}$$

$$\tag{251}$$

where for some countable set A the sum over $\mathbb{SP}^i_j A$ denotes the sum over all symmetric permutation of j elements in A indexed by $i_1, ..., i_j$. The introduced sum collapses into a product

$$\sum_{\mathbb{SP}^l_n\{\boldsymbol{k}_1,...,\boldsymbol{k}_n\}} \frac{1}{\sum_{j=1}^n D(\boldsymbol{k}_{l_j})}\cdot\frac{1}{\sum_{j=1}^{n-1} D(\boldsymbol{k}_{l_j})}\cdot ...\cdot\frac{1}{D(\boldsymbol{k}_{l_1})} = \prod_{i=1}^n \frac{1}{D(\boldsymbol{k}_i)} \tag{252}$$

and hence

$$|\Psi_\kappa(g)\rangle \;=\; \sum_{n=0}^\infty \frac{(-g)^n}{n!}\int d^3k_1 \,...\int d^3k_n \prod_{i=1}^n \frac{I_\kappa(\boldsymbol{k}_i)}{D(\boldsymbol{k}_i)}a^\dagger_{\boldsymbol{k}_i}|0\rangle \tag{253}$$

$$=\; e^{-g\int d^3k\,\frac{I_\kappa(\boldsymbol{k})}{D(\boldsymbol{k})}a^\dagger_{\boldsymbol{k}}}|0\rangle \tag{254}$$

so we can read off the projector

$$\boxed{P_\kappa(g) = e^{-g\int d^3k\ \frac{I_\kappa(k)}{D(k)} a_k^\dagger}} \tag{255}$$

which is a well-defined operator on $\mathcal{F}_{mes}$ even in the limit $\kappa \to \infty$ as we shall see in the proceeding subsection. So the eigenvalue of the ground state can be computed easily by

$$\begin{aligned}
E_\kappa(g) &= \frac{\langle 0|T_\kappa P_\kappa(g)|0\rangle}{\langle 0|P_\kappa(g)|0\rangle} = E_0 + g\frac{\langle 0|T_{I_\kappa} P_\kappa(g)|0\rangle}{\langle 0|P_\kappa(g)|0\rangle} &(256)\\
&= 0 + g\frac{\langle 0|\int d^3k\ I_\kappa(\boldsymbol{k})(a_{\boldsymbol{k}} + a_{\boldsymbol{k}}^\dagger)P_\kappa(g)|0\rangle}{\langle 0|P_\kappa(g)|0\rangle} &(257)\\
&= g\frac{\langle 0|\int d^3k\ I_\kappa(\boldsymbol{k})a_{\boldsymbol{k}}P_\kappa(g)|0\rangle}{\langle 0|P_\kappa(g)|0\rangle} &(258)\\
&= -g^2\int d^3k\ \frac{I_\kappa^2(\boldsymbol{k})}{D(\boldsymbol{k})}\frac{\langle 0|P_\kappa(g)|0\rangle}{\langle 0|P_\kappa(g)|0\rangle} &(259)\\
&= -g^2\int d^3k\ \frac{I_\kappa^2(\boldsymbol{k})}{D(\boldsymbol{k})} &(260)
\end{aligned}$$

since

$$\begin{aligned}
[\int d^3k\ I_\kappa(\boldsymbol{k})a_{\boldsymbol{k}}, P_\kappa(g)] &= \left[\int d^3k\ I_\kappa(\boldsymbol{k})a_{\boldsymbol{k}}, e^{-g\int d^3l\ \frac{I_\kappa(l)}{D(l)}a_l^\dagger}\right] &(261)\\
&= \left[\int d^3k\ I_\kappa(\boldsymbol{k})a_{\boldsymbol{k}}, \sum_{n=0}^\infty \frac{(-g)^n}{n!}\left(\int d^3l\ \frac{I_\kappa(\boldsymbol{l})}{D(\boldsymbol{l})}a_{\boldsymbol{l}}^\dagger\right)^n\right] &(262)\\
&= \int d^3k\ I_\kappa(\boldsymbol{k})\sum_{n=0}^\infty \frac{(-g)^n}{n!}\left[a_{\boldsymbol{k}}, \left(\int d^3l\ \frac{I_\kappa(\boldsymbol{l})}{D(\boldsymbol{l})}a_{\boldsymbol{l}}^\dagger\right)^n\right] &(263)\\
&= \int d^3k\ I_\kappa(\boldsymbol{k})\sum_{n=0}^\infty \frac{(-g)^n}{n!}\cdot n\frac{I_\kappa(\boldsymbol{k})}{D(\boldsymbol{k})} &(264)\\
&= -g\int d^3k\ \frac{I_\kappa^2(\boldsymbol{k})}{D(\boldsymbol{k})} &(265)
\end{aligned}$$

Hence for the ground state $|\Psi\rangle_\kappa(g)$ of T_κ we find

$$\boxed{T_\kappa\,|\Psi_\kappa(g)\rangle = E_\kappa(g)\,|\Psi_\kappa(g)\rangle = -g^2\int d^3k\ \frac{I_\kappa^2(\boldsymbol{k})}{D(\boldsymbol{k})}\,|\Psi_\kappa(g)\rangle} \tag{266}$$

4.3.4 The renormalized model Hamiltonian $\widehat{T}$

In the last subsection we have calculated

$$P_\kappa(g) = \frac{1}{2\pi i}\oint_{\mathcal{C}(E_0)} d\zeta\frac{1}{T_0 + gT_{I_\kappa} - \zeta} = e^{-g\int d^3k\ \frac{I_\kappa(k)}{D(k)}a_k^\dagger} \tag{267}$$

for $|0\rangle$. In order to yield a projection onto correctly normalized states we define

$$\begin{aligned}
U_\kappa^\dagger(g) &:= e^{-g\int d^3k\ \frac{I_\kappa(k)}{D(k)}(a_k^\dagger - a_k)} &(268)\\
&=: U_\kappa^{-1}(g) &(269)
\end{aligned}$$

and its adjoint

$$U_\kappa(g) = e^{g\int d^3k\ \frac{I_\kappa(k)}{D(k)}(a_k^\dagger - a_k)} \tag{270}$$

such that

$$U_\kappa^\dagger(g)U_\kappa(g) = \mathbb{1}_{id}^{\mathcal{F}_{mes}} \tag{271}$$

Since by the requirements (230p) we have $\frac{I_\kappa(\boldsymbol{k})}{D(\boldsymbol{k})} \in \mathcal{L}_2(\mathbb{R}^3)$ even in the limit $\kappa \to \infty$. Hence the operators $U_\kappa(g), U_\kappa^\dagger(g)$ induce a unitary transformation on $\mathcal{F}_{mes}$ for $\kappa \leq \infty$ - see [3, Appendix IV] or [15, Lemma 1] with $\beta_\kappa = \frac{I_\kappa(\boldsymbol{k})}{D(\boldsymbol{k})}$ for a proof. This kind of transformation is commonly known under the name Bogoliubov transformation. Note that

$$\begin{aligned}
U_\kappa^{-1}(g)\,|0\rangle &= e^{-g\int d^3k\ \frac{I_\kappa(\boldsymbol{k})}{D(\boldsymbol{k})}(a_{\boldsymbol{k}}^\dagger - a_{\boldsymbol{k}})}\,|0\rangle && (272)\\
&= e^{-\frac{g^2}{2}\int d^3k\ |\frac{I_\kappa}{D(\boldsymbol{k})}|^2} e^{-g\int d^3k\ \frac{I_\kappa(\boldsymbol{k})}{D(\boldsymbol{k})}a_{\boldsymbol{k}}^\dagger}\,|0\rangle && (273)\\
&= e^{-\frac{g^2}{2}\int d^3k\ |\frac{I_\kappa}{D(\boldsymbol{k})}|^2} P_\kappa(g)\,|0\rangle && (274)\\
&= e^{-\frac{g^2}{2}\int d^3k\ |\frac{I_\kappa}{D(\boldsymbol{k})}|^2}\,|\Psi_\kappa(g)\rangle && (275)
\end{aligned}$$

by the Baker-Hausdorf identity[16]. This means we have found an unitary transformation that projects the ground state of T_0 to the correctly normalized ground state of T_κ and so lies near that this transformation may diagonalize the toy model Hamiltonian T_κ. The following computation shows that this intuition is correct. For convenience we define

$$\beta_\kappa(\boldsymbol{k}) := -g\frac{I_\kappa(\boldsymbol{k})}{D(\boldsymbol{k})} \tag{276}$$

and find[17]

$$\begin{aligned}
U_\kappa(g)a_{\boldsymbol{k}}^\dagger U_\kappa(g)^{-1} &= a_{\boldsymbol{k}}^\dagger + \beta(\boldsymbol{k}) && (277)\\
U_\kappa(g)a_{\boldsymbol{k}} U_\kappa(g)^{-1} &= a_{\boldsymbol{k}} + \beta(\boldsymbol{k}) && (278)
\end{aligned}$$

and hence for a suitable domain for finite κ the toy model Hamiltonian transforms according to

$$\begin{aligned}
T_\kappa' &:= U_\kappa(g)T_\kappa U_\kappa^{-1}(g) && (279)\\
&= \int d^3k\ (a_{\boldsymbol{k}}^\dagger + \beta_\kappa(\boldsymbol{k}))D(\boldsymbol{k})(a_{\boldsymbol{k}} + \beta_\kappa(\boldsymbol{k})) + && (280)\\
&\quad +g\int d^3k\ I_\kappa(\boldsymbol{k})(a_{\boldsymbol{k}}^\dagger + a_{\boldsymbol{k}} + 2\beta_\kappa(\boldsymbol{k})) && (281)\\
&\quad +g^2\int d^3k\ \frac{I^2(\boldsymbol{k})}{D(\boldsymbol{k})} && (282)\\
&= \int d^3k\ a_{\boldsymbol{k}}^\dagger D(\boldsymbol{k})a_{\boldsymbol{k}} + && (283)\\
&\quad +g\int d^3k\ \left(I_\kappa(\boldsymbol{k}) + D(\boldsymbol{k})\frac{\beta(\boldsymbol{k})}{g}\right)(a_{\boldsymbol{k}}^\dagger + a_{\boldsymbol{k}}) + && (284)\\
&\quad +g^2\int d^3k\ \left(\frac{I^2(\boldsymbol{k})}{D(\boldsymbol{k})} + 2\beta_\kappa(\boldsymbol{k})I_\kappa(\boldsymbol{k})\right) && (285)
\end{aligned}$$

plugging the above definition of $\beta_\kappa(\boldsymbol{k}) = I_\kappa(\boldsymbol{k})/D(\boldsymbol{k})$ in the equation and with

$$V_{SE}^\kappa \ := \ -g^2\int d^3k\ \frac{I_\kappa^2(\boldsymbol{k})}{D(\boldsymbol{k})} \tag{286}$$

[16] Let A,B be linear operators on a common subset of Hilbert-space with $[A,[A,B]] = [B,[A,B]] = 0$ then $e^{A+B} = e^A e^B e^{\frac{1}{2}[A,B]}$.

[17] One could say U_κ mediates between two representations of the commutator algebra or in a classical sense induces a shift in the oscillator amplitudes. This is an example of the discussed unitary equivalent representations of the commutator algebra.

we can write the transformed toy model Hamiltonian like

$$T'_\kappa = \int d^3k \; a^\dagger_{\boldsymbol{k}} D(\boldsymbol{k}) a_{\boldsymbol{k}} + V^\kappa_{SE} \tag{287}$$

which is obviously diagonal. Since V^κ_{SE} may diverge in general in the limit $\kappa \to \infty$ and thus may generate problems we define a new operator

$$\widehat{T}' = T'_\kappa - V^\kappa_{SE} = \int d^3k \; a^\dagger_{\boldsymbol{k}} D(\boldsymbol{k}) a_{\boldsymbol{k}} \tag{288}$$

which is self-adjoint on $\mathcal{F}_{mes}$ with the core $\mathcal{D}(\mathcal{N}^D_{mes})$ for all $\kappa \leq \infty$. Hence we can obtain an operator $\widehat{T}_\kappa := U^{-1}_\kappa(g)\widehat{T}' U_\kappa(g)$ on $\mathcal{F}_{mes}$ with the core $U^{-1}_\kappa(g)(\mathcal{D}(\mathcal{N}^D_{mes}))$ as the following. From the self-adjointness we get

$$\widehat{T}' \left|\psi\right\rangle \quad \in \quad \mathcal{F}_{mes} \tag{289}$$

so using the unitary of U_κ, U^{-1}_κ

$$\widehat{T}' U_\kappa(g) U^{-1}_\kappa(g) \left|\psi\right\rangle \quad \in \quad \mathcal{F}_{mes} \tag{290}$$

and since $U_\kappa(g)$ is well-defined on the Fock-space

$$\underbrace{U^{-1}_\kappa(g)\widehat{T}' U_\kappa(g)}_{=\widehat{T}} U^{-1}_\kappa(g) \left|\psi\right\rangle \quad \in \quad \mathcal{F}_{mes} \tag{291}$$

The operator $\widehat{T} := U_\kappa(g)^{-1}\widehat{T}' U_\kappa(g)$ is obviously symmetric since $\widehat{T}'$ is and the domain of it and its adjoint are equal. Hence $\widehat{T}_\kappa$ is self-adjoint on $\mathcal{F}_{mes}$ with the core $U_\kappa(g)^{-1}(\mathcal{D}(\mathcal{N}^D_{mes}))$. The transformation $U_\kappa(g), U^{-1}_\kappa(g)$ stays unitary for all $\kappa \leq \infty$ and $\widehat{T}'$ is independent of κ hence

$$s - \lim_{\kappa\to\infty} U_\kappa(g)^{-1}\widehat{T}' U_\kappa(g) = s - \lim_{\kappa\to\infty}(T_\kappa - V^\kappa_{SE}) = \widehat{T}_\infty =: \widehat{T} \tag{292}$$

and from the self-adjointness on $\mathcal{F}_{mes}$ we get the convergence of one parameter unitary group

$$s - \lim_{\kappa\to\infty} e^{iU_\kappa(g)^{-1}\widehat{T}' U_\kappa(g)t} = s - \lim_{\kappa\to\infty} e^{i(T_\kappa - V^\kappa_{SE})t} = e^{i\widehat{T}t} \tag{293}$$

We shall call $\widehat{T}$ the renormalized toy model Hamiltonian. Now we arrived at the same stage like after the proof of Nelson's theorem 2.6.1 in the dynamic quantum Nelson model except that now we know the domain of the renormalized toy model Hamiltonian $\widehat{T}$.

$$\boxed{\mathcal{D}(\widehat{T}) = U^{-1}_\infty(\mathcal{D}(\mathcal{N}^D_{mes}))} \tag{294}$$

Furthermore by a theorem of Trotter [17, VIII.21] we get the strong convergence of the resolvent of $T_\kappa - V^\kappa_{SE}$ from the strong convergence of the one parameter unitary group of T_κ. Hence $s - \lim_{\kappa\to\infty} P_\kappa(g)$ and so $s - \lim_{\kappa\to\infty} U^{\pm 1}_\kappa(g)$ are well-defined operators on $\mathcal{F}_{mes}$. We denote them by $P_\infty(g)$ and $U^{\pm 1}_\infty(g)$. That means there exists a ground state in $\mathcal{F}_{mes}$ even in the limit $\kappa \to \infty$. That has the immediate consequence that

$$\begin{aligned} |\widehat{\Psi}(g)\rangle \quad &:= \quad (s - \lim_{\kappa\to\infty} U^{-1}_\kappa(g)) \left|0\right\rangle && (295) \\ &= \quad \lim_{\kappa\to\infty} (U^{-1}_\kappa(g)) \left|0\right\rangle) \in \mathcal{F}_{mes} && (296) \end{aligned}$$

is the ground state of $\widehat{T}$ since $|\Psi_\kappa(g)\rangle$ is the ground state of both operators T_κ and $(T_\kappa - V^\kappa_{SE})$ which the following consideration shows

$$\begin{aligned} T_\kappa \left|\Psi_\kappa(g)\right\rangle \quad &= \quad E_\kappa(g) \left|\Psi_\kappa(g)\right\rangle && (297) \\ (T_\kappa - V^\kappa_{SE}) \left|\Psi_\kappa(g)\right\rangle \quad &= \quad (E_\kappa(g) - V^\kappa_{SE}) \left|\Psi_\kappa(g)\right\rangle && (298) \end{aligned}$$

This yields

$$\boxed{\widehat{T}\,|\widehat{\Psi}(g)\rangle = s - \lim_{\kappa\to\infty}(T_\kappa - V_{SE}^\kappa)\,|\widehat{\Psi}(g)\rangle = \lim_{\kappa\to\infty}(E_\kappa(g) - V_{SE}^\kappa)\,|\widehat{\Psi}(g)\rangle} \tag{299}$$

We define

$$\widehat{E}(g) := \lim_{\kappa\to\infty}(E_\kappa(g) - V_{SE}^\kappa) \tag{300}$$

which has to be finite according to the existence of the resolvent and indeed by inserting $E_\kappa(g)$ from subsection 4.3.3 we find that $\widehat{E}(g) = 0$.

4.3.5 What does $\widehat{T}$ look like?

From our computation in the last subsection we have found that

$$\widehat{T} := s - \lim_{\kappa\to\infty} U_\kappa(g)^{-1}\widehat{T}' U_\kappa(g) = s - \lim_{\kappa\to\infty}(T_\kappa - V_{SE}^\kappa) \tag{301}$$

and hence the answer concerning an explicit well-defined expression for the operator is

$$\widehat{T} := U_\infty(g)^{-1}\widehat{T}' U_\infty(g) \tag{302}$$

However $U_\infty(g)$ is only that explicit because we were able to calculate the projector $P_\kappa(g)$ in the perturbation theory of subsection 4.3.3 on which it depends. This could be done that easily because we chose the toy model in such a way that it has a simple structured ground state what is basically due to the fact that we nailed down the source density at the origin. In the dynamic quantum Nelson model however it will turn out to be hard to explicitly calculate the projector in terms of regular perturbation theory. So we have to get along with the formal expression

$$\widehat{T} := s - \lim_{\kappa\to\infty}(T_\kappa - V_{SE}^\kappa) \tag{303}$$

But since V_{SE}^κ may diverge in general in the limit $\kappa \to \infty$ how can we give this expression a mathematical sense? The following example shows that this can be done in what we shall call a weak sense. This means the following. At first the formal action of $(T_\kappa - V_{SE}^\kappa)$ on an element in $\mathcal{D}(\widehat{T})$ is computed. Elements in this domain will be of such a kind that then the, in the limit $\kappa \to \infty$, appearing divergent terms cancel each other in that expression such that remains a well-defined mathematical object. In contrary for all other Fock-vectors the divergent terms will in general not cancel so it will not be possible to obtain a well-defined expression. Although this sounds adventurous it is already the standard way to deal with initially ill-defined differential operators in many areas of mathematical physics. The following example illustrates how the Sturm-Liouville theory deals in a similar way with singular differential equations.

Example 4.3.1. *Let T be a linear operator formally given by*

$$T := a\frac{d^2}{dx^2} + b\frac{1}{x^2} \tag{304}$$

for e.g. $a, b \in \mathbb{R}$, which shall be defined on a subspace of $\mathcal{L}_2(\mathbb{R})$. Clearly the operator is initially not well-defined at $x = 0$ as it stands. Say $f \in \mathcal{L}_2(\mathbb{R})$ then from the action of T on f

$$(Tf)(x) = af''(x) + b\frac{f(x)}{x^2} \tag{305}$$

we can read of the properties the function f must have in order to yield a well-defined expression for all $x \in \mathbb{R}$. How can we then make sense out of T without excluding the point 0? Now on the one hand it can be shown that $\frac{b}{x^2}$ is a small perturbation in the sense of Kato to the self-adjoint operator $a\frac{d^2}{dx^2}$ on $\mathcal{D}(\frac{d^2}{dx^2}) := \{\varphi \in \mathcal{L}_2(\mathbb{R}^2) | k^2(F\varphi(k)) \in \mathcal{L}_2(\mathbb{R})\}$, where F denotes the Fourier *transform. On the other hand we know from the theory of singular Sturm-Liouville operators like T that these operators are self-adjoint on various other domains - see [14, Sec. 32]. Boundary conditions are used to fix the domains of the Sturm-Liouville operators. In order to get back to our simple example mainly two things can happen when we want to give T a meaning as an operator on a subset of $\mathcal{L}_2(\mathbb{R})$. If we consider functions $f \in \mathcal{D}(\frac{d^2}{dx^2})$, hence two times differentiable, then we have to demand that at least $f(x) \sim |x|^2$ near $|x| = 0$ so that the second term $b\frac{f(x)}{x^2}$ stays finite near $|x| \to 0$. A typical representative is*

$$f(x) := x^2 e^{-x^2} \tag{306}$$

with f and Tf in $\mathcal{L}_2(\mathbb{R})$. In the second case the singular behavior of $af''(x)$ has to cancel out with the one of $b\frac{f(x)}{x^2}$ at $|x| \to 0$. For the case $a > 4b$ we can easily give an example where this happens.

$$f(x) := |x|^{(1/2-\sqrt{1/4-b/a})} e^{-x^2} \tag{307}$$

for which f and Tf are again $\mathcal{L}_2(\mathbb{R})$ but now both terms $f''(x)$ and $\frac{f(x)}{x^2}$ diverge as $|x| \to \infty$. However their sum $(Tf)(x) \to 0$ is non-divergent in the limit $x \to 0$. In fact what happens in our toy model or in the Nelson model is of the latter kind. Of course these are just two examples of possible functions for which T can get a mathematical meaning as a linear operator on some subset of $\mathcal{L}_2(\mathbb{R})$. In order to yield a self-adjoint operator on $\mathcal{L}_2(\mathbb{R})$ one had to find a subset that lies dense in $\mathcal{L}_2(\mathbb{R})$. That in fact can be done as mentioned above but for now the purpose is just to understand that we are actually used to formal definitions of operators like in the case of the Schrödinger operators with singular potentials. So the moral is formal expressions of operators get a special mathematical meaning by making special choice for their domain.

On the basis of this example we can argue that it is not necessarily a well-defined expression of the operator $\widehat{T}$ that we seek but a special set $\mathcal{D} \subset \mathcal{F}_{mes}$ for which the formal action

$$\lim_{\kappa\to\infty} [(T_\kappa - V_{SE}^\kappa) |\varphi\rangle] = \widehat{T} |\varphi\rangle \tag{308}$$

for any $|\varphi\rangle \in \mathcal{D}$ is well-defined. More precisely the formal equality $\widehat{T} = s - \lim_{\kappa\to\infty}(T_\kappa - V_{SE}^\kappa)$ can be giving a mathematical meaning on a special set $\mathcal{D}$ in this weak sense. If this set lies dense in $\mathcal{F}_{mes}$ and in addition $\widehat{T}$ is self-adjoint on it we can define $\widehat{T}$ to be its closure with respect to the core $\mathcal{D}$ and in this way obtain a self-adjoint operator on $\mathcal{F}_{mes}$. In subsection 4.3.7 we compute the formal action of $\widehat{T}$ on states of its domain and argue that in our toy model $\mathcal{D} = \mathcal{D}(\widehat{T})$.

4.3.6 What does $\mathcal{D}(\widehat{T})$ look like?

We have found that $\mathcal{D}(\widehat{T}) = U_\infty^{-1}(\mathcal{D}(N_{mes}^D))$. $\mathcal{D}(N_{mes}^D)$ we know very well. It is the set of all $|\psi\rangle \in \mathcal{F}_{mes}$ such that

$$\sum_{i=1}^{\infty} \int d^3k_1 \, ... \int d^3k_n \, |D(\boldsymbol{k}_i) \langle ...; \boldsymbol{k}_1, ..., \boldsymbol{k}_n|\psi\rangle|^2 < \infty \tag{309}$$

More explicitly let $\mathcal{M}$ be the set of all finite linear combinations of n-meson Fock-vectors

$$|\psi_n\rangle := \int d^3k_1 \, ... \int d^3k_n \;\; \langle \boldsymbol{k}_1, ..., \boldsymbol{k}_n|\psi_n\rangle \, |\boldsymbol{k}_1, ..., \boldsymbol{k}_n\rangle \tag{310}$$

for which $\sum_{i=1}^n D(\boldsymbol{k}_i) \langle \boldsymbol{k}_1, ..., \boldsymbol{k}_n|\psi_n\rangle \in L_2(\mathbb{R}^{3n})$. Then the closure of $\mathcal{M}$ in $\mathcal{F}_{mes}$ denoted by $\overline{\mathcal{M}} = \mathcal{D}(\mathcal{N}_{mes}^D)$. For any Fock-vector $|\psi\rangle \in \mathcal{D}(\mathcal{N}_{mes}^D)$ the transformed Fock-vector $U_\infty^{-1} |\psi\rangle \in \mathcal{D}(\widehat{T})$.

For the above n-meson Fock-vector we obtain

$$
\begin{aligned}
U_\infty^{-1}|\psi_n\rangle &:= \int d^3k_1 \ldots \int d^3k_n \; \langle \boldsymbol{k}_1,\ldots,\boldsymbol{k}_n|\psi_n\rangle \, U_\infty^{-1}|\psi_n\rangle \, |\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n\rangle && (311)\\
&= \int d^3k_1 \ldots \int d^3k_n \; \langle \boldsymbol{k}_1,\ldots,\boldsymbol{k}_n|\psi_n\rangle \, U_\infty^{-1}\frac{1}{\sqrt{n!}} a^\dagger_{\boldsymbol{k}_1}\ldots a^\dagger_{\boldsymbol{k}_n}|0\rangle && (312)\\
&= \int d^3k_1 \ldots \int d^3k_n \; \langle \boldsymbol{k}_1,\ldots,\boldsymbol{k}_n|\psi_n\rangle \, \frac{1}{\sqrt{n!}} \times && (313)\\
&\qquad \times U_\infty^{-1}a^\dagger_{\boldsymbol{k}_1}U_\infty U_\infty^{-1}a^\dagger_{\boldsymbol{k}_2}U_\infty U_\infty^{-1}\ldots U_\infty U_\infty^{-1}a^\dagger_{\boldsymbol{k}_n}U_\infty U_\infty^{-1}|0\rangle && (314)\\
&= \int d^3k_1 \ldots \int d^3k_n \; \langle \boldsymbol{k}_1,\ldots,\boldsymbol{k}_n|\psi_n\rangle \prod_{i=1}^{n}(a^\dagger_{\boldsymbol{k}_i} - \beta_\infty(\boldsymbol{k}_i))U_\infty^{-1}|0\rangle && (315)\\
& && (316)
\end{aligned}
$$

where $\beta_\infty = I_\infty(\boldsymbol{k})/D(\boldsymbol{k}) \in \mathcal{L}_2(\mathbb{R}^3)$ by (230). Let $\mathcal{M}'$ be the set of all finite linear combinations of these transformed n-meson Fock-vectors $U_\infty^{-1}|\psi_n\rangle$ then $\mathcal{D}(\widehat{T}) = \overline{\mathcal{M}'}$. Please recall that $U_\infty^{-1}|0\rangle$ is the correctly normalized ground state of $\widehat{T}$. However as we have mentioned in the last subsection the object U_∞^{-1} is not as explicitly given in the case of the dynamic quantum Nelson model as it is here in the toy model. That is why we will not be able to do a similar computation of the domain in the dynamic quantum Nelson model. Still there is one thing we can learn from the above that holds even in the dynamic quantum Nelson model. Every excitation of the ground state $U_\infty^{-1}|0\rangle$ by finite linear combinations of the operators $\int d^3k \; f(\boldsymbol{k})a^\dagger_{\boldsymbol{k}}$ and $\int d^3k \; f(\boldsymbol{k})a_{\boldsymbol{k}}$ for any functions f such that $D(\boldsymbol{k})f(\boldsymbol{k}) \in \mathcal{L}_2(\mathbb{R}^3)$ is again in the domain of $\widehat{T}$. Excited states like these are briefly examined in the following subsection.

