THE GRIFFITHS GROUP OF THE GENERIC ABELIAN 3-FOLD.

ANDREAS ROSENSCHON AND V. SRINIVAS

Abstract. Let $\mathrm{J}(C)$ be the Jacobian of the generic complex curve of genus 3. We show that the image of the Ceresa cycle in the Griffiths group $\mathrm{Griff}^2(\mathrm{J}(C)) \otimes \mathbb{Z}/\ell$ is non-zero for all but finitely many primes ℓ. We show further that for the generic principally polarised abelian variety A of dimension 3 the Griffiths group $\mathrm{Griff}^2(A) \otimes \mathbb{Z}/\ell$ is infinite for all but finitely many primes ℓ.

1. Introduction

Let X be a smooth projective variety over an algebraically closed subfield $k \subseteq \mathbb{C}$, and let $\mathrm{CH}^p(X)$ be the Chow group of codimension p cycles modulo rational equivalence. The Griffiths group $\mathrm{Griff}^p(X)$ is defined as the quotient of codimension p cycles which are homologically trivial modulo cycles which are algebraically equivalent to zero. By definition, there is an exact sequence

$$0 \to \mathrm{A}^p(X) \to \mathrm{CH}^p_{\mathrm{hom}}(X) \to \mathrm{Griff}^p(X) \to 0,$$

where $\mathrm{CH}^p(X)_{\mathrm{hom}} \subseteq \mathrm{CH}^p(X)$ is the subgroup of homologically trivial cycles and $\mathrm{A}^p(X) \subseteq \mathrm{CH}^p_{\mathrm{hom}}(X)$ is the subgroup of cycles which are algebraically equivalent to zero. The group $\mathrm{A}^p(X)$ is generated by the images of correspondences coming from Jacobians of curves over k. Since k is algebraically closed these Jacobians are divisible, thus $\mathrm{A}^p(X) \otimes \mathbb{Z}/\ell = 0$, and therefore

$$\mathrm{CH}^p_{\mathrm{hom}}(X) \otimes \mathbb{Z}/\ell = \mathrm{Griff}^p(X) \otimes \mathbb{Z}/\ell.$$

Let $\mathrm{J}(C)$ be the Jacobian of a curve C of genus 3 over $\mathbb{C}$. If c_0 is a fixed point of C, we have the standard embedding $\rho : C \to \mathrm{J}(C)$, $c \mapsto c - c_0$. Let $[-1]_*$ be the morphism on cycle groups induced by the natural involution on $\mathrm{J}(C)$. The *Ceresa cycle* we consider is the codimension 2 cycle on $\mathrm{J}(C)$

$$\Xi = \rho_*(C) - [-1]_*\rho_*(C).$$

Since $[-1]_*$ acts as the identity on $\mathrm{H}^4(\mathrm{J}(C), \mathbb{Z})$, the Ceresa cycle is homologically trivial and defines a class $[\Xi]$ in $\mathrm{Griff}^2(\mathrm{J}(C))$ which is independent of the choice of the base point. Ceresa has shown in [7] that for C generic the class of Ξ in $\mathrm{Griff}^2(\mathrm{J}(C)) \otimes \mathbb{Q}$ is non-zero.

We first consider the image of Ξ in the Griffiths group modulo ℓ.

Theorem 1.1. *Let $\mathrm{J}(C)$ be the Jacobian of the generic curve of genus 3 over $\mathbb{C}$. Then the image of $[\Xi]$ in $\mathrm{CH}^2_{\mathrm{hom}}(\mathrm{J}(C)) \otimes \mathbb{Z}/\ell = \mathrm{Griff}^2(\mathrm{J}(C)) \otimes \mathbb{Z}/\ell$ is non-zero for all but finitely many primes ℓ.*

To our knowledge, this result provides the first example in the literature of a homologically trivial cycle in a Chow group of codimension 2 over an algebraically closed field which is not divisible for all but finitely many primes. The first example of such a cycle which is not divisible for some primes has been given by Bloch-Esnault [4]; in their example $k = \overline{\mathbb{Q}}$. Other examples of non-divisible cycles have been constructed by Schoen. He showed that for an elliptic curve E/k the Chow group $\mathrm{CH}^2_{\mathrm{hom}}(E^3_k) \otimes \mathbb{Z}/\ell$ is non-zero in the following cases: E is general, $k = \mathbb{C}$ and $\ell \in \{5, 7, 11, 13, 17\}$ (see [20]), or E is the Fermat curve, $k = \overline{\mathbb{Q}}$ and $\ell \equiv 1 \bmod 3$ (see [21]). For similar results for triple products of elliptic curves over p-adic fields, see also [18].

In [16] Nori used the Ceresa cycle to prove that for the generic abelian variety A of dimension 3 the Griffiths group $\mathrm{Griff}^2(A)\otimes\mathbb{Q}$ is infinite-dimensional. To construct cycles he considers isogenies $h : B \to A$ with B principally polarised. Thus $B \cong \mathrm{J}(C)$ and B carries a Ceresa cycle Ξ_B whose image $h_*(\Xi_B)$ in $\mathrm{Griff}^2(A)\otimes\mathbb{Q}$ is non-trivial, because of Ceresa's theorem. Nori shows that there are infinitely many choices for h such that the resulting cycles $h_*(\Xi_B)$ are linearly independent, since they twist by different characters of a suitable Galois group.

We construct isogenies from modular correspondence to adapt his argument to our setting and use Theorem 1.1 to prove the following.

Theorem 1.2. *Let A be the generic principally polarised abelian 3-fold over the complex numbers $\mathbb{C}$. Then for all but finitely many primes ℓ:*

$$\#(\mathrm{CH}^2_{\mathrm{hom}}(A) \otimes \mathbb{Z}/\ell) = \#(\mathrm{Griff}^2(A) \otimes \mathbb{Z}/\ell) = \infty.$$

If A is as in Theorem 1.2, it follows from the projective bundle formula that for $d \geq 3$ the complex variety $A \times \mathbb{P}^{d-3}$ has the analogous property.

Corollary 1.3. *For $d \geq 3$ there exists a smooth projective variety $X/\mathbb{C}$ of dimension d such that $\#(\mathrm{CH}^p(X) \otimes \mathbb{Z}/\ell) = \infty$ for $2 \leq p \leq d-1$ and all but finitely many primes ℓ.*

Let $\mathrm{CH}^p(X)[\ell]$ be the ℓ-torsion subgroup of $\mathrm{CH}^p(X)$, i.e. the kernel of the map 'multiplication by ℓ'. Then Corollary 1.3, combined with a result on external product maps due to Schoen [19, Theorem 0.2], implies the following.

Corollary 1.4. *For $d \geq 4$ there exists a smooth projective variety $X/\mathbb{C}$ of dimension d such that $\#(\mathrm{CH}^p(X)[\ell]) = \infty$ for $3 \leq p \leq d-1$ and all but finitely many primes ℓ.*

We remark that the bounds in the above Corollaries are sharp: In characteristic 0 and for any prime ℓ the Chow group $\mathrm{CH}^p(X) \otimes \mathbb{Z}/\ell$ is finite for $p = 0, 1$ and d; this is clear in codimension 0 and 1, and follows for zero cycles from Roitman's theorem [17], [3, 4.2]. The same arguments show the finiteness of $\mathrm{CH}^p(X)[\ell]$ for $p = 0, 1$ and d; for the remaining case $p = 2$ this is a consequence of the Merkurjev-Suslin theorem [13].

Remark. We note that during the Colloquium, the first author stated results analogous to Theorems 1.1. and 1.2, but for a different abelian 3-fold, namely the one considered by Schoen in [20]. The argument presented in the talk was incomplete. We do believe our results are correct for Schoen's abelian 3-folds as well, but different arguments seem to be needed. We hope to return to this elsewhere.

Acknowledgements. The authors thank the referee for pointing out an inaccuracy and numerous suggestions which helped to improve the exposition of the paper. The first author gratefully acknowledges hospitality of the Tata Institute of Fundamental Research.

2. Proof of Theorem 1.1

We first recall some basic properties of moduli and prove an analogue of the Bloch-Esnault theorem for the Jacobian of a generic curve; our general reference for moduli spaces is [15].

Let ζ_3 be a fixed primitive complex cube root of unity. By a full symplectic level 3 structure on a curve C of genus g we mean a basis $\{u_1,..,u_g,v_1,..,v_g\}$ of the 3-torsion subgroup $\mathrm{J}(C)[3]$ of the Jacobian $\mathrm{J}(C)$ of C such that with respect to the Weil-pairing $e_3 : \mathrm{J}(C)[3]\times \mathrm{J}(C)[3] \to \mu_3$ we have $(u_i,v_i)=\zeta_3$ for all i and $(u_i,v_j)=0$ for $i\neq j$.