4.3.7 What does the action of $\widehat{T}$ on elements in $\mathcal{D}(\widehat{T})$ look like?

We have argued in subsection 4.3.5 that the expression

$$
\widehat{T} := s - \lim_{\kappa\to\infty}(T_\kappa - V_{SE}^\kappa) \tag{317}
$$

can be given a mathematical meaning on some set $\mathcal{D}$ in the sense that when formally acting on elements in $\mathcal{D}$ all divergent terms that occur cancel each other and the remaining expression is well-defined. Clearly the ground state must be in $\mathcal{D}$ and so at a first glance it seems reasonable that $\mathcal{D}(\widehat{T}) \subset \mathcal{D}$. That this is the case shall be examined in the following. Let us start with the analysis of the action of $\widehat{T}$ on the ground state. We take the above definition of $\widehat{T}$ in the weak sense

$$
\langle \boldsymbol{k}_1,\ldots,\boldsymbol{k}_n|\,\widehat{T}\,|\Psi(g)\rangle \quad = \quad \lim_{\kappa\to\infty}[\langle \boldsymbol{k}_1,\ldots,\boldsymbol{k}_n|\,(T_\kappa - V_{SE}^\kappa)\,|\Psi_\kappa(g)\rangle] \tag{318}
$$

as we have discussed in 4.3.5. Hence

$$\widehat{T}\,|\widehat{\Psi}(g)\rangle = \lim_{\kappa\to\infty}\left(\sum_{i=1}^{n} D(\boldsymbol{k}_i) - g^2\int d^3k\;\frac{I_\kappa^2(\boldsymbol{k})}{D(\boldsymbol{k})}\right)\langle \boldsymbol{k}_1,...,\boldsymbol{k}_n|\Psi_\kappa(g)\rangle + \tag{319}$$

$$+g\int d^3k\; I_\kappa(\boldsymbol{k})\sqrt{n+1}\,\langle \boldsymbol{k}_1,...,\boldsymbol{k}_n,\boldsymbol{k}|\Psi_\kappa(g)\rangle + \tag{320}$$

$$+g\sum_{i=1}^{n}\frac{I_\kappa(\boldsymbol{k})}{\sqrt{n}}\,\langle \boldsymbol{k}_1,...,\widehat{\boldsymbol{k}}_i,...,\boldsymbol{k}_n|\Psi_\kappa(g)\rangle \tag{321}$$

$$= \lim_{\kappa\to\infty}\left(\sum_{i=1}^{n} D(\boldsymbol{k}_i) - g^2\int d^3k\;\frac{I_\kappa^2(\boldsymbol{k})}{D(\boldsymbol{k})} + \right. \tag{322}$$

$$+g\int d^3k\; I_\kappa(\boldsymbol{k})\sqrt{n+1}\,\frac{\langle \boldsymbol{k}_1,...,\boldsymbol{k}_n,\boldsymbol{k}|\Psi_\kappa(g)\rangle}{\langle \boldsymbol{k}_1,...,\boldsymbol{k}_n|\widehat{\Psi}_\kappa(g)\rangle} + \tag{323}$$

$$\left. +g\sum_{i=1}^{n}\frac{I_\kappa(\boldsymbol{k})}{\sqrt{n}}\,\frac{\langle \boldsymbol{k}_1,...,\widehat{\boldsymbol{k}}_i,...,\boldsymbol{k}_n|\Psi_\kappa(g)\rangle}{\langle \boldsymbol{k}_1,...,\boldsymbol{k}_n|\Psi(g)\rangle}\right)\langle \boldsymbol{k}_1,...,\boldsymbol{k}_n|\Psi_\kappa(g)\rangle \tag{324}$$

the n-meson wave function of the ground state is by direct computation from line (253)

$$\langle \boldsymbol{k}_1,...,\boldsymbol{k}_n|\Psi_\kappa(g)\rangle = \frac{(-g)^n}{\sqrt{n!}}\prod_{i=1}^{n}\beta_\kappa(\boldsymbol{k}_i) \tag{325}$$

so

$$\langle \boldsymbol{k}_1,...,\boldsymbol{k}_n|\,\widehat{T}\,|\widehat{\Psi}(g)\rangle = \lim_{\kappa\to\infty}\left(\sum_{i=1}^{n} D(\boldsymbol{k}_i) - g^2\int d^3k\;\frac{I_\kappa^2(\boldsymbol{k})}{D(\boldsymbol{k})} + \right. \tag{326}$$

$$-g^2\int d^3k\; I_\kappa(\boldsymbol{k})\beta_\kappa(\boldsymbol{k}) + \tag{327}$$

$$\left. -\sum_{i=1}^{n}\frac{I_\kappa(\boldsymbol{k})}{\beta_\kappa(\boldsymbol{k}_i)}\right)\langle \boldsymbol{k}_1,...,\boldsymbol{k}_n|\Psi_\kappa(g)\rangle \tag{328}$$

Now $\beta_\infty(\boldsymbol{k}) = I_\infty/D(\boldsymbol{k})$. This means that the limit can not be taken term by term because from the requirements (230) the function $I_\infty^2(\boldsymbol{k})/D(\boldsymbol{k}) = I_\infty(\boldsymbol{k})\beta_\infty(\boldsymbol{k})$ does not have to be integrable. So both appearing integrals will in general fail to converge in the limit. Fortunately the ground state is of such a kind that both integrals diverge equally strong and have the opposite sign. Hence they cancel each other and we find

$$\langle \boldsymbol{k}_1,...,\boldsymbol{k}_n|\,\widehat{T}\,|\Psi(g)\rangle = \left(\sum_{i=1}^{n} D(\boldsymbol{k}_i) - \sum_{i=1}^{n}\frac{I_\infty(\boldsymbol{k})}{\beta_\infty(\boldsymbol{k}_i)}\right)\langle \boldsymbol{k}_1,...,\boldsymbol{k}_n|\widehat{\Psi}(g)\rangle \tag{329}$$

$$= 0\cdot\langle \boldsymbol{k}_1,...,\boldsymbol{k}_n|\widehat{\Psi}(g)\rangle \tag{330}$$

$$= \widehat{E}(g)\,\langle \boldsymbol{k}_1,...,\boldsymbol{k}_n|\widehat{\Psi}(g)\rangle \tag{331}$$

The eigenvalue equal zero was expected since we have already computed that eigenvalue in line (300) and found the same result. So our intuition was correct for the ground state of $\widehat{T}$ and the formal expression $\widehat{T} := s - \lim_{\kappa\to\infty}(T_\kappa - V_{SE}^\kappa)$ has a mathematical meaning for this ground state. It can be expected that the mechanism will be the same for other states in its domain. So let us take a look at excited states like the ones we have discussed in subsection 4.3.6. We take the first excited state $\int d^3k\;\varphi(\boldsymbol{k})a_{\boldsymbol{k}}^\dagger|0\rangle$ in $\mathcal{D}(\mathcal{N}_{mes}^D)$ as an example. Please note that this means $\varphi(\boldsymbol{k})D(\boldsymbol{k}) \in \mathcal{L}_2(\mathbb{R}^3)$. From subsection 4.3.6 we know that $|\varphi\rangle := U_\infty^{-1}\int d^3k\;\varphi(\boldsymbol{k})a_{\boldsymbol{k}}^\dagger|0\rangle$ is in $\mathcal{D}(\widehat{T})$ so let us examine the action of $\widehat{T}$ on this excited state

$$\widehat{T}\,|\varphi\rangle = \widehat{T}U_\infty^{-1}\int d^3k\;\varphi(\boldsymbol{k})a_{\boldsymbol{k}}^\dagger|0\rangle = \widehat{T}U_\infty^{-1}U_\infty\int d^3k\;\varphi(\boldsymbol{k})U_\infty^{-1}a_{\boldsymbol{k}}^\dagger U_\infty U_\infty^{-1}|0\rangle \tag{332}$$

$$= \widehat{T}\int d^3k\;\varphi(\boldsymbol{k})(a_{\boldsymbol{k}}^\dagger + g\beta_\infty(\boldsymbol{k}))U_\infty^{-1}|0\rangle \tag{333}$$

and let us therefore compute the commutator

$$[\widehat{T}, \int d^3l\; \varphi(\boldsymbol{l}) a_{\boldsymbol{l}}^\dagger] = \lim_{\kappa\to\infty} \Big[\int d^3k\; a_{\boldsymbol{k}}^\dagger D(\boldsymbol{k}) a_{\boldsymbol{k}} + \tag{334}$$

$$+ g \int d^3k\; I_\kappa(\boldsymbol{k})(a_{\boldsymbol{k}} + a_{\boldsymbol{k}}^\dagger) - V_{SE}^\kappa, \int d^3l\; \varphi(\boldsymbol{l}) a_{\boldsymbol{l}}^\dagger \Big] \tag{335}$$

$$= \int d^3k\; D(\boldsymbol{k})\varphi(\boldsymbol{k}) a_{\boldsymbol{k}}^\dagger + g \int d^3k\; I_\infty(\boldsymbol{k})\varphi(\boldsymbol{k}) \tag{336}$$

which is a well-defined operator on $\mathcal{D}(\sqrt{\mathcal{N}_{mes}^1})$ since $D(\boldsymbol{k})\varphi(\boldsymbol{k}) \in \mathcal{L}_2(\mathbb{R}^3)$ as we mentioned above and by Schwartz inequality $I_\infty(\boldsymbol{k})\varphi(\boldsymbol{k}) = \frac{I_\infty(\boldsymbol{k})}{D(\boldsymbol{k})} \cdot D(\boldsymbol{k})\varphi(\boldsymbol{k})$ is integrable - recalling the properties of D, I_∞ in lines (230p) where we demanded that $\frac{I_\infty(\boldsymbol{k})}{D(\boldsymbol{k})} \in \mathcal{L}_2(\mathbb{R}^3)$. Hence

$$\widehat{T}\,|\varphi\rangle = \Big([\widehat{T}, \int d^3k\; \varphi(\boldsymbol{k}) a_{\boldsymbol{k}}^\dagger] + \int d^3k\; (\varphi(\boldsymbol{k}) a_{\boldsymbol{k}}^\dagger + g\beta_\infty(\boldsymbol{k}))\widehat{T} \Big) U_{\beta_\infty}^{-1}\,|0\rangle \tag{337}$$

$$= \Big(\int d^3k\; D(\boldsymbol{k})\varphi(\boldsymbol{k}) a_{\boldsymbol{k}}^\dagger + g \int d^3k\; I_\infty(\boldsymbol{k})\varphi(\boldsymbol{k}) + \tag{338}$$

$$+ \int d^3k\; (\varphi(\boldsymbol{k}) a_{\boldsymbol{k}}^\dagger + g\beta_\infty(\boldsymbol{k}))\widehat{T} \Big)\,|\Psi(g)\rangle \tag{339}$$

$$= \Big(\int d^3k\; D(\boldsymbol{k})\varphi(\boldsymbol{k}) a_{\boldsymbol{k}}^\dagger + g \int d^3k\; I_\infty(\boldsymbol{k})\varphi(\boldsymbol{k}) \Big)\,|\Psi(g)\rangle + 0 \cdot |\varphi\rangle \tag{340}$$

The zero again comes from the eigenvalue $\widehat{E}(g)$. Since $\widehat{T}$ is the generator of the unitary group of a time evolution we can say that in our case where the ground state eigenvalue is zero this excited state is very unstable. $\widehat{T}$ causes immediately a non-zero transition amplitude back to the ground state. It is interesting to note that no transitions to another excitation of $|\varphi\rangle$ occur.

Note 4.3.1. *Please note that if we had not subtracted V_{SE}^∞ but for example $V_{SE}^\infty - \alpha$ for some $\alpha \in \mathbb{C}$ from the operator $\widehat{T}$ in order to renormalize it we would now yield a stability of the state with a probability somehow proportional to $|\alpha|^2$ depending on the choice of the probability measure. In the folklore vectors in $U_\infty^{-1}(\mathcal{D}(\mathcal{N}_{mes}^D))$ are often called 'heavy quanta', 'dressed' or 'clothed' states in order to distinguish them from 'bare' or 'virtual quanta', which here would be vectors in $\mathcal{D}(\mathcal{N}_{mes}^D)$. The picture is that U_∞^{-1} attaches a meson 'cloud' to the sources, i.e. the nucleons, in momentum space and that the source together with its 'cloud' has something to do with the presence of a 'physical' particle. If the renormalization concept applied here is of any sense to physical application the arbitrariness in choosing the renormalization constant - on the footing that we can only measure the energy relatively - excludes pictures like that because depending on our choice we would find some 'physical' particles and or not. We will not go into this any further but refer to the introduction of this section, where we discussed the absence of a connection between mathematical objects in a theory and* elements of reality*. Until this connection is not established all these states are, either bare or dressed, just what they are, vectors in a Hilbert-space.*

4.3.8 A physical interpretation for this toy model

The toy model Hamiltonian T could be seen as a Hamiltonian for a field with dispersion relation D interacting with a source fixed at the origin. In order to give it a physical sense we could for example choose

$$D(\boldsymbol{k}) := M + \frac{\boldsymbol{k}^2}{2M} \approx \sqrt{\boldsymbol{k}^2 + M^2} \tag{341}$$

as an dispersion relation for a field, which is close to the Schrödinger field. Now going through a similar motivation for the Hamiltonian $H_\kappa = T_\kappa$ as we did in subsection 4.1 we would typically yield

$$I_\kappa(\boldsymbol{k}) := \frac{\widehat{\rho}_\kappa(\boldsymbol{k})}{\sqrt{2D(\boldsymbol{k})}} \tag{342}$$

For this choice the above requirements for D and I_κ are fulfilled and every statements holds for this model. Here it is very interesting to note that the resulting self-energy

$$V_{SE}^\kappa := -g^2 \int d^3k \, \frac{\widehat{\rho}_\kappa(\boldsymbol{k})}{2D^2(\boldsymbol{k})} \tag{343}$$

does not diverge for $\kappa \to \infty$. That means that here $H_\kappa = T_\kappa$ can be given a meaning as an self-adjoint operator on $\mathcal{F}_{mes}$ even without subtracting a divergent renormalization constant.

Thus we have found a physical more or less meaningful model where the self-energy V_{SE}^κ does not diverge. Hence we must conclude that although in general V_{SE}^κ diverges in the limit $\kappa \to \infty$ this behavior is not generic and therefore the subtraction of it from the model Hamiltonian $T_\kappa - V_{SE}^\kappa$ can not be seen as the main remedy for obtaining a self-adjoint operator $\widehat{T}$. In fact even T_κ can be given a mathematical meaning as it stands only by choosing some appropriate set $\mathcal{D}$. However V_{SE}^κ will be divergent in the cases of the static and dynamic quantum Nelson model and there it is of course necessary to subtract it because an operator T_κ carrying a divergent constant will remain ill-defined on whatever space one considers. We conclude that the subtraction of V_{SE}^κ is an ingredient but the spirit of the here examined renormalization concept lies chiefly in the way we make sense out of $s - \lim_{\kappa\to\infty}(T_\kappa - V_{SE}^\kappa)$ by restriction to a special set $\mathcal{D} \subset \mathcal{F}_{mes}$ on which the formal action is well-defined.

4.4 The static quantum Nelson model

In the beginning of this section we have motivated the static quantum Nelson Hamiltonian (215). Here we shall treat only one nucleon and so set $N = 1$. In this case the static quantum Nelson model is somehow similar to the toy model except that now the source does not lie in the origin but at some position $\boldsymbol{x}$. Unfortunately it does not fulfill the requirement $I_\infty(\boldsymbol{k})/D(\boldsymbol{k}) \in \mathcal{L}_2(\mathbb{R}^3)$ we imposed on the toy model in order to be able to renormalize it. For this reason it fails to have a ground state in $\mathcal{F}_1$ for $\kappa \to \infty$ and the renormalization concept applied to the toy model can not be applied here. We still want to present some facts about it because it reveals the static Yukawa theory as we have already seen in subsection 4.2.3. For convenience we have added the rest mass term M to the Hamiltonian, which only causes an energy shift but does not affect the dynamics.

$$H_\kappa \quad := \quad M + \int d^3k \; a^\dagger_{\boldsymbol{k}} \omega_\kappa(\boldsymbol{k}) a_{\boldsymbol{k}} + g \int d^3k \; \gamma_\kappa(\boldsymbol{k}) \left(a_{\boldsymbol{k}} e^{i\boldsymbol{k}\widehat{\boldsymbol{x}}} + a^\dagger_{\boldsymbol{k}} e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}} \right) \tag{344}$$

For $\kappa < \infty$ this Hamiltonian is self-adjoint and conserves the total momentum by trivial corollaries of theorems 2.4.1 and 2.5.1. So $[\mathcal{P}, H_\kappa] = 0$ and there exists a common family of eigenvectors of both operators. The eigenvectors of $\mathcal{P}$ are not in $\mathcal{F}_1$ and we shall therefore call them generalized eigenvectors with respect to the total momentum operator. These generalized eigenvectors are vector-valued functions of the total momentum, say $\boldsymbol{p}$, and have the property that they are in $\mathcal{F}_1$ when e.g. regularized with a function $\varphi \in \mathcal{S}(\mathbb{R}^3)$. Decomposed with respect to eigenvectors of the total number operator we can write every generalized total momentum eigenvector like

$$\mathcal{P}\,|\Psi(\boldsymbol{p})\rangle \quad = \quad \boldsymbol{p}\,|\Psi(\boldsymbol{p})\rangle \tag{345}$$

as

$$|\Psi(\boldsymbol{p})\rangle \quad := \quad \sum_{n=0}^{\infty} \int d^3q \int d^3k_1 \, ... \int d^3k_n \;\; \langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n | \Psi(\boldsymbol{p}) \rangle \, |\boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n\rangle \tag{346}$$

Line (345) already defines one basic property of the functions $\langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n | \Psi(\boldsymbol{p})\rangle$, which can be seen by inserting identities $e^{-i\mathcal{P}\boldsymbol{d}} e^{i\mathcal{P}\boldsymbol{d}}$ of the unitary group of the total momentum operator $\mathcal{P}$ in

$$\begin{aligned}
\langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n | \Psi(\boldsymbol{p})\rangle &= \frac{1}{\sqrt{n!}} \langle \boldsymbol{q}; 0| a_{\boldsymbol{k}_1} ... a_{\boldsymbol{k}_n} |\Psi(\boldsymbol{p})\rangle && (347)\\
&= \frac{1}{\sqrt{n!}} \langle \boldsymbol{q}; 0| e^{-i\mathcal{P}\boldsymbol{d}} e^{i\mathcal{P}\boldsymbol{d}} a_{\boldsymbol{k}_1} e^{-i\mathcal{P}\boldsymbol{d}} e^{i\mathcal{P}\boldsymbol{d}} ... && (348)\\
&\qquad ... e^{-i\mathcal{P}\boldsymbol{d}} e^{i\mathcal{P}\boldsymbol{d}} a_{\boldsymbol{k}_n} e^{-i\mathcal{P}\boldsymbol{d}} e^{i\mathcal{P}\boldsymbol{d}} |\Psi(\boldsymbol{p})\rangle && (349)\\
&= \frac{e^{i(\boldsymbol{p}-\boldsymbol{q}-\sum_{i=1}^{N} \boldsymbol{k}_i)\boldsymbol{d}}}{\sqrt{n!}} \langle \boldsymbol{q}; 0| a_{\boldsymbol{k}_1} ... a_{\boldsymbol{k}_n} |\Psi(\boldsymbol{p})\rangle && (350)
\end{aligned}$$

which must hold for any $\boldsymbol{d} \in \mathbb{R}^3$. Hence we can separate the tempered delta distribution which carries the $\boldsymbol{p}$ behavior

$$\langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n | \Psi(\boldsymbol{p})\rangle = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q} - \sum_{i=1}^{N} \boldsymbol{k}_i) \psi(\boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n) \tag{351}$$

where $\psi \in \mathcal{L}_2(\mathbb{R}^3)$. Among these $|\Psi(\boldsymbol{p})\rangle$ we look for a family, which are also eigenvectors of H_κ. Hence we look for a solution of the eigenvector equation of H_κ, which has the properties of the above $|\Psi(\boldsymbol{p})\rangle$. Let E_κ be an eigenvalue of H_κ then for some generalized eigenvector of the total momentum operator $|\Psi_\kappa(\boldsymbol{p})\rangle$ we have

$$H_\kappa |\Psi_\kappa(\boldsymbol{p})\rangle = E_\kappa |\Psi_\kappa(\boldsymbol{p})\rangle \tag{352}$$

So for the n-meson wave function of this vector we get

$$\begin{aligned}
\langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n | H_\kappa |\Psi_\kappa(\boldsymbol{p})\rangle &= (M + \sum_{i=1}^{n} \omega_{\boldsymbol{k}_i}) |\Psi_\kappa(\boldsymbol{p})\rangle + && (353)\\
&+g \int d^3k\, \gamma_\kappa(\boldsymbol{k}) \sqrt{n+1} \langle \boldsymbol{q} - \boldsymbol{k}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n, \boldsymbol{k} | \Psi_\kappa(\boldsymbol{p})\rangle + && (354)\\
&+g \sum_{i=1}^{n} \frac{\gamma_\kappa(\boldsymbol{k}_i)}{\sqrt{n}} \langle \boldsymbol{q} + \boldsymbol{k}_i; \boldsymbol{k}_1, ..., \widehat{\boldsymbol{k}_i}, ..., \boldsymbol{k}_n | \Psi_\kappa(\boldsymbol{p})\rangle && (355)
\end{aligned}$$

In [9] a special ansatz was used by Greenberg and Schweber to find a solution of the eigenvector equation. This can be done but with the perturbation theory we have developed for the toy model in section 4.3.3 and which is applicable for finite κ to the model at hand we already know the answer

$$\psi(\boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n) := \psi(\boldsymbol{k}_1, ..., \boldsymbol{k}_n) := \frac{(-g)^n}{\sqrt{n!}} \prod_{i=1}^{n} \beta_\kappa(\boldsymbol{k}_i) \tag{356}$$

with $\beta_\kappa(\boldsymbol{k}) = I_\kappa(\boldsymbol{k})/D(\boldsymbol{k}) = \frac{\gamma_\kappa(\boldsymbol{k})}{\omega_{\boldsymbol{k}}}$. We plug that into above equation (353) which then reads

$$\begin{aligned}
\langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n | H_\kappa |\Psi_\kappa(\boldsymbol{p})\rangle &= \Bigg(M + \sum_{i=1}^{n} \omega_{\boldsymbol{k}_i} + && (357)\\
&\quad -g^2 \int d^3k\, \gamma_\kappa(\boldsymbol{k}) \beta(\boldsymbol{k}) + && (358)\\
&\quad - \sum_{i=1}^{n} \frac{\gamma_\kappa(\boldsymbol{k})}{\beta(\boldsymbol{k}_i)} \Bigg) \langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n | \Psi_\kappa(\boldsymbol{p})\rangle && (359)\\
&= \left(M - g^2 \int d^3k\, \gamma_\kappa^2(\boldsymbol{k}) \beta(\boldsymbol{k}) \right) \langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n | \Psi_\kappa(\boldsymbol{p})\rangle && (360)\\
&=: \left(M + V_{SE}^\kappa \right) \langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n | \Psi_\kappa(\boldsymbol{p})\rangle && (361)
\end{aligned}$$

Hence the generalized Fock-vector

$$|\Psi_\kappa(\boldsymbol{p})\rangle \quad := \quad e^{-g\int d^3k\ \frac{\gamma_\kappa(\boldsymbol{k})}{\omega_k} a_k^\dagger e^{-ik\hat{x}}} |\boldsymbol{p};0\rangle \tag{362}$$

is an eigenvector of H_κ with the eigenvalue $M + V_{SE}^\kappa$. V_{SE}^∞ is formally identical to the self-energy term V_{SE} we have found in the classical case in subsection 3.3. In subsection 4.2.3 we have shown that for $\kappa \to \infty$ it diverges and if M is finite the eigenvalue $M - V_{SE}^\kappa$ diverges. But even worse in the limit $|\Psi_\kappa(\boldsymbol{p})\rangle$ is not even a generalized eigenvector anymore because $\frac{\gamma_\infty(\boldsymbol{k})}{\omega_k} \notin \mathcal{L}_2(\mathbb{R}^3)$. The infinite energy eigenvalue can be remedied by assuming that the initial parameter M is infinite such that $M + V_{SE}^\kappa = m \in \mathbb{R}_+$. This is the reason for the name mass renormalization although energy renormalization is more appropriate. Since we can only measure quantities relative to each other there is no way to determine M. Only m must coincide with the actual mass one would measure in an experiment. Looking at it in this way we could obtain an finite energy eigenvalue however since $|\Psi_\kappa(\boldsymbol{p})\rangle$ is still not a generalized Fock-vector it seems that the only way to make sense out of the given Hamiltonian is to define it on another space. In fact we shall see that we find the same result for the dynamic quantum Nelson model if we give the nucleons a relativistic dispersion relation. That means the renormalization concept applied to the toy model has to be extended to a bigger space than the Fock-space that hosts the model ground state even if it does not lie in Fock-space anymore. This will be subject of further investigations.

4.5 The dynamic quantum Nelson model

Things get more complicated than in the toy model or the static quantum Nelson model when we introduce a momentum dependent dispersion relation for the nucleons. The main difficulty lies in the fact that the eigenvectors can not be computed as easily and explicitly as it was done for the toy model and the static quantum Nelson model. However we shall see that the renormalization concept applied to the toy model can be applied in the same fashion to the dynamic quantum Nelson model. Therefore we shall proceed in the same manner as we did in the case of the toy model and at first recall our central questions about the renormalized model Hamiltonian $\widehat{H}$, which we have posed in subsection 2.8:

1. What does $\widehat{H}$ look like?

2. What does $\mathcal{D}(\widehat{H})$ look like?

3. What does the action of $\widehat{H}$ on elements in $\mathcal{D}(\widehat{H})$ look like?

The definition of the model was already given in subsection 2.2. The self-adjointness of the model Hamiltonian was proven by theorem 2.4.1. The existence of a renormalized self-adjoint Hamiltonian $\widehat{H}$ that induces a well-defined time evolution was proven by Nelson's theorem 2.6.1. So let us continue with the perturbation theory of this model and see how explicit one can write down the ground state in order to obtain a similar transformation, which we have called $U_\kappa(g)$ in the toy model.

4.5.1 Perturbation theory

First steps in understanding the divergence of the dynamic quantum Nelson Model without cutoff were achieved by Tomonaga and later by Gross and are based on observations made in the polaron model. It had been realized that the divergence in this model is of a very simple kind. Simple in the sense of that in regular perturbation theory in the coupling constant g the only divergent term in the power series of the energy is the second order one. Renormalization may then be done by simple substraction of this second order term from the model Hamiltonian and performing the removal of the cutoff in the limit $\kappa \to \infty$. This led Gross [10] to the unitary transformation of the model Hamiltonian with cutoff, the Gross transformation namely, which extracts exactly that second order term. This transformation is of the same kind like the one we have applied to the toy model. Unfortunately this time it fails to diagonalize the model Hamiltonian but still proves to be a helpful tool in showing the existence of the renormalized Hamiltonian $\widehat{H}$, which was done by Nelson and whose proof we have already gone through in subsection 2.6.

In this subsection we at first want to show how in terms of regular perturbation theory this divergent second order term shows up and that the energy power series converges if that second order term is taken out like it was also the case for the toy model. Therefore we need to check that the model Hamiltonian is of such a kind that regular perturbation theory is applicable. To get a feeling about how the perturbation theory works we explicitly compute the Rayleigh-Schrödinger correction terms of the energy up to the sixth order for the dynamic quantum Nelson Hamiltonian (20) restricted to the one nucleon sector $\mathcal{F}_1$. The understanding we hereby gain will enable us to find a simple recursion formula for the wave function correction terms and with them the energy correction terms for arbitrary high order. In that way we can form a more explicit expression for the perturbed ground state compared to the resolvent representation although it has the disadvantage that we need a bound on the coupling constant. Since the projector of this model is not as easy to compute as it was the case in the toy model we have to get along without it and use the representation of the perturbed ground state in terms of wave functions instead. It has to be remarked that all computations made in the following are completely general and hold for any one nucleon system coupled to a real, scalar, bosonic field, except that one has to insert the appropriate dispersion relations for the nucleon and the field. In our discussion of the properties of the correction terms we will use the dispersion relations of the one nucleon dynamic quantum Nelson model, which will help us to understand how the renormalization concept of this model works. Further along the way by taking the limit $\boldsymbol{p}^2 \ll M^2$ and adding a rest mass M, which loosely speaking turns off the nucleon dispersion term in the Hamiltonian, we reveal the static quantum Nelson model. That enables us to distinguish between correction terms that are only due to the meson field terms in the model Hamiltonian and the ones, which arise from the nucleon dispersion term.

Again like for the case of the toy model we need to show analyticity properties in g, which we shall recall in the following. In order to apply regular perturbation theory of the ground state of an operator $H_\kappa = H_0 + gH_{I_\kappa}$ with respect to the coupling constant g we need to prove analyticity of this operator, i.e. it has to be possible to expand the perturbed ground state and its energy eigenvalue in a Taylor series in the coupling constant g with some non-zero radius of convergence. We are interested in the ground state of H_κ so following [19, XII] we need a property that is called analyticity of type (A) near $g = 0$ in the perturbation theory of linear operators. This means the following. Let $E_0 \in \mathbb{R}$ be the ground state eigenvalue of H_0 then analyticity of type (A) ensures that if E_0 is an isolated and non-degenerate point in the spectrum $\sigma(H_0)$ there exists a unique point $E(g) \in \sigma(H_\kappa)$, the eigenvalue of the perturbed ground state, which is again isolated and non-degenerate and moreover then $E_\kappa(g)$ and the ground state itself are analytic near $g = 0$.

On $\mathcal{F}_1$ the operators H_0 and H_{I_κ} and for finite κ have the following explicit form in terms of the total momentum operator $\mathcal{P}$

$$H_0 \quad := \quad \frac{(\mathcal{P} - \int d^3k\ a^\dagger_{\boldsymbol{k}} \boldsymbol{k} a_{\boldsymbol{k}})^2}{2M} + \int d^3k\ a^\dagger_{\boldsymbol{k}} \omega_{\boldsymbol{k}} a_{\boldsymbol{k}} \tag{363}$$

$$H_{I_\kappa} \quad := \quad \int d^3k\ \gamma_\kappa(\boldsymbol{k})(a_{\boldsymbol{k}} e^{i\boldsymbol{k}\widehat{\boldsymbol{x}}} + a^\dagger_{\boldsymbol{k}} e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}}) \tag{364}$$

Having in mind the conservation of the total momentum (theorem 2.5.1) we introduce the following convention

$$|\boldsymbol{k}_1, ..., \boldsymbol{k}_n\rangle_{\boldsymbol{p}} \quad := \quad |\boldsymbol{p} - \sum_{i=1}^{n} \boldsymbol{k}_i; \boldsymbol{k}_1, ..., \boldsymbol{k}_n\rangle \tag{365}$$

and the joint spectral decomposition of $\mathcal{F}_1$ with respect to the operator $\mathcal{P}$ given by the constant fibre direct integral

$$\mathcal{F}_1 = \int^{\oplus} d^3p \, \mathcal{F}_1^{\boldsymbol{p}} \tag{366}$$

where we indicate the $\boldsymbol{p}$ fibre of $|\psi\rangle \in \mathcal{F}_1$ with $|\psi\rangle_{\boldsymbol{p}}$, which is naturally a generalized eigenvector of $\mathcal{P}$, hence we can write[18] $\langle \boldsymbol{k}_1, ..., \boldsymbol{k}_n|_{\boldsymbol{p}'} |\psi\rangle_{\boldsymbol{p}} = \delta^3(\boldsymbol{p}' - \boldsymbol{p})\psi_{\boldsymbol{p}}(\boldsymbol{k}_1, ..., \boldsymbol{k}_n)$ with $\psi_{\boldsymbol{p}}$ representing the meson wave function in $\mathcal{L}_2(\mathbb{R}^{3n})$. The space $\mathcal{F}_1^{\boldsymbol{p}}$ is isomorphic to $\mathcal{F}_1$ with the scalar product given by

$$\begin{aligned}
(|\varphi\rangle_{\boldsymbol{p}}, |\phi\rangle_{\boldsymbol{p}})_{\mathcal{F}_1^{\boldsymbol{p}}} &:= \sum_{n=0}^{\infty} \int d^3k_1 \, ... \int d^3k_n \, \varphi_{\boldsymbol{p}}^*(\boldsymbol{k}_1, ..., \boldsymbol{k}_n)\phi_{\boldsymbol{p}}(\boldsymbol{k}_1, ..., \boldsymbol{k}_n) && (367)\\
&=: \langle \varphi|_{\boldsymbol{p}} |\phi\rangle_{\boldsymbol{p}} && (368)
\end{aligned}$$

Now any $|\psi\rangle_{\boldsymbol{p}}$ can be regarded as a generalized eigenvector of $\mathcal{P}$ in the one nucleon sector $\mathcal{F}_1$ or as an proper eigenvector of $\mathcal{P} \restriction \mathcal{F}_1^{\boldsymbol{p}}$ in each fibre $\mathcal{F}_1^{\boldsymbol{p}}$. We usually drop the subscript of the scalar product - whenever both Fock-vectors are indexed by $\boldsymbol{p}$ like in $\langle \varphi|_{\boldsymbol{p}} |\phi\rangle_{\boldsymbol{p}}$ we mean the scalar product in $\mathcal{F}_1^{\boldsymbol{p}}$. Furthermore we define the fibre bundle for any $B \subset \mathbb{R}^3$

$$\mathcal{F}_1^B := \bigcup_{\boldsymbol{p} \in B} \mathcal{F}_1^{\boldsymbol{p}} \tag{369}$$

and indicate the restriction of an operator A on $\mathcal{D}(A) \subset \mathcal{F}_1$ to the fibre $\mathcal{F}_1^{\boldsymbol{p}}$ with $A^{\boldsymbol{p}}$.

Theorem 4.5.1. *For $\kappa < \infty$, $|\boldsymbol{p}| < \sqrt{2M\mu}$ and $\mu, M \in \mathbb{R}_+$ the Hamiltonians $H_\kappa^{\boldsymbol{p}} = H_0^{\boldsymbol{p}} + gH_{I_\kappa}^{\boldsymbol{p}}$ are an analytic family in $g \in \mathbb{R}$ of type (A) near $g = 0$ and $\inf \sigma(H_0^{\boldsymbol{p}})$ is an isolated point in $\sigma(H_0^{\boldsymbol{p}})$.*

Proof. By [19, XII.9] in order to get the analyticity we only need to show

1. $\mathcal{D}(H_{I_\kappa}) \supset \mathcal{D}(H_0)$
2. for some $a, b \in \mathbb{R}$ and $\forall\, |\psi\rangle \in \mathcal{D}(H_0)$, $||gH_{I_\kappa}|| \leq a||H_0 |\psi\rangle || + b|| |\psi\rangle ||$

(1.) is given by line (37) and (2.) by line (39). All generalized eigenvectors of $H_0^{\boldsymbol{p}}$ are given by $|\boldsymbol{k}_1, ..., \boldsymbol{k}_n\rangle_{\boldsymbol{p}}$ for all $n \geq 0$ because of the completeness $\mathcal{F}_1^{\boldsymbol{p}} = \text{span}\{|\boldsymbol{k}_1, ..., \boldsymbol{k}_n\rangle_{\boldsymbol{p}}\}_{n \in \mathbb{N}_0}$.

$$\begin{aligned}
H_0^{\boldsymbol{p}} |\boldsymbol{k}_1, ..., \boldsymbol{k}_n\rangle_{\boldsymbol{p}} &= \left(\frac{(\boldsymbol{p} - \sum_{i=1}^n \boldsymbol{k}_i)^2}{2M} + \sum_{i=1}^n \omega_{\boldsymbol{k}_i} \right) |\boldsymbol{k}_1, ..., \boldsymbol{k}_n\rangle_{\boldsymbol{p}} && (370)\\
&=: E_0^{\boldsymbol{p}}(\boldsymbol{k}_1, ..., \boldsymbol{k}_n) \, |\boldsymbol{k}_1, ..., \boldsymbol{k}_n\rangle_{\boldsymbol{p}} && (371)
\end{aligned}$$

So we estimate $E_0^{\boldsymbol{p}}(\boldsymbol{k}_1, ..., \boldsymbol{k}_n)$ for all $\boldsymbol{p}, \boldsymbol{k}_1, ..., \boldsymbol{k}_n \in \mathbb{R}^3$ by

$$E_0^{\boldsymbol{p}}(\boldsymbol{k}_1, ..., \boldsymbol{k}_n) \geq \sum_{i=1}^n \omega_{\boldsymbol{k}_i} \geq n\mu \tag{372}$$

Hence there exist $\boldsymbol{p} \in \mathbb{R}^3$ such that $\inf \sigma(H_0^{\boldsymbol{p}}) = E_0^{\boldsymbol{p}}$, which is the eigenvalue of the meson vacuum Fock-vector $|0\rangle_{\boldsymbol{p}}$. In order to have a gap in the spectrum we need to fix those $\boldsymbol{p}$ and so need to fulfill

$$E_0^{\boldsymbol{p}}(\boldsymbol{k}_1, ..., \boldsymbol{k}_n) - E_0^{\boldsymbol{p}} \quad > \quad 0 \tag{373}$$

[18] See discussion in the static case from line (345) on.

for all $n \in \mathbb{N}$. One way to find appropriate $\boldsymbol{p}$ is to solve

$$\frac{|\sum_{i=1}^{n} \boldsymbol{k}_i|^2}{2M} - \frac{\boldsymbol{p}|\sum_{i=1}^{n} \boldsymbol{k}_i|}{M} + C = 0 \tag{374}$$

for $k = |\sum_{i=1}^{n} \boldsymbol{k}_i|$ with $C \in R_+$. Clearly for $k = 0$ the term is positive and if there exits no solution to the above equation for special $\boldsymbol{p} \in \mathbb{R}^3$ then by continuity we have a gap. But the solution only exists if $\sqrt{\frac{\boldsymbol{p}^2}{M^2} - \frac{2C}{M}}$ is real, which is only the case for $|\boldsymbol{p}| < \sqrt{2M \cdot C}$. Hence for this restriction on $\boldsymbol{p}$ we find a lower bound for the gap

$$E_0^{\boldsymbol{p}}(\boldsymbol{k}_1, ..., \boldsymbol{k}_n) - E_0^{\boldsymbol{p}} \geq \frac{|\sum_{i=1}^{n} \boldsymbol{k}_i|^2}{2M} - \frac{\boldsymbol{p}|\sum_{i=1}^{n} \boldsymbol{k}_i|}{M} + n\mu > n\mu - C \tag{375}$$

Since $n\mu - C > 0$ must hold for all $n \in \mathbb{N}$ in order to have a gap in the spectrum we chose $C := \mu$ and yield $|\boldsymbol{p}| < \sqrt{2M\mu}$. □

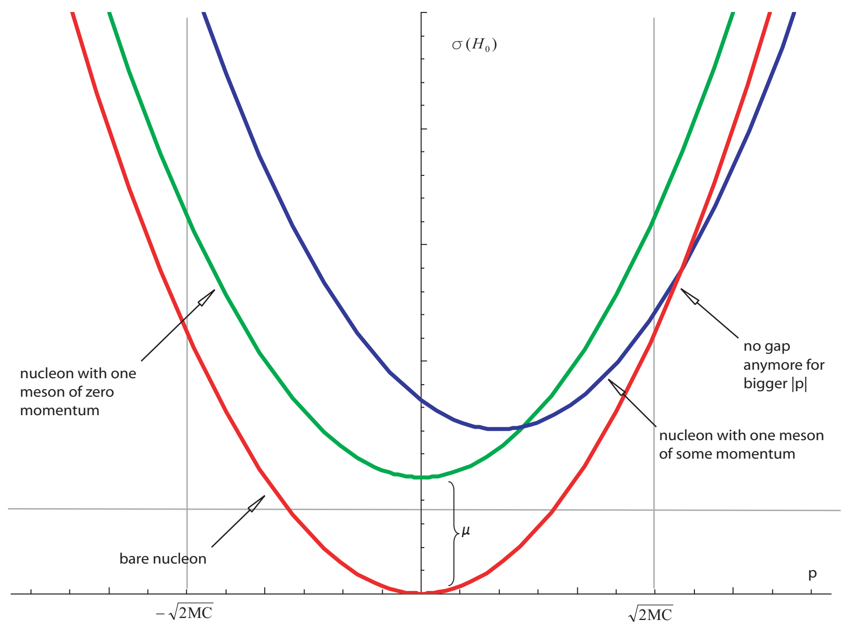

Figure 3: Spectrum of H_0.

Thus we may expand the eigenvalue $E(g)$ and its corresponding ground state eigenvector in a power series around $g = 0$. In the calculation below we will use the spectral decomposition of H_κ with respect to the total momentum operator $\mathcal{P}$ and therefore index $|\Psi(g)\rangle_{\boldsymbol{p}}$ and $E^{\boldsymbol{p}}(g)$ by sub- or superscript $\boldsymbol{p}$. Although all occurring correction terms are κ dependent we shall only index $|\Psi_\kappa(g)\rangle_{\boldsymbol{p}}$ and $E_\kappa^{\boldsymbol{p}}(g)$ with a subscript κ. However the reader is asked to bear the κ dependence in mind everywhere we consider the limit $\kappa \to \infty$. In contrary to the toy model we will not try to compute the projector from the ground state to the perturbed ground state but use the ansatz of Rayleigh-Schrödinger that suggests to directly expand $|\Psi_\kappa(g)\rangle$ and $E_\kappa^{\boldsymbol{p}}(g)$ in a power series in g for finite κ.

Unperturbed ground state

$$H_0^{\boldsymbol{p}} |\Psi_0\rangle_{\boldsymbol{p}} = E_0^{\boldsymbol{p}} |\Psi_0\rangle_{\boldsymbol{p}} \tag{376}$$
$$|\Psi_0\rangle_{\boldsymbol{p}} := |0\rangle_{\boldsymbol{p}} \tag{377}$$
$$E_0^{\boldsymbol{p}} := \frac{\boldsymbol{p}^2}{2M} \tag{378}$$

Perturbed ground state

$$H^{\boldsymbol{p}}_\kappa \left|\Psi(g)\right\rangle_{\boldsymbol{p}} = (H^{\boldsymbol{p}}_0 + gH^{\boldsymbol{p}}_{I_\kappa}) \left|\Psi_\kappa(g)\right\rangle_{\boldsymbol{p}} = E^{\boldsymbol{p}}_\kappa(g) \left|\Psi_\kappa(g)\right\rangle_{\boldsymbol{p}} \tag{379}$$

$$\left|\Psi_\kappa(g)\right\rangle_{\boldsymbol{p}} =: \sum_{n=0}^{\infty} g^n \left|\Psi_n\right\rangle_{\boldsymbol{p}} \tag{380}$$

$$E^{\boldsymbol{p}}_\kappa(g) =: \sum_{n=0}^{\infty} g^n E^{\boldsymbol{p}}_n \tag{381}$$

For the vector $\left|\Psi_\kappa(g)\right\rangle_{\boldsymbol{p}}$ we choose the normalization

$$\left\langle\Psi_0\right|_{\boldsymbol{p}} \left|\Psi_0\right\rangle_{\boldsymbol{p}} = \left\langle\Psi_0\right|_{\boldsymbol{p}} \left|\Psi_\kappa(g)\right\rangle_{\boldsymbol{p}} = \left\langle 0\right|_{\boldsymbol{p}} \left|0\right\rangle_{\boldsymbol{p}} = 1 \tag{382}$$

The above has the immediate consequence that

$$\left\langle\Psi_0\right|_{\boldsymbol{p}} \left|\Psi_n\right\rangle_{\boldsymbol{p}} = 0 \;\; \forall n \geq 1 \tag{383}$$

From (379) and by defining $\left|\Psi_{-1}\right\rangle_{\boldsymbol{p}} := 0$ we get

$$\sum_{n=0}^{\infty} g^n (H^{\boldsymbol{p}}_0 \left|\Psi_n\right\rangle_{\boldsymbol{p}} + H^{\boldsymbol{p}}_{I_\kappa} \left|\Psi_{n-1}\right\rangle_{\boldsymbol{p}}) = \sum_{n,m=0}^{\infty} g^{n+m} E^{\boldsymbol{p}}_n \left|\Psi_m\right\rangle_{\boldsymbol{p}} \tag{384}$$

If we compare the terms of equal order of g we get

$$H^{\boldsymbol{p}}_0 \left|\Psi_n\right\rangle_{\boldsymbol{p}} + H^{\boldsymbol{p}}_{I_\kappa} \left|\Psi_{n-1}\right\rangle_{\boldsymbol{p}} = \sum_{i=0}^{n} E^{\boldsymbol{p}}_i \left|\Psi_{n-i}\right\rangle_{\boldsymbol{p}} \tag{385}$$

by multiplying $\left\langle\Psi_0\right|$ from the left and inserting the expression for $H^{\boldsymbol{p}}_{I_\kappa}$ we can write the energy corrections $E^{\boldsymbol{p}}_n$ as

$$E^{\boldsymbol{p}}_n = \frac{\left\langle\Psi_0\right|_{\boldsymbol{p}} H^{\boldsymbol{p}}_{I_\kappa} \left|\Psi_{n-1}\right\rangle_{\boldsymbol{p}}}{\left\langle\Psi_0\right|_{\boldsymbol{p}} \left|\Psi_0\right\rangle_{\boldsymbol{p}}} = \int d^3k \; \gamma_\kappa(\boldsymbol{k}) \left\langle\boldsymbol{k}\right|_{\boldsymbol{p}} \left|\Psi_{n-1}\right\rangle_{\boldsymbol{p}} \tag{386}$$

and finally by expanding $\left|\Psi_n\right\rangle_{\boldsymbol{p}}$ in terms of the eigenvectors of $H^{\boldsymbol{p}}_0$ for $m \geq 1$

$$\left\langle\boldsymbol{k}_1, ..., \boldsymbol{k}_m\right|_{\boldsymbol{p}} \left|\Psi_n\right\rangle_{\boldsymbol{p}} = \frac{\sum_{i=1}^{n} E^{\boldsymbol{p}}_i \left\langle\boldsymbol{k}_1, ..., \boldsymbol{k}_m\right|_{\boldsymbol{p}} \left|\Psi_{n-i}\right\rangle_{\boldsymbol{p}} - \left\langle\boldsymbol{k}_1, ..., \boldsymbol{k}_m\right|_{\boldsymbol{p}} H^{\boldsymbol{p}}_{I_\kappa} \left|\Psi_{n-1}\right\rangle_{\boldsymbol{p}}}{E^{\boldsymbol{p}}_0(\boldsymbol{k}_1, ..., \boldsymbol{k}_m) - E^{\boldsymbol{p}}_0} \tag{387}$$

with $E^{\boldsymbol{p}}_0$ given by line (378) and $E^{\boldsymbol{p}}_0(\boldsymbol{k}_1, ..., \boldsymbol{k}_n)$ by (371). Let us define

$$\boxed{\eta(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n) := E^{\boldsymbol{p}}_0(\boldsymbol{k}_1, ..., \boldsymbol{k}_n) - E^{\boldsymbol{p}}_0} \tag{388}$$

$$\tag{389}$$

and with $H^{\boldsymbol{p}}_{I_\kappa}$ acting to the left side we obtain

$$\left\langle\boldsymbol{k}_1, ..., \boldsymbol{k}_m\right|_{\boldsymbol{p}} \left|\Psi_n\right\rangle_{\boldsymbol{p}} = \frac{1}{\eta(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_m)} \times \tag{390}$$

$$\times \Bigg[\sum_{i=1}^{n} \int d^3k \; \gamma_\kappa(\boldsymbol{k}) \left\langle\boldsymbol{k}\right|_{\boldsymbol{p}} \left|\Psi_{i-1}\right\rangle_{\boldsymbol{p}} \left\langle\boldsymbol{k}_1, ..., \boldsymbol{k}_m\right|_{\boldsymbol{p}} \left|\Psi_{n-i}\right\rangle_{\boldsymbol{p}} + \tag{391}$$

$$- \int d^3k \; \gamma_\kappa(\boldsymbol{k}) \sqrt{m+1} \left\langle\boldsymbol{k}_1, ..., \boldsymbol{k}_m, \boldsymbol{k}\right|_{\boldsymbol{p}} \left|\Psi_{n-1}\right\rangle_{\boldsymbol{p}} + \tag{392}$$

$$- \sum_{j=1}^{m} \frac{\gamma_\kappa(\boldsymbol{k}_j)}{\sqrt{m}} \left\langle\boldsymbol{k}_1, ..., \widehat{\boldsymbol{k}_j}, ..., \boldsymbol{k}_m\right|_{\boldsymbol{p}} \left|\Psi_{n-1}\right\rangle_{\boldsymbol{p}} \Bigg] \tag{393}$$

We hereby use the decomposition of the Fock-vector with respect to the generalized eigenvectors of $\mathcal{N}^1_{mes}$.

$$|\Psi_n\rangle_{\boldsymbol{p}} = \sum_{m=0}^{\infty} \int d^3k_1 \ldots \int d^3k_m \; \langle \boldsymbol{k}_1, \ldots, \boldsymbol{k}_m|_{\boldsymbol{p}} \, |\Psi_n\rangle_{\boldsymbol{p}} \, |\boldsymbol{k}_1, \ldots, \boldsymbol{k}_m\rangle_{\boldsymbol{p}} \tag{394}$$

However direct application of the recursion (390) shows that

$$|\Psi_n\rangle_{\boldsymbol{p}} = \sum_{m=0}^{n} \int d^3k_1 \ldots \int d^3k_m \; \langle \boldsymbol{k}_1, \ldots, \boldsymbol{k}_m|_{\boldsymbol{p}} \, |\Psi_n\rangle_{\boldsymbol{p}} \, |\boldsymbol{k}_1, \ldots, \boldsymbol{k}_m\rangle_{\boldsymbol{p}} \tag{395}$$

because for all $m > n$ the m-meson wave functions $\langle \boldsymbol{k}_1, \ldots, \boldsymbol{k}_m|_{\boldsymbol{p}} \, |\Psi_n\rangle_{\boldsymbol{p}} = 0$. In the following paragraphs we present the correction terms explicitly up to sixth order. After each first equal sign the expressions are very close to (390) and after the second one we ordered them in a way that makes it easier to recognize the recursive structure. In each step we introduce conventions and definitions in order to simplify the formulas. In the end we melt these definitions together in a single, generalized one, which holds for all orders.

First order

$$E_1^{\boldsymbol{p}} = 0 \tag{396}$$

$$|\Psi_1\rangle_{\boldsymbol{p}} = \int d^3k_1 \, \frac{1}{\eta(\boldsymbol{p};\boldsymbol{k}_1)} \left[-\frac{\gamma_\kappa(\boldsymbol{k}_1)}{\sqrt{1!}} \, \langle 0|_{\boldsymbol{p}} \, |\Psi_0\rangle_{\boldsymbol{p}} \right] |\boldsymbol{k}_1\rangle_{\boldsymbol{p}} \tag{397}$$

$$= \int d^3k_1 \, \frac{(-1)\gamma_\kappa(\boldsymbol{k}_1)}{\sqrt{1!}} \frac{1}{\eta(\boldsymbol{p};\boldsymbol{k}_1)} \, |\boldsymbol{k}_1\rangle_{\boldsymbol{p}} \tag{398}$$

Second order

$$E_2^{\boldsymbol{p}} = \frac{\langle \Psi_0|_{\boldsymbol{p}} \, H_{I_\kappa} \, |\Psi_1\rangle_{\boldsymbol{p}}}{\langle \Psi_0|_{\boldsymbol{P}} \, |\Psi_0\rangle_{\boldsymbol{p}}} \tag{399}$$

$$= \int d^3l_1 \; \gamma_\kappa(\boldsymbol{l}_1) \, \langle \boldsymbol{l}_1|_{\boldsymbol{p}} \, |\Psi_1\rangle_{\boldsymbol{p}} = -\int d^3l_1 \, \frac{\gamma_\kappa^2(\boldsymbol{l}_1)}{\eta(\boldsymbol{p};\boldsymbol{l}_1)} \tag{400}$$

$$= -\int d^3l_1 \, \frac{\gamma_\kappa^2(\boldsymbol{l}_1)}{\frac{\boldsymbol{l}_1^2 - \boldsymbol{p}\boldsymbol{l}_1}{2M} + \omega_{\boldsymbol{l}_1}} \tag{401}$$

$$|\Psi_2\rangle_{\boldsymbol{p}} = \int d^3k_1 \int d^3k_2 \, \frac{(-1)^2 \gamma_\kappa(\boldsymbol{k}_1)\gamma_\kappa(\boldsymbol{k}_2)}{\sqrt{2!} \; \eta(\boldsymbol{p};\boldsymbol{k}_1,\boldsymbol{k}_2)} \left[\frac{1}{\eta(\boldsymbol{p};\boldsymbol{k}_1)} + \frac{1}{\eta(\boldsymbol{p};\boldsymbol{k}_2)} \right] |\boldsymbol{k}_1,\boldsymbol{k}_2\rangle_{\boldsymbol{p}} \tag{402}$$

Please note the similarity of (401) with the definition of R_κ in line (66) and hence that it diverges logarithmically for $\kappa \to \infty$. We shall at a later point see that only this term causes the energy eigenvalue to diverge in the limit but for now we continue with finite κ. For the sake of legibility we shall define a function ξ recursively, which we will later call contraction of zero-th order

$$\boxed{\xi(\boldsymbol{p};\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n) := \frac{1}{\eta(\boldsymbol{p};\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)} \sum_{i=1}^{n} \xi(\boldsymbol{p};\boldsymbol{k}_1,\ldots,\widehat{\boldsymbol{k}_i},\ldots,\boldsymbol{k}_n)} \tag{403}$$

and the termination of this recursion

$$\boxed{\xi(\boldsymbol{p};\boldsymbol{k}_1) := \frac{1}{\eta(\boldsymbol{p};\boldsymbol{k}_1)}} \tag{404}$$

Since $\eta(\boldsymbol{p};\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)$ is symmetric in the arguments $\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n$ the function $\xi(\boldsymbol{p};\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)$ is, too, by construction. Furthermore for $|\boldsymbol{p}| \leq \sqrt{2M\mu}$ the functions ξ have no singularities on whole $\mathbb{R}^{3n}$

since the η do not. In addition let

$$\Gamma_\kappa(\boldsymbol{k}_1, ..., \boldsymbol{k}_n) \quad := \quad \frac{(-1)^n \prod_{i=1}^n \gamma_\kappa(\boldsymbol{k}_i)}{\sqrt{n!}} \tag{405}$$

Thus in the new notation we have

$$\begin{aligned}
|\Psi_1\rangle_{\boldsymbol{p}} &= \int d^3k_1 \; \Gamma_\kappa(\boldsymbol{k}_1)\xi(\boldsymbol{p}; \boldsymbol{k}_1) \, |\boldsymbol{k}_1\rangle_{\boldsymbol{p}} && (406)\\
E_2^{\boldsymbol{p}} &= -\int d^3l_1 \; \gamma_\kappa^2(\boldsymbol{l}_1)\xi(\boldsymbol{p}; \boldsymbol{l}_1) && (407)\\
|\Psi_2\rangle_{\boldsymbol{p}} &= \int d^3k_1 \int d^3k_2 \; \Gamma_\kappa(\boldsymbol{k}_1, \boldsymbol{k}_2)\xi(\boldsymbol{p}; \boldsymbol{k}_1, \boldsymbol{k}_2) \, |\boldsymbol{k}_1, \boldsymbol{k}_2\rangle_{\boldsymbol{p}} && (408)
\end{aligned}$$

Third order

$$\begin{aligned}
E_3^{\boldsymbol{p}} &= 0 && (409)\\
|\Psi_3\rangle_{\boldsymbol{p}} &= \int d^3k_1 \, \frac{1}{\eta(\boldsymbol{p}; \boldsymbol{k}_1)} \Big[\int d^3k \; \gamma_\kappa(\boldsymbol{k}) \, \langle \boldsymbol{k}|_{\boldsymbol{p}} \, |\Psi_1\rangle_{\boldsymbol{p}} \, \langle \boldsymbol{k}_1|_{\boldsymbol{p}} \, |\Psi_1\rangle_{\boldsymbol{p}} + && (410)\\
&\qquad - \int d^3k \; \gamma_\kappa(\boldsymbol{k}) \sqrt{2} \, \langle \boldsymbol{k}_1, \boldsymbol{k}|_{\boldsymbol{p}} \, |\Psi_2\rangle_{\boldsymbol{p}} \Big] \, |\boldsymbol{k}_1\rangle_{\boldsymbol{p}} + && (411)\\
&\int d^3k_1 \, ... \int d^3k_3 \; \Gamma_\kappa(\boldsymbol{k}_1, ..., \boldsymbol{k}_3)\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_3) \, |\boldsymbol{k}_1, ..., \boldsymbol{k}_3\rangle_{\boldsymbol{p}} && (412)\\
&= \int d^3k_1 \, \frac{\Gamma_\kappa(\boldsymbol{k}_1)}{\eta(\boldsymbol{p}; \boldsymbol{k}_1)} \int d^3k \; \gamma_\kappa^2(\boldsymbol{k}) \big[-\xi(\boldsymbol{p}; \boldsymbol{k}_1)\xi(\boldsymbol{p}; \boldsymbol{k}) + \xi(\boldsymbol{p}; \boldsymbol{k}_1, \boldsymbol{k}) \big] \, |\boldsymbol{k}_1\rangle_{\boldsymbol{p}} + && (413)\\
&\quad + \int d^3k_1 \, ... \int d^3k_3 \; \Gamma_\kappa(\boldsymbol{k}_1, ..., \boldsymbol{k}_3)\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_3) \, |\boldsymbol{k}_1, ..., \boldsymbol{k}_3\rangle_{\boldsymbol{p}} && (414)
\end{aligned}$$