We denote by M be the moduli space of curves of genus 3 with a level 3 structure defined as above with respect to a fixed primitive cube root ζ_3. In particular, M is a smooth quasi-projective integral scheme over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{3},\zeta_3])$.

Let $\mathcal{C}\to M$ be the universal family of curves, and $J(\mathcal{C})_M \to M$ the corresponding family of Jacobians. We may also regard $\mathcal{C}$ as the moduli space of curves of genus 3 with the level 3 structure as above, together with a marked point. Set $T=\mathcal{C}_{\overline{\mathbb{Q}}}$; note that T is an integral scheme over $\overline{\mathbb{Q}}$. Regarding T as the $\overline{\mathbb{Q}}$-moduli scheme for curves of genus 3 with level 3 structure and a marked point, there is a universal family $\mathcal{C}_T$ over T, together with a distinguished section. Let $L=\overline{\mathbb{Q}}(T)$ be the function field, and let C_L denote the generic fiber of $\mathcal{C}_T\to T$. The distinguished section of $\mathcal{C}_T\to T$ determines a rational point of C_L, which we use to define the Ceresa cycle

$$\Xi_L \in Z^2_{\mathrm{hom}}(\mathrm{J}(C_L)).$$

We will abuse notation and write $J(C)_L$ (resp. $J(C)_{\overline{L}}$) in place of $J(C_L)$ (resp. $J(C_{\overline{L}})$). If C is generic over $\mathbb{C}$, we can construct a commutative diagram of cartesian squares

$$(1)\qquad \begin{array}{ccccccc} \mathrm{J}(C) & \to & \mathrm{J}(C)_L & \to & \mathrm{J}(\mathcal{C}_T) & \to & \mathrm{J}(\mathcal{C})_M \\ {\scriptstyle f_{\mathbb{C}}}\downarrow & & {\scriptstyle f_L}\downarrow & & \downarrow{\scriptstyle f_T} & & \downarrow{\scriptstyle f_M} \\ \mathrm{Spec}(\mathbb{C}) & \to & \mathrm{Spec}(L) & \xrightarrow{i} & T & \to & M \end{array}$$

where $\mathrm{J}(\mathcal{C})_M \to M$ is the family of Jacobians for the universal family of curves $\mathcal{C} \to M$, and the remaining families are base changes of this family.

Let $\overline{L}$ be an algebraic closure of L and denote by $[\Xi_{\overline{L}}]$ the image of $[\Xi_L]$ in $\mathrm{CH}^2_{\mathrm{hom}}(\mathrm{J}(C)_{\overline{L}})$. Since base change along extensions of algebraically closed fields induces an isomorphism on Chow groups modulo ℓ [11], to prove Theorem 1.1 it suffices to show $[\Xi_{\overline{L}}]$ is non-zero in $\mathrm{CH}^2_{\mathrm{hom}}(\mathrm{J}(C)_{\overline{L}}) \otimes \mathbb{Z}/\ell$ for all but finitely many ℓ.

Recall that the primitive part $\mathrm{PH}^3(\mathrm{J}(C), \mathbb{Q}(2))$ of the singular cohomology group $\mathrm{H}^3(\mathrm{J}(C), \mathbb{Q}(2))$ is defined as the cokernel of the injective cup-product map $\cup[\Theta] : \mathrm{H}^1(\mathrm{J}(C), \mathbb{Q}(1)) \to \mathrm{H}^3(\mathrm{J}(C), \mathbb{Q}(2))$, where $[\Theta] \in \mathrm{H}^2(\mathrm{J}(C), \mathbb{Q}(1))$ is the class of the theta divisor. The resulting primitive decomposition

$$\mathrm{H}^3(\mathrm{J}(C), \mathbb{Q}(2)) = \mathrm{PH}^3(\mathrm{J}(C), \mathbb{Q}(2)) \oplus \mathrm{H}^1(\mathrm{J}(C), \mathbb{Q}(1)) \tag{2}$$

is the decomposition of $\mathrm{H}^3(\mathrm{J}(C), \mathbb{Q}(2))$ into irreducible $\mathrm{Sp}_6(\mathbb{Z})$-modules, see [8, pg. 121]. Let P be a self-correspondence of $\mathrm{J}(C)_L$ with the property that the image $P_* \mathrm{H}^3(\mathrm{J}(C), \mathbb{Q}(2))$ is the primitive part $\mathrm{PH}^3(\mathrm{J}(C), \mathbb{Q}(2))$.

The standard comparison theorems imply that there is a finite set S of primes such that for all $\ell \notin S$ the following holds: There is a decomposition

$$\mathrm{H}^3_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}/\ell(2)) = \mathrm{PH}^3_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}/\ell(2)) \oplus \mathrm{H}^1_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}/\ell(1)), \tag{3}$$

which is the decomposition of $\mathrm{H}^3_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}/\ell(2))$ into irreducible $\mathrm{Gal}(\overline{L}/L)$-modules, and the image of the map induced by P is the primitive part

$$P_* \mathrm{H}^3_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}/\ell(2)) = \mathrm{PH}^3_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}/\ell(2)).$$

Let $\mathrm{CH}^2(\mathrm{J}(C)_{\overline{L}})[\ell]$ be the ℓ-torsion subgroup of $\mathrm{CH}^2(\mathrm{J}(C)_{\overline{L}})$. The next lemma shows that for P as above we can control the image $P_* \mathrm{CH}^2(\mathrm{J}(C)_{\overline{L}})[\ell]$.

Lemma 2.1. $P_* \mathrm{CH}^2(\mathrm{J}(C)_{\overline{L}})[\ell] = 0$ *for all but finitely many primes* ℓ.

Proof. For any ℓ we have by [3, 3.5] and [21, Proposition 6.1] an isomorphism

$$P_* \mathrm{CH}^2(\mathrm{J}(C)_{\overline{L}})[\ell] \cong \mathrm{N}^1(\mathrm{PH}^3_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}/\ell(2)), \tag{4}$$

where N^1 denotes the first level of the coniveau filtration. We will show $\mathrm{N}^1(\mathrm{PH}^3_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}/\ell(2)) = 0$ for $\ell \notin S$, where S is a finite set of primes as above. Recall that the Bloch-Esnault theorem [4, Theorem 1.2] states the following: If V is a smooth projective variety over a complete discrete valuation field K with valuation ring R and perfect residue field k, of mixed characteristic $(0, \ell)$, then

$$\mathrm{N}^1 \mathrm{H}^3_{\text{ét}}(V_{\overline{K}}, \mathbb{Z}/\ell(2)) \neq \mathrm{H}^3_{\text{ét}}(V_{\overline{K}}, \mathbb{Z}/\ell(2)), \tag{5}$$

provided (i) V has good ordinary reduction, i.e. there exist cartesian squares

$$\begin{array}{ccccc} V & \to & X & \leftarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \to & \mathrm{Spec}(R) & \leftarrow & \mathrm{Spec}(k) \end{array}$$

with X smooth and proper over $\mathrm{Spec}(R)$, and Y ordinary over $\mathrm{Spec}(k)$, (ii) $\dim Y < (l-1)/\gcd(e, \ell-1)$ (e is the absolute ramification index), and (iii) $\Gamma(Y, \Omega^3_Y) \neq 0$. We note that we can omit the assumption that R has perfect residue field and instead assume in (i) that X is smooth and proper over $\mathrm{Spec}(R)$, and $Y_{\overline{k}}$ is ordinary. Indeed, let R be a discrete valuation ring of mixed characteristic $(0, \ell)$ with residue field k and maximal ideal M_R. By [12, Theorem 29.1] there exists a discrete valuation ring R' containing R with maximal ideal $M_R \cdot R'$ and residue field k^{perf}, the perfect closure of k. The completion $\widehat{R}'$ of R' has perfect residue field and maximal ideal $M_R \cdot \widehat{R}'$. Thus R and $\widehat{R}'$ have the same absolute ramification index, and base change along the algebraic closures of the valuation fields of R and $\widehat{R}'$ induces an isomorphism in étale cohomology. In a similar fashion, we may omit the condition that R is complete, since base change along the algebraic closures of the valuation fields of R and of its completion $\widehat{R}$ again induces an isomorphism in étale cohomology.