Next we define an operation recursively on the $\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n)$, which we like to call contraction of first order by

$$\begin{aligned}
\overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n, \boldsymbol{l}_1)}^{l_1} \quad &:= \quad \frac{1}{\eta(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n)} \Big[\int d^3k \; \gamma_\kappa^2(\boldsymbol{l}_1) \times && (415)\\
&\qquad \times \big[\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n, \boldsymbol{l}_1) - \xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n)\xi(\boldsymbol{p}; \boldsymbol{l}_1) \big] + && (416)\\
&\qquad + \sum_{i=1}^n \overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n)}^{k_i} \Big] && (417)
\end{aligned}$$

and its termination by

$$\overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, \boldsymbol{l}_1)}^{l_1} \quad := \quad \frac{1}{\eta(\boldsymbol{p}; \boldsymbol{k}_1)} \int d^3l \; \gamma_\kappa^2(\boldsymbol{l}_1) \Big[\xi(\boldsymbol{p}; \boldsymbol{k}_1, \boldsymbol{l}_1) - \xi(\boldsymbol{p}; \boldsymbol{k}_1)\xi(\boldsymbol{p}; \boldsymbol{l}_1) \Big] \tag{418}$$

As long as we leave κ finite the integral exists and the recursion together with (403) and (404) is well-defined. So we can write

$$\begin{aligned}
|\Psi_3\rangle_{\boldsymbol{p}} &= \int d^3k_1 \; \Gamma_\kappa(\boldsymbol{k}_1) \overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, \boldsymbol{l}_1)}^{l_1} \, |\boldsymbol{k}_1\rangle_{\boldsymbol{p}} && (419)\\
&\quad + \int d^3k_1 \, ... \int d^3k_3 \; \Gamma_\kappa(\boldsymbol{k}_1, ..., \boldsymbol{k}_3)\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_3) \, |\boldsymbol{k}_1, ..., \boldsymbol{k}_3\rangle_{\boldsymbol{p}} && (420)
\end{aligned}$$

Fourth order

$$E_4^{\boldsymbol{p}} = -\int d^3l_1\ \gamma_\kappa^2(\boldsymbol{l}_1)\overbrace{\xi(\boldsymbol{p};\boldsymbol{l}_1,\boldsymbol{l}_2)}^{\boldsymbol{l}_2} \tag{421}$$

$$|\Psi_4\rangle_{\boldsymbol{p}} = \int d^3k_1 \int d^3k_2\ \frac{1}{\eta(\boldsymbol{p};\boldsymbol{k}_1,\boldsymbol{k}_2)}\Bigg[\int d^3k\ \gamma_\kappa(\boldsymbol{k})\,\langle\boldsymbol{k}|_{\boldsymbol{p}}\,|\Psi_1\rangle_{\boldsymbol{p}} \times \tag{422}$$

$$\times\,\langle\boldsymbol{k}_1,\boldsymbol{k}_2|_{\boldsymbol{p}}\,|\Psi_3\rangle_{\boldsymbol{p}} - \int d^3k\ \gamma_\kappa(\boldsymbol{k})\sqrt{3}\,\langle\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}|_{\boldsymbol{p}}\,|\Psi_3\rangle_{\boldsymbol{p}} + \tag{423}$$

$$-\frac{\gamma_\kappa(\boldsymbol{k}_1)}{\sqrt{2}}\,\langle\boldsymbol{k}_2|_{\boldsymbol{p}}\,|\Psi_3\rangle_{\boldsymbol{p}} - \frac{\gamma_\kappa(\boldsymbol{k}_2)}{\sqrt{2}}\,\langle\boldsymbol{k}_1|_{\boldsymbol{p}}\,|\Psi_3\rangle_{\boldsymbol{p}}\Bigg]\,|\boldsymbol{k}_1,\boldsymbol{k}_2\rangle_{\boldsymbol{p}} \tag{424}$$

$$+\int d^3k_1 \ldots \int d^3k_4\ \Gamma_\kappa(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)\xi(\boldsymbol{p};\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)\,|\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4\rangle_{\boldsymbol{p}} \tag{425}$$

$$= \int d^3k_1 \ldots \int d^3k_2\ \Gamma_\kappa(\boldsymbol{k}_1,\boldsymbol{k}_2)\overbrace{\xi(\boldsymbol{p};\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{l}_1)}^{\boldsymbol{l}_1}\,|\boldsymbol{k}_1,\boldsymbol{k}_2\rangle_{\boldsymbol{p}} + \tag{426}$$

$$+\int d^3k_1 \ldots \int d^3k_4\ \Gamma_\kappa(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)\xi(\boldsymbol{p};\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)\,|\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4\rangle_{\boldsymbol{p}} \tag{427}$$

Fifth order

$$E_5^{\boldsymbol{p}} = 0 \tag{428}$$

$$|\Psi_5\rangle_{\boldsymbol{p}} = \int d^3k_1\ \frac{1}{\eta(\boldsymbol{p};\boldsymbol{k}_1)}\Bigg[\int d^3k\ \gamma_\kappa(\boldsymbol{k})\,\langle k|_{\boldsymbol{p}}\,|\Psi_1\rangle_{\boldsymbol{p}} \times \tag{429}$$

$$\times\,\langle\boldsymbol{k}_1|_{\boldsymbol{p}}\,|\Psi_3\rangle_{\boldsymbol{p}} + \int d^3k\ \gamma_\kappa(\boldsymbol{k})\,\langle k|_{\boldsymbol{p}}\,|\Psi_3\rangle_{\boldsymbol{p}}\,\langle\boldsymbol{k}_1|_{\boldsymbol{p}}\,|\Psi_1\rangle_{\boldsymbol{p}} + \tag{430}$$

$$-\int d^3k\ \gamma_\kappa(\boldsymbol{k})\sqrt{2}\,\langle\boldsymbol{k}_1,\boldsymbol{k}|_{\boldsymbol{p}}\,|\Psi_4\rangle_{\boldsymbol{p}}\Bigg]\,|\boldsymbol{k}_1\rangle_{\boldsymbol{p}} + \tag{431}$$

$$+\int d^3k_1 \ldots \int d^3k_1\ \frac{1}{\eta(\boldsymbol{p};\boldsymbol{k}_1,\ldots,\boldsymbol{k}_3)}\Bigg[\int d^3k\ \gamma_\kappa(\boldsymbol{k})\,\langle\boldsymbol{k}|_{\boldsymbol{p}}\,|\Psi_1\rangle_{\boldsymbol{p}} \times \tag{432}$$

$$\times\,\langle\boldsymbol{k}_1,\ldots,\boldsymbol{k}_3|_{\boldsymbol{p}}\,|\Psi_3\rangle_{\boldsymbol{p}} - \int d^3k\ \gamma_\kappa(\boldsymbol{k})\sqrt{4}\,\langle\boldsymbol{k}_1,\ldots,\boldsymbol{k}_3,\boldsymbol{k}|_{\boldsymbol{p}}\,|\Psi_4\rangle_{\boldsymbol{p}} + \tag{433}$$

$$-\sum_{i=1}^{3}\frac{\gamma_\kappa(\boldsymbol{k}_i)}{\sqrt{3}}\,\langle\boldsymbol{k}_1,\ldots,\widehat{\boldsymbol{k}_i},\ldots,\boldsymbol{k}_3|_{\boldsymbol{p}}\,|\Psi_4\rangle_{\boldsymbol{p}}\Bigg]\,|\boldsymbol{k}_1,\ldots,\boldsymbol{k}_3\rangle_{\boldsymbol{p}} + \tag{434}$$

$$+\int d^3k_1 \ldots \int d^3k_5\ \Gamma_\kappa(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_5)\xi(\boldsymbol{p};\boldsymbol{k}_1,\ldots,\boldsymbol{k}_5)\,|\boldsymbol{k}_1,\ldots,\boldsymbol{k}_5\rangle_{\boldsymbol{p}} \tag{435}$$

$$= \int d^3k_1\ \frac{\Gamma_\kappa(\boldsymbol{k}_1)}{\eta(\boldsymbol{p};\boldsymbol{k}_1)}\Bigg[\int d^3k\ \gamma_\kappa^2(\boldsymbol{k})\big[-\overbrace{\xi(\boldsymbol{p};\boldsymbol{k}_1,\boldsymbol{l})}^{\boldsymbol{l}}\xi(\boldsymbol{p};\boldsymbol{k}) + \overbrace{\xi(\boldsymbol{p};\boldsymbol{k}_1,\boldsymbol{k},\boldsymbol{l})}^{\boldsymbol{l}}\big] + \tag{436}$$

$$+E_4^{\boldsymbol{p}}\xi(\boldsymbol{p};\boldsymbol{k}_1)\Bigg]\,|\boldsymbol{k}_1\rangle_{\boldsymbol{p}} + \tag{437}$$

$$+\int d^3k_1 \ldots \int d^3k_3\ \Gamma_\kappa(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_3)\overbrace{\xi(\boldsymbol{p},\boldsymbol{p};\boldsymbol{k}_1,\ldots,\boldsymbol{k}_3,\boldsymbol{l}_1)}^{\boldsymbol{l}_1}\,|\boldsymbol{k}_1,\ldots,\boldsymbol{k}_3\rangle_{\boldsymbol{p}} + \tag{438}$$

$$+\int d^3k_1 \ldots \int d^3k_5\ \Gamma_\kappa(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_5)\xi(\boldsymbol{p};\boldsymbol{k}_1,\ldots,\boldsymbol{k}_5)\,|\boldsymbol{k}_1,\ldots,\boldsymbol{k}_5\rangle_{\boldsymbol{p}} \tag{439}$$

Sixth order

$$E_6^{\boldsymbol{p}} = -\int d^3l_1\ \gamma_\kappa^2(\boldsymbol{l}_1)\frac{1}{\eta(\boldsymbol{p};\boldsymbol{l}_1)}\Bigg[\int d^3l_2\ \gamma_\kappa^2(\boldsymbol{l}_2)\big[-\overbrace{\xi(\boldsymbol{p};\boldsymbol{l}_2,\boldsymbol{l}_3)}^{\boldsymbol{l}_3}\xi(\boldsymbol{p};\boldsymbol{l}_1) + \overbrace{\xi(\boldsymbol{p};\boldsymbol{l}_1,\boldsymbol{l}_2,\boldsymbol{l}_3)}^{\boldsymbol{l}_3}\big] \tag{440}$$

$$+E_4^{\boldsymbol{p}}\xi(\boldsymbol{p};\boldsymbol{l}_1)\Bigg] \tag{441}$$

Now to arrive at a general formula for the wave function and the energy correction terms it is necessary to generalize the contraction operation to the m-th order. We can define the m-th order contraction by induction and give it the symbolical form $\overbrace{\xi(\boldsymbol{p}; ..., \boldsymbol{l}_1, ..., \boldsymbol{l}_m)}^{\boldsymbol{l}_1,...,\boldsymbol{l}_m}$. The nature of the contraction of arbitrary order is naturally given by nothing else then equation (390), so for all $n \in \mathbb{N}$, $m \in \mathbb{N}_0$ such that $n - 2m \geq 1$ we start with

$$\overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_{n-2m}, \boldsymbol{l}_1, ..., \boldsymbol{l}_m)}^{\boldsymbol{l}_1,...,\boldsymbol{l}_m} := \frac{\langle \boldsymbol{k}_1, ..., \boldsymbol{k}_{n-2m}|_{\boldsymbol{p}} |\Psi_n\rangle}{\Gamma_\kappa(\boldsymbol{k}_1, ..., \boldsymbol{k}_{n-2m})} \tag{442}$$

We have already introduced the zero-th order ($m = 0$) in lines (403), (404) and first order ($m = 1$) in lines (415), (418). Let us now compute the m-th order

$$\overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_{n-2m}, \boldsymbol{l}_1, ..., \boldsymbol{l}_m)}^{\boldsymbol{l}_1,...,\boldsymbol{l}_m} = \frac{\Gamma_\kappa^{-1}(\boldsymbol{k}_1, ..., \boldsymbol{k}_{n-2m})}{\eta(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_{n-2m})} \times \tag{443}$$

$$\times \Bigg[E_2^{\boldsymbol{p}} \langle \boldsymbol{k}_1, ..., \boldsymbol{k}_{n-2m}|_{\boldsymbol{p}} |\Psi_{n-2}\rangle_{\boldsymbol{p}} + \tag{444}$$

$$- \int d^3k\ \gamma_\kappa(\boldsymbol{k}) \sqrt{n - 2m + 1}\, \langle \boldsymbol{k}_1, ..., \boldsymbol{k}_{n-2m}, \boldsymbol{k}|_{\boldsymbol{p}} |\Psi_{n-1}\rangle_{\boldsymbol{p}} + \tag{445}$$

$$+ \sum_{i=2}^{m} E_{2i}^{\boldsymbol{p}} \langle \boldsymbol{k}_1, ..., \boldsymbol{k}_{n-2m}|_{\boldsymbol{p}} |\Psi_{n-2i}\rangle_{\boldsymbol{p}} + \tag{446}$$

$$- \sum_{i=1}^{n-2m} \frac{\gamma_\kappa(\boldsymbol{k}_i)}{\sqrt{n - 2m}} \langle \boldsymbol{k}_1, ..., \widehat{\boldsymbol{k}_i}, ..., \boldsymbol{k}_{n-2m}|_{\boldsymbol{p}} |\Psi_{n-1}\rangle_{\boldsymbol{p}} \Bigg] \tag{447}$$

$$= \frac{1}{\eta(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_{n-2m})} \Bigg[\int d^3k\ \gamma_\kappa^2(\boldsymbol{k}) \big[\overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_{n-2m}, \boldsymbol{k}, \boldsymbol{l}_1, ..., \boldsymbol{l}_{m-1})}^{\boldsymbol{l}_1,...,\boldsymbol{l}_{m-1}} + \tag{448}$$

$$- \overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_{n-2m}, \boldsymbol{l}_1, ..., \boldsymbol{l}_{m-1})}^{\boldsymbol{l}_1,...,\boldsymbol{l}_{m-1}} \xi(\boldsymbol{p}; \boldsymbol{k}) \big] + \tag{449}$$

$$+ \sum_{i=2}^{m} E_{2i}^{\boldsymbol{p}} \overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_{n-2m}, \boldsymbol{l}_1, ..., \boldsymbol{l}_{m-i})}^{\boldsymbol{l}_1,...,\boldsymbol{l}_{m-i}} + \tag{450}$$

$$+ \sum_{i=1}^{n-2m} \overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \widehat{\boldsymbol{k}_i}, ..., \boldsymbol{k}_{n-2m}, \boldsymbol{l}_1, ... \boldsymbol{l}_m)}^{\boldsymbol{l}_1,...\boldsymbol{l}_m} \Bigg] \tag{451}$$

or more general for all $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$

$$\overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n, \boldsymbol{l}_1, ..., \boldsymbol{l}_m)}^{\boldsymbol{l}_1,...,\boldsymbol{l}_m} = \frac{1}{\eta(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n)} \times \tag{452}$$

$$\times \Bigg[\int d^3k\ \gamma_\kappa^2(\boldsymbol{k}) \big[\overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n, \boldsymbol{k}, \boldsymbol{l}_1, ..., \boldsymbol{l}_{m-1})}^{\boldsymbol{l}_1,...,\boldsymbol{l}_{m-1}} + \tag{453}$$

$$- \overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n, \boldsymbol{l}_1, ..., \boldsymbol{l}_{m-1})}^{\boldsymbol{l}_1,...,\boldsymbol{l}_{m-1}} \xi(\boldsymbol{p}; \boldsymbol{k}) \big] + \tag{454}$$

$$+ \sum_{i=2}^{m} E_{2i}^{\boldsymbol{p}} \overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n, \boldsymbol{l}_1, ..., \boldsymbol{l}_{m-i})}^{\boldsymbol{l}_1,...,\boldsymbol{l}_{m-i}} + \tag{455}$$

$$+ \sum_{i=1}^{n} \overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \widehat{\boldsymbol{k}_i}, ..., \boldsymbol{k}_n, \boldsymbol{l}_1, ... \boldsymbol{l}_m)}^{\boldsymbol{l}_1,...\boldsymbol{l}_m} \Bigg] \tag{456}$$

Again for finite κ all integrals in this definition appearing directly or indirectly in the constants $E_{2i}^{\boldsymbol{p}}$ exist and so the contraction of arbitrary order on the ξ together with (404) is well-defined.

We can now write the n-th order wave function and energy correction term for $n \geq 1$ as

$$
\begin{aligned}
|\Psi_n\rangle_{\boldsymbol{p}} &= \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \int d^3k_1 \,...\int d^3k_{n-2i}\; \Gamma_\kappa(\boldsymbol{k}_1, ..., \boldsymbol{k}_{n-2i}) \overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_{n-2i}, \boldsymbol{l}_1, ..., \boldsymbol{l}_i)}^{\boldsymbol{l}_1,...,\boldsymbol{l}_i} \times \quad (457)\\
&\quad \times |\boldsymbol{k}_1, ..., \boldsymbol{k}_{n-2i}\rangle_{\boldsymbol{p}} \quad (458)\\
E_n^{\boldsymbol{p}} &= \begin{cases} -\int d^3l \; \gamma_\kappa^2(\boldsymbol{l}) \overbrace{\xi(\boldsymbol{p}; \boldsymbol{l}, \boldsymbol{l}_1, ..., \boldsymbol{l}_{n/2})}^{\boldsymbol{l}_1,...,\boldsymbol{l}_{n/2}} \text{ for } n \text{ even} \\ 0 \text{ for } n \text{ odd} \end{cases} \quad (459)
\end{aligned}
$$

and the ground state and its energy for finite κ and small enough g as

$$
\begin{aligned}
|\Psi_\kappa(g)\rangle_{\boldsymbol{p}} &= |0\rangle_{\boldsymbol{p}} + \sum_{n=1}^{\infty} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} g^n \int d^3k_1 \,...\int d^3k_{n-2i}\; \Gamma_\kappa(\boldsymbol{k}_1, ..., \boldsymbol{k}_{n-2i}) \times \quad (460)\\
&\quad \times \overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_{n-2i}, \boldsymbol{l}_1, ..., \boldsymbol{l}_i)}^{\boldsymbol{l}_1,...,\boldsymbol{l}_i} |\boldsymbol{k}_1, ..., \boldsymbol{k}_{n-2i}\rangle_{\boldsymbol{p}} \quad (461)\\
E_\kappa^{\boldsymbol{p}}(g) &= \frac{\boldsymbol{p}^2}{2M} - \sum_{n=1}^{\infty} g^{2n} \int d^3l \; \gamma_\kappa^2(\boldsymbol{l}) \overbrace{\xi(\boldsymbol{p}; \boldsymbol{l}, \boldsymbol{l}_1, ..., \boldsymbol{l}_{(n-1)})}^{\boldsymbol{l}_1,...,\boldsymbol{l}_{(n-1)}} \quad (462)
\end{aligned}
$$

since these series are ensured to converge to a Fock-vector and a finite energy respectively by theorem 4.5.1. Here $\lfloor x \rfloor$ means the largest natural number smaller than x. The n-th meson wave function of the ground state $|\Psi_\kappa(g)\rangle_{\boldsymbol{p}}$ is then

$$
\begin{aligned}
\langle \boldsymbol{k}_1, ..., \boldsymbol{k}_n|_{\boldsymbol{p}} |\Psi_\kappa(g)\rangle_{\boldsymbol{p}} &= \sum_{\substack{m=n\\ m+=2}}^{\infty} g^m \Gamma_\kappa(\boldsymbol{k}_1, ..., \boldsymbol{k}_n) \overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n, \boldsymbol{l}_1, ..., \boldsymbol{l}_{(m-n)/2})}^{\boldsymbol{l}_1,...,\boldsymbol{l}_{(m-n)/2}} \quad (463)\\
&= \sum_{m=0}^{\infty} g^{n+2m} \Gamma_\kappa(\boldsymbol{k}_1, ..., \boldsymbol{k}_n) \overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n, \boldsymbol{l}_1, ..., \boldsymbol{l}_m)}^{\boldsymbol{l}_1,...,\boldsymbol{l}_m} \quad (464)
\end{aligned}
$$

where $m+ = 2$ in the sum denotes the increment in the sum is two, i.e. its summands are of the type $m = n$, $m = n + 2$, $m = n + 4$,...

We now turn to the question which terms in the perturbation series disappear when we turn off the nucleon dispersion term in the model Hamiltonian. We will find that in the absence of a nucleon dispersion term our formulas for the ground state and its energy eigenvalue collapse to the simple formulas we have already found for the static quantum Nelson model in subsection 4.4. This will help us to understand the effect the Schrödinger dispersion relation of the nucleons has on the n-meson wave functions of the ground state. At a later point we shall see that this effect causes the ground state to remain in Fock-space even in the limit $\kappa \to \infty$ in contrary to the static quantum Nelson model.

Static quantum Nelson model revisited As we discussed in subsection 4.4 the interaction term stays the same but $H_0^{\boldsymbol{p}}$ becomes

$$H_0^{\boldsymbol{p}} := M + \int d^3k \; a_{\boldsymbol{k}}^\dagger \omega_{\boldsymbol{k}} a_{\boldsymbol{k}} \quad (465)$$

For this case the properties of theorem 4.5.1 trivially hold and can even be extended to $\boldsymbol{p} \in \mathbb{R}^3$. Thus the same kind of perturbation theory will be applicable and in fact all the above results of computed correction orders will be the same except that now

$$E_0^{\boldsymbol{p}} = M \tag{466}$$

$$E_0^{\boldsymbol{p}}(\boldsymbol{k}_1, ..., \boldsymbol{k}_n) = M + \sum_{i=0}^{n} \omega_{\boldsymbol{k}_i} \tag{467}$$

$$\eta(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n) = \frac{1}{E_0^{\boldsymbol{p}}(\boldsymbol{k}_1, ..., \boldsymbol{k}_n) - E_0^{\boldsymbol{p}}} = \frac{1}{\sum_{i=0}^{n} \omega_{\boldsymbol{k}_i}} \tag{468}$$

and if we pull this expression through we find

$$\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n) = \prod_{i=1}^{n} \xi(\boldsymbol{p}; \boldsymbol{k}_i) = \prod_{i=1}^{n} \frac{1}{\omega_{\boldsymbol{k}_i}} \tag{469}$$

Immediately we obtain $\overbrace{\xi(\boldsymbol{p}; ..., \boldsymbol{l}_1, ..., \boldsymbol{l}_j)}^{\boldsymbol{l}_1, ..., \boldsymbol{l}_j} = 0$ for every $j \in \mathbb{N}$, which eases the pain of calculating all orders of perturbation theory immensely since just the leading terms survive. Thus lines (457) and (459) collapse to

$$\begin{aligned}
|\Psi_n\rangle_{\boldsymbol{p}} &= \frac{(-1)^n}{\sqrt{n!}} \int d^3k_1 \, ... \int d^3k_n \, \prod_{i=1}^{n} \gamma_\kappa(\boldsymbol{k}_i) \xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n) \, |\boldsymbol{k}_1, ..., \boldsymbol{k}_n\rangle_{\boldsymbol{p}} && (470)\\
&= \frac{(-1)^n}{\sqrt{n!}} \int d^3k_1 \, ... \int d^3k_n \, \prod_{i=1}^{n} \gamma_\kappa(\boldsymbol{k}_i) \xi(\boldsymbol{p}; \boldsymbol{k}_i) \, |\boldsymbol{k}_1, ..., \boldsymbol{k}_n\rangle_{\boldsymbol{p}} && (471)\\
E_2^{\boldsymbol{p}} &= -\int d^3l_1 \; \gamma_\kappa^2(\boldsymbol{l}_1) \xi(\boldsymbol{p}; \boldsymbol{l}_1) && (472)\\
E_{n\neq 2}^{\boldsymbol{p}} &= 0 && (473)
\end{aligned}$$

As one would have expected the second order term again carries the divergent behavior for $\kappa \to \infty$. This time the divergence is linear and so even stronger than in the dynamic case. Furthermore for $\kappa \to \infty$ the $\prod_{i=1}^{n} \gamma_\kappa(\boldsymbol{k}_i) \xi(\boldsymbol{p}; \boldsymbol{k}_i)$ are not in $\mathcal{L}_2$ and the ground state does not lie in Fock-space any more. This is why the static case is said to be more singular than the dynamic one. Both of these properties are obviously effects of the missing free Schrödinger dispersion term of the nucleons in the model Hamiltonian. The formulas for the exact the exact ground state $|\Psi_\kappa(g)\rangle_{\boldsymbol{p}}$ and energy $E_\kappa^{\boldsymbol{p}}(g)$ are hence

$$\begin{aligned}
|\Psi_\kappa(g)\rangle_{\boldsymbol{p}} &= \sum_{n=0}^{\infty} g^n \, |\Psi_n\rangle_{\boldsymbol{p}} && (474)\\
&= \sum_{n=0}^{\infty} \int d^3k_1 \, ... \int d^3k_n \, \frac{(-g)^n}{\sqrt{n!}} \prod_{i=1}^{n} \frac{\gamma_\kappa(\boldsymbol{k}_i)}{\omega_{\boldsymbol{k}_i}} \, |\boldsymbol{k}_1, ..., \boldsymbol{k}_n\rangle_{\boldsymbol{p}} && (475)\\
&= \sum_{n=0}^{\infty} \int d^3k_1 \, ... \int d^3k_n \, \frac{(-g)^n}{n!} \prod_{i=1}^{n} \frac{\gamma_\kappa(\boldsymbol{k}_i)}{\omega_{\boldsymbol{k}_i}} a_{\boldsymbol{k}_1}^\dagger e^{-i\boldsymbol{k}_1\boldsymbol{x}} ... a_{\boldsymbol{k}_n}^\dagger e^{-i\boldsymbol{k}_n\boldsymbol{x}} \, |0\rangle_{\boldsymbol{p}} && (476)\\
&= e^{-g\int d^3k \, \frac{\gamma_\kappa(\boldsymbol{k})}{\omega_{\boldsymbol{k}}} a_{\boldsymbol{k}}^\dagger e^{-i\boldsymbol{k}\boldsymbol{x}}} \, |0\rangle_{\boldsymbol{p}} && (477)\\
E_\kappa^{\boldsymbol{p}}(g) &= \sum_{n=0}^{\infty} g^n E_n^{\boldsymbol{p}} = E_0^{\boldsymbol{p}} + g^2 E_2^{\boldsymbol{p}} && (478)\\
&= M - g^2 \int d^3k_1 \, \frac{\gamma_\kappa^2(\boldsymbol{k}_1)}{\omega_{\boldsymbol{k}_1}} && (479)
\end{aligned}$$

which are of course exactly the same result as the one we achieved in subsection 4.4. So what we have learned from this computation is that the divergence of the energy eigenvalue is caused by the field. In fact it is the same divergence like the one we have found for the classical version of this model and hence one could say it is a common and generical problem of the point particle limit in all classical and quantum field theories of this kind. Secondly we have learned that the ground state in the dynamic quantum model is only a Fock-vector because of the presence of the Schrödinger nucleon dispersion term in the Hamiltonian.

Note 4.5.1. *Let us phenomenologically analyze where the second order term comes from.* $|\Psi_1\rangle_{\boldsymbol{p}}$ *is a Fock-vector fibre with one meson wave function* $\gamma_\kappa(\boldsymbol{k})/\omega(\boldsymbol{k})$*. We have seen that* $E_2^{\boldsymbol{p}}$ *is proportional to the transition amplitude*

$$|\,\langle 0|_{\boldsymbol{p}}\, H_\kappa^{\boldsymbol{p}}\, |\Psi_1\rangle_{\boldsymbol{p}}\,| = |\,\langle 0|_{\boldsymbol{p}}\, H_{I_\kappa}^{\boldsymbol{p}}\, |\Psi_1\rangle_{\boldsymbol{p}}\,| \tag{480}$$

The probability of the process of a one meson annihilation induced by the whole Hamiltonian $H_\kappa^{\boldsymbol{p}}$ *is, depending on the 'choice' of the position probability measure, somehow a functional of this transition amplitude. One could again argue that it is of the exact type of divergence as the classical one because* $E_2^{\boldsymbol{p}}$ *is for* $\boldsymbol{p}=0$ *exactly equal to the renormalization constant we have found for the static classical Nelson model in section 3.3. Of course only formally we may interpret this self-energy as the Yukawa potential evaluated at radius* $\to 0$ *in the following sense*

$$\lim_{\kappa\to\infty} |\,\langle 0|_{\boldsymbol{p}}\, H_\kappa\, |\Psi_1\rangle_{\boldsymbol{p}}\,| \quad = \quad \lim_{\kappa\to\infty}\int d^3k_1\, \frac{\gamma_\kappa^2(\boldsymbol{k}_1)}{\omega_{\boldsymbol{k}_1}} \text{"="} \int d^3k_1\, \frac{1}{2\omega_{\boldsymbol{k}}^2} \tag{481}$$

$$\text{"="} \quad \lim_{|\boldsymbol{x}|\to 0}\int d^3k_1\, \frac{e^{-i\boldsymbol{k}_1\boldsymbol{x}}}{2\omega_{\boldsymbol{k}}^2} \text{"="} \lim_{|\boldsymbol{x}|\to 0}\frac{1}{4\pi}\frac{e^{-\mu|\boldsymbol{x}|}}{|\boldsymbol{x}|} \tag{482}$$

Hence one could say $|\Psi_1\rangle_{\boldsymbol{p}} \to |0\rangle$ *is a problematic process - similarly* $|0\rangle \to |\Psi_1\rangle_{\boldsymbol{p}}$ *of course. This corresponds to the classical picture one would have, i.e. that the meson is annihilated at the position of the nucleon or that a meson is created at the position of the nucleon where the field is not well-defined.*

4.5.2 The renormalized model Hamiltonian $\widehat{H}$ and its ground state

The existence of the renormalized model Hamiltonian $\widehat{H}$ has been proven by Nelson's theorem 2.6.1 so following the steps we did in the toy model we need now to establish the existence of the ground state for $\widehat{H}$ in order to obtain one element in its yet unknown domain $\mathcal{D}(\widehat{H})$. As we shall see in the following the situation for the dynamic quantum Nelson model Hamiltonian H_κ, although more complicated, is slightly better behaved like the static one when taking the limit $\kappa\to\infty$ - i.e. its ground state will remain in Fock-space even in the limit although its energy eigenvalue diverges. Furthermore we shall find that the divergent behavior of the energy eigenvalue is only carried by the second order correction term $E_2^{\boldsymbol{p}}$. This we have also found to be true for the toy model and so we will try to define the renormalized model Hamiltonian $\widehat{H}$ as the limit of $H_\kappa - R_\kappa$ for $\kappa\to\infty$ in this weak sense, which has successfully been done in the case of the toy model. So let us start with the discussion of the energy corrections terms first. The limit will only concern the γ_κ functions. Recall that $\gamma_\infty(\boldsymbol{k}) \sim |\boldsymbol{k}|^{-1/2}$. The first non-zero energy correction after $E_0^{\boldsymbol{p}}$ is

$$E_2^{\boldsymbol{p}} = -\int d^3l_1\, \underbrace{\gamma_\infty^2(\boldsymbol{l}_1)}_{\sim|\boldsymbol{l}_1|^{-1}}\underbrace{\xi(\boldsymbol{p};\boldsymbol{l}_1)}_{\sim|\boldsymbol{l}_1|^{-2}} \tag{483}$$

but since $|\xi(\boldsymbol{p};\boldsymbol{l}_1)| \sim |\boldsymbol{l}_1|^{-2}$ by line (404) the integral diverges logarithmically in the limit $\kappa\to\infty$. The divergence is hence not as severe as in the static case but still remains to be a problem.

Already the next higher non-zero energy correction is finite as the following calculation shows

$$
\begin{aligned}
E_4^{\boldsymbol{p}} &= -\int d^3l_1\ \gamma_\infty^2(\boldsymbol{l}_1)\overbrace{\xi(\boldsymbol{p};\boldsymbol{l}_1,\boldsymbol{l}_2)}^{\boldsymbol{l}_2} && (484)\\
&= -\int d^3l_1\ \frac{\gamma_\infty^2(\boldsymbol{l}_1)}{\eta(\boldsymbol{p};\boldsymbol{l}_1)}\int d^3l_2\ \gamma_\infty^2(\boldsymbol{l}_2)\Big[\xi(\boldsymbol{p};\boldsymbol{l}_1,\boldsymbol{l}_2)-\xi(\boldsymbol{p};\boldsymbol{l}_1)\xi(\boldsymbol{p};\boldsymbol{l}_2)\Big] && (485)\\
&= -\int d^3l_1\ \frac{\gamma_\infty^2(\boldsymbol{l}_1)}{\eta(\boldsymbol{p};\boldsymbol{l}_1)}\int d^3l_2\ \gamma_\infty^2(\boldsymbol{l}_2)\Big[\frac{1}{\eta(\boldsymbol{p};\boldsymbol{l}_1,\boldsymbol{l}_2)}\big[\xi(\boldsymbol{p};\boldsymbol{l}_1)+\xi(\boldsymbol{p};\boldsymbol{l}_2)\big]+ && (486)\\
&\qquad -\xi(\boldsymbol{p};\boldsymbol{l}_1)\xi(\boldsymbol{p};\boldsymbol{l}_2)\Big] && (487)\\
&= -\int d^3l_1\ \gamma_\infty^2(\boldsymbol{l}_1)\int d^3l_2\ \gamma_\infty^2(\boldsymbol{l}_2)\Big[\frac{1}{\eta(\boldsymbol{p};\boldsymbol{l}_1,\boldsymbol{l}_2)\eta(\boldsymbol{p};\boldsymbol{l}_1)\eta(\boldsymbol{p};\boldsymbol{l}_2)}+ && (488)\\
&\qquad +\frac{1}{\eta^2(\boldsymbol{p};\boldsymbol{l}_1)}\big[\frac{1}{\eta(\boldsymbol{p};\boldsymbol{l}_1,\boldsymbol{l}_2)}-\frac{1}{\eta(\boldsymbol{p};\boldsymbol{l}_2)}\big]\Big] && (489)
\end{aligned}
$$

In order to estimate the integral we make use of Young's inequality for convolutions which states that for $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=2$ and $p,q,r\geq 1$

$$\int dx\int dy\ f(x)g(x+y)h(x)\leq \|f\|_{\mathcal{L}_p}\|g\|_{\mathcal{L}_q}\|h\|_{\mathcal{L}_r} \tag{490}$$

With this we may estimate the first term of the sum by

$$\int d^3l_1\ \int d^3l_2\ \frac{\gamma_\infty^2(\boldsymbol{l}_1)}{\eta(\boldsymbol{p};\boldsymbol{l}_1)}\frac{\gamma_\infty^2(\boldsymbol{l}_2)}{\eta(\boldsymbol{p};\boldsymbol{l}_1)}\frac{1}{\eta(\boldsymbol{p};\boldsymbol{l}_1,\boldsymbol{l}_2)}\leq\left\|\frac{\gamma_\infty^2}{\eta(\boldsymbol{p};\cdot)}\right\|_{\mathcal{L}_{\frac{4}{3}}}^2\left\|\frac{1}{\eta(\boldsymbol{p};\cdot,0)}\right\|_{\mathcal{L}_2}<\infty \tag{491}$$

The second term gives

$$
\begin{aligned}
&\int d^3l_1\ \frac{\gamma_\infty^2(\boldsymbol{l}_1)}{\eta(\boldsymbol{p};\boldsymbol{l}_1)^2}\int d^3l_2\ \gamma_\infty^2(\boldsymbol{l}_2)\Big[\frac{1}{\eta(\boldsymbol{p};\boldsymbol{l}_2)}-\frac{1}{\eta(\boldsymbol{p};\boldsymbol{l}_1,\boldsymbol{l}_2)}\Big] && (492)\\
&=\int d^3l_1\ \frac{\gamma_\infty^2(\boldsymbol{l}_1)}{\eta(\boldsymbol{p};\boldsymbol{l}_1)^2}\int d^3l_2\ \frac{\gamma_\infty^2(\boldsymbol{l}_2)}{\eta(\boldsymbol{p};\boldsymbol{l}_2)}\Big[\frac{1}{2M}\big[\boldsymbol{l}_1^2+2\boldsymbol{l}_1(p-\boldsymbol{l}_2)\big]-w_{\boldsymbol{l}_2}\Big]\frac{1}{\eta(\boldsymbol{p};\boldsymbol{l}_1,\boldsymbol{l}_2)} && (493)\\
&\leq c_1\int d^3l_1\ \int d^3l_2\ \frac{\gamma_\infty^2(\boldsymbol{l}_1)\|\boldsymbol{l}_1\|^2}{\eta(\boldsymbol{p};\boldsymbol{l}_1)^2}\frac{\gamma_\infty^2(\boldsymbol{l}_2)}{\eta(\boldsymbol{p};\boldsymbol{l}_2)}\frac{1}{\eta(\boldsymbol{p};\boldsymbol{l}_1,\boldsymbol{l}_2)} && (494)\\
&+c_2\int d^3l_1\ \int d^3l_2\ \frac{\gamma_\infty^2(\boldsymbol{l}_1)\|\boldsymbol{l}_1\|}{\eta(\boldsymbol{p};\boldsymbol{l}_1)^2}\frac{\gamma_\infty^2(\boldsymbol{l}_2)}{\eta(\boldsymbol{p};\boldsymbol{l}_2)}\frac{1}{\eta(\boldsymbol{p};\boldsymbol{l}_1,\boldsymbol{l}_2)} && (495)\\
&+c_3\int d^3l_1\ \int d^3l_2\ \frac{\gamma_\infty^2(\boldsymbol{l}_1)\|\boldsymbol{l}_1\|}{\eta(\boldsymbol{p};\boldsymbol{l}_1)^2}\frac{\gamma_\infty^2(\boldsymbol{l}_2)\|\boldsymbol{l}_2\|}{\eta(\boldsymbol{p};\boldsymbol{l}_2)}\frac{1}{\eta(\boldsymbol{p};\boldsymbol{l}_1,\boldsymbol{l}_2)} && (496)\\
&+c_4\int d^3l_1\ \int d^3l_2\ \frac{\gamma_\infty^2(\boldsymbol{l}_1)}{\eta(\boldsymbol{p};\boldsymbol{l}_1)^2}\frac{\gamma_\infty^2(\boldsymbol{l}_2)\|\boldsymbol{l}_2\|}{\eta(\boldsymbol{p};\boldsymbol{l}_2)}\frac{1}{\eta(\boldsymbol{p};\boldsymbol{l}_1,\boldsymbol{l}_2)} && (497)
\end{aligned}
$$

where c_2 depends on the fixed $\boldsymbol{p}$. Again using Young's inequality we get the estimate

$$
\begin{aligned}
\ldots\leq c_1&\left\|\frac{\gamma_\infty^2\|\cdot\|^2}{\eta(\boldsymbol{p};\cdot)^2}\right\|_{\mathcal{L}_{\frac{4}{3}}}\left\|\frac{\gamma_\infty^2}{\eta(\boldsymbol{p};\cdot)}\right\|_{\mathcal{L}_{\frac{4}{3}}}\left\|\frac{1}{\eta(\boldsymbol{p};\cdot,0)}\right\|_{\mathcal{L}_2} && (498)\\
+c_2&\left\|\frac{\gamma_\infty^2\|\cdot\|}{\eta(\boldsymbol{p};\cdot)^2}\right\|_{\mathcal{L}_1}\left\|\frac{\gamma_\infty^2}{\eta(\boldsymbol{p};\cdot)}\right\|_{\mathcal{L}_{\frac{4}{3}}}\left\|\frac{1}{\eta(\boldsymbol{p};\cdot,0)}\right\|_{\mathcal{L}_4} && (499)\\
+c_3&\left\|\frac{\gamma_\infty^2\|\cdot\|}{\eta(\boldsymbol{p};\cdot)^2}\right\|_{\mathcal{L}_1}\left\|\frac{\gamma_\infty^2\|\cdot\|}{\eta(\boldsymbol{p};\cdot)}\right\|_{\mathcal{L}_2}\left\|\frac{1}{\eta(\boldsymbol{p};\cdot,0)}\right\|_{\mathcal{L}_2} && (500)\\
+c_4&\left\|\frac{\gamma_\infty^2}{\eta(\boldsymbol{p};\cdot)^2}\right\|_{\mathcal{L}_1}\left\|\frac{\gamma_\infty^2\|\cdot\|}{\eta(\boldsymbol{p};\cdot)}\right\|_{\mathcal{L}_2}\left\|\frac{1}{\eta(\boldsymbol{p};\cdot,0)}\right\|_{\mathcal{L}_2}<\infty && (501)\\
& && (502)
\end{aligned}
$$

Note that the cancelations in the sum in line (492) are needed in order for the integral to be finite. Integration over the two summands separately leads to divergent integrals.

By direct computation one finds that the sixth order energy correction is again finite and in fact the cancellation of the infinities takes place in a very similar way like the one we have observed computing $E_4^{\boldsymbol{p}}$. So one gets the feeling that the only divergent energy correction term is the one of second order. Furthermore one would put up the question if it is possible to construct a ground state for the dynamic quantum Nelson model with the same perturbation theory even with the cutoff removed if we subtract the divergent $-g^2 E_2^{\boldsymbol{p}}$ from the Hamiltonian $H_\kappa^{\boldsymbol{p}}$. Nelson's theorem 2.6.1 already answers in the affirmative, which is spelled out in the following simple corollary of it.

Corollary 4.5.2. *Given $\kappa \leq \infty$ (!), $\mu, M \in \mathbb{R}_+$, $|\boldsymbol{p}| \leq \sqrt{2MC}$, $0 \leq C < \mu$ and the renormalization constant formally by $R_\kappa(g) := -g^2 \int d^3k \, \frac{\gamma_\kappa^2(\boldsymbol{k})}{\frac{p^2}{2M}+\omega_{\boldsymbol{k}}} = -g^2 E_2^{\boldsymbol{p}=0}$ from Nelson's theorem 2.6.1, then the Hamiltonian*

$$\widehat{H}_\kappa^{\boldsymbol{p}}(g) \quad := \quad H_\kappa^{\boldsymbol{p}}(g) - R_\kappa(g) \tag{503}$$

has a well-defined resolvent

$$Res_{\kappa,g}^{\boldsymbol{p}}(\zeta) \quad := \quad \frac{1}{\widehat{H}_\kappa^{\boldsymbol{p}}(g) - \zeta} \tag{504}$$

on $\mathcal{F}_1^{\boldsymbol{p}}$ and the perturbed ground state is given by

$$|\widehat{\Psi}_\kappa(g)\rangle_{\boldsymbol{p}} \quad := \quad \oint_{\mathcal{C}(E_0^{\boldsymbol{p}})} d\zeta \;\; Res_{\kappa,g}^{\boldsymbol{p}}(\zeta)\, |0\rangle_{\boldsymbol{p}} \in \mathcal{D}(\widehat{H}_\kappa^{\boldsymbol{p}}(g)) \tag{505}$$

also for $\kappa \leq \infty$, where $\mathcal{C}(E_0^{\boldsymbol{p}})$ is a circle in the complex plane around $E_0^{\boldsymbol{p}}$ with a radius smaller than the energy gap $\mu - C$. In fact for sufficiently small $g \in \mathbb{R}$

1. *$|\widehat{\Psi}_\kappa(g)\rangle_{\boldsymbol{p}} = |\Psi_\kappa(g)\rangle_{\boldsymbol{p}}$, the perturbed ground state of $H_\kappa^{\boldsymbol{p}}$ from line (460), for all $\kappa \leq \infty$ and furthermore*

2. *$\widehat{E}^{\boldsymbol{p}}(g)_\kappa = E_\kappa^{\boldsymbol{p}}(g) - R_\kappa$, where $E_\kappa^{\boldsymbol{p}}(g)$ is the ground state energy eigenvalue from line (462) for all $\kappa \leq \infty$.*

Proof. In Nelson's theorem 2.6.1 it was shown that $e^{-i(H_\kappa - R_\kappa)t}$ converges strongly in the limit of $\kappa \to \infty$ to $e^{-i\hat{H}t}$ for all $t \in \mathbb{R}$. By a theorem of Trotter [17, VIII.21] we immediately get the strong resolvent convergence and $s - \lim_{\kappa\to\infty} Res_\kappa(\zeta) =: Res_\infty(\zeta)$ is well-defined. Let us define $Res_\kappa^{\boldsymbol{p}} := Res_\kappa \restriction \mathcal{F}_1^{\boldsymbol{p}}$. The existence of the limit resolvent enables us to write the perturbed ground state like in line (505) for all $\kappa \leq \infty$. Hence the limit $\lim_{\kappa\to\infty} |\widehat{\Psi}_\kappa(g)\rangle_{\boldsymbol{p}}$ exists in $\mathcal{F}_1^{\boldsymbol{p}}$ and is the ground state and has a finite energy eigenvalue $\lim_{\kappa\to\infty} \widehat{E}_\kappa^{\boldsymbol{p}}$. The rest can be shown by simple calculation, which we do in the following. It is a simple corollary to theorem 4.5.1 that shows that the resolvent can be expanded in a Taylor series around $g = 0$ for finite κ since the additional multiplication operator $R_\kappa(g)$ is finite for finite κ and so it can be dragged into the self-adjoint H_0 part of the Hamiltonian. Then we can perform the integral and extract terms of equal order of g and yield similar formulas for the Taylor coefficients (386) and (387) for $H_\kappa^{\boldsymbol{p}}$ but now taking into account that also $R_\kappa(g)$ depends on g^2. Since we want to compare the perturbation series of $H_\kappa^{\boldsymbol{p}}$ with $\widehat{H}_\kappa^{\boldsymbol{p}}$ we denote all coefficients belonging to the perturbation expansion for $\widehat{H}_\kappa^{\boldsymbol{p}}$ with an hat to distinguish them to the ones we have computed for $H_\kappa^{\boldsymbol{p}}$ in the beginning of this subsection. With the choice of normalization $\langle\widehat{\Psi}_0|_{\boldsymbol{p}}\,|\widehat{\Psi}\rangle_{\boldsymbol{p}} = \langle\widehat{\Psi}_0|_{\boldsymbol{p}}\,|\widehat{\Psi}_0\rangle_{\boldsymbol{p}} = 1$ the Taylor coefficients for $\kappa < \infty$ are

$$\widehat{E}_n^{\boldsymbol{p}} = \frac{\langle\widehat{\Psi}_0|_{\boldsymbol{p}}\, H_{I_\kappa}^{\boldsymbol{p}}\,|\widehat{\Psi}_{n-1}\rangle - \frac{R_\kappa}{g^2}\langle\widehat{\Psi}_0|_{\boldsymbol{p}}\,|\widehat{\Psi}_{n-2}\rangle_{\boldsymbol{p}}}{\langle\widehat{\Psi}_0|_{\boldsymbol{p}}\,|\widehat{\Psi}_0\rangle_{\boldsymbol{p}}} \tag{506}$$

$$\langle\boldsymbol{k}_1,...,\boldsymbol{k}_m|_{\boldsymbol{p}}\,|\widehat{\Psi}_n\rangle_{\boldsymbol{p}} = \frac{1}{\widehat{E}_0^{\boldsymbol{p}}(\boldsymbol{k}_1,...,\boldsymbol{k}_m) - \widehat{E}_0^{\boldsymbol{p}}}\Bigg[\sum_{i=1}^{n}\widehat{E}_i^{\boldsymbol{p}}\,\langle\boldsymbol{k}_1,...,\boldsymbol{k}_m|_{\boldsymbol{p}}\,|\widehat{\Psi}_{n-i}\rangle_{\boldsymbol{p}} \tag{507}$$

$$+\frac{R_\kappa}{g^2}\langle\boldsymbol{k}_1,...,\boldsymbol{k}_m|_{\boldsymbol{p}}\,|\widehat{\Psi}_{n-2}\rangle_{\boldsymbol{p}} - \langle\boldsymbol{k}_1,...,\boldsymbol{k}_m|_{\boldsymbol{p}}\, H_{I_\kappa}^{\boldsymbol{p}}\,|\widehat{\Psi}_{n-1}\rangle_{\boldsymbol{p}}\Bigg] \tag{508}$$

defining $|\widehat{\Psi}_{-1}\rangle_{\boldsymbol{p}} = |\widehat{\Psi}_{-2}\rangle_{\boldsymbol{p}} = 0$. Again only for reason of brevity we did not spell out that every coefficient of the expansions is depended of κ and g. Of course since the free Hamiltonian is the same as in the case of $H_\kappa^{\boldsymbol{p}}$, $|\widehat{\Psi}_0\rangle_{\boldsymbol{p}} = |\Psi_0\rangle_{\boldsymbol{p}}$, $\widehat{E}_0^{\boldsymbol{p}} = E_0^{\boldsymbol{p}}$ and $\widehat{E}_0^{\boldsymbol{p}}(\boldsymbol{k}_1,...,\boldsymbol{k}_n) = E_0^{\boldsymbol{p}}(\boldsymbol{k}_1,...,\boldsymbol{k}_n)$. For $n \neq 2$ the energy correction term is given by

$$\widehat{E}_n^{\boldsymbol{p}} = \frac{\langle\widehat{\Psi}_0|_{\boldsymbol{p}}\, H_{I_\kappa^{\boldsymbol{p}}}\,|\widehat{\Psi}_{n-1}\rangle}{\langle\widehat{\Psi}_0|_{\boldsymbol{p}}\,|\widehat{\Psi}_0\rangle_{\boldsymbol{p}}} \tag{509}$$

for $n = 2$ since we find $\langle\boldsymbol{k}_1,...,\boldsymbol{k}_m|_{\boldsymbol{p}}\,|\widehat{\Psi}_1\rangle_{\boldsymbol{p}} = \langle\boldsymbol{k}_1,...,\boldsymbol{k}_m|_{\boldsymbol{p}}\,|\Psi_1\rangle_{\boldsymbol{p}}$

$$\widehat{E}_2^{\boldsymbol{p}} = \frac{\langle\widehat{\Psi}_0|_{\boldsymbol{p}}\, H_{I_\kappa}^{\boldsymbol{p}}\,|\widehat{\Psi}_1\rangle}{\langle\widehat{\Psi}_0|_{\boldsymbol{p}}\,|\widehat{\Psi}_0\rangle_{\boldsymbol{p}}} - \frac{R_\kappa}{g^2} = E_2^{\boldsymbol{p}} - \frac{R_\kappa}{g^2} \tag{510}$$

$$= g^2\int d^3k\;\gamma_\kappa^2(\boldsymbol{k})\left(\frac{1}{\frac{\boldsymbol{k}^2}{2M}+\omega_{\boldsymbol{k}}} - \frac{1}{\frac{(\boldsymbol{p}-\boldsymbol{k})^2-\boldsymbol{p}^2}{2M}+\omega_{\boldsymbol{k}}}\right) \tag{511}$$

Note that even in the limit $\kappa \to \infty$ this expression stays finite for all $\boldsymbol{p} \in \mathbb{R}^3$ since $\gamma_\infty^2(\boldsymbol{k}) \sim |\boldsymbol{k}|^{-1}$. Let us now insert the $\widehat{E}_2^{\boldsymbol{p}}$ into the formula for the wave function corrections

$$\langle\boldsymbol{k}_1,...,\boldsymbol{k}_m|_{\boldsymbol{p}}\,|\widehat{\Psi}_n\rangle_{\boldsymbol{p}} = \frac{1}{\widehat{E}_0^{\boldsymbol{p}}(\boldsymbol{k}_1,...,\boldsymbol{k}_m) - \widehat{E}_0^{\boldsymbol{p}}}\Bigg[\sum_{i\neq 2}^{n}\widehat{E}_i^{\boldsymbol{p}}\,\langle\boldsymbol{k}_1,...,\boldsymbol{k}_m|_{\boldsymbol{p}}\,|\widehat{\Psi}_{n-i}\rangle_{\boldsymbol{p}} + \tag{512}$$

$$+(\widehat{E}_2^{\boldsymbol{p}} + \frac{R_\kappa}{g^2})\,\langle\boldsymbol{k}_1,...,\boldsymbol{k}_m|_{\boldsymbol{p}}\,|\widehat{\Psi}_{n-2}\rangle_{\boldsymbol{p}} + \tag{513}$$

$$-\langle\boldsymbol{k}_1,...,\boldsymbol{k}_m|_{\boldsymbol{p}}\, H_{I_\kappa}^{\boldsymbol{p}}\,|\widehat{\Psi}_{n-1}\rangle_{\boldsymbol{p}}\Bigg] \tag{514}$$

$$= \frac{1}{\widehat{E}_0^{\boldsymbol{p}}(\boldsymbol{k}_1,...,\boldsymbol{k}_m) - \widehat{E}_0^{\boldsymbol{p}}}\Bigg[\sum_{i\neq 2}^{n}\widehat{E}_i^{\boldsymbol{p}}\,\langle\boldsymbol{k}_1,...,\boldsymbol{k}_m|_{\boldsymbol{p}}\,|\widehat{\Psi}_{n-i}\rangle_{\boldsymbol{p}} + \tag{515}$$

$$+E_2^{\boldsymbol{p}}\,\langle\boldsymbol{k}_1,...,\boldsymbol{k}_m|_{\boldsymbol{p}}\,|\widehat{\Psi}_{n-2}\rangle_{\boldsymbol{p}} - \langle\boldsymbol{k}_1,...,\boldsymbol{k}_m|_{\boldsymbol{p}}\, H_{I_\kappa}^{\boldsymbol{p}}\,|\widehat{\Psi}_{n-1}\rangle_{\boldsymbol{p}}\Bigg] \tag{516}$$

Hence $\widehat{E}_n^{\boldsymbol{p}} = E_n^{\boldsymbol{p}}$ except $\widehat{E}_2^{\boldsymbol{p}} = E_2^{\boldsymbol{p}} - \frac{R_\kappa}{g^2}$ and $\langle\boldsymbol{k}_1,...,\boldsymbol{k}_m|_{\boldsymbol{p}}\,|\widehat{\Psi}_n\rangle_{\boldsymbol{p}} = \langle\boldsymbol{k}_1,...,\boldsymbol{k}_m|_{\boldsymbol{p}}\,|\Psi_n\rangle_{\boldsymbol{p}}$ for all $n \in \mathbb{N}_0$, which yields

$$|\widehat{\Psi}_\kappa(g)\rangle_{\boldsymbol{p}} = |\Psi_\kappa(g)\rangle_{\boldsymbol{p}} \tag{517}$$

$$\widehat{E}_\kappa^{\boldsymbol{p}}(g) = E_\kappa^{\boldsymbol{p}}(g) - R_\kappa \tag{518}$$

□

This partly answers the question if $E_2^{\boldsymbol{p}}$ is the only divergent energy correction term for the Hamiltonian $H_\kappa^{\boldsymbol{p}}$ because we have found in the above corollary 4.5.2 that $\widehat{E}_n^{\boldsymbol{p}} = E_n^{\boldsymbol{p}}$ for all $n \neq 2$. Since by the above corollary $E_\infty^{\boldsymbol{p}}(g) - R_\infty(g)$ is finite $\lim_{\kappa\to\infty}(E_\kappa^{\boldsymbol{p}}(g) - g^2E_2^{\boldsymbol{p}})$ is also finite - recall line (511). However this does not mean that all energy correction terms are bounded but if they are not the divergencies cancel out within the sum. For a boundness criterion on every single energy correction term we would have to show that either the convergence is uniform in κ

$$\lim_{m\to\infty}\sup_{\kappa\in\mathbb{R}_+}\{\sum_{n=m}^{\infty} g^n E_n^{\boldsymbol{p}}\} = 0 \tag{519}$$

in order to interchange the κ limit with the summation in the energy power series or that $\widehat{E}_{\infty}^{\boldsymbol{p}}(g)$ is analytic around $g = 0$. Analyticity of $\widehat{E}^{\boldsymbol{p}}(g)$ would allow us to expand it in a Taylor series with respect to g and to check if the Taylor coefficients are equal to the term by term limit $\lim_{\kappa\to\infty}\widehat{E}_n^{\boldsymbol{p}}$. It seems reasonable that this can be done. Why the perturbation series of the ground state $|\widehat{\Psi}_\kappa(g)\rangle_{\boldsymbol{p}}$ of $\widehat{H}_\kappa^{\boldsymbol{p}}$ and the ground state $|\Psi_\kappa(g)\rangle_{\boldsymbol{p}}$ of $H_\kappa^{\boldsymbol{p}}$ are identical can be seen easily for finite κ by the following consideration. We have found

$$H_\kappa^{\boldsymbol{p}}(g)\,|\Psi_\kappa(g)\rangle_{\boldsymbol{p}} \quad = \quad E_\kappa^{\boldsymbol{p}}(g)\,|\Psi_\kappa(g)\rangle_{\boldsymbol{p}} \tag{520}$$

hence

$$(H_\kappa^{\boldsymbol{p}}(g) - R_\kappa(g))\,|\Psi_\kappa(g)\rangle_{\boldsymbol{p}} \quad = \quad (E_\kappa^{\boldsymbol{p}}(g) - R_\kappa)\,|\Psi_\kappa(g)\rangle_{\boldsymbol{p}} \tag{521}$$

$$\tag{522}$$

but now because we have the existence of the limit resolvent of the operator $(H_\kappa^{\boldsymbol{p}} - R_\kappa(g))$ this holds even for $\kappa \to \infty$. Hence

$$\boxed{\lim_{\kappa\to\infty}(H_\kappa^{\boldsymbol{p}}(g) - R_\kappa(g))\,|\Psi_\kappa(g)\rangle_{\boldsymbol{p}} = \widehat{E}^{\boldsymbol{p}}(g)\lim_{\kappa\to\infty}|\Psi_\kappa(g)\rangle_{\boldsymbol{p}} = \widehat{E}^{\boldsymbol{p}}(g)\,|\widehat{\Psi}(g)\rangle_{\boldsymbol{p}}} \tag{523}$$

So even in the limit $\kappa \to \infty$ the ground state of $H_\kappa^{\boldsymbol{p}}$, namely $|\Psi_\kappa(g)\rangle_{\boldsymbol{p}}$ stays in $\mathcal{F}_1^{\boldsymbol{p}}$. If we recall the definition of its n-meson wave functions (387) for finite κ we recognize that the misbehaving $E_2^{\boldsymbol{p}}$ term is part of the sum in that line. Taking the limit $\kappa \to \infty$ will unavoidably produce an infinite constant in the formula of each n-meson wave function that will throw it out of $\mathcal{L}_2$ if there is no mechanism that prevents it from doing that. This mechanism can be observed in the way we wrote the wave function a bit later in terms of contractions over ξ, line (452). The $E_2^{\boldsymbol{p}}$ term was taken out of the sum (455) and dragged under the integral (454), which is due to the meson creation operator in the interaction term of the Hamiltonian. For the third order wave function correction we more or less computed this kind of integral of line (453) and (454) when we showed in the above that $E_4^{\boldsymbol{p}}$ stays finite in the limit $\kappa \to \infty$. Based on this observation and of course only in a loose way of speaking one again could argue that the only problematic process in this model is again the one that creates a meson field mode out of the vacuum - recall the discussion in the note 4.5.1.