To apply this to our setting, recall that M is an irreducible smooth quasi-projective scheme over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{3}, \zeta_3])$. Consider the scheme

$$\mathrm{J}(C)_\ell := \mathrm{J}(\mathcal{C})_M \times_M \mathrm{Spec}(\mathcal{O}_{M,x}),$$

where x is the generic point of the fiber of $M \to \mathrm{Spec}(\mathbb{Z}[\frac{1}{3}, \zeta_3])$ over a prime lying over ℓ (so that $\mathcal{O}_{M,x}$ is a discrete valuation ring of mixed characteristic $(0, \ell)$). Then we have (i) since C is generic, in (ii) $e = 1$ and the condition holds for $\ell \geq 5$, and (iii) holds trivially. Thus for $\ell \geq 5$ the geometric generic fiber of the universal family of Jacobians satisfies (5), and

$$\mathrm{N}^1\,\mathrm{H}^3_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}/\ell(2)) \neq \mathrm{H}^3_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}/\ell(2)), \quad \ell \geq 5. \tag{6}$$

Modifying S if necessary, we may assume the primes $2, 3$ are contained in S. The factor $\mathrm{H}^1_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}/\ell(1))$ in (3) is supported on the theta divisor and contained in $\mathrm{N}^1\,\mathrm{H}^3_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}/\ell(2))$. Thus it follows from (6) that for $\ell \notin S$

$$\mathrm{N}^1\,\mathrm{PH}^3_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}/\ell(2)) \neq \mathrm{PH}^3_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}/\ell(2)). \tag{7}$$

Since $\mathrm{PH}^3_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}/\ell(2))$ is an irreducible $\mathrm{Gal}(\overline{L}/L)$-module, (7) implies $\mathrm{N}^1\,\mathrm{PH}^3_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}/\ell(2) = 0$ for $\ell \notin S$. The lemma follows now from (4). □

We are ready to prove Theorem 1.1. We modify the Bloch-Esnault method [4] and use Hain's proof of Ceresa's theorem [8] which shows that the Ceresa cycle over $\mathbb{C}$ has non-torsion image in a finitely generated abelian group.

Proof. (of Theorem 1.1) We show the image of $[\Xi_{\overline{L}}]$ in $\mathrm{CH}^2(\mathrm{J}(C)_{\overline{L}}) \otimes \mathbb{Z}/\ell$ is non-trivial for all but finitely many ℓ. We begin with the construction of a commutative diagram of cycle maps. Let ℓ be an arbitrary prime, and let

$$\eta^2_L : \mathrm{CH}^2_{\mathrm{hom}}(\mathrm{J}(C)_L) \to \mathrm{H}^1(G_L, \mathrm{H}^3_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}_\ell(2)))$$

be the ℓ-adic Abel-Jacobi map. This map is induced from the cycle class map with values in $\mathrm{H}^4_{\text{ét}}(\mathrm{J}(C)_L, \mathbb{Z}_\ell(2))$, using the filtration on this group coming from the Hochschild-Serre spectral sequence. For details of the construction and general properties, see for example, [2], [9, §9] and [6, §1]. We write

$$\nu_L^2 : \mathrm{CH}^2_{\mathrm{hom}}(\mathrm{J}(C)_L) \to \mathrm{H}^1(G_L, \mathrm{PH}^3_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}_\ell(2)))$$

for the composition of η_L^2 with map induced by the natural quotient map $\mathrm{H}^3_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}_\ell(2)) \to \mathrm{PH}^3_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}_\ell(2))$; similarly we have $\eta^2_{L'}$ and $\nu^2_{L'}$ for any finite field extension L'/L.

If $f_B : C_B \to B$ is a family of smooth curves of genus 3, and $\mathrm{J}(C)_B \to B$ is the corresponding family of Jacobians, the Leray spectral sequence associated to f_B induces a filtration on $\mathrm{H}^4_{\text{ét}}(\mathrm{J}(C)_B, \mathbb{Z}_\ell(2))$. Define $\mathrm{CH}^2_{\mathrm{hom}}(\mathrm{J}(C)_B)$ to be the kernel of the natural map $\mathrm{CH}^2(\mathrm{J}(C)_B) \to \mathrm{H}^0(B, \mathrm{R}^4(f_B)_*\mathbb{Z}_\ell(2))$. There is an induced Abel-Jacobi map

$$\eta_B^2 : \mathrm{CH}^2_{\mathrm{hom}}(\mathrm{J}(C)_B) \to \mathrm{H}^1(B, \mathrm{R}^3(f_B)_*\mathbb{Z}_\ell(2))$$

which for $B = \mathrm{Spec}(L)$ coincides with map η_L^2 defined above. If $B = B_{\mathbb{C}}$ is a smooth complex variety, the same construction applied to the cycle map $\mathrm{CH}^2(\mathrm{J}(C)_{B_{\mathbb{C}}}) \to \mathrm{H}^4(\mathrm{J}(C)_{B_{\mathbb{C}}}, \mathbb{Z}(2))$ with values in singular cohomology gives rise to a similar map

$$\eta^2_{B^{\mathrm{an}}_{\mathbb{C}}} : \mathrm{CH}^2_{\mathrm{hom}}(\mathrm{J}(C)_{B_{\mathbb{C}}}) \to \mathrm{H}^1(B^{\mathrm{an}}_{\mathbb{C}}, \mathrm{R}^3(f^{\mathrm{an}}_{B_{\mathbb{C}}})_*\mathbb{Z}(2)).$$

Let $\mathrm{R}^1(f_B)_*\mathbb{Z}_\ell(1) \to \mathrm{R}^3(f_B)_*\mathbb{Z}_\ell(2)$ be the map induced by cup-product with the theta-divisor, and define $\mathrm{PR}^3(f_B)_*\mathbb{Z}_\ell(2)$ to be the cokernel of this map. Thus $\mathrm{PR}^3(f_B)_*\mathbb{Z}_\ell(2)$ is the local system on B corresponding the primitive 3rd cohomology of the fibers (with $\mathbb{Z}_\ell(2)$-coefficients). Similarly we have the local system $\mathrm{PR}^3(f^{\mathrm{an}}_{B_{\mathbb{C}}})_*\mathbb{Z}(2))$ in the analytic topology. The composition of the Abel-Jacobi map η_B^2 with the evident quotient map defines the map

$$\nu_B^2 : \mathrm{CH}^2_{\mathrm{hom}}(\mathrm{J}(C)_B) \to \mathrm{H}^1(B, \mathrm{PR}^3(f_B)_*\mathbb{Z}_\ell(2));$$

similarly we have $\nu^2_{B_{\mathbb{C}}}$ and $\nu^2_{B^{\mathrm{an}}_{\mathbb{C}}}$. We claim the following diagram commutes.

$$\begin{array}{ccc}
\mathrm{CH}^2_{\mathrm{hom}}(\mathrm{J}(C)_L) & \xrightarrow{\nu_L^2} & \mathrm{H}^1(G_L, \mathrm{PH}^3_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}_\ell(2))) \\
\uparrow & & \uparrow {\scriptstyle i^*} \\
\mathrm{CH}^2_{\mathrm{hom}}(\mathrm{J}(C)_T) & \xrightarrow{\nu_T^2} & \mathrm{H}^1(T, \mathrm{PR}^3(f_T)_*\mathbb{Z}_\ell(2)) \\
\downarrow & & {\scriptstyle \cong}\downarrow {\scriptstyle \gamma} \\
\mathrm{CH}^2_{\mathrm{hom}}(\mathrm{J}(C)_{T_{\mathbb{C}}}) & \xrightarrow{\nu^2_{T_{\mathbb{C}}}} & \mathrm{H}^1(T_{\mathbb{C}}, \mathrm{PR}^3(f_{T_{\mathbb{C}}})_*\mathbb{Z}_\ell(2)) \\
{\scriptstyle \nu^2_{T^{\mathrm{an}}_{\mathbb{C}}}}\downarrow & & {\scriptstyle \cong}\downarrow {\scriptstyle \delta} \\
\mathrm{H}^1(T^{\mathrm{an}}_{\mathbb{C}}, \mathrm{PR}^3(f^{\mathrm{an}}_{T_{\mathbb{C}}})_*\mathbb{Z}(2)) & \xrightarrow{\iota} & \mathrm{H}^1(T^{\mathrm{an}}_{\mathbb{C}}, \mathrm{PR}^3(f^{\mathrm{an}}_{T_{\mathbb{C}}})_*\mathbb{Z}(2)) \otimes \mathbb{Z}_\ell
\end{array} \tag{8}$$

Here the top square is induced from the center square in (1) and the middle square is the base change map induced from the inclusion $\overline{\mathbb{Q}} \to \mathbb{C}$ (recall that $T = \mathcal{C}_{\overline{\mathbb{Q}}}$ and $T_{\mathbb{C}} = \mathcal{C}_{\mathbb{C}}$ thought of as moduli spaces over $\overline{\mathbb{Q}}$ and $\mathbb{C}$

of curves with a marked point). These squares commute since the cycle maps ν^2_L, ν^2_T and $\nu^2_{T_\mathbb{C}}$ are functorial under pullback. By [1, Corollaire 1.6] étale cohomology is invariant under extensions of algebraically closed fields, which implies that the right vertical map γ is an isomorphism. The map δ in the bottom is an isomorphism by the comparison theorem between étale and analytic cohomology [1, Theoreme 4.1]. The map ι is the natural inclusion; this square commutes since the cycle classes of the total spaces in singular and ℓ-adic cohomology coincide under the comparison isomorphism, and the corresponding Leray spectral sequences are compatible with this map.