Unfortunately because we did not explicitly calculate the projector to the eigenvectors of the $\widehat{H}$, which turns out to be rather lengthy, we can not give an as explicit expression for the renormalized model Hamiltonian as it was possible for the toy model. However we recall that in the toy model it was also possible to forget about the explicit expression we have found for $\widehat{T}$ and to only give the formal expression $s - \lim_{\kappa\to\infty}(T_\kappa - V_{SE}^\kappa)$ a mathematical meaning on a special subset of the Fock-space T_κ was defined on. The domain of $\widehat{T}$ turned out to be at least a subset of this set. Hence the observations in this subsection and the experiences with toy model suggest that it might be possible to give the formal expression $s - \lim_{\kappa\to\infty}(H_\kappa^{\boldsymbol{p}} - g^2 E_2^{\boldsymbol{p}})$ a mathematical meaning as renormalized model Hamiltonian on the $\boldsymbol{p}$ fibre on some subset of $\mathcal{F}_1^{\boldsymbol{p}}$ in this weak sense. How this can be done and how it can be extended to $\mathcal{F}_N$ shall be discussed in next subsection. Finally we like to again emphasize that by comparison of the dynamic and the static quantum Nelson model one observes that it is only the presence of the free Schödinger dispersion relation of the nucleons in the dynamic quantum Nelson model that smoothes out the n-meson wave functions of the ground state in such a way that the ground state stays a $\mathcal{F}_1^{\boldsymbol{p}}$ vector even in the limit $\kappa \to \infty$. Moreover all energy corrections other than $E_2^{\boldsymbol{p}}$ arise only from the free Schödinger dispersion relation of the nucleons since they do not appear in the static case anymore. It is interesting to note that for a relativistic dispersion relation of the nucleons that might be $\sim |\boldsymbol{p}|$ like in the Dirac case instead of $\sim |\boldsymbol{p}|^2$ for the Schrödinger case all wave function correction terms in the limit $\kappa \to \infty$ will not lie in $\mathcal{L}_2$ anymore and so the ground state will not lie in $\mathcal{F}_1^{\boldsymbol{p}}$. So the renormalization concept applied here would fail. Does that mean that the Poincaré symmetry preserving point particle limit $\kappa \to \infty$ can only be reasonably worked out for a non-relativistic dispersion relation of the nucleons? This indeed sounds odd and gives one the feeling that the energy renormalization concept in our case only works because of a mathematical coincidence. Again we have to conclude that this renormalization concept has to be extended to a bigger space than Fock-space that hosts the model ground state even if it does not lie in Fock-space anymore and as already mentioned this will be subject of further investigation.

4.5.3 What does $\widehat{H}$ look like?

According to the discussion in the end of last subsection one would naturally try to take $s - \lim_{\kappa\to\infty}(H_\kappa^{\boldsymbol{p}} - E_2^{\boldsymbol{p}})$ to be the renormalized Hamiltonian on $\mathcal{F}_1^{\boldsymbol{p}}$ in the same sense as we did in the case of the toy model 4.3.5. The problem with that choice is that even for finite κ the integrand in

$$E_2^{\boldsymbol{p}} := -g^2 \int d^3k \; \frac{\gamma_\kappa^2(\boldsymbol{k})}{\frac{\boldsymbol{k}^2 - 2\boldsymbol{p}\boldsymbol{k}}{2M} + \omega_{\boldsymbol{k}}} \tag{524}$$

is only non-singular on the fibre bundle $\mathcal{F}_1^B$ with $B := \{\boldsymbol{p} \in \mathbb{R}^3$ such that $(\frac{\boldsymbol{k}^2-2\boldsymbol{p}\boldsymbol{k}}{2M} + \omega_{\boldsymbol{k}}) > 0 \;\; \forall \boldsymbol{k} \in \mathbb{R}^3\}$. The only way to renormalize $H_\kappa \upharpoonright \mathcal{F}_1$ on whole $\mathcal{F}_1$ is to make the family of renormalization constants R_κ independent of $\boldsymbol{p}$ under the constraint that $\lim_{\kappa\to\infty}(E_{\boldsymbol{p}}^2 - R_\kappa) < \infty$ holds. That can be done easily for example by

$$R_\kappa := -g^2 \int d^3k \; \frac{\gamma_\kappa^2(\boldsymbol{k})}{\frac{\boldsymbol{k}^2}{2M} + \omega_{\boldsymbol{k}}} \tag{525}$$

which is Nelson's choice - see line (66). Indeed as Nelson proved for the more general case of N nucleons there exist a self-adjoint Hamiltonian $\widehat{H}$ on an unknown domain $\mathcal{D}(\widehat{H}) \subset \mathcal{D}(\sqrt{H_0})$ for which the unitary group

$$s - \lim_{\kappa\to\infty} e^{it(H_\kappa - NR_\kappa)} \upharpoonright \mathcal{F}_N = e^{it\widehat{H}} \tag{526}$$

converges, where H_κ is given by line (6). The ambiguity in the choice of R_κ will only be relevant if we consider absolute energy values. Even for the unitary time evolution $e^{-i\widehat{H}t}$ this ambiguity is irrelevant because a physical state is represented by ray of vectors in the Hilbert-space, i.e. $|\Psi\rangle \in \mathcal{F}_N$ is physically equivalent to $z\,|\Psi\rangle$ for all $z \in \mathbb{C}$ since all these vectors generate the same dynamics. However one will get severe difficulties if one intends to extend the renormalization concept to whole $\mathcal{F}$. As soon as we deal with Fock-vectors, which are not eigenvectors of the nucleon number operator, we will get a different arbitrariness in each of the N nucleon sectors of $\mathcal{F}$. On top of that it has to be emphasized that not even $E_2^{\boldsymbol{p}}$ is the pure self-energy but is a mix between the self-energy and effects taking in account the free motion of the N nucleons in each N-nucleon sector. Hence it is not only the choice of R_κ, which brings in ambiguities but there is a natural ambiguity between the renormalization of the say N-th and M-th nucleon sector for $M \neq M$. In the following we will use the renormalization constant R_κ from line (525) and try to find a set $\mathcal{D} \subset \mathcal{F}_1$ on which the formal expression

$$\boxed{\widehat{H} := s - \lim_{\kappa\to\infty}(H_\kappa - R_\kappa)} \tag{527}$$

can be given a meaning as well-defined operator in the weak sense as we did in the case of the toy model.

4.5.4 What does $\mathcal{D}(\widehat{H})$ look like?

Since we have no explicit expression for the operator $\widehat{H}$ like in the case of the toy model it will not be possible to infer information about the domain that easily from (527). However we know one vector in $\mathcal{D}(\widehat{H}^{\boldsymbol{p}} = \widehat{H} \restriction \mathcal{F}_1^{\boldsymbol{p}})$ already, i.e. the ground state $|\widehat{\Psi}(g)\rangle$ - in fact we shall only concentrate on the $\boldsymbol{p}$ fibre in the following discussion. We now may copy what we have done in the toy model case and create other vectors in $\mathcal{D}(\widehat{H}^{\boldsymbol{p}})$ with respect to that ground state by using the operators $\int d^3k\ f(\boldsymbol{k})a^\dagger_{\boldsymbol{k}}$ and $\int d^3k\ f(\boldsymbol{k})a_{\boldsymbol{k}}$ for a special choice of $f \in \mathcal{L}_2(\mathbb{R}^3)$. For the toy model we have seen that f had to fulfill $D(\boldsymbol{k})f(\boldsymbol{k}) \in \mathcal{L}_2(\mathbb{R}^3)$, where $D(\boldsymbol{k})$ was the dispersion relation of the mesons. For our case we will find by the following lemma that f needs to fulfill $(\frac{\boldsymbol{k}^2}{2M} + \omega_{\boldsymbol{k}})f(\boldsymbol{k}) \in \mathcal{L}_2(\mathbb{R}^3)$, where $\omega_{\boldsymbol{k}}$ is the dispersion of the mesons and in addition $\frac{\boldsymbol{k}^2}{2M}$ is of the same form as the dispersion relation of the nucleons.

Lemma 4.5.3. *Let $\mu, M \in \mathbb{R}_+$, $|\boldsymbol{p}| < \sqrt{2M\mu}$, g sufficiently small, $|\widehat{\Psi}(g)\rangle_{\boldsymbol{p}}$ the ground state of $\widehat{H}^{\boldsymbol{p}} = \widehat{H} \restriction \mathcal{F}_1^{\boldsymbol{p}}$ with its energy eigenvalue $\widehat{E}^{\boldsymbol{p}}(g)$ given by corollary 4.5.2 and*

$$f \in \mathcal{D}_0 := \left\{ \varphi \in \mathcal{L}_2(\mathbb{R}^3) \middle| (\frac{\boldsymbol{k}^2}{2M} + \omega_{\boldsymbol{k}})\varphi(\boldsymbol{k}) \in \mathcal{L}_2(\mathbb{R}^3) \right\} \tag{528}$$

then

$$\int d^3k\ f(\boldsymbol{k})a^\dagger_{\boldsymbol{k}} e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}} |\Psi(g)\rangle_{\boldsymbol{p}} \quad \in \quad \mathcal{D}(\widehat{H}^{\boldsymbol{p}}) \tag{529}$$

$$\int d^3k\ f(\boldsymbol{k})a_{\boldsymbol{k}} e^{i\boldsymbol{k}\widehat{\boldsymbol{x}}} |\Psi(g)\rangle_{\boldsymbol{p}} \quad \in \quad \mathcal{D}(\widehat{H}^{\boldsymbol{p}}) \tag{530}$$

Proof. Let us calculate the commutator and bear in mind the weak sense definition of $\widehat{H}$

$$\left[\widehat{H}, \int d^3k\ f(\boldsymbol{k})a^\dagger_{\boldsymbol{k}}\right] = \left[\lim_{\kappa\to\infty}(H_\kappa - R_\kappa), \int d^3k\ f(\boldsymbol{k})a^\dagger_{\boldsymbol{k}} e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}}\right] \tag{531}$$

$$= \lim_{\kappa\to\infty}\left[\frac{\widehat{\boldsymbol{p}}^2}{2M} + \int d^3k\ a^\dagger_{\boldsymbol{k}}\omega_{\boldsymbol{k}}a_{\boldsymbol{k}} + \right. \tag{532}$$

$$\left. +g\int d^3k\ \gamma_\kappa(\boldsymbol{k})(a_{\boldsymbol{k}}e^{i\boldsymbol{k}\widehat{\boldsymbol{x}}} + a^\dagger_{\boldsymbol{k}}e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}}), \int d^3k\ f(\boldsymbol{k})a^\dagger_{\boldsymbol{k}}e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}}\right] \tag{533}$$

$$= \int d^3k\ (\frac{\boldsymbol{k}^2}{2M} + \omega_{\boldsymbol{k}})f(\boldsymbol{k})a^\dagger_{\boldsymbol{k}}e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}} + g\int d^3k\ \gamma_\infty(\boldsymbol{k})f(\boldsymbol{k}) \tag{534}$$

Now $f \in \mathcal{D}_0$ implies that the first term is a well-defined operator on $\mathcal{D}(\sqrt{\mathcal{N}^1_{mes}})$, which is dense such that the closure of this operator can be define on whole $\mathcal{F}_1$. The integral over the second term exists because $f \in \mathcal{L}_2(\mathbb{R}^3)$ and $\gamma_\infty(\boldsymbol{k}) \sim |\boldsymbol{k}|^{-1/2}$ are non-singular. So the operator resulting from the commutator can be defined on $\mathcal{F}_1$ and since this operator obviously conserves the total momentum we can decompose it with respect to the total momentum operator $\mathcal{P}$. Hence

$$\widehat{H}^{\boldsymbol{p}}\int d^3k\ f(\boldsymbol{k})a^\dagger_{\boldsymbol{k}}e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}}|\widehat{\Psi}(g)\rangle_{\boldsymbol{p}} = \left(\left[\widehat{H}^{\boldsymbol{p}}, \int d^3k\ f(\boldsymbol{k})a^\dagger_{\boldsymbol{k}}e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}}\right] + \right. \tag{535}$$

$$\left. + \int d^3k\ f(\boldsymbol{k})a^\dagger_{\boldsymbol{k}}e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}}\widehat{H}^{\boldsymbol{p}}\right)|\widehat{\Psi}(g)\rangle_{\boldsymbol{p}} \tag{536}$$

$$= \int d^3k\ \left(\frac{\boldsymbol{k}^2}{2M} + \omega_{\boldsymbol{k}} + \widehat{E}^{\boldsymbol{p}}(g)\right)f(\boldsymbol{k})a^\dagger_{\boldsymbol{k}}e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}}|\widehat{\Psi}(g)\rangle_{\boldsymbol{p}} \tag{537}$$

$$+g\int d^3k\ \gamma_\infty(\boldsymbol{k})f(\boldsymbol{k})\,|\widehat{\Psi}(g)\rangle_{\boldsymbol{p}} \tag{538}$$

and for the other case we similarly yield

$$\widehat{H}^{\boldsymbol{p}}\int d^3k\ f(\boldsymbol{k})a_{\boldsymbol{k}}e^{i\boldsymbol{k}\widehat{\boldsymbol{x}}}\,|\widehat{\Psi}(g)\rangle_{\boldsymbol{p}} = \int d^3k\left(\widehat{E}^{\boldsymbol{p}}(g)-\frac{\boldsymbol{k}^2}{2M}-\omega_{\boldsymbol{k}}\right)f(\boldsymbol{k})a_{\boldsymbol{k}}e^{i\boldsymbol{k}\widehat{\boldsymbol{x}}}\,|\widehat{\Psi}(g)\rangle_{\boldsymbol{p}} \tag{539}$$

$$-g\int d^3k\ \gamma_\infty(\boldsymbol{k})f(\boldsymbol{k})\,|\widehat{\Psi}(g)\rangle_{\boldsymbol{p}} \tag{540}$$

□

By multiple application of the commutator computed in the proof to the above lemma we find that for functions $f_1,...,f_n\in\mathcal{D}_0$ and finite n

$$|\psi_n\rangle = \int d^3k_1 \ldots \int d^3k_n \prod_{i=1}^{n} f_i(\boldsymbol{k}_i)a_{\boldsymbol{k}_i}\,|\widehat{\Psi}(g)\rangle \in \mathcal{D}(\widehat{H}^{\boldsymbol{p}}) \tag{541}$$

$$|\psi_n^\dagger\rangle = \int d^3k_1 \ldots \int d^3k_n \prod_{i=1}^{n} f_i(\boldsymbol{k}_i)a^\dagger_{\boldsymbol{k}_i}\,|\widehat{\Psi}(g)\rangle \in \mathcal{D}(\widehat{H}^{\boldsymbol{p}}) \tag{542}$$

Let now $\mathcal{M}$ be the set of all finite linear combinations of $|\psi_n\rangle$ and $|\psi_n^\dagger\rangle$ then $\mathcal{M}$ is a non-empty set and

$$\boxed{\mathcal{M}\subset\mathcal{D}(\widehat{H}^{\boldsymbol{p}})} \tag{543}$$

However this is only a subset of $\mathcal{D}(\widehat{H}^{\boldsymbol{p}})$. In order to get the whole $\mathcal{D}(\widehat{H}^{\boldsymbol{p}})$ and with it the domain $\mathcal{D}(\widehat{H})$ from Nelson's proof we probably need to do something similar that we have done in the toy model, where we have found an unitary transformation that diagonalizes the renormalized Hamiltonian $\widehat{H}$ - see subsection 4.3.6.

4.5.5 What does the action of $\widehat{H}$ on elements in $\mathcal{D}(\widehat{H})$ look like?

Following the preceding subsections we have decided to take the formal expression $\widehat{H} := s-\lim_{\kappa\to\infty}(H_\kappa - R_\kappa)$ as the definition of the operator $\widehat{H}$ on some special set $\mathcal{D}\subset\mathcal{F}_N$ in what we have called the weak sense. So if we let $(H_\kappa - R_\kappa)$ act on a Fock-vector in $\mathcal{D}$ and take the limit $\kappa\to\infty$ then the so obtained expression is expected to remain a well-defined Fock-vector - please also recall our example 4.3.1 about the analogy to the definition of singular differential equations. In this subsection we shall observe that this is indeed what happens. At first we analyze the action of $\widehat{H}$ on a general Fock-vector in $\mathcal{F}_1^{\boldsymbol{p}}$ and deduce properties of the elements in $\mathcal{D}$. Later we argue that the ground state $\boldsymbol{p}$ fibre $|\widehat{\Psi}(g)\rangle_{\boldsymbol{p}}$ of $\widehat{H}$ has exactly these properties, which by last subsection means that at least $\mathcal{D}(\widehat{H})\subset\mathcal{D}$ since $|\widehat{\Psi}(g)\rangle_{\boldsymbol{p}}\in\mathcal{D}(\widehat{H}^{\boldsymbol{p}})$ like it is the case for all other eigenvectors of $\widehat{H}^{\boldsymbol{p}}$ with isolated and non-degenerate eigenvectors because we have shown the existence of the limit resolvent (504) of $\widehat{H}^{\boldsymbol{p}}$. Of course this is what one would expect because otherwise $\widehat{H}$ from the Nelson's theorem 2.6.1 would have nothing to do with the weak sense definition $\widehat{H} := s-\lim_{\kappa\to\infty}(H_\kappa - R_\kappa)$. Although $\mathcal{D}(\widehat{H})$ may be only a subset of $\mathcal{D}$ it is sufficient to only concentrate on it and forget about the possible rest of $\mathcal{D}$ because $\mathcal{D}(\widehat{H})$ lies dense in $\mathcal{F}_N$ and hence suffices to define $\widehat{H}$ on whole $\mathcal{F}_N$.

At first we observe how $s-\lim_{\kappa\to\infty}(H_\kappa - R_\kappa)$ acts in the weak sense on an arbitrary $|\psi\rangle\in\mathcal{F}_1$ independent of κ, so again bearing in mind the weak sense definition of $\widehat{H}$

$$\langle\boldsymbol{k}_1,...,\boldsymbol{k}_n|\widehat{H}|\psi\rangle = \langle\boldsymbol{q};\boldsymbol{k}_1,...,\boldsymbol{k}_n|s-\lim_{\kappa\to\infty}(H_\kappa - R_\kappa)|\psi\rangle \tag{544}$$

$$= \lim_{\kappa\to\infty}\langle\boldsymbol{q};\boldsymbol{k}_1,...,\boldsymbol{k}_n|H_\kappa - R_\kappa|\psi\rangle \tag{545}$$

That the so obtain expression is still an element of $\mathcal{F}_1$ can naturally be checked by the finiteness of its norm in $\mathcal{F}_1$

$$\sum_{n=0}^{\infty} \int d^3q \int d^3k_1 ... \int d^3k_n \, | \lim_{\kappa\to\infty} \langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n | H_\kappa - R_\kappa | \psi \rangle \, |^2 \quad < \quad \infty \tag{546}$$

which means that at least all n-meson wave functions $\lim_{\kappa\to\infty} \langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n | H_\kappa - R_\kappa | \psi \rangle$ have to be in $\mathcal{L}_2(\mathbb{R}^{3(n+1)})$. Therefrom we recognize the main properties a Fock-vector $|\psi\rangle$ should have to be an element in $\mathcal{D}(\widehat{H})$. Note that this is just a necessary but not sufficient condition for (546) but it will do the job for the first main observations.

$$\lim_{\kappa\to\infty} \langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n | H_\kappa - R_\kappa | \psi \rangle \quad = \quad \lim_{\kappa\to\infty} \Bigg[\left(\frac{\boldsymbol{q}^2}{2M} + \sum_{i=1}^{n} \omega_{k_i} - R_\kappa \right) \langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n | \psi \rangle + \tag{547}$$

$$+ g \int d^3k \; \gamma_\kappa(\boldsymbol{k}) \sqrt{n+1} \, \langle \boldsymbol{q} - \boldsymbol{k}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n, \boldsymbol{k} | \psi \rangle + \tag{548}$$

$$+ g \sum_{i=1}^{n} \gamma_\kappa(\boldsymbol{k}_i) \frac{1}{\sqrt{n}} \, \langle \boldsymbol{q} + \boldsymbol{k}_i; \boldsymbol{k}_1, ..., \widehat{\boldsymbol{k}_i}, ..., \boldsymbol{k}_n | \psi \rangle \Bigg] \tag{549}$$

$$= \quad \lim_{\kappa\to\infty} \left(\frac{\boldsymbol{q}^2}{2M} + \sum_{i=1}^{n} \omega_{k_i} - R_\kappa + g \int d^3k \; \gamma_\kappa(\boldsymbol{k}) \frac{\langle \boldsymbol{q} - \boldsymbol{k}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n, \boldsymbol{k} | \psi \rangle}{\langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n | \psi \rangle} \sqrt{n+1} + \right. \tag{550}$$

$$\left. + g \sum_{i=1}^{n} \gamma_\kappa(\boldsymbol{k}_i) \frac{\langle \boldsymbol{q} + \boldsymbol{k}_i; \boldsymbol{k}_1, ..., \widehat{\boldsymbol{k}_i}, ..., \boldsymbol{k}_n | \psi \rangle}{\langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n | \psi \rangle} \frac{1}{\sqrt{n}} \right) \langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n | \psi \rangle \tag{551}$$

Recall that the renormalization constant is defined as $R_\kappa := \int d^3k \; \mathcal{R}_\kappa(\boldsymbol{k})$ and according to Nelson's theorem 2.6.1, $\mathcal{R}(\boldsymbol{k}) := -g^2 \frac{\gamma_\kappa^2(\boldsymbol{k})}{\frac{k^2}{2M} + \omega_{\boldsymbol{k}}}$, which has the asymptotic behavior $\mathcal{R}_\kappa(\boldsymbol{k}) \sim \frac{1}{|\boldsymbol{k}|^3}$ and has no singularities for $\mu \in \mathbb{R}_+$. Now having an idea of how the action of $\widehat{H}$ may formally look like we can conclude some properties $|\psi\rangle$ should have to be in $\mathcal{D}(\widehat{H})$.

Observation 4.5.1. *Since we know from Nelson's proof that the subtraction of R_κ is the remedy, which makes is possible to obtain a well-defined, self-adjoint operator $\widehat{H}$ the first thing to do is to find the diverging terms which R_κ will compensate. Because we take the absolute value in the Fock-norm condition (546) we know that this compensation has to take place in every n-meson amplitude respectively. The only constant other than R_κ in the above equation is the one resulting from the integral in line (550). All other terms are functions of $\boldsymbol{q}, \boldsymbol{k}_1, ..., \boldsymbol{k}_n$. If both terms should sum up to something smaller than infinity in the limit $\kappa \to \infty$ the integral in line (550) must have the same asymptotic behavior like R_κ which means that*

$$\gamma_\infty(\boldsymbol{k}) \frac{\langle \boldsymbol{q} - \boldsymbol{k}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n, \boldsymbol{k} | \psi \rangle}{\langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n | \psi \rangle} \sim \mathcal{R}_\infty(\boldsymbol{k}) \tag{552}$$

Only that way we can arrive at

$$C_{RSE}(\boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n) := g \int d^3k \left(\gamma_\infty(\boldsymbol{k}) \frac{\langle \boldsymbol{q} - \boldsymbol{k}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n, \boldsymbol{k} | \psi \rangle}{\langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n | \psi \rangle} \sqrt{n+1} - \mathcal{R}_\infty(\boldsymbol{k}) \right) < \infty \tag{553}$$

Let RSE stand for rest self energy.

Observation 4.5.2. *Because we take the absolute value in the Fock-norm condition (546) we know that this compensation has to take place in every n-meson amplitude $\lim_{\kappa\to\infty} \langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n | \, (H_\kappa - R_\kappa) \, |\psi\rangle$. In other words the above observation (552) must hold for any $n \in \mathbb{N}$ and hence using the symmetry in the meson coordinates we get*

$$\frac{\langle \boldsymbol{q} - \boldsymbol{k}_i; \boldsymbol{k}_1, ..., \boldsymbol{k}_i, ..., \boldsymbol{k}_n | \psi \rangle}{\langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \widehat{\boldsymbol{k}_i}, ..., \boldsymbol{k}_n | \psi \rangle} \sim \frac{\mathcal{R}_\infty(\boldsymbol{k}_i)}{\gamma_\infty(\boldsymbol{k}_i)} \sim \frac{1}{|\boldsymbol{k}_i|^{5/2}} \tag{554}$$

as an rough estimate of the asymptotic behavior in the $\boldsymbol{k}_i$ coordinate for all $i = 1, ..., n$. Explicitly this means that none of the n meson components of $|\psi\rangle$ may be e.g. zero. They all have to be functions of $\boldsymbol{q}, \boldsymbol{k}_1, ..., \boldsymbol{k}_n$ with the above asymptotic behavior. Moreover we learn from (554) that obviously $|\psi\rangle$ cannot be in $\mathcal{D}(H_0) := \mathcal{D}(\boldsymbol{p}^2) \bigcap \mathcal{D}(\mathcal{N}^{\omega}_{mes})$ since $\omega_{\boldsymbol{k}} \mathcal{R}_\infty(\boldsymbol{k})/\gamma_\infty(\boldsymbol{k})$ is not in $\mathcal{L}_2(\mathbb{R}^3)$. In fact it cannot even be in the form domain $\mathcal{D}(\sqrt{H_0})$. That raises the question, which is in this form a quote from the very last sentence in Nelson's paper [15]: "Is $\mathcal{D}(\widehat{H}) \bigcap \mathcal{D}(\sqrt{H_0}) = 0$?". This is for example true on the $\boldsymbol{p}$ fibre for the set $\mathcal{M} \subset \mathcal{D}(\widehat{H}^{\boldsymbol{p}})$, which we have found in subsection 4.5.4, since the ground state is not in $\mathcal{D}(\sqrt{H_0^{\boldsymbol{p}}})$ and all other Fock-vectors in $\mathcal{M}$ are constructed with respect to this ground state by finite application of smeared out creation and annihilation operators.

Observation 4.5.3. *Finally we have a look at line (551). As $\gamma_\kappa(\boldsymbol{k}_i) \langle \boldsymbol{q} + \boldsymbol{k}_i; \boldsymbol{k}_1, ..., \widehat{\boldsymbol{k}}_i, ..., \boldsymbol{k}_n|\psi\rangle$ is only in $\mathcal{L}_2(\mathbb{R}^{3n})$ for $\kappa < \infty$ because $\gamma_\infty(\boldsymbol{k}) \notin L_2(\mathbb{R}^{3(n+1)})$ something has to be done about it or we will again get divergent terms in the Fock-norm (546). The only hope one can have from now on is that all remaining terms sum up to a function $\mathcal{E} := \mathcal{E}(\boldsymbol{q}, \boldsymbol{k}_1, ..., \boldsymbol{k}_n)$ with the properties*

$$\mathcal{E}(\boldsymbol{q}, \boldsymbol{k}_1, ..., \boldsymbol{k}_n) \quad := \quad \frac{\boldsymbol{q}^2}{2M} + \sum_{i=1}^{n} \omega_{\boldsymbol{k}_i} + C_{RSE}(\boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n) \tag{555}$$

$$+ \sum_{i=1}^{n} g\gamma_\infty(\boldsymbol{k}_i) \frac{\langle \boldsymbol{q} + \boldsymbol{k}_i; \boldsymbol{k}_1, ..., \widehat{\boldsymbol{k}}_i, ..., \boldsymbol{k}_n|\psi\rangle}{\langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n|\psi\rangle} \frac{1}{\sqrt{n}} \tag{556}$$

and

$$\mathcal{E}(\boldsymbol{q}, \boldsymbol{k}_1, ..., \boldsymbol{k}_n) \langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n|\psi\rangle \in \mathcal{L}_2(\mathbb{R}^{3(n+1)}) \;\; \forall n \in \mathbb{N} \tag{557}$$

By inverting line (554) we can partly give the asymptotic behavior of $\mathcal{E}(\boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n)$.

$$\frac{\langle \boldsymbol{q} + \boldsymbol{k}_i; \boldsymbol{k}_1, ..., \widehat{\boldsymbol{k}}_i, ..., \boldsymbol{k}_n|\psi\rangle}{\langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n|\psi\rangle} \quad = \quad \left(\frac{\langle (\boldsymbol{q} + \boldsymbol{k}_i) - \boldsymbol{k}_i; \boldsymbol{k}_1, ..., \boldsymbol{k}_i, ..., \boldsymbol{k}_n|\psi\rangle}{\langle (\boldsymbol{q} + \boldsymbol{k}_i); \boldsymbol{k}_1, ..., \widehat{\boldsymbol{k}}_i, ..., \boldsymbol{k}_n|\psi\rangle} \right)^{-1} \sim |\boldsymbol{k}_i|^{5/2} \tag{558}$$

thus by inserting $\omega_{\boldsymbol{k}} \sim |\boldsymbol{k}|$ and $\gamma_\infty(\boldsymbol{k}) \sim \frac{1}{|\boldsymbol{k}|^{1/2}}$

$$\mathcal{E}(\boldsymbol{q}, \boldsymbol{k}_1, ..., \boldsymbol{k}_n) \quad := \quad \frac{\boldsymbol{q}^2}{2M} + \sum_{i=1}^{n} \underbrace{\omega_{\boldsymbol{k}_i}}_{\sim|\boldsymbol{k}_i|} + C_{RSE}(\boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n) \tag{559}$$

$$+ \sum_{i=1}^{n} g \underbrace{\gamma_\infty(\boldsymbol{k}_i) \frac{\langle \boldsymbol{q} + \boldsymbol{k}_i; \boldsymbol{k}_1, ..., \widehat{\boldsymbol{k}}_i, ..., \boldsymbol{k}_n|\psi\rangle}{\langle \boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n|\psi\rangle}}_{\sim \underbrace{\frac{1}{|\boldsymbol{k}|^{1/2} + \epsilon'} (|\boldsymbol{k}_i|^{5/2} + \epsilon)}_{\sim|\boldsymbol{k}_i|^2}} \frac{1}{\sqrt{n}} \tag{560}$$

But $|\boldsymbol{k}_i|^m \langle \boldsymbol{q} + \boldsymbol{k}_i; \boldsymbol{k}_1, ..., \widehat{\boldsymbol{k}}_i, ..., \boldsymbol{k}_n|\psi\rangle$ is not in $\mathcal{L}_2(\mathbb{R}^{3n})$ for all $m \geq 1$ if (554). Hence the only chance is that these terms sum up to something which nevertheless fulfills (557). In the following we analyze how this is possible and we shall construct a Fock-vector in the form domain, and examine the ground state in the domain of $\widehat{H}$.

Based on the above observations we now try to fix some basic conditions, which such a Fock-vector, say $|\psi\rangle$, typically would fulfill. Since $[H_\kappa, \mathcal{P}] = 0$ there exists a common family of simultaneous eigenvectors of H_κ and $\mathcal{P}$. We wish to concentrate on generalized eigenvectors $|\psi\rangle_{\boldsymbol{p}}$ of $\mathcal{P}$ with

$$\langle \boldsymbol{k}_1, ..., \boldsymbol{k}_n|_{\boldsymbol{p}'} |\psi\rangle_{\boldsymbol{p}} = \delta^3(\boldsymbol{p}' - \boldsymbol{p}) \psi_{\boldsymbol{p}}(\boldsymbol{k}_1, ..., \boldsymbol{k}_n) \tag{561}$$

from which we infer that

$$\langle \boldsymbol{k}_1, ..., \boldsymbol{k}_n|_{\boldsymbol{p}'} (H_\kappa - R_\kappa) |\psi\rangle_{\boldsymbol{p}} = \Bigg(\frac{(\boldsymbol{p} - \sum_{i=1}^n \boldsymbol{k}_i)^2}{2M} + \sum_{i=1}^n \omega_{\boldsymbol{k}_i} + \tag{562}$$

$$+ g \sum_{i=1}^n \frac{\gamma_\kappa(\boldsymbol{k}_i)}{\sqrt{n}} \frac{\psi(\boldsymbol{k}_1, ..., \widehat{\boldsymbol{k}}_i, ..., \boldsymbol{k}_n)}{\psi(\boldsymbol{k}_1, ..., \boldsymbol{k}_n)} \tag{563}$$

$$-R_\kappa + g \int d^3k\; \gamma_\kappa(\boldsymbol{k}) \sqrt{n+1} \frac{\psi(\boldsymbol{k}_1, ..., \boldsymbol{k}_n, \boldsymbol{k})}{\psi(\boldsymbol{k}_1, ..., \boldsymbol{k}_n)} \Bigg) \times \tag{564}$$

$$\times \langle \boldsymbol{k}_1, ..., \boldsymbol{k}_n|_{\boldsymbol{p}'} |\psi\rangle_{\boldsymbol{p}} \tag{565}$$

In order to simplify the formula we write $\langle \boldsymbol{k}_1, ..., \boldsymbol{k}_n|_{\boldsymbol{p}'} |\psi\rangle_{\boldsymbol{p}}$ as

$$\langle \boldsymbol{k}_1, ..., \boldsymbol{k}_n|_{\boldsymbol{p}'} |\psi\rangle_{\boldsymbol{p}} = \delta^3(\boldsymbol{p}' - \boldsymbol{p}) \frac{(-g)^n}{\sqrt{n!}} \psi_{\boldsymbol{p}}(\boldsymbol{k}_1, ..., \boldsymbol{k}_n) \tag{566}$$

which yields

$$\langle \boldsymbol{k}_1, ..., \boldsymbol{k}_n|_{\boldsymbol{p}'} (H_\kappa - R_\kappa) |\psi\rangle_{\boldsymbol{p}} = \Bigg(\frac{(\boldsymbol{p} - \sum_{i=1}^n \boldsymbol{k}_i)^2}{2M} + \sum_{i=1}^n \omega_{\boldsymbol{k}_i} + \tag{567}$$

$$- \sum_{i=1}^n \gamma_\kappa(\boldsymbol{k}_i) \frac{\psi(\boldsymbol{k}_1, ..., \widehat{\boldsymbol{k}}_i, ..., \boldsymbol{k}_n)}{\psi(\boldsymbol{k}_1, ..., \boldsymbol{k}_n)} \tag{568}$$

$$-R_\kappa - g^2 \int d^3k\; \gamma_\kappa(\boldsymbol{k}) \frac{\psi(\boldsymbol{k}_1, ..., \boldsymbol{k}_n, \boldsymbol{k})}{\psi(\boldsymbol{k}_1, ..., \boldsymbol{k}_n)} \Bigg) \times \tag{569}$$

$$\times \langle \boldsymbol{k}_1, ..., \boldsymbol{k}_n|_{\boldsymbol{p}'} |\psi\rangle_{\boldsymbol{p}} \tag{570}$$

Now it is time to delimit the ψ functions without violating the above observations. In the proof of Nelson's theorem 2.6.1 the Gross transformation e^{T_κ} is the tool that extracts the divergent energy term out of the Hamiltonian. The Gross transformation is a Weyl operator, i.e. an unitary transformation that turns the meson vacuum Fock-vector into a coherent[19] Fock-vector in the meson components. So the ansatz lies near that in order to get a Fock-vector in the form domain or domain of $\widehat{H}$ we should assume $|\psi\rangle_{\boldsymbol{p}}$ to be coherent in the meson coordinates. In doing so we note that then all $(-g)^n \psi_{\boldsymbol{p}}(\boldsymbol{k}_1, ..., \boldsymbol{k}_n)$ fall into a product $\prod_{i=1}^n (-g) f(\boldsymbol{k}_i)$ for some $f \in \mathcal{L}_2(\mathbb{R}^3)$. This assumption yields

$$\langle \boldsymbol{k}_1, ..., \boldsymbol{k}_n|_{\boldsymbol{p}'} (H_\kappa - R_\kappa) |\psi\rangle_{\boldsymbol{p}} = \Bigg(\frac{(\boldsymbol{p} - \sum_{i=1}^n \boldsymbol{k}_i)^2}{2M} + \sum_{i=1}^n \omega_{\boldsymbol{k}_i} - \sum_{i=1}^n \frac{\gamma_\kappa(\boldsymbol{k}_i)}{f(\boldsymbol{k}_i)} \tag{571}$$

$$-R_\kappa - g^2 \int d^3k\; \gamma_\kappa(\boldsymbol{k}) f(\boldsymbol{k}) \Bigg) \langle \boldsymbol{k}_1, ..., \boldsymbol{k}_n|_{\boldsymbol{p}'} |\psi\rangle_{\boldsymbol{p}} \tag{572}$$

Observation 4.5.1 would then give us a very direct hint about how f must look like with

$$C_{RSE} := -R_\kappa - g^2 \int d^3k\; \gamma_\kappa(\boldsymbol{k}) f(\boldsymbol{k}) = g^2 \int d^3k\; \gamma_\kappa(\boldsymbol{k}) \left[\frac{\gamma_\kappa(\boldsymbol{k})}{\frac{\boldsymbol{k}^2}{2M} + \omega_{\boldsymbol{k}}} - f(\boldsymbol{k}) \right] \quad < \quad \infty \tag{573}$$

Obviously $f(\boldsymbol{k}) = \frac{\gamma_\kappa(\boldsymbol{k})}{\frac{\boldsymbol{k}^2}{2M} + \omega_{\boldsymbol{k}}}$ will do the job. For that special choice we get

$$\langle \boldsymbol{k}_1, ..., \boldsymbol{k}_n|_{\boldsymbol{p}'} (H_\kappa - R_\kappa) |\psi\rangle_{\boldsymbol{p}} = \Bigg(\underbrace{\frac{(\boldsymbol{p} - \sum_{i=1}^n \boldsymbol{k}_i)^2}{2M} - \frac{\sum_{i=1}^n \boldsymbol{k}_i^2}{2M}}_{=:\mathcal{E}(\boldsymbol{q}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n)} \Bigg) \langle \boldsymbol{k}_1, ..., \boldsymbol{k}_n|_{\boldsymbol{p}'} |\psi\rangle_{\boldsymbol{p}} \tag{574}$$

[19] A coherent Fock-vector $|\varphi\rangle$ is defined by $a_{\boldsymbol{k}} |\varphi\rangle = f(k) |\varphi\rangle$, where f is some function in $\mathcal{L}_2(\mathbb{R}^3)$. Correctly normalized coherent Fock-vectors can be constructed with the help of the Weyl operator $\mathcal{W}(f) = e^{\int d^3k\; f(k)(a_{\boldsymbol{k}}^\dagger - a_{\boldsymbol{k}})}$ acting on the vacuum Fock-vector. There is an example that explains how this works in the proof of lemma 4.5.4.

In fact now the Fock-vector $|\psi\rangle_{\boldsymbol{p}}$ under concern is the Gross transformed meson vacuum Fock-vector $|0\rangle_{\boldsymbol{p}} = |\boldsymbol{p};0\rangle \in \mathcal{F}_1^{\boldsymbol{p}}$. Unfortunately $\widehat{H}^{\boldsymbol{p}}\,|\psi\rangle_{\boldsymbol{p}} \notin \mathcal{F}_1^{\boldsymbol{p}}$ because based on our observations $\mathcal{E}(\boldsymbol{q};\boldsymbol{k}_1,...,\boldsymbol{k}_n)\cdot \langle \boldsymbol{k}_1,...,\boldsymbol{k}_n|_{\boldsymbol{p}'}\,|\psi\rangle_{\boldsymbol{p}} \sim |\boldsymbol{k}_i|\cdot|\boldsymbol{k}_i|^{-5/2}$ for $i=1,..,n$. However it is in its the form domain $\mathcal{Q}(\widehat{H}^{\boldsymbol{p}})$ as we shall show with the next lemma.

Lemma 4.5.4. *Let $\kappa \le \infty$ (!), $g\in\mathbb{R}$, $\mu, M \in \mathbb{R}_+$, $\varphi\in\mathcal{S}(\mathbb{R}^3)$, $\beta_\kappa(\boldsymbol{k}) := -g\frac{\gamma_\kappa(\boldsymbol{k})}{\frac{\boldsymbol{k}^2}{2M}+\omega_{\boldsymbol{k}}}$ and the Weyl operator restricted to $\mathcal{F}_1$*

$$\mathcal{W}(f) := e^{\int d^3k\ f(\boldsymbol{k})(a_{\boldsymbol{k}}^\dagger e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}} - a_{\boldsymbol{k}} e^{i\boldsymbol{k}\widehat{\boldsymbol{x}}})} \tag{575}$$

then

1. $|\psi\rangle_{\boldsymbol{p}} := \mathcal{W}(\beta_\kappa)\,|0\rangle_{\boldsymbol{p}} \in \mathcal{F}_1^{\boldsymbol{p}}$ *for all* $\boldsymbol{p}\in\mathbb{R}^3$ *and*
2. $|\psi_\varphi\rangle := \int d^3p\ \varphi(\boldsymbol{p})\,|\psi\rangle_{\boldsymbol{p}} \in \mathcal{D}(\sqrt{\widehat{H}}) = \mathcal{Q}(\widehat{H})$
3. *the quadratic form* $\langle\psi_\varphi|\,(H_\kappa - R_\kappa)\,|\psi_\varphi\rangle = \langle\psi_\varphi| - \frac{\nabla^2}{2M} - \int d^3k\ a_{\boldsymbol{k}}^\dagger \frac{\boldsymbol{k}^2}{2M} a_{\boldsymbol{k}}\,|\psi_\varphi\rangle$

Note that $\mathcal{W}(\beta_\kappa)$ is equal to the Gross transformation e^{T_κ}.

Proof. We begin with the first point. Since $\mathcal{W}(\beta_\kappa) = e^{T_\kappa}\restriction \mathcal{F}_1^{\boldsymbol{p}}$, the Gross transformation restricted to $\mathcal{F}_1^{\boldsymbol{p}}$, is unitary $|\psi\rangle_{\boldsymbol{p}}$ obviously in $\mathcal{F}_1^{\boldsymbol{p}}$. We still want to give a more elementary proof that shows the basics of handling coherent Fock-vectors.

$$|\psi\rangle_{\boldsymbol{p}} \;:=\; \mathcal{W}(\beta_\kappa)\,|0\rangle_{\boldsymbol{p}} = e^{\int d^3k\ \beta_\kappa(\boldsymbol{k})(a_{\boldsymbol{k}}^\dagger - a_{\boldsymbol{k}})}\,|0\rangle_{\boldsymbol{p}} \tag{576}$$
$$\tag{577}$$

using the Baker-Hausdorf identity

$$= e^{-\frac{1}{2}\int d^3k\ |\beta_\kappa(\boldsymbol{k})|^2}\cdot e^{\int d^3k\ \beta_\kappa(\boldsymbol{k}) a_{\boldsymbol{k}}^\dagger}\,|0\rangle_{\boldsymbol{p}} \tag{578}$$

which yields

$$\langle \boldsymbol{k}_1,...,\boldsymbol{k}_n|_{\boldsymbol{p}}\,|\psi\rangle_{\boldsymbol{p}} = e^{-\frac{1}{2}\int d^3k\ |\beta_\kappa(\boldsymbol{k})|^2}\frac{1}{\sqrt{n!}}\prod_{i=1}^n \beta_\kappa(\boldsymbol{k}_i) \tag{579}$$

hence

$$\begin{aligned} \langle\psi|_{\boldsymbol{p}}\,|\psi\rangle_{\boldsymbol{p}} &= e^{-\int d^3k\ |\beta_\kappa(\boldsymbol{k})|^2}\cdot\sum_{n=0}^\infty \int d^3k_1 \ldots \int d^3k_n\ \langle\psi|_{\boldsymbol{p}}\,|\boldsymbol{k}_1,...,\boldsymbol{k}_n\rangle_{\boldsymbol{p}}\,\langle \boldsymbol{k}_1,...,\boldsymbol{k}_n|_{\boldsymbol{p}}\,|\psi\rangle_{\boldsymbol{p}} &(580)\\ &= e^{-\int d^3k\ |\beta_\kappa(\boldsymbol{k})|^2}\cdot\sum_{n=0}^\infty \frac{1}{n!}(\int d^3k\ |\beta_\kappa(\boldsymbol{k})|^2)^n &(581)\\ &= 1 &(582)\end{aligned}$$

so $|\psi\rangle_{\boldsymbol{p}} \in \mathcal{F}_1^{\boldsymbol{p}}$ and that for all $\kappa\le\infty$ because $\beta_\infty(\boldsymbol{k}) \in L_2(\mathbb{R}^3)$.

Now the second point. For every positive operator A on $\mathcal{D}(A)\subset\mathcal{F}_1$ we have the equivalence

$$|\psi_\varphi\rangle := \int d^3p\ \varphi(\boldsymbol{p})\,|\psi\rangle_{\boldsymbol{p}} \in \mathcal{D}(\sqrt{A}) \Leftrightarrow \int d^3p' \int d^3p\ \varphi(\boldsymbol{p}')\varphi(\boldsymbol{p})\,\langle\psi|_{\boldsymbol{p}'}\,A\,|\psi\rangle_{\boldsymbol{p}} < \infty \tag{583}$$

Therefore we compute

$$\begin{aligned} \int d^3p' \int d^3p\ \varphi(\boldsymbol{p}')\varphi(\boldsymbol{p})\,\langle\psi|_{\boldsymbol{p}}\,\widehat{H}\,|\psi\rangle_{\boldsymbol{p}} &= \lim_{\kappa\to\infty}\int d^3p' \int d^3p\ \varphi(\boldsymbol{p}')\varphi(\boldsymbol{p})\times &(584)\\ &\qquad\times\langle\psi|_{\boldsymbol{p}}\,(H_\kappa - R_\kappa)\,|\psi\rangle_{\boldsymbol{p}} &(585)\end{aligned}$$

$$= \lim_{\kappa\to\infty} \int d^3p'' \int d^3p' \int d^3p \; \varphi(\boldsymbol{p}')\varphi(\boldsymbol{p}) \sum_{n=0}^{\infty} \int d^3k_1 \, ... \int d^3k_n \; \times \tag{586}$$

$$\times \left\langle \psi \right|_{\boldsymbol{p}} \left| \boldsymbol{k}_1, ..., \boldsymbol{k}_n \right\rangle_{\boldsymbol{p}''} \left\langle \boldsymbol{k}_1, ..., \boldsymbol{k}_n \right|_{\boldsymbol{p}''} (H_\kappa - R_\kappa) \left| \psi \right\rangle_{\boldsymbol{p}} \tag{587}$$

by (574)

$$= \lim_{\kappa\to\infty} \int d^3p'' \int d^3p' \int d^3p \; \varphi(\boldsymbol{p}')\varphi(\boldsymbol{p}) \sum_{n=0}^{\infty} \int d^3k_1 \, ... \int d^3k_n \; \times \tag{588}$$

$$\times \left\langle \psi \right|_{\boldsymbol{p}'} \left| \boldsymbol{k}_1, ..., \boldsymbol{k}_n \right\rangle_{\boldsymbol{p}''} \left\langle \boldsymbol{k}_1, ..., \boldsymbol{k}_n \right|_{\boldsymbol{p}''} \left| \psi \right\rangle_{\boldsymbol{p}} \times \tag{589}$$

$$\times \left(\frac{(\boldsymbol{p}'' - \sum_{i=1}^n \boldsymbol{k}_i)^2}{2M} - \frac{\sum_{i=1}^n \boldsymbol{k}_i^2}{2M} \right) \tag{590}$$

$$= \lim_{\kappa\to\infty} e^{-\int d^3k \, |\beta_\kappa(\boldsymbol{k})|^2} \int d^3p \; |\varphi(\boldsymbol{p})|^2 \sum_{n=0}^{\infty} \int d^3k_1 \, ... \int d^3k_n \; \frac{1}{n!} \prod_{i=1}^n |\beta_\kappa(\boldsymbol{k}_i)|^2 \times \tag{591}$$

$$\times \left(\frac{\boldsymbol{p}^2}{2M} - \frac{\boldsymbol{p} \sum_{i=1}^n \boldsymbol{k}_i}{M} + \frac{\sum_{i\neq j}^n \boldsymbol{k}_i \boldsymbol{k}_j}{2M} \right) \tag{592}$$

$$\leq \lim_{\kappa\to\infty} e^{-\int d^3k \, |\beta_\kappa(\boldsymbol{k})|^2} \int d^3p \; |\varphi(\boldsymbol{p})|^2 \sum_{n=0}^{\infty} \int d^3k_1 \, ... \int d^3k_n \; \frac{1}{n!} \prod_{i=1}^n |\beta_\kappa(\boldsymbol{k}_i)|^2 \times \tag{593}$$

$$\times \left(\frac{\boldsymbol{p}^2}{2M} - \frac{|\boldsymbol{p}| \sum_{i=1}^n |\boldsymbol{k}_i|}{M} + \frac{\sum_{i\neq j}^n |\boldsymbol{k}_i||\boldsymbol{k}_j|}{2M} \right) \tag{594}$$

The aim is now to retain the product structure of a coherent state in this expression. The only term that prevents of from doing so is the one in the parenthesis. We therefore introduce $\zeta_\kappa(\boldsymbol{k}) := \sqrt{(1+\sqrt{2}|\boldsymbol{k}|)}\beta_\kappa(\boldsymbol{k})$ and dominate $\beta_\kappa(\boldsymbol{k})$ times the term in the parenthesis with the help of it. By induction

$$\prod_{i=1}^n |\zeta_\kappa(\boldsymbol{k}_i)|^2 \quad = \quad \prod_{i=1}^n |\beta_\kappa(\boldsymbol{k}_i)|^2 \sum_{j=0}^n (\sqrt{2})^j \sum_{\mathbb{SP}^l_{n-j}\{1,2,...,n\}} \prod_{m=1}^{n-j} |\boldsymbol{k}_{l_m}| \tag{595}$$

For $n = 1$ the above equation holds trivially. We need to show $n \to (n+1)$

$$\prod_{i=1}^{n+1} |\zeta_\kappa(\boldsymbol{k}_i)|^2 \quad = \quad \prod_{i=1}^{n} |\zeta_\kappa(\boldsymbol{k}_i)|^2 \cdot |\zeta_\kappa(\boldsymbol{k}_{n+1})|^2 (1+\sqrt{2}|\boldsymbol{k}_{n+1}|) \tag{596}$$

$$= \quad \prod_{i=1}^{n+1} |\beta_\kappa(\boldsymbol{k}_i)|^2 \bigg(\sum_{j=0}^n (\sqrt{2})^j \sum_{\mathbb{SP}^l_{n-j}\{1,...,n\}} \prod_{m=1}^{n-j} |\boldsymbol{k}_{l_m}| + \tag{597}$$

$$+ \sum_{j=1}^n (\sqrt{2})^j \sum_{\mathbb{SP}^l_{n-j-1}\{1,...,n\}} \prod_{m=1}^{n-j-1} |\boldsymbol{k}_{l_m}| \cdot |\boldsymbol{k}_{n+1}| \bigg) \tag{598}$$

$$= \quad \prod_{i=1}^{n+1} |\beta_\kappa(\boldsymbol{k}_i)|^2 \sum_{j=0}^{n+1} (\sqrt{2})^j \sum_{\mathbb{SP}^l_{n+1-j}\{1,...,n+1\}} \prod_{m=1}^{n+1-j} |\boldsymbol{k}_{l_m}| \tag{599}$$

because

$$\sum_{\mathbb{SP}^l_j\{1,...,n\}} \prod_{m=1}^{j} |\boldsymbol{k}_{l_m}| + \sum_{\mathbb{SP}^l_{j-1}\{1,...,n\}} \prod_{m=1}^{j-1} |\boldsymbol{k}_{l_m}| \cdot |\boldsymbol{k}_{n+1}| = \sum_{\mathbb{SP}^l_j\{1,...,n+1\}} \prod_{m=1}^{j} |\boldsymbol{k}_{l_m}| \tag{600}$$

Now we got a function dominating all the terms in line (594) since

$$\prod_{i=1}^{n} |\beta_\kappa(\boldsymbol{k}_i)|^2 \sum_{j=1}^{n} |\boldsymbol{k}_j| \quad < \quad \prod_{i=1}^{n} |\zeta_\kappa(\boldsymbol{k}_i)|^2 \tag{601}$$

$$\prod_{i=1}^{n} |\beta_\kappa(\boldsymbol{k}_i)|^2 \sum_{j\neq l}^{n} |\boldsymbol{k}_j||\boldsymbol{k}_l| \quad < \quad \prod_{i=1}^{n} |\zeta_\kappa(\boldsymbol{k}_i)|^2 \tag{602}$$

$$\tag{603}$$

and can continue our estimate

$$(594) \quad < \quad \lim_{\kappa\to\infty} e^{-\int d^3k\, |\beta_\kappa(\boldsymbol{k})|^2} \int d^3p\, |\varphi(\boldsymbol{p})|^2 \sum_{n=0}^{\infty} \int d^3k_1 \ldots \int d^3k_n\, \frac{1}{n!} \prod_{i=1}^{n} |\zeta_\kappa(\boldsymbol{k}_i)|^2 \times \tag{604}$$

$$\times \left(\frac{\boldsymbol{p}^2}{2M} - \frac{|\boldsymbol{p}|}{M} + \frac{1}{2M} \right) \tag{605}$$

$$= \quad \lim_{\kappa\to\infty} e^{\int d^3k\, (|\zeta_\kappa(\boldsymbol{k})|^2 - |\beta_\kappa(\boldsymbol{k})|^2)} \int d^3p\, |\varphi(\boldsymbol{p})|^2 \left(\frac{\boldsymbol{p}^2}{2M} - \frac{|\boldsymbol{p}|}{M} + \frac{1}{2M} \right) \tag{606}$$

Now for $\mu, M \in \mathbb{R}_+$ the functions asymptotically behave like $\beta_\infty(\boldsymbol{k}) \sim |\boldsymbol{k}|^{-5/2}$ and by definition $\zeta_\infty(\boldsymbol{k}) \sim |\boldsymbol{k}|^{-2}$ without any singularities on whole $\mathbb{R}^3$ and hence are both in $\mathcal{L}_2(\mathbb{R}^3)$. The $\boldsymbol{p}$ integral converges also for $\varphi \in \mathcal{S}(\mathbb{R}^3)$ and therefore $(606) < \infty$. The third point is just an application of the second point. □

Now we turn to vectors in the domain of $\widehat{H}$. From observation 4.5.3 we recall that the crucial point is to choose a Fock-vector for which simultaneously the properties of $\mathcal{E}$ from line (557) and the asymptotic behavior of the n-meson wave functions $\langle \boldsymbol{q}; \boldsymbol{k}_1, \ldots, \boldsymbol{k}_n|_{\boldsymbol{p}'}\, |\psi\rangle_{\boldsymbol{p}}$ are fulfilled. Since we have already computed the ground state of the $\boldsymbol{p}$ fibre of $\widehat{H}$ we at first examine for that special case how $\widehat{H}^{\boldsymbol{p}}$ acts on this Fock-vector in the weak sense and how divergent terms cancel out in every n-meson wave function. However we will not claim to prove anything but only try to motivate an understanding why $\widehat{H}^{\boldsymbol{p}}$ does not take the n-meson wave functions out of $\mathcal{L}_2(\mathbb{R}^{3n})$. So let again $|\widehat{\Psi}(g)\rangle_{\boldsymbol{p}}$ be the ground state given by the limit (523) of the power series in g in line (460) for finite κ and we obtain

$$\langle \boldsymbol{k}_1, \ldots, \boldsymbol{k}_n|_{\boldsymbol{p}}\, (H_\kappa - R_\kappa)\, |\widehat{\Psi}(g)\rangle_{\boldsymbol{p}} \quad = \quad \sum_{m=0}^{\infty} g^{n+2m} \Bigg[\frac{(\boldsymbol{p} - \sum_{i=1}^{n} \boldsymbol{k}_i)^2}{2M} + \sum_{i=1}^{n} \omega_{\boldsymbol{k}_i} - R_\kappa + \tag{607}$$

$$-g^2 \int d^3k\; \gamma_\kappa(\boldsymbol{k}) \frac{\overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{k}, \boldsymbol{l}_1, \ldots, \boldsymbol{l}_m)}^{l_1,\ldots,l_m}}{\overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{l}_1, \ldots, \boldsymbol{l}_m)}^{l_1,\ldots,l_m}} + \tag{608}$$

$$-\sum_{i=1}^{n} \frac{\overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, \ldots, \widehat{\boldsymbol{k}}_i, \ldots, \boldsymbol{k}_n, \boldsymbol{l}_1, \ldots, \boldsymbol{l}_m)}^{l_1,\ldots,l_m}}{\overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{l}_1, \ldots, \boldsymbol{l}_m)}^{l_1,\ldots,l_m}} \Bigg] \times \tag{609}$$

$$\times \Gamma_\kappa(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n) \overbrace{\xi(\boldsymbol{p}; \boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{l}_1, \ldots, \boldsymbol{l}_m)}^{l_1,\ldots,l_m} \tag{610}$$