Consider the class of the Ceresa cycle $[\Xi_{T_\mathbb{C}}]$ in $\mathrm{CH}^2_{\mathrm{hom}}(\mathrm{J}(C)_{T_\mathbb{C}})$. It follows from Hain's proof of Ceresa's theorem that the image of the Ceresa cycle

$$\nu^2_{T^{\mathrm{an}}_\mathbb{C}}([\Xi_{T_\mathbb{C}}]) \in \mathrm{H}^1(T_\mathbb{C}, \mathrm{PR}^3(f^{\mathrm{an}}_{T_\mathbb{C}})_*\mathbb{Z}(2))$$

defines an element of infinite order. Indeed, consider $T_\mathbb{C}$ as the quotient $\mathfrak{X}_3/\Gamma(3)$, where $\mathfrak{X}_3$ is the appropriate Teichmüller space and $\Gamma(3)$ is the level 3 subgroup of the Mapping Class group. Since $\mathfrak{X}_3$ is simply connected and $\Gamma(3)$ acts freely and properly discontinuously, there is an isomorphism

$$\mathrm{H}^1(T^{\mathrm{an}}_\mathbb{C}, \mathrm{PR}^3(f^{\mathrm{an}}_{T_\mathbb{C}})_*\mathbb{Z}(2)) \cong \mathrm{H}^1(\Gamma(3), \mathrm{PR}^3(f^{\mathrm{an}}_{T_\mathbb{C}})_*\mathbb{Z}(2))).$$

It follows from [8, proof of Theorem 8.2] that the Ceresa cycle defines a class of infinite order in $\mathrm{H}^1(\Gamma(3), \mathrm{PR}^3(f^{\mathrm{an}}_{T_\mathbb{C}})_*\mathbb{Z}(2)))$ which coincides with the class $\nu^2_{T_\mathbb{C}}([\Xi_{T_\mathbb{C}}])$; see also [8, Corollary 10.4].

Assume now that S is a finite set primes such that for $\ell \notin S$ the conclusion of lemma 2.1 holds, i.e. $P_* \mathrm{CH}^2(\mathrm{J}(C)_{\overline{L}}[\ell] = 0$. Fix such a prime $\ell \notin S$ and assume there is a cycle $[\Xi'_\ell] \in \mathrm{CH}^2(\mathrm{J}(C)_{\overline{L}})$ with the property that

$$\ell \cdot [\Xi'_\ell] = [\Xi_{\overline{L}}] \text{ in } \mathrm{CH}^2_{\mathrm{hom}}(\mathrm{J}(C)_{\overline{L}}). \tag{9}$$

We show this implies $\nu^2_{T_\mathbb{C}}([\Xi_{T_\mathbb{C}}])$ is divisible by the prime ℓ. Since $\nu^2_{T_\mathbb{C}}([\Xi_{T_\mathbb{C}}])$ is an element of infinite order in a finitely generated abelian group, it is divisible only by a finite number of primes. Therefore $[\Xi_{\overline{L}}]$ is divisible by at most a finite number of primes ℓ, i.e. this will prove our theorem.

Assume (9). Since the cohomology of $\mathrm{J}(C)_{\overline{L}}$ is torsion free, it follows that $[\Xi'_\ell]$ is homologically trivial. Let L'/L be a finite Galois extension such that the cycle $[\Xi'_\ell]$ is defined over the field L', and consider $[\Xi'_\ell]$ as an element of the Chow group $\mathrm{CH}^2_{\mathrm{hom}}(\mathrm{J}(C)_{L'})$. If $\sigma \in \mathrm{Gal}(L'/L)$, by (9) and lemma 2.1,

$$P_*([\Xi'_l] - \sigma[\Xi'_l]) \in P_* \mathrm{CH}^2(\mathrm{J}(C)_{\overline{L}})[\ell] = 0$$

which implies

$$P_*[\Xi'_l] = P_*\sigma[\Xi'_l]. \tag{10}$$

Since the ℓ-adic Abel-Jacobi map is compatible with the action of correspondences, it follows from (10) that $\nu^2_{L'}([\Xi'_\ell])$ is invariant under the action of $\mathrm{Gal}(L'/L)$. The proof of [4, Proposition 4.1] gives an isomorphism

$$\mathrm{H}^1(G_L, \mathrm{PH}^3_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}_\ell(2))) \cong \mathrm{H}^1(G_{L'}, \mathrm{PH}^3_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}_\ell(2)))^{\mathrm{Gal}(L'/L)}$$

so that independently of the field of definition L' of $[\Xi'_\ell]$ we have the relation

$$\ell \cdot \nu^2_{L'}([\Xi'_\ell]) = \nu^2_L([\Xi_L]) \quad \text{in} \ \mathrm{H}^1(G_L, \mathrm{PH}^3_{\text{ét}}(\mathrm{J}(C)_{\overline{L}}, \mathbb{Z}_\ell(2))). \tag{11}$$

Consider the map i^* in (8). If $\emptyset \neq U \subset T$ is a Zariski open subset, the localisation sequence in étale cohomology induces an exact sequence

$$0 \to \mathrm{H}^1(T, \mathrm{PR}^3(f_T)_*\mathbb{Z}_\ell(2)) \to \mathrm{H}^1(U, \mathrm{PR}^3(f_U)_*\mathbb{Z}_\ell(2)) \to \oplus\, \mathrm{PH}^3_{\text{ét}}(\mathrm{J}(C)_t, \mathbb{Z}_\ell(1)),$$

where the sum is taken over a finite number of geometric points $t \in T \setminus U$. Note that the groups $\mathrm{PH}^3_{\text{ét}}(\mathrm{J}(C)_t, \mathbb{Z}_\ell(2))$ are torsion free. Taking the colimit over all such U shows that the map i^* in (8) is an injective map whose cokernel is torsion free. In particular, we can view (11) as

$$\ell \cdot \nu^2_{L'}([\Xi'_\ell]) = \nu^2_L([\Xi_L]) \ \ \text{in} \ \ \mathrm{H}^1(T, \mathrm{PR}^3(f_T)_*\mathbb{Z}_\ell(2)). \tag{12}$$

Let $[\Xi'_{\ell,T_\mathbb{C}}]$ denote the image of $[\Xi'_\ell]$ under the base change map induced by the inclusion $L' \to \mathbb{C}$, and let $[\Xi_{T_\mathbb{C}}]$ be the image of $[\Xi_L]$ under the map induced by $L \to \mathbb{C}$. By (12) we can consider $\ell \cdot \nu^2_{L'}([\Xi'_\ell])$ and $\nu^2_L([\Xi_L])$ as elements of $\mathrm{H}^1(T, \mathrm{PR}^3(f_T)_*\mathbb{Z}_\ell(2))$, which allows us to conclude that

$$\ell \cdot \nu^2_{T_\mathbb{C}}([\Xi'_{\ell,T_\mathbb{C}}] = (\delta \circ \gamma)(\ell \cdot \nu^2_{L'}([\Xi'_\ell])) = (\delta \circ \gamma)(\nu^2_L([\Xi_L])) = \nu^2_{T_\mathbb{C}}([\Xi_{T_\mathbb{C}}]) \tag{13}$$

in $\mathrm{H}^1(T_\mathbb{C}, \mathrm{PR}^3(f_{T_\mathbb{C}})_*\mathbb{Z}_\ell(2))$. To finish the proof, note that (13) combined with the lower square in (8) immediately implies the claimed 'integral' relation

$$\ell \cdot \nu^2_{T^{\text{an}}_\mathbb{C}}([\Xi'_{\ell,T_\mathbb{C}}]) = \nu^2_{T^{\text{an}}_\mathbb{C}}([\Xi_{T_\mathbb{C}}]) \ \text{in} \ \mathrm{H}^1(T^{\text{an}}_\mathbb{C}, \mathrm{PR}^3(f^{\text{an}}_{T_\mathbb{C}})_*\mathbb{Z}(2)). \tag{14}$$

□

3. Proof of Theorem 1.2

We prove Theorem 1.2 in a series of lemmas following the strategy of Nori's proof of the infinite generation of $\mathrm{Griff}^2(A) \otimes \mathbb{Q}$ for the generic complex abelian variety A of dimension 3. Instead of the isogenies used by Nori we work with modular correspondences which arise from Atkin-Lehner type of involutions coming from the structure of the underlying moduli; the construction of these correspondences is similar to [20].

Let X be the fine moduli space of principally polarised abelian varieties of dimension 3 with a full symplectic level 3 structure with respect to the Weil pairing and a fixed cube root of unity ζ, considered as an irreducible smooth complex variety. Let $F = \mathbb{C}(X)$ be the function field of X, and write $A = A_F$ for the generic fiber. Similar we have the moduli space M of curves of genus 3 with a full symplectic level 3 structure with respect to ζ, now viewed as an irreducible smooth complex variety. Let $E = \mathbb{C}(M)$ be the function field and $C = C_E$ the generic fiber. Then E/F is a quadratic field extension and there is a cartesian square (see, for example, [16, pg. 192, II])

$$\begin{array}{ccc} \mathrm{J}(C) & \rightarrow & A \\ \downarrow & & \downarrow \\ \mathrm{Spec}(E) & \rightarrow & \mathrm{Spec}(F) \end{array}$$

which implies $\mathrm{J}(C) \cong A_E$; in particular $\mathrm{J}(C)_{\overline{F}} \cong A_{\overline{F}}$ for a fixed algebraic closure $\overline{F}$ of F. Let $f_{\overline{F}} : \mathrm{J}(C)_{\overline{F}} \to A_{\overline{F}}$ denote this isomorphism. The Ceresa cycle $\Xi_{\overline{F}}$ on $J(C)_{\overline{F}}$ defines a class $[\Xi_{\overline{F}}]$ in $\mathrm{Griff}^2(\mathrm{J}(C)_{\overline{F}})$. Consider the cycle

$$[\Theta_{\overline{F}}] = f_{\overline{F}*}([\Xi'_{\overline{F}}]) \in \mathrm{Griff}^2(A_{\overline{F}}).$$

The proof of Theorem 1.1 shows that the image of $[\Theta_{\overline{F}}]$ in $\mathrm{Griff}^2(A_{\overline{F}})\otimes\mathbb{Z}/\ell$ is non-trivial for all but finitely many primes ℓ. We show that for any such ℓ the quotient $\mathrm{Griff}^2(A_{\overline{F}}) \otimes \mathbb{Z}/\ell$ is infinite. Since base change along the inclusion $\overline{F} \to \mathbb{C}$ induces an isomorphism on Chow groups modulo ℓ [11], this immediately implies Theorem 1.2.

For the remaining part of this section we fix a prime ℓ with the property that the image of $[\Theta_{\overline{F}}]$ is non-trivial in the Griffiths group $\mathrm{Griff}^2(A_{\overline{F}}) \otimes \mathbb{Z}/\ell$.

Lemma 3.1. *Let $p > 3$ be a prime, $p \neq \ell$. There exists an isomorphism*

$$\Gamma_p : \mathrm{Griff}^2(A_{\overline{F}}) \otimes \mathbb{Z}/\ell \to \mathrm{Griff}^2(A_{\overline{F}}) \otimes \mathbb{Z}/\ell.$$

Proof. Let $p > 3$ be a fixed prime, and let $X(3, p)$ be the fine moduli space of abelian varieties of dimension 3 with principal polarisation $\mathcal{L}$ (up to algebraic equivalence), a full symplectic level 3 structure with respect to a fixed cube root, and a subgroup of the p-torsion subgroup of order p^3 which is maximal isotropic for the $e_p^{\mathcal{L}}$-pairing induced by the principal polarisation $\mathcal{L}$.

Let $(A_+, [\mathcal{L}], P) \to X(3, p)$ be the universal polarised abelian variety with a distinguished subgroup of order p^3. Then A_+ is the pullback of A along the ‘forget map’ $X(3, p) \to X$. Consider the quotient map $q : A_+ \to A'_+ = A_+/P$. The abelian variety A'_+ is principally polarised in a unique way such that the pullback of the polarisation $[\mathcal{L}']$ on A'_+ to A_+ is $[\mathcal{L}^{\otimes p}]$. Note that $\mathcal{L}'$ is well-defined only up to algebraic equivalence; it depends on the choice of a character of the isotropic subgroup scheme P and is uniquely determined only up to tensoring by a torsion line bundle. Note also that for algebraically equivalent line bundles $\mathcal{L} \approx \widetilde{\mathcal{L}}$ and any integer m, we have $e_m^{\mathcal{L}} = e_m^{\widetilde{\mathcal{L}}}$.

The image of the full symplectic level 3 structure $\{u_1, u_2, u_3, v_1, v_2, v_3\}$ on A_+ under the quotient map $A_+ \to A'_+$ determines a full symplectic level 3 structure on A'_+, given by $\{-q(pv_1), -q(pv_2), -q(pv_3), q(u_1), q(u_2), q(u_3)\}$ (where pv_j denotes the p-th multiple of v_j for the group structure; this is just $\pm v_j$, since v_j has order 3). This slightly tricky definition of the level 3 structure on the quotient is made so as to be consistent with the formula (20) below.

Let P' be the image of the p-torsion subgroup scheme $A_+[p]$ under the quotient map q. Then P' is a subgroup scheme of $A'_+[p]$ of order p^3 which is isotropic for the $e_p^{\mathcal{L}'}$-pairing. In fact, for an integer $m \geq 1$ and x, y in $A_+[m]$ with images x', y' in $A'_+[m]$ we have from [14, pg. 228] the following formulae

$$e_m^{\mathcal{L}'}(x', y') = e_m^{\mathcal{L}^{\otimes m}}(x, y) = e_m^{\mathcal{L}}(x, my)$$

which imply that our prescription above does give a full symplectic level 3 structure for A'_+.

Thus the triple $(A'_+, [\mathcal{L}'], P')$ defines a point in the fine moduli $X(3,p)$ and there exists a unique morphism $\omega_p : X(3,p) \to X(3,p)$ such that A'_+ is the pullback of A_+ with respect to ω_p. Consider the commutative diagram

$$\begin{array}{ccccc} A_+ & \xrightarrow{q} & A'_+ & \xrightarrow{\omega'_p} & A_+ \\ \downarrow & & \downarrow & & \downarrow \\ X(3,p) & \xrightarrow{=} & X(3,p) & \xrightarrow{\omega_p} & X(3,p) \end{array} \tag{15}$$

where the right square is cartesian, and the top left map q is the canonical quotient map. Define $u_p : A_+ \to A_+$ to be the composition $\omega'_p \circ q$.

Applying the above 'quotient construction' to $(A'_+, [\mathcal{L}'], P')$, we obtain a triple $(A''_+, [\mathcal{L}''], P'')$, and an isogeny $q' : A'_+ \to A''_+$, together with a commutative diagram, with Cartesian squares

$$\begin{array}{ccccccc} A_+ & \xrightarrow{q'\circ q} & A''_+ & \xrightarrow{\omega''_p} & A'_+ & \xrightarrow{\omega'_p} & A_+ \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X(3,p) & \xrightarrow{=} & X(3,p) & \xrightarrow{\omega_p} & X(3,p) & \xrightarrow{\omega_p} & X(3,p) \end{array}$$

The kernel of the composite isogeny $q' \circ q : A_+ \to A''_+$ is the p-torsion of A_+, and so we may identify A''_+ with A_+, such that $q' \circ q$ is identified with the map $[p] : A_+ \to A_+$ given by multiplication by p. Further, the principal polarisation and the distinguished subgroup of A_+, obtained from the isomorphism with A''_+, coincide with the ones we started with, and the level 3 structure is that determined by $\{-pu_1, -pu_2, -pu_3, -pv_1, -pv_2, -pv_3\}$, which is isomorphic to the original one.

Since $X(3,p)$ is a fine moduli space, we must have that $\omega_p \circ \omega_p = \mathrm{id}$. Composing the quotient map $A_+ \to A_+/A_+[p]$ with the above isomorphism, we see that $u_p \circ u_p$ is multiplication by $\epsilon(p)p$, where $\epsilon(p) = -1$ if $p \equiv 1 \bmod 3$ and $\epsilon(p) = 1$ if $p \equiv -1 \bmod 3$.