Since this is a power series in g the cancellations claimed in the above observations have to take place for every m respectively. We shall only look at the case $m = 0$ since the cancellation mechanism is the same for all m and we can then forget about the here unimportant cross terms which appear in contractions of higher order than zero over ξ. So the $m = 0$ summand looks like

$$g^n \Bigg[\frac{(\boldsymbol{p} - \sum_{i=1}^n \boldsymbol{k}_i)^2}{2M} + \sum_{i=1}^n \omega_{\boldsymbol{k}_i} - \underbrace{R_\kappa}_{(*)} - \underbrace{g^2 \int d^3k \; \gamma_\kappa(\boldsymbol{k}) \frac{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n, \boldsymbol{k})}{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n)}}_{(**)} + \tag{611}$$

$$- \frac{\sum_{i=1}^n \xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \widehat{\boldsymbol{k}_i}, ..., \boldsymbol{k}_n)}{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n)} \Bigg] \Gamma_\kappa(\boldsymbol{k}_1, ..., \boldsymbol{k}_n) \xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n) \tag{612}$$

Recall by definition of R_κ in line (66) and ξ in line (452) that in the limit $\kappa \to \infty$ both $(*)$ and $(**)$ diverge. However they diverge with opposite sign and we shall in the following see that fortunately both divergences cancel each other out and the remaining expression is well-defined. Plugging in the definition of the contraction of zero-th order from line (452) we find

$$... = g^n \Bigg[\frac{(\boldsymbol{p} - \sum_{i=1}^n \boldsymbol{k}_i)^2}{2M} + \sum_{i=1}^n \omega_{\boldsymbol{k}_i} - R_\kappa + \tag{613}$$

$$-g^2 \int d^3k \; \gamma_\kappa^2(\boldsymbol{k}) \frac{1}{\eta(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n, \boldsymbol{k})} \left(1 + \sum_{i=1}^n \frac{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \widehat{\boldsymbol{k}_i}, ..., \boldsymbol{k}_n, \boldsymbol{k})}{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n)} \right) + \tag{614}$$

$$-\eta(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n) \Bigg] \Gamma_\kappa(\boldsymbol{k}_1, ..., \boldsymbol{k}_n) \xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n) \tag{615}$$

Referring to our observations a term from line (614) will be cancelled by R_κ and the rest of the terms sum up to a function we have called $\mathcal{E}$ in line (555). We rewrite the above

$$... = g^n \Bigg[\frac{(\boldsymbol{p} - \sum_{i=1}^n \boldsymbol{k}_i)^2}{2M} + \sum_{i=1}^n \omega_{\boldsymbol{k}_i} - \eta(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n) \tag{616}$$

$$-R_\kappa - g^2 \int d^3k \; \gamma_\kappa^2(\boldsymbol{k}) + \tag{617}$$

$$-g^2 \int d^3k \; \gamma_\kappa^2(\boldsymbol{k}) \frac{1}{\eta(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n, \boldsymbol{k})} \sum_{i=1}^n \frac{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \widehat{\boldsymbol{k}_i}, ..., \boldsymbol{k}_n, \boldsymbol{k})}{\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n)} \Bigg] \times \tag{618}$$

$$\times \Gamma_\kappa(\boldsymbol{k}_1, ..., \boldsymbol{k}_n) \xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n) \tag{619}$$

and plug in the definition of η from (388) and R_κ from Nelson's theorem 2.6.1

$$... = g^n \Bigg[\frac{\boldsymbol{p}^2}{2M} + g^2 \int d^3k \; \gamma_\kappa^2(\boldsymbol{k}) \Bigg(\underbrace{\frac{1}{\eta(\boldsymbol{p}; \boldsymbol{k})} - \frac{1}{\eta(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n, \boldsymbol{k})}}_{\sim |\boldsymbol{k}|^{-3}} \Bigg) + \tag{620}$$

$$-g^2 \int d^3k \; \gamma_\kappa^2(\boldsymbol{k}) \underbrace{\frac{\sum_{i=1}^n \xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \widehat{\boldsymbol{k}_i}, ..., \boldsymbol{k}_n, \boldsymbol{k})}{\eta(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n, \boldsymbol{k}) \xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n)}}_{\sim |\boldsymbol{k}|^{-4}} \Bigg] \times \tag{621}$$

$$\times \Gamma_\kappa(\boldsymbol{k}_1, ..., \boldsymbol{k}_n) \xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n) \tag{622}$$

Now even in the limit $\kappa \to \infty$ for $\mu, M \in \mathbb{R}_+$ and $|\boldsymbol{p}| < \sqrt{2M\mu}$ both integrals exist because the integrands are non-singular and $\gamma_\infty^2(\boldsymbol{k}) \sim |\boldsymbol{k}|^{-1}$. Hence in worst case the terms in the parentheses sum up to a non-singular function asymptotically approaching $\frac{\boldsymbol{p}^2}{2M} + const$ for big $|\boldsymbol{k}_1|, ..., |\boldsymbol{k}_n|$. So as long as $\Gamma_\kappa(\boldsymbol{k}_1, ..., \boldsymbol{k}_n)\xi(\boldsymbol{p}; \boldsymbol{k}_1, ..., \boldsymbol{k}_n)$ is in $\mathcal{L}_2(\mathbb{R}^{3n})$ the whole above expression will again be in $\mathcal{L}_2(\mathbb{R}^{3n})$. Hence it is very probably that all these terms sum up to a function we have called $\mathcal{E}$ in line (555) with the properties (557). As we have said we shall not prove anything here but be satisfied with this basic understanding of how the cancellations take place.

4.5.6 The idea of a sea model

Let us finally have a brief look at the time evolution of these excited states. Here our two example states shall be

$$|\varphi_{a^\dagger}\rangle_{\boldsymbol{p}} \quad := \quad \int d^3k\ f(\boldsymbol{k}) a^\dagger_{\boldsymbol{k}} e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}} \, |\widehat{\Psi}(g)\rangle_{\boldsymbol{p}} \tag{623}$$

$$|\varphi_{pair}\rangle_{\boldsymbol{p}} \quad := \quad \int d^3k\ f(\boldsymbol{k}) \Big(a_{\boldsymbol{k}} e^{i\boldsymbol{k}\widehat{\boldsymbol{x}}} + a^\dagger_{\boldsymbol{k}} e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}} \Big) \, |\widehat{\Psi}(g)\rangle_{\boldsymbol{p}} \tag{624}$$

in an infinitesimal small time step dt such that $e^{-i\widehat{H}^{\boldsymbol{p}} dt} \approx 1 - i\widehat{H}^{\boldsymbol{p}} dt$

$$\begin{aligned}
(1 - i\widehat{H}^{\boldsymbol{p}} dt) \, |\varphi_{a^\dagger}\rangle_{\boldsymbol{p}} \quad &= \quad (1 - i\widehat{E}^{\boldsymbol{p}}(g) dt) \, |\varphi_{a^\dagger}\rangle_{\boldsymbol{p}} + &(625)\\
&\quad -idtg \int d^3k\ \gamma_\infty(\boldsymbol{k}) f(\boldsymbol{k}) \, |\widehat{\Psi}(g)\rangle_{\boldsymbol{p}} + &(626)\\
&\quad -idt \int d^3k\ \left(\frac{\boldsymbol{k}^2}{2M} + \omega_{\boldsymbol{k}} \right) f(\boldsymbol{k}) a^\dagger_{\boldsymbol{k}} e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}} \, |\widehat{\Psi}(g)\rangle_{\boldsymbol{p}} &(627)
\end{aligned}$$

Hence we immediately find a non-zero transition amplitude back to the ground state. For the other state

$$\begin{aligned}
(1 - i\widehat{H}^{\boldsymbol{p}} dt) \, |\varphi_{pair}\rangle_{\boldsymbol{p}} \quad &= \quad (1 - i\widehat{E}^{\boldsymbol{p}}(g) dt) \, |\varphi_{pair}\rangle_{\boldsymbol{p}} + &(628)\\
&\quad +idt \int d^3k\ \left(\frac{\boldsymbol{k}^2}{2M} + \omega_{\boldsymbol{k}} \right) f(\boldsymbol{k}) a_{\boldsymbol{k}} e^{i\boldsymbol{k}\widehat{\boldsymbol{x}}} \, |\widehat{\Psi}(g)\rangle_{\boldsymbol{p}} + &(629)\\
&\quad -idt \int d^3k\ \left(\frac{\boldsymbol{k}^2}{2M} + \omega_{\boldsymbol{k}} \right) f(\boldsymbol{k}) a^\dagger_{\boldsymbol{k}} e^{-i\boldsymbol{k}\widehat{\boldsymbol{x}}} \, |\widehat{\Psi}(g)\rangle_{\boldsymbol{p}} &(630)
\end{aligned}$$

the transition amplitude back to the ground state is always zero. It seems that the pair just evolves in time and will never be annihilated. For the case of a such a pair state it would be interesting to analyze the dynamic Nelson model in the two nucleon sector. Furthermore it seems that the $a_{\boldsymbol{k}}$ part of this state evolves like the $a^\dagger_{\boldsymbol{k}}$ part but back in time. Based on these observations one could put up the question if we can interpret the ground state as a meson *sea*, where we can deal with the excited states as mesons and meson *holes* in that *sea* and forget about the *sea* Fock-vector $|\widehat{\Psi}(g)\rangle$, i.e. the ground state. Especially for the relativistic models for which we have argued that the ground state does not lie in the Fock-space anymore this point of view would be advantageous. In the end one would this way yield a free field theory with respect to the sea and not to the vacuum $|0\rangle$. Can this be an explanation why coefficients of the perturbation series of an interacting field theory can be computed by a certain recipe in terms of vacuum expectation values of a related free field theory? - recall the quote in 4.1. Of course until now it is far too early to give an answer or even a statement about the sea idea. Nevertheless the time evolution and this idea will be subject of further investigation.

5 Conclusion

In order to introduce local, or often called relativistic, interactions in mechanical theories we have found the field to be a helpful mathematical tool. Along the way we have motivated several scalar field interaction models and have analyzed arising mathematical difficulties of the classical and its quantum version and ways to deal with them. Based on the observations made in this work we conclude with the following three points.

1. **The point particle limit** Probably the biggest mathematical problem of field theories, classical or quantum ones, is that we need to treat the particles as points with no geometrical extend, such that the resulting theory fundamentally respects the Poincaré symmetry. Exactly this point particle limit of field theories generates divergences in the definition of the fields itself and therefore in the equations of motion and makes it so hard to write down a complete mechanical theory involving interactions described by these fields. These divergences persist in the quantum field theories and are, as we have seen in the example of the total energy of the static Nelson model, even identical. However in the classical as in the quantum case we have presented ways to make sense out of the equations of motion, which do not conflict with their original physical meaning and are known under the name of renormalization theories.

2. **The renormalization concept** As we have seen, the field amplitudes, the ones we have called γ or I, are in general not in $\mathcal{L}_2(\mathbb{R}^3)$. Hence the fields are, if no cutoff is imposed, not well-defined operators on the Fock-space. So the Hamiltonians being functionals of the fields are in general not well-defined operators on the Fock-space. However in the case of the toy model and the dynamic Nelson model we have suggested that the resulting Hamiltonians can nevertheless be given a meaning as self-adjoint operators on Fock-space when restricting their formal operator expressions to a special but dense subset of the Fock-space on which their action is well-defined. This is possible only because we have chosen a special dispersion relation D of the nucleons such that $\gamma/D \in \mathcal{L}_2(\mathbb{R}^3)$, for example in the dynamic quantum Model D was of a Schrödinger type $\sim \boldsymbol{p}^2$. For most relativistic dispersion relations of the nucleons, e.g. a Dirac type $\sim |\boldsymbol{p}|$ or even the simple static quantum Nelson model $\sim |\boldsymbol{p}|^0$, the here considered renormalization concept can no longer be applied since the ground state does not lie in Fock-space anymore. So the question stands if it is possible to extend this renormalization concept for the relativistic case, which would be more relevant for today's high-energy physics. In other words can this renormalization concept be extended to a bigger space than Fock-space in a physical meaningful way? Furthermore we have discussed that it is only possible to renormalize on the N-th nucleon sector of the Fock-space. As soon as we like to take a varying nucleon number into account, e.g. models describing pair creation, this renormalization concept again fails and it thus has to also be extended in this way.

3. **The missing link to what we see** Until now it is far from clear what the quantum version of these field theories are about. Most Hamiltonians are motivated, as we have gone through, with the help of a correspondence principle starting with a classical Hamiltonian. Of course like for any other fundamental physical law there will most probably never be a rigorous derivation of the field equations in the same way as there will probably never be one for e.g. Newton's gravitation force law. Only symmetry considerations and simplicity arguments will help to motivate one or another Hamiltonian. The only thing we can do is to accept these Hamiltonians to be the ones to deal with and so to be the physically relevant ones in quantum field theories as well. However there is still one bit missing. In a classical theory the Hamiltonian is all we need in order to arrive at a complete mechanical theory because somehow everyone intuitively knows that $\boldsymbol{q}_i(t)$ corresponds to the trajectory of the i-th particle that we can track with our eyes - the so-called *element of reality*. The stationarity principle uniquely defines the position probability measure and we are ready to do statistics on these trajectories, whose probability outcomes agree to a great accuracy in the classical regime as nowadays experiments in statistical mechanics and hydrodynamics show.

In Bohmian mechanics the situation is quite the same. $\boldsymbol{q}_i(t)$ corresponds to the trajectories of the i-th particle, only now given as an integral curve to a different vector field as the one in the classical case. Here the equivariance principle [6] gives us the position probability measure such that we are ready to speak about statistics on the Bohmian trajectories. The probability outcomes of this theory agree astonishingly precise with the ones found in the quantum regime in nowadays experiments. In quantum field theories however this connection to an *element of reality* is exactly the one missing. What is our theory about? More precisely if we long to talk about particles then which mathematical object in our theory corresponds to the position measurement performed by some detector? This missing link is for the latter case a position probability measure. Having a position probability measure on a Hilbert-space we can connect the abstract vectors in this Hilbert-space to the positions of the particles, which we like to describe, and are again ready to compute statistics. At least in the case of quantum electrodynamics is seems very promising that this question can be answered in the near future as there already exist algorithms and recipes to arrive at *numbers* that are also in astonishing agreement with the ones *found* in nowadays high-energy experiments.

A Appendix

A.1 Fourier transformation of the Yukawa potential

Let $\widehat{V}(\boldsymbol{k}) := \frac{1}{(2\pi)^{3/2}} \frac{1}{\boldsymbol{k}^2+\mu^2}$ be the Yukawa potential in momentum space then

$$V(\boldsymbol{x}) \quad := \quad \int đ^3k\ \widehat{V}(\boldsymbol{k}) e^{i\boldsymbol{k}\boldsymbol{x}} \tag{631}$$

is the Yukawa potential in position space. In order to compute the integral we change to spherical coordinates with

$$\boldsymbol{k} = k \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix} \tag{632}$$

such that $\boldsymbol{k}\boldsymbol{x} = kx\cos\theta$ which gives

$$V(x) \quad := \quad \frac{1}{(2\pi)^{3/2}} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^{\infty} dk\ k^2 \sin\theta \frac{e^{ikx\cos\theta}}{(2\pi)^{3/2}(k^2+\mu^2)} \tag{633}$$

$$= \quad \frac{1}{(2\pi)^2} \int_0^{\infty} dk \frac{k^2}{k^2+\mu^2} \underbrace{\int_0^{\pi} d\theta \sin\theta e^{ikx\cos\theta}}_{(*)} \tag{634}$$

$$\tag{635}$$

The θ integration gives

$$(*) = \int_{-1}^{1} dt\ e^{-ikxt} = \frac{e^{ikx} - e^{-ikx}}{ikx} \tag{636}$$

and obtain

$$V(x) \quad = \quad \frac{1}{(2\pi)^2 ix} \int_0^{\infty} dk\ \frac{|k|(e^{ikx} - e^{-ikx})}{k^2+\mu^2} \tag{637}$$

$$= \quad \frac{1}{(2\pi)^2 ix} \int_{-\infty}^{\infty} dk\ \frac{ke^{ikx}}{(k+i\mu)(k-i\mu))} \tag{638}$$

We can now close the path of integration along the real axis in the upper complex plane without changing the value of the integral since $ikr \to -\infty$ in the upper half of complex plane for $|k| \to \infty$. The integration path hence includes one singularity at $k = i\mu$ and we use the theorem of residues to compute

$$V(\boldsymbol{x}) \quad = \quad \frac{e^{-\mu||\boldsymbol{x}||}}{4\pi||\boldsymbol{x}||} \tag{639}$$

B Reference of conventions and symbols

Object/Symbol	**Convention/Lookup table**
bold letters like $\boldsymbol{x}$	vectors $\in \mathbb{R}^3$
$\int d^3x$	$\int_{\mathbb{R}^3} d^3x$
$\oint_{\mathcal{C}} d\gamma$	closed integral along the anti-clockwise path $\mathcal{C}$ parameterize by γ
dashed symbols like $đ^n x$ or $\bar{đ}^n x$	$đ^n x = d^n x/(2\pi)^{n/2}$ i.e. $\bar{đ}^n x = d^n x/(2\pi)^n$!!! except traditionally $\hbar = 1$

fourier transformation F	$F(f(\boldsymbol{x}),\boldsymbol{x})(\boldsymbol{k}) := \int d\!\!\!{}^-{}^3 x f(x) e^{i\boldsymbol{kx}}$ $F^{-1}(\hat{f}(\boldsymbol{k}),\boldsymbol{k})(\boldsymbol{x}) := \int d\!\!\!{}^-{}^3 k \hat{f}(k) e^{-i\boldsymbol{kx}}$
$\mathbb{N}$	all natural numbers from 1 on, i.e. $\{1,2,3,...\}$
$\mathbb{N}_0$	$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$
$\mathbb{R}_+$	$\mathbb{R}_+ := \{x \in \mathbb{R} \vert x > 0\}$
$\mathcal{L}_2(\Omega,\mathcal{K},d\mu)$	the Hilbert-space of square integrable functions $\Omega \to K$ with respect to measure $d\mu$ - if abbreviated with $\mathcal{L}_2$ or similarly then usually $(\Omega,\mathcal{K},d\mu) = (\mathbb{R}^3,\mathbb{C},d^3x)$
$\mathcal{C}_c^n(\Omega,\mathcal{K})$	the space of n times differentiable functions $\Omega \to \mathbb{C}$ with compact support - if abbreviated with $\mathcal{C}_c^n$ or similarly then usually $(\Omega,\mathcal{K}) = (\mathbb{R}^3,\mathbb{C})$
$\mathcal{H}_{nuc}^{\otimes N}$	N nucleon Hilbert-space (definition 2.1.3 in subsection 2.1)
$\mathcal{F}_{nuc}$, $\mathcal{F}_{mes}$	nucleon and meson Fock-space respectively (definition 2.1.5 in subsection 2.1)
$\mathcal{F}_N$	$\mathcal{H}_{nuc}^{\otimes N} \otimes \mathcal{F}_{mes}$ (definition 2.1.5 in subsection 2.1)
$\nabla_{\boldsymbol{x}}$, $\nabla_{\boldsymbol{x}}^2$	gradient and Laplace differential operator with respect to the coordinates $\boldsymbol{x}$
$\Box_{(\boldsymbol{x},t)}$	the D'Alembert differential operator $\Box_{(\boldsymbol{x},t)} = \frac{\partial^2}{\partial t^2} - \nabla_{\boldsymbol{x}}^2$
$\widehat{\boldsymbol{x}}$	is the position operator, $\widehat{\boldsymbol{x}} = -i\nabla_{\boldsymbol{p}}$ in momentum representation on $\mathcal{H}_{nuc}^{\otimes 1}$
$\widehat{\boldsymbol{p}}$	is the momentum operator, $\widehat{\boldsymbol{p}} = -i\nabla_{\boldsymbol{x}}$ in position representation on $\mathcal{H}_{nuc}^{\otimes 1}$
∇^2	is short for $\nabla_{\boldsymbol{x}}^2 \otimes \mathbb{1}_{id}^{\mathcal{F}_{mes}}$ on $\mathcal{H}_{nuc}^{\otimes 1} \otimes \mathcal{F}_{mes} = \mathcal{F}_1$
∇_i^2	is short for $\mathbb{1}_{id}^{\mathcal{H}_{nuc}^{\otimes 1}} \otimes ... \otimes \nabla_{\boldsymbol{x}}^2 \otimes ... \otimes \mathbb{1}_{id}^{\mathcal{H}_{nuc}^{\otimes 1}} \otimes \mathbb{1}_{id}^{\mathcal{F}_{mes}}$ on $\mathcal{F}_N$ with $\nabla_{\boldsymbol{x}}^2$ being in the i-th position
field operators	$\psi^\dagger, \psi$ are the nucleon and $a^\dagger, a$ the meson field operators in position representation with parentheses, e.g. $a^\dagger(\boldsymbol{x})$, $a(\boldsymbol{x})$ and in momentum representation without, e.g. $a_{\boldsymbol{k}}^\dagger$, $a_{\boldsymbol{k}}$. $a_{\boldsymbol{k}}^\dagger = \int d^3x\ \langle \boldsymbol{x} \vert \boldsymbol{k} \rangle\, a^\dagger(\boldsymbol{x}) = \int d\!\!\!{}^-{}^3 x\ e^{i\boldsymbol{kx}} a^\dagger(\boldsymbol{x})$ $a_{\boldsymbol{k}} = \int d^3x\ \langle \boldsymbol{k} \vert \boldsymbol{x} \rangle\, a(\boldsymbol{x}) = \int d\!\!\!{}^-{}^3 x\ e^{-i\boldsymbol{kx}} a(\boldsymbol{x})$ $a^\dagger(\boldsymbol{x}) = \int d^3k\ \langle \boldsymbol{k} \vert \boldsymbol{x} \rangle\, a_{\boldsymbol{k}}^\dagger = \int d\!\!\!{}^-{}^3 k\ e^{-i\boldsymbol{kx}} a_{\boldsymbol{k}}^\dagger$ $a(\boldsymbol{x}) = \int d^3k\ \langle \boldsymbol{x} \vert \boldsymbol{k} \rangle\, a_{\boldsymbol{k}} = \int d\!\!\!{}^-{}^3 k\ e^{i\boldsymbol{kx}} a_{\boldsymbol{k}}$ $e^{i\boldsymbol{lx}} a_{\boldsymbol{k}}^\dagger = a_{\boldsymbol{k}+\boldsymbol{l}}^\dagger$ for $\boldsymbol{x}$ being the position operator $e^{i\boldsymbol{lx}} a_{\boldsymbol{k}} = a_{\boldsymbol{k}-\boldsymbol{l}}$ $e^{i\boldsymbol{pd}} a_{\boldsymbol{x}}^\dagger = a_{\boldsymbol{x}-\boldsymbol{d}}^\dagger$ for $\boldsymbol{p}$ being the momentum operator $e^{i\boldsymbol{pd}} a_{\boldsymbol{x}} = a_{\boldsymbol{x}+\boldsymbol{d}}$
span$\mathcal{M}$	the vector-space spanned by the elements in the set M
$\mathcal{D}(T)$, $\mathcal{Q}(T)$	domain and form domain of an operator T
$\overline{T}$	the closure of an operator T
$T \restriction \mathcal{M}$	the restriction of the operator T to the set $\mathcal{M}$
$f(x,y) \sim g(x)$	For functions $f : \Omega \times \Omega' \to \mathcal{K}$ and $g : \Omega \to \mathcal{K}$ we write $f(x,y) \sim g(x)$ for $x \in \Omega, y \in \Omega'$ if $\lim_{\vert\vert x \vert\vert_\Omega \to \infty} \vert\vert f(x,y)\big\vert_{y=const} g^{-1}(x) \vert\vert_{\mathcal{K}} = C = const$ and *at least* $f(x,y) \sim g(x)$ if C is a constant or even zero,

	which means that $f(x, y)$ would fall off equal fast or faster than $g(x)$
$\lfloor x \rfloor$	floor of the real number x, i.e. the largest natural number smaller than x.
$\sum_{\mathbb{SP}^i_j A}$	sum over all symmetric permutations $\{i_1, ..., i_j\} \subset A$
mod	denotes the modulus of a natural number
g	coupling constant $\in \mathbb{R}_+$
M	nucleon mass $\in \mathbb{R}_+$
μ	meson mass $\in \mathbb{R}_+$
N	number of nucleons $\in \mathbb{N}$

References

[1] Arai A.: Ground state of the massless Nelson model without infrared cutoff in a non-Fock representation, Rev. Math. Phys. **13**, 1075-1094 (2001)

[2] Bauer G.: Ein Existenzsatz für die Wheeler-Feynman Elektrodynamik, München, Diss., Utz, Wiss., Mathematik (1997)

[3] Bogoliubov N., Shirkov V.: Quantum Fields, The Benjamin/Cumming Publishing Company, Inc. (1983)

[4] Betz V., Hiroshima F., Lörinczi J., Minlos R.A., and Spohn H.: Ground state properties of the Nelson Hamiltonian - a Gibbs measure-based approach, Rev. Math. Phys. **14**, 173-198 (2002)

[5] Dirac P.A.M.: Classical Theory of Radiating Electrons, Proc. Roy. Soc. **A178**, 148 (1938)

[6] Dürr D., Goldstein S., Zanghì N.: Quantum Equilibrium and the Origin of Absolute Uncertainty, Journal of Statistical Physics **67**, 843-907 (1992)

[7] Fröhlich J.: Existence of dressed one electron states in a class of persistent models. Fortschr. Phys. **22**, 159-198 (1974)

[8] Gradstein, Ryshik: Sum, Product and Integral Tables

[9] Greenberg O. W., Schweber S. S.: Clothed Partcile Operators in Simple Models of Quantum Field Theory, Nuovo Cimento **8**, 3, 1306 (1958)

[10] Gross E. P.: Particle-Like Solutions in Field Theory, Ann. Phys. **19**, 219-233 (1962)

[11] Henley E.M., Thirring W.: Elementary Quantum Field Theory

[12] Hirokawa M., Hiroshima F., and Spohn H.: Ground state for point particles interacting through a massless scalar Bose field, to appear in Advances in Mathematics, math-ph/0211050

[13] Kato T.: Perturbation Theory of Linear Operators, A Series of Comprehensive Studies in Mathematics, second edition

[14] Kemble E.C.: Fundamental Principles of Quantum Mechanics, McGRAW-HILL BOOK COMPANY, Inc. (1937)

[15] Nelson E.: Interaction of Nonrelativistic Particles with a Quantized Scalar Field, Journ. of Math. Phys. **5**, Nr. 9, 1190 (1964)

[16] Pizzo A.: One Particle (improper) states and Scattering States in Nelson's Massless Model, math-ph/0010043 (2000)

[17] Reed M., Simon B.: Methods of Modern Mathematical Physics I, Functional Analysis

[18] Reed M., Simon B.: Methods of Modern Mathematical Physics II, Fourier Analysis, Self-Adjointness

[19] Reed M., Simon B.: Methods of Modern Mathematical Physics IV, Analysis of operators

[20] Rohrlich F.: Classical Charged Particles, Addison-Wesley (1965)

[21] Schwinger J.: Selected Papers on Quantum Electrodynamics, Dover Publications, Inc., New York (1958)

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Selbstständigkeitserklärung

Hiermit erkläre ich, daß ich die vorliegende Arbeit selbständig verfaßt und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Dirk - André Deckert