Let N be a squarefree product of odd primes $p > 3$, where $(p, \ell) = 1$. Let $X(3,N)$ be the fine moduli space of principally polarised abelian varieties of dimension 3 with a distinguished subgroup of order N^3 which is maximal isotropic for the e_N-pairing. We have a natural isomorphism

$$X(3,N) \cong \underset{p|N}{\times} X(3,p),$$

where the fiber product is taken over X. Let $F_N = \mathbb{C}(X(3,N))$ be the complex function field of $X(3,N)$. For each p dividing N the pullback of the left square in (15) along the map $X(3,N) \to X(3,p)$ produces a diagram

$$\begin{array}{ccccc} A_+(N) & \xrightarrow{q,N} & A'_+(N) & \xrightarrow{\omega'_{p,N}} & A_+(N) \\ \downarrow & & \downarrow & & \downarrow \\ X(3,N) & \xrightarrow{=} & X(3,N) & \xrightarrow{\omega_{p,N}} & X(3,N) \end{array} \tag{16}$$

where $A_+(N) = A_+ \times_{X(3,p)} X(3,N)$, and $A'_+(N) = A'_+ \times_{X(3,p)} X(3,N)$. The morphism $\omega_{p,N} : X(3,N) \to X(3,N)$ is an involution, and $u_{p,N} = \omega'_{p,N} \circ q,N$ is an isogeny; these maps are compatible with the maps ω_p and u_p defined earlier.

If $N|N'$, the morphism $X(3,N') \to X(3,N)$ induces a field extension $F_N \subset F_{N'}$ which is compatible with the involutions $\omega_{p,N}$ and $\omega_{p,N'}$ in the sense that $\omega_{p,N'}$ on $F_{N'}$ induces $\omega_{p,N}$ on F_N. Let $F_\infty = \cup F_N$, where the union is taken over all squarefree odd $N > 3$ prime to ℓ. The $\omega_{p,N}$ give rise to a well-defined involution $\omega_p \in \mathrm{Aut}(F_\infty)$; let $\overline{\omega}_p$ be a fixed lifting of ω_p to an element of $\mathrm{Aut}(\overline{F})$. Thus we have for any p dividing N an endomorphism

$$\gamma_p = \gamma_{p,N} : A_{\overline{F}} = A_+(N) \times \mathrm{Spec}(\overline{F}) \overset{u_{p,N}\times\overline{\omega}_p}{\to} A_+(N) \times \mathrm{Spec}(\overline{F}) = A_{\overline{F}} \tag{17}$$

which is independent of the choice of N. From γ_p we obtain maps on Chow groups and étale cohomology groups which are compatible with the cycle map. In particular, pullback along γ_p induces a homomorphism

$$\Gamma_p = \gamma_p^* : \mathrm{Griff}^2_{\mathrm{hom}}(A_{\overline{F}}) \otimes \mathbb{Z}/\ell \to \mathrm{Griff}^2_{\mathrm{hom}}(A_{\overline{F}}) \otimes \mathbb{Z}/\ell.$$

Since $u_p \circ u_p = \epsilon(p)p$ on $A_{\overline{F}}$ and $\ell \neq p$, the proof of [20, Lemma 4.7] implies that Γ_p is an isomorphism. □

Consider the tower of field extensions $F \subset F_\infty \subset \overline{F}$. For N a squarefree product of odd primes $p > 3$ prime to ℓ, set $M(3,N) = M \times_X X(3,N)$, let E_N be its complex function field, and denote by $\mathrm{J}(C(N))$ the generic fiber. Thus E_N is a quadratic extension of F_N and there is a cartesian square

$$\begin{array}{ccc} \mathrm{J}(C(N)) & \to & A_+(N) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(E_N) & \to & \mathrm{Spec}(F_N) \end{array}$$

If $E_\infty = \cup_N E_N$, where the union is again taken over all squarefree odd $N > 3$ prime to ℓ, we obtain a quadratic extension E_∞/F_∞. Let χ is the composition $\mathrm{Gal}(\overline{F}/F_\infty) \to \mathrm{Gal}(E_\infty/F_\infty) = \{\pm 1\}$. Then the action of the Galois group $\mathrm{Gal}(\overline{F}/F_\infty)$ on the class $[\Theta_{\overline{F}}]$ is given by the following simple formula.

Lemma 3.2. *For all* $g \in \mathrm{Gal}(\overline{F}/F_\infty) : g \cdot [\Theta_{\overline{F}}] = \chi(g) \cdot [\Theta_{\overline{F}}]$.

Proof. Similar to [16, Proposition 1]. □

Let $\mathcal{P} = \{p \mid p \text{ prime} > 3,\ p \neq \ell\}$, so that each $p \in \mathcal{P}$ defines a non-trivial cycle $\Gamma_p([\Theta_{\overline{F}}])$. We show the set $\{\Gamma_p([\Theta_{\overline{F}}]) \mid p \in \mathcal{P}\} \subset \mathrm{Griff}^2(A_{\overline{F}}) \otimes \mathbb{Z}/\ell$ is linearly independent. For $p \in \mathcal{P}$ let $\overline{\omega_p} \in \mathrm{Aut}(\overline{F})$ be as in the proof of Lemma 3.1. Define

$$\chi^{\overline{\omega}_p}(g) = \chi(\overline{\omega}_p g \overline{\omega}_p^{-1}), \tag{18}$$

where $\chi : \mathrm{Gal}(\overline{F}/F_\infty) \to \mathrm{Gal}(E_\infty/F_\infty) = \{\pm 1\}$ is as above.

Lemma 3.3. *For $p \in \mathcal{P}$ and $g \in \mathrm{Gal}(\overline{F}/F_\infty)$:*

$$g \cdot \Gamma_p([\Theta_{\overline{F}}]) = \chi^{\overline{\omega}_p}(g) \cdot \Gamma_p([\Theta_{\overline{F}}]).$$

Proof. By definition $\Gamma_p = \gamma_p^*$, where $\gamma_p = u_p \times \overline{\omega}_p : A_{\overline{F}} \to A_{\overline{F}}$. Let $\widetilde{\gamma}_p = u_p \times \overline{\omega}_p^{-1} : A_{\overline{F}} \to A_{\overline{F}}$. The proof of Lemma 3.1 shows $u_p \circ u_p = \epsilon(p)p$, so that

$$\gamma_p g \widetilde{\gamma}_p = (u_p \times \overline{\omega}_p) \circ (\mathrm{id} \times g) \circ (u_p \times \overline{\omega}_p^{-1}) = (u_p \circ u_p) \times \overline{\omega}_p g \overline{\omega}_p^{-1} = \epsilon(p)p \times \overline{\omega}_p g \overline{\omega}_p^{-1}.$$

Since $p \neq \ell$ it follows from [5, §4, Proposition] that $\epsilon(p)p$ acts as the identity on $\mathrm{Griff}^2(A_{\overline{F}}) \otimes \mathbb{Z}/\ell$. In particular, the above equation applied with $g = \mathrm{id}$ shows that $\widetilde{\gamma}_p$ acts as the inverse to γ_p on $\mathrm{Griff}^2(A_{\overline{F}}) \otimes \mathbb{Z}/\ell$. Therefore

$$(\Gamma_p)^{-1} g \Gamma_p = (\gamma_p^*)^{-1} g \gamma_p^* = (\gamma_p g \widetilde{\gamma}_p)^* = (\overline{\omega}_p g \overline{\omega}_p^{-1})^*.$$

The claim follows from Lemma 3.2 by applying the last equation to $[\Theta_{\overline{F}}]$. □

The following lemma completes the proof of Theorem 1.2.

Lemma 3.4. *For $p \in \mathcal{P}$ the $\chi^{\overline{\omega}_p}$ are distinct characters of* $\mathrm{Gal}(\overline{F}/F_\infty)$.

Proof. We follow here the discussion in [16, in particular, pg. 194]. For an integer $N \geq 3$ consider the moduli space $X(N)$ of principally polarised complex abelian varieties of dimension 3 with a full symplectic level N structure. Here, we fix the standard complex primitive N-root of unity $e^{2\pi i/N}$ in considering full level structure.

For $N|N'$ there are natural maps $X(N') \to X(N)$. Thus there is a field $\widetilde{F}$ (denoted by F in [16]) obtained as the union of the fields $\mathbb{C}(X(N))$. If N is a product of distinct primes > 3, we also have a (finite) map $X(3N) \to X(3, N)$ determined by associating to an abelian variety with full symplectic level $3N$ structure $\{u_1, u_2, u_3, v_1, v_2, v_3\}$ the same abelian variety, with induced full symplectic level 3 structure, and the distinguished subgroup generated by $\{3v_1, 3v_2, 3v_3\}$. This is compatible with the natural maps obtained for $N|N'$, where N' is a square-free product of primes > 3.

Hence we may regard our field F_∞ as a subfield of the field $\widetilde{F}$ in a natural way, which is an algebraic extension, and thus identify $\overline{F}$ with the algebraic closure of $\widetilde{F}$. In particular, we may view $\mathrm{Gal}(\overline{F}/\widetilde{F})$ as a subgroup of $\mathrm{Gal}(\overline{F}/F_\infty)$, and so our character χ determines a character $\widetilde{\chi} : \mathrm{Gal}(\overline{F}/\widetilde{F}) \to \mathrm{Gal}(E_\infty \widetilde{F}/\widetilde{F}) = \{\pm 1\}$. This is the character considered by Nori in [16, pg. 193] (denoted there by χ). In a similar fashion, the characters $\chi^{\overline{\omega}_p}$ determine characters $\widetilde{\chi}^{\overline{\omega}_p}$ of $\mathrm{Gal}(\overline{F}/\widetilde{F})$ by restriction; it clearly suffices the prove that these characters are all distinct.

We may view $X(N)$ as the quotient of the Siegel half-space $\mathfrak{S}_3$ by the action of the principal congruence subgroup $\Gamma(N)$ of level N in $\mathrm{Sp}_6(\mathbb{Z})$. Let $\widetilde{\mathrm{Sp}}_6(\mathbb{R})$ be the subgroup of $\mathrm{GL}_6(\mathbb{R})$ generated by $\mathrm{Sp}_6(\mathbb{R})$ and the scalar matrices. Set $\widetilde{\mathrm{Sp}}_6(\mathbb{Q}) = \widetilde{\mathrm{Sp}}_6(\mathbb{R}) \cap \mathrm{GL}_6(\mathbb{Q})$. There is an action of $\widetilde{\mathrm{Sp}}_6(\mathbb{R})/\mathbb{R}^\times$ on $\mathfrak{S}_3$ and each $g \in \widetilde{\mathrm{Sp}}_6(\mathbb{Q}) \cap \mathrm{M}_6(\mathbb{Z})$ induces an automorphism $\rho_1(g)$ of $\mathrm{Spec}(\widetilde{F})$ and an endomorphism $\rho_2(g)$ of $A_{\widetilde{F}}$ which are compatible with the structure map $A_{\widetilde{F}} \to \operatorname{Spec}(\widetilde{F})$, see [16, pg. 194] (where our $\widetilde{F}$ is Nori's F). By abuse of notation we also write $j : \operatorname{Spec}(\overline{F}) \to \operatorname{Spec}(\widetilde{F})$ for the morphism corresponding to the given embedding $j : \widetilde{F} \to \overline{F}$. Consider the group

$$G = \{(\alpha, g) \in \operatorname{Aut}(\overline{F}) \times \widetilde{\operatorname{Sp}_6}(\mathbb{Q}) \mid \rho_1(g) \circ j = j \circ \alpha\}$$

which fits into the following exact sequence

$$1 \to \operatorname{Gal}(\overline{F}/\widetilde{F}) \to G \to \widetilde{\operatorname{Sp}_6}(\mathbb{Q}) \to 1.$$

Assume now $\{r_i\}_{i\in I} \subset \widetilde{\operatorname{Sp}_6}(\mathbb{Q})$ is a system of distinct coset-representatives of $\widetilde{\operatorname{Sp}_6}(\mathbb{Q})/\operatorname{Sp}_6(\mathbb{Z})$, and $s_i \in G$ is a lift of $r_i \in \widetilde{\operatorname{Sp}_6}(\mathbb{Q})$. For $\widetilde{\chi} : \operatorname{Gal}(\overline{F}/\widetilde{F}) \to \operatorname{Gal}(E_\infty \widetilde{F}/\widetilde{F}) = \{\pm 1\}$ as above (denoted χ in [16]), and $g \in \operatorname{Gal}(\overline{F}/\widetilde{F})$, set

$$\widetilde{\chi}^{s_i}(g) = \widetilde{\chi}(s_i g s_i^{-1}). \tag{19}$$

Nori has shown [16, pg. 195] that the above assumption on the $\{r_i\}$ implies that the corresponding $\widetilde{\chi}^{s_i}$ define distinct characters of $\operatorname{Gal}(\overline{F}/\widetilde{F})$. In particular, to prove the lemma, it suffices to show that for $p \in \mathcal{P}$ the characters $\widetilde{\chi}^{\omega_p}$ are of this form. This is done below, using some explicit calculations on the Siegel space $\mathfrak{S}_3$, and on the universal family over it.

We use the notations and conventions of [10]. Recall that the Siegel space $\mathfrak{S}_3$ is the space of complex 3×3 matrices Ω which are symmetric, and have positive definite imaginary part. The universal family of abelian 3-folds over $\mathfrak{S}_3$ has fiber over Ω equal to the abelian variety $A(\Omega) = \mathbb{C}^3/\Omega\mathbb{Z}^3 + \mathbb{Z}^3$. This is regarded as principally polarised, with the polarisation being given by the unimodular symplectic form on the lattice, determined by taking the 3 columns of Ω, followed by the 3 basis vectors of $\mathbb{Z}^3$, as a symplectic basis. Equivalently, the map $\mathbb{Z}^6 \to \mathbb{C}^3$ given by $\mathbb{Z}^6 = \mathbb{Z}^3 \oplus \mathbb{Z}^3 \to \Omega\mathbb{Z}^3 + \mathbb{Z}^3$ transports the standard symplectic form on $\mathbb{Z}^6 = \mathbb{Z}^3 \oplus \mathbb{Z}^3$ to the chosen one on the lattice.

The abelian variety $A(\Omega)$ may be endowed with a full symplectic level N structure as follows. The N-torsion subgroup of $A(\Omega)$ is the image of $\frac{1}{N}\Omega\mathbb{Z}^3 + \frac{1}{N}\mathbb{Z}^3$; the e_N-pairing is determined by $(\alpha, \beta) = e^{2\pi i N <\alpha,\beta>}$, where $\alpha, \beta \in \frac{1}{N}\Omega\mathbb{Z}^3 + \frac{1}{N}\mathbb{Z}^3$, and $<,>$ is the induced rational symplectic form on $\Omega\mathbb{Q}^3 + \mathbb{Q}^3 = (\Omega\mathbb{Z}^3 + \mathbb{Z}^3) \otimes \mathbb{Q}$. Now we see that if e_1, e_2, e_3 are the basis vectors of $\mathbb{Z}^3 \subset \mathbb{C}^3$, $u_i = \frac{1}{N}\Omega e_i$ and $v_i = \frac{1}{N} e_i$, then $\{u_1, u_2, u_3, v_1, v_2, v_3\}$ gives a symplectic basis for the N-torsion. This symplectic level N structure on $A(\Omega)$, for each $\Omega \in \mathfrak{S}_3$, determines a map $\mathfrak{S}_3 \to X(N)$, identifying $X(N)$ (as a complex space) as the quotient of $\mathfrak{S}_3$ by the principal congruence subgroup of level N, as noted earlier while reviewing Nori's constructions.

In particular, if $p \in \mathcal{P}$, the composite $\mathfrak{S}_3 \to X(3p) \to X(3, p)$ associates to each $\Omega \in \mathfrak{S}_3$ the choice of the full symplectic level 3 structure given above, together with the distinguished subgroup of order p^3 of $A(\Omega)[p]$ given by

$$P(\Omega) = \Omega\mathbb{Z}^3 + \frac{1}{p}\mathbb{Z}^3 (\operatorname{mod} \Omega\mathbb{Z}^3 + \mathbb{Z}^3).$$

Hence $\mathfrak{S}_3 \to X(3,p) \stackrel{\omega_p}{\to} X(3,p)$ takes $A(\Omega)$ to the quotient abelian variety $A(\Omega)/P(\Omega)$, endowed with an induced full symplectic level 3 structure (given by our earlier recipe), and a distinguished subgroup of order p^3.

On the other hand, the action of $\widetilde{\mathrm{Sp}}_6(\mathbb{R})$ on $\mathfrak{S}_3$ is given by the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\Omega) = (a\Omega + b)(c\Omega + d)^{-1},$$

where the components a, b, c, d in the above matrix in $\widetilde{\mathrm{Sp}}_6(\mathbb{R}) \subset \mathrm{M}_6(\mathbb{R})$ are elements of $\mathrm{M}_3(\mathbb{R})$ (compare [10], pg. 137). We associate to $p \in \mathcal{P}$ the matrix

$$\Upsilon(p) = \begin{bmatrix} 0 & I_3 \\ -pI_3 & 0 \end{bmatrix} \in \widetilde{\mathrm{Sp}}_6(\mathbb{Q}) \cap \mathrm{M}_6(\mathbb{Z}), \tag{20}$$

where I_3 denotes the 3×3 identity matrix. The matrix $\Upsilon(p)$ induces a holomorphic map on the Siegel space $\mathfrak{S}_3 \to \mathfrak{S}_3$ given by the formula

$$\Omega \mapsto -p^{-1}\Omega^{-1}.$$

This is visibly an involution on $\mathfrak{S}_3$, and is reminiscent of the formula for the analogous classical Atkin-Lehner involution associated to elliptic curves,

$$\tau \mapsto \frac{-1}{p\tau}.$$

We verify, by explicit calculation, that $\Upsilon(p)$ fits into a commutative diagram

$$\begin{array}{ccc} \mathfrak{S}_3 & \stackrel{\Upsilon(p)}{\longrightarrow} & \mathfrak{S}_3 \\ \downarrow & & \downarrow \\ X(3,p) & \stackrel{\omega_p}{\longrightarrow} & X(3,p) \end{array} \tag{21}$$

where the two vertical arrows are given by the map $\mathfrak{S}_3 \to X(3,p)$ described above, i.e. we show that the abelian varieties $A(\Omega)/P(\Omega)$ and $A(\Upsilon(p)(\Omega))$, together with the additional structure, are compatibly isomorphic.

The quotient variety $A(\Omega)/P(\Omega)$, as a complex abelian variety, is just

$$\mathbb{C}^3/\Omega\mathbb{Z}^3 + \frac{1}{p}\mathbb{Z}^3,$$

where the quotient map from $A(\Omega)$ is induced by the identity on $\mathbb{C}^3$. We may rescale, and write this quotient as

$$A(p\Omega) = \mathbb{C}^3/p\Omega\mathbb{Z}^3 + \mathbb{Z}^3;$$

now the quotient map $A(\Omega) \to A(p\Omega)$ is induced by multiplication by p on $\mathbb{C}^3$. With this rescaled choice, the pullback to $A(\Omega)$ of the principal polarisation of $A(p\Omega)$ coincides with p times the principal polarisation of $A(\Omega)$. Hence variety $A(p\Omega)$ is the "correct" quotient of $A(\Omega)$ by $P(\Omega)$, as a principally polarised abelian variety.

However, the "quotient" level 3 structure, and the distinguished subgroup of order p^3, as defined earlier, do not agree with the choices made for $A(p\Omega)$.

For example, the distinguished subgroup for the quotient structure is

$$\Omega\mathbb{Z}^3 + \mathbb{Z}^3 (\operatorname{mod} p\Omega\mathbb{Z}^3 + \mathbb{Z}^3),$$

while our chosen one for $A(p\Omega)$ is

$$p\Omega\mathbb{Z}^3 + \frac{1}{p}\mathbb{Z}^3 (\operatorname{mod} p\Omega\mathbb{Z}^3 + \mathbb{Z}^3).$$

Hence we need to find another point of $\mathfrak{S}_3$ which is $\mathrm{Sp}_6(\mathbb{Z})$-equivalent to $p\Omega$, so that for the corresponding abelian variety, these extra data are also compatible. This can be achieved by considering the action of the matrix

$$\begin{bmatrix} 0 & I_3 \\ -I_3 & 0 \end{bmatrix}$$

on Siegel space. The action of this matrix corresponds on the lattice to the operation $\{u_1, u_2, u_3, v_1, v_2, v_3\} \mapsto \{-v_1, -v_2, -v_3, u_1, u_2, u_3\}$ on the symplectic basis, thus we see that the corresponding abelian variety $A(-p^{-1}\Omega^{-1})$, which is isomorphic to $A(p\Omega) = A(\Omega)/P(\Omega)$ as a principally polarised abelian variety, also has a compatible distinguished subgroup of order p^3.

Next, we compute that the full level 3 structure are compatible as well: if $\{\bar{u_1}, \bar{u_2}, \bar{u_3}, \bar{v_1}, \bar{v_2}, \bar{v_3}\}$ are the images of the symplectic basis for 3-torsion under the quotient map $A(\Omega) \to A(\Upsilon(p)(\Omega))$, then $\{-p\bar{v_1}, -p\bar{v_2}, -p\bar{v_3}, \bar{u_1}, \bar{u_2}\bar{u_3}\}$ coincides with the chosen symplectic basis for 3-torsion in $A(\Upsilon(p)(\Omega))$. This is just the formula used to specify the "quotient" full symplectic level 3 structure in our definition of the map $\omega_p : X(3,p) \to X(3,p)$, thus $A(\Omega)/P(\Omega) \cong A(\Upsilon(p)(\Omega))$, compatibly with the additional data, as claimed.

Since $\Upsilon(p) \in \widetilde{\mathrm{Sp}_6}(\mathbb{Q}) \cap \mathrm{M}_6(\mathbb{Z})$, the matrix $\Upsilon(p)$ determines an automorphism $\widehat{\omega_p} = \rho_1(\Upsilon(p))$ of $\mathrm{Spec}(\widetilde{F})$. Let $\overline{\omega}'_p \in \mathrm{Aut}(\mathrm{Spec}(\overline{F}))$ be a fixed lift of $\widehat{\omega_p}$, compatible with the embedding $j : \widetilde{F} \to \overline{F}$. Then $s_p = (\overline{\omega}'_p, \Upsilon(p))$ defines a lift of $\Upsilon(p)$ to G. Consider the corresponding character $\widetilde{\chi}^{s_p}$. Because

$$s_p g s_p^{-1} = (\overline{\omega}'_p \times \Upsilon(p)) \circ (g \times \mathrm{id}) \circ ((\overline{\omega}'_p \times \Upsilon(p))^{-1}) = \overline{\omega}'_p g \overline{\omega}'^{-1}_p,$$

we have $\widetilde{\chi}^{s_p} = \widetilde{\chi}^{\overline{\omega}'_p}$ as characters of $\mathrm{Gal}(\overline{F}/\widetilde{F})$. For distinct primes $p, q \in \mathcal{P}$ the matrix $\Upsilon(p)\Upsilon(q)^{-1}$ is not contained in $\mathrm{Sp}_6(\mathbb{Z})$, thus the set $\{\Upsilon(p) \,|p \in \mathcal{P}\}$ defines a system of distinct coset representatives of $\widetilde{\mathrm{Sp}_6}(\mathbb{Q})/\mathrm{Sp}_6(\mathbb{Z})$, and the $\widetilde{\chi}^{s_p} = \widetilde{\chi}^{\overline{\omega}'_p}$ are distinct as characters of $\mathrm{Gal}(\overline{F}/\widetilde{F})$.

To complete the proof, we need to show that $\widetilde{\chi}^{\overline{\omega}_p} = \widetilde{\chi}^{\overline{\omega}'_p}$ as characters of $\mathrm{Gal}(\overline{F}/\widetilde{F})$. Indeed, the original character $\widetilde{\chi}$, of which both are twists, was obtained by restriction of a character χ of $\mathrm{Gal}(\overline{F}/F)$, corresponding to a quadratic extension $\mathbb{C}(M)$ of the field $F = \mathbb{C}(X)$. This quadratic extension may (by Kummer theory, say) be viewed as obtained by adjoining a square root of an element $h \in F^\times$ (determined up to $F^{\times 2}$); then χ is determined by the formula $\chi(g) = g(h)/h = \pm 1$. The twisted characters are then determined by adjoining square roots of $\overline{\omega}_p^{-1}(h)$ and $(\overline{\omega}'_p)^{-1}(h)$ respectively to $\widetilde{F}$. But from the diagram (21), both $\overline{\omega}_p$ and $\overline{\omega}'_p$ restrict to $\omega_p = \omega_p^{-1}$ on the subfield $\mathbb{C}(X(3,p) = F_p \subset \widetilde{F}$, and $h \in F \subset F_p$, so that $\overline{\varpi}_p^{-1}(h) = (\overline{\varpi}'_p)^{-1}(h)$ as elements of $\widetilde{F}^{\times}$, and so we are adjoining the same square root in both cases.

□

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Mathematisches Institut, Ludwigs-Maximilians Universität, München, Germany

E-mail address: axr@math.lmu.de

School of Mathematics, Tata Institute of Fundamental Research, Mumbai, India

E-mail address: srinivas@math.tifr.res.